

ADM FORMULATION OF GENERIC MASSLESS SPIN-2 GRAVITY

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
MIDDLE EAST TECHNICAL UNIVERSITY

MÜSLÜM DOĞRU

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF MASTER OF SCIENCE
IN
PHYSICS

JUNE 2018

Approval of the thesis:

ADM FORMULATION OF GENERIC MASSLESS SPIN-2 GRAVITY

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ABSTRACT

ADM FORMULATION OF GENERIC MASSLESS SPIN-2 GRAVITY

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June 2018, 83 pages

We give a review of the article [1], which uses the Dirac constraint analysis and $3 + 1$ split of geometrodynamics[2] to create the Hamiltonian formulation of some gravitational actions. We apply the suggested methods to, Born-Infeld extension of New Massive Gravity, (BINMG) theory and find out the constraints and their classifications except for two constraints. The number of degrees of freedom in BINMG depends on the classification of these two constraints. We could not determine the classes of these two constraints but if some conditions, collected in chapter 4, are satisfied then BINMG is ghost-free at the nonlinear level. If these conditions are not satisfied then, BINMG has a third degree of freedom which does not appear at the linearized studies.

Keywords: ADM, Hamiltonian formulation, constraint analysis, geometrodynamics, BINMG

ÖZ

GENEL KÜTLESİZ SPİN 2 YERÇEKİMİ TEORİLERİNİN ADM FORMULASYONU

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Tez Yöneticisi : Prof. Dr. Bayram Tekin

Haziran 2018 , 83 sayfa

Dirac'ın kısıtlama analizini ve uzay-zamanın $3 + 1$ ayrışmasını[2] kullanan bir makalenin [1] incelemesi yapıp bazı gravitasyonel eylemlerin Hamilton formülasyonları bulundu. Bu makalede önerilen metodlar BINMG theorisine uygulayıp kısıtlamalar ve kısıtlamaların sınıflandırılması bulundu, sadece iki adet kısıtlama hariç. Dördüncü bölümde belirtilen bazı koşulların sağlanması halinde BINMG theorisinin sadece iki dinamik moda sahip olduğu gösterildi. Eğer bu koşulları sağlanmıyorsa BINMG theorisinin üç dinamik modu olduğu gösterildi.

Anahtar Kelimeler: ADM, Hamiltonian formülasyonu, kısıtlama analizi, uzay-zaman dinamiđi, BINMG

Aileme

ACKNOWLEDGMENTS

First of all, I want to thank my advisor Prof. Dr. Bayram Tekin whose classes I joined the most and enjoyed most of the time. His style of teaching, where he focuses on the principles more than the calculations makes the lectures always lasting and thought provoking. As for this thesis, which I do not like all that much, might be just the thing I needed, since after learning GR, differential geometry, Lie groups etc. rigourosly in our group meetings all I could think about my old notes was that they were sloppy. So, I think all this effort will lead me to become a more careful person in my future studies. The tools that I learned to complete the thesis are also very valuable.

I also wish to thank my family and friends who did not ignore me even when I ignored them. I value them the most even though I would never show them that.

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CHAPTER 1

INTRODUCTION

To make a quantum theory of an existing classical theory, one has two main methods. One can use the path integral formalism or one can convert the action into the Hamiltonian form and work in the phase space and use Dirac's correspondence principle. However, in some theories, gauge symmetry might lead to infinity when calculating the partition function. This is encountered in electromagnetism and the problem is solved by introducing gauge conditions. According to the works initiated by Dirac and Bergmann and later improved by other physicists, the gauge symmetry is a symmetry that exists in the constrained Hamiltonian systems [3]. Since quantum field theories usually have this type of symmetry, it is important to understand the ways to deal with them. In this thesis, we review an article [1], perform the calculations in the article and apply the methods suggested in the article to Born-Infeld Extension of New Massive Gravity which was worked in detail in [4],[5]. One minor difference will be that instead of the method suggested in the article [6], we will stick to the Dirac's methods [3]. Both of these methods give the same results as is well known [7]. We will refer to the theory as BINMG in short. In this thesis, the goal is to find the number of the degrees of freedom of the BINMG theory at the nonlinear level. The number of the degrees of freedom was previously calculated in [8] at the linear level.

1.1 Structure of the Thesis

There are some tools we need to understand really well to be able to use the methods given in the article [1]. Appendices A and B give short summaries for the tools needed. Appendix C gives some of the main ideas for Born-Infeld

extension of a theory. Once the tools are mastered, the thesis can be treated as a step by step application of Dirac's ideas in the context of constrained systems. The equations are really long and some of the steps are also cumbersome. We will not show all of the calculations explicitly. However, we will give the sample calculations that show the characteristics and difficulties of the step that is being performed. We will explain the notations as they are needed.

CHAPTER 2

$f(R_{\alpha\beta\mu\nu})$ ACTION AND THE METHOD OF AUXILIARY VARIABLES

We will find the Hamiltonian formulation for all actions that can be classified as $f(R_{\alpha\beta\mu\nu})$ action, that is

$$S[g_{\mu\nu}] = \frac{1}{2} \int_{\mathcal{M}} d^D x \sqrt{-g} f(R_{\alpha\beta\mu\nu}). \quad (2.1)$$

Now, for many gravitational actions this expression is going to be really simple. However, some actions including the BINMG action are not that simple and a straightforward application for finding the canonical Hamiltonian is not possible. This is due to the reason that we cannot invert the tensor relations of higher than first order. This inversion is necessary for replacing the time derivatives of configuration space variables with the phase space variables. One can opt to not invert them at all and make it work in some cases by working with algebraic expressions but it would be extremely laborious. So we use a well known method called auxiliary variables method as suggested in the article [1] to make a more amenable action. Let us describe this method first.

2.1 Auxiliary Variables Method

Let us suppose we are given a general classical field theory action:

$$S[\phi] = \int d^D x f[\Omega_i[\phi]] \quad (2.2)$$

where Ω_i is a function of the configuration field variables $\phi = \{\phi_1, \phi_2, \dots, \phi_n\}$ and f a function of the Ω_i . Let us also separate the collective index i into two groups of collective indices, namely $\{i\} = \{\mu\} \cup \{A\}$. Now, instead of the action (2.2) we will write an action with the configuration space variables $\{\phi\}, \{A_\mu\}, \{B_\mu\}$

which gives the same equations of motion.

$$S[\phi, A, B] = \int d^D x \left\{ f[\Omega_A[\phi], A_\mu] + B^\mu(\Omega_\mu - A_\mu) \right\}. \quad (2.3)$$

The equations of motion for (2.2) are given by

$$\begin{aligned} \delta S &= 0 \\ \frac{\delta f}{\delta \phi} &= \frac{\delta f}{\delta \Omega_i} \frac{\delta \Omega_i}{\delta \phi} = 0 \\ &= \frac{\delta f}{\delta \Omega_A} \frac{\delta \Omega_A}{\delta \phi} + \frac{\delta f}{\delta \Omega_\mu} \frac{\delta \Omega_\mu}{\delta \phi} = 0. \end{aligned} \quad (2.4)$$

Equations of motion for the action (2.3) are

$$\begin{aligned} \delta S &= 0 \\ \frac{\delta S}{\delta B^\mu} &= \Omega_\mu - A_\mu = 0 \\ \frac{\delta S}{\delta A^\mu} &= \frac{\delta f}{\delta A^\mu} - B_\mu = 0 \\ \frac{\delta S}{\delta \phi} &= \frac{\delta f}{\delta \Omega_A} \frac{\delta \Omega_A}{\delta \phi} + B^\mu \frac{\delta \Omega_\mu}{\delta \phi} = 0. \end{aligned} \quad (2.5)$$

These equations give the same equations of motion but allows us to deal with otherwise hard calculations. One interesting thing to note is that in applying the auxiliary variables method, we actually insert second class constraints into our theory. Later, we will use this method to get rid of a second order time derivative in the $f(R_{\alpha\beta\mu\nu})$ theory, by transferring the second derivative onto an auxiliary variable. Now, that sort of manipulation might turn some of the second class constraints into a first class constraint. Since, there is no consensus on how to quantize such a constrained Hamiltonian system that contains a first class constraint, one can expect that the action with the auxiliary variables might give a different quantum theory than the original action. We will come back to this warning later, when we do that manipulation.

2.2 $f(R_{\alpha\beta\mu\nu})$ Action and BINMG Action

We will study gravitational actions that can be written as

$$S[g_{\mu\nu}] = \frac{1}{2} \int_{\mathcal{M}} d^D x \sqrt{-g} f(R_{\alpha\beta\mu\nu}). \quad (2.6)$$

The equivalent action that uses the auxiliary fields and gives the same equations

of motion is

$$S[g_{\mu\nu}, A_{\mu\nu\rho\sigma}, B^{\mu\nu\rho\sigma}] = \frac{1}{2} \int_{\mathcal{M}} d^D x \sqrt{-g} \left[f(A_{\mu\nu\rho\sigma}) + B^{\mu\nu\rho\sigma} (R_{\mu\nu\rho\sigma} - A_{\mu\nu\rho\sigma}) \right]. \quad (2.7)$$

The equations of motion are given in [1]. Let us also give the BINMG action [5],[4] which is a 2 + 1 dimensional theory:

$$S_{BINMG} = \frac{-4m}{\kappa^2} \int_{\mathcal{M}} d^3 x \sqrt{-g} F(R, K, S) \quad (2.8)$$

where

$$F(R, K, S) = \sqrt{1 - \frac{\sigma}{2m^2} \left(R + \frac{\sigma}{2m^2} K - \frac{S}{12m^4} \right) - \left(1 - \frac{\lambda}{2} \right)},$$

$$R = g^{\mu\nu} R_{\mu\nu},$$

$$K = R_{\mu\nu} R^{\mu\nu} - \frac{R^2}{2},$$

$$S = 8R_{\mu\alpha} R_{\nu}^{\alpha} R^{\mu\nu} - 6RR_{\mu\nu} R^{\mu\nu} + R^3.$$

To have the same action with the article [5] we should choose $\sigma = -1$.

2.3 Decomposing the Action

Before any sort of Hamiltonian can be written, we must choose a direction the gradient of which must be a time-like one-form. This is not necessary in many theories where the time is explicitly given. In gravitation theories time is a part of space-time and the separation of time is a geometrical process. This is given as a summary in Appendix A, while the references given in [9], [10] deal with it in more detail. Decomposition of the action must be done in a case by case analysis and usually involves nothing more than multiplication. In this section we will decompose the action for BINMG. First let us explain a few notations that will be used throughout the thesis. We will deal with mainly three tensors in this section. These will be the Riemann tensor and two auxiliary variables explained previously. Since we will decompose the action, these tensors will be further separated into independent projections. To understand the notation at a single glance, we will use a notation slightly different from than the one in the

article [1]. We will write the undecomposed action as

$$S[g_{\mu\nu}, \mathcal{A}_{\mu\nu\rho\sigma}, \mathcal{B}^{\mu\nu\rho\sigma}] = \frac{1}{2} \int_{\mathcal{M}} d^D x \sqrt{-g} \left[f(\mathcal{A}_{\mu\nu\rho\sigma}) + \mathcal{B}^{\mu\nu\rho\sigma} (R_{\mu\nu\rho\sigma} - \mathcal{B}_{\mu\nu\rho\sigma}) \right]. \quad (2.9)$$

${}^{(D)}R_{\alpha\beta\mu\nu}$ will denote the Riemann tensor for the D dimensional space-time while ${}^{(D)}R_{\alpha\beta}$ and ${}^{(D)}R$ will denote the Ricci tensor and Ricci scalar respectively for the D dimensional space-time. We will change the superscript to (d) for the spacelike slices where $d = D - 1$. These superscripts $(D), (d)$ will only be used when the tensor in question might be confused with another tensor. The script font letter $\mathcal{R}_{\alpha\beta\mu\nu}$ will denote the part of the Riemann tensor fully projected onto the spacelike slices. Notice that this is the left-hand side of the equation, namely the Gauss relation, in (A.52). $\mathcal{R}_{\alpha\mu\nu}$ will denote the part of the Riemann tensor that is projected once onto the normal direction in the second index and three times onto spatial directions in the other indices. Note that it has three indices and it is the left-hand side of the Codazzi relation, (A.56). $\mathcal{R}_{\alpha\beta}$ will denote the part of the Riemann tensor that is projected twice onto the normal direction in the second and the fourth indices and twice projected onto the spacelike slices on the first and third indices. Notice that this is the left-hand side of (A.62). All functions of the Riemann tensor can be written in terms of the left-hand sides of the Gauss, Codazzi, Ricci relations and the normal vector to the space-like hypersurface. For the auxiliary fields, we will adopt the opposite style of notation, that is the script font versions will denote the unprojected tensors while normal versions will denote the projected versions. This is because we will use the projected parts of these tensors a lot. $\mathcal{A}_{\alpha\beta\mu\nu}, \mathcal{B}_{\alpha\beta\mu\nu}$ will denote the unprojected auxiliary fields. $A_{\alpha\beta\mu\nu}, B_{\alpha\beta\mu\nu}$ will denote the parts of the tensors \mathcal{A}, \mathcal{B} that are fully projected onto the spatial surface. $A_{\alpha\mu\nu}, B_{\alpha\mu\nu}$ will denote the parts of the tensors \mathcal{A}, \mathcal{B} that are once projected onto the normal direction in the second index and three times projected onto the spatial surface in the remaining indices. $A_{\alpha\beta}, B_{\alpha\beta}$ will denote the parts of the tensors \mathcal{A}, \mathcal{B} that are twice projected onto the normal direction in the second and fourth indices and two times projected onto the spatial surface in the remaining indices. Other than these three tensors we will not need to introduce any notation for the

decomposition. Let us give the relations for these notations.

$$A_{\alpha\beta\mu\nu} = \gamma^\rho_\alpha \gamma^\lambda_\beta \gamma^\sigma_\mu \gamma^\eta_\nu \mathcal{A}_{\rho\lambda\sigma\eta}, \quad (2.10)$$

$$A_{\alpha\mu\nu} = \gamma^\rho_\alpha n^\lambda \gamma^\sigma_\mu \gamma^\eta_\nu \mathcal{A}_{\rho\lambda\sigma\eta}, \quad (2.11)$$

$$A_{\alpha\beta} = \gamma^\rho_\alpha n^\lambda \gamma^\sigma_\beta n^\eta \mathcal{A}_{\rho\lambda\sigma\eta}, \quad (2.12)$$

$$\mathcal{R}_{\alpha\beta\mu\nu} = \gamma^\rho_\alpha \gamma^\lambda_\beta \gamma^\sigma_\mu \gamma^\eta_\nu R_{\rho\lambda\sigma\eta}, \quad (2.13)$$

$$\mathcal{R}_{\alpha\mu\nu} = \gamma^\rho_\alpha n^\lambda \gamma^\sigma_\mu \gamma^\eta_\nu R_{\rho\lambda\sigma\eta}, \quad (2.14)$$

$$\mathcal{R}_{\alpha\beta} = \gamma^\rho_\alpha n^\lambda \gamma^\sigma_\beta n^\eta R_{\rho\lambda\sigma\eta}, \quad (2.15)$$

where the same relations can be written for \mathcal{B}, B by changing them with the \mathcal{A}, A respectively. Notice that the left-hand side of the above equations are all spatial tensors.

2.3.1 Decomposition of the Riemann tensor

Riemann tensor has a decomposition in terms of $\mathcal{R}_{\alpha\beta\mu\nu}, \mathcal{R}_{\alpha\mu\nu}, \mathcal{R}_{\alpha\beta}, n_\alpha$. Let us find it.

$$\begin{aligned} R_{\mu\nu\rho\sigma} &= \delta_\mu^\alpha \delta_\nu^\beta \delta_\rho^\lambda \delta_\sigma^\eta R_{\alpha\beta\lambda\eta} \\ &= (\gamma^\alpha_\mu - n^\alpha n_\mu)(\gamma^\beta_\nu - n^\beta n_\nu)(\gamma^\lambda_\rho - n^\lambda n_\rho)(\gamma^\eta_\sigma - n^\eta n_\sigma) R_{\alpha\beta\lambda\eta}. \end{aligned} \quad (2.16)$$

Due to the symmetries of the Riemann tensor, we can only have the Riemann tensor contract with the normal vector at most two times. So, we will not include terms like $n^\alpha n_\mu n^\beta n_\nu n^\lambda n_\rho \gamma^\eta_\sigma R_{\alpha\beta\lambda\eta}$ since these terms are automatically zero. The result of the above multiplication is then

$$\begin{aligned} R_{\mu\nu\rho\sigma} &= [\gamma^\alpha_\mu \gamma^\beta_\nu \gamma^\lambda_\rho \gamma^\eta_\sigma - \gamma^\alpha_\mu \gamma^\beta_\nu n^\lambda n_\rho \gamma^\eta_\sigma - \gamma^\alpha_\mu \gamma^\beta_\nu \gamma^\lambda_\rho n^\eta n_\sigma + \gamma^\alpha_\mu \gamma^\beta_\nu n^\lambda n_\rho n^\eta n_\sigma \\ &\quad - \gamma^\alpha_\mu n^\beta n_\nu \gamma^\lambda_\rho \gamma^\eta_\sigma + \gamma^\alpha_\mu n^\beta n_\nu n^\lambda n_\rho \gamma^\eta_\sigma + \gamma^\alpha_\mu n^\beta n_\nu \gamma^\lambda_\rho n^\eta n_\sigma - n^\alpha n_\mu \gamma^\beta_\nu \gamma^\lambda_\rho \gamma^\eta_\sigma \\ &\quad + n^\alpha n_\mu \gamma^\beta_\nu n^\lambda n_\rho \gamma^\eta_\sigma + n^\alpha n_\mu \gamma^\beta_\nu \gamma^\lambda_\rho n^\eta n_\sigma + n^\alpha n_\mu n^\beta n_\nu \gamma^\lambda_\rho \gamma^\eta_\sigma] R_{\alpha\beta\lambda\eta}, \end{aligned} \quad (2.17)$$

where the fourth and eleventh terms are again zero since the Riemann tensor is antisymmetric in its first and the last two indices. Using the symmetries of the Riemann tensor, making some dummy index changes and using the short notations (2.13), (2.14), (2.15), one can reach the result:

$$\begin{aligned} R_{\mu\nu\rho\sigma} &= \mathcal{R}_{\mu\nu\rho\sigma} - n_\rho \mathcal{R}_{\sigma\nu\mu} - n_\sigma \mathcal{R}_{\rho\mu\nu} - n_\nu \mathcal{R}_{\mu\rho\sigma} - n_\nu n_\rho \mathcal{R}_{\mu\sigma} \\ &\quad + n_\nu n_\sigma \mathcal{R}_{\mu\rho} + n_\mu \mathcal{R}_{\nu\rho\sigma} + n_\mu n_\rho \mathcal{R}_{\nu\sigma} - n_\mu n_\sigma \mathcal{R}_{\nu\rho}. \end{aligned} \quad (2.18)$$

Any tensor showing the symmetries of the Riemann tensor which includes the auxiliary fields that we introduced in (2.9) can be decomposed as in the last equation. One simply changes R, \mathcal{R} with \mathcal{A}, A or \mathcal{B}, B respectively.

2.4 Decomposition of $\mathcal{B}^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma}$

This is simply a multiplication and using the spatial property of projected parts and the relation $n_\alpha n^\alpha = -1$ one gets:

$$\begin{aligned}
R_{\mu\nu\rho\sigma} \mathcal{B}^{\mu\nu\rho\sigma} &= [\mathcal{R}_{\mu\nu\rho\sigma} - n_\rho \mathcal{R}_{\sigma\nu\mu} - n_\sigma \mathcal{R}_{\rho\mu\nu} - n_\nu \mathcal{R}_{\mu\rho\sigma} - n_\nu n_\rho \mathcal{R}_{\mu\sigma} \\
&\quad + n_\nu n_\sigma \mathcal{R}_{\mu\rho} + n_\mu \mathcal{R}_{\nu\rho\sigma} + n_\mu n_\rho \mathcal{R}_{\nu\sigma} - n_\mu n_\sigma \mathcal{R}_{\nu\rho}] \\
&\quad \times [B^{\mu\nu\rho\sigma} - n^\rho B^{\sigma\nu\mu} - n^\sigma B^{\rho\mu\nu} - n^\nu B^{\mu\rho\sigma} - n^\nu n^\rho B^{\mu\sigma} \\
&\quad + n^\nu n^\sigma B^{\mu\rho} + n^\mu B^{\nu\rho\sigma} + n^\mu n^\rho B^{\nu\sigma} - n^\mu n^\sigma B^{\nu\rho}] \\
&= \mathcal{R}_{\mu\nu\rho\sigma} B^{\mu\nu\rho\sigma} - 4\mathcal{R}_{\mu\nu\rho} B^{\mu\nu\rho} + 4\mathcal{R}_{\mu\nu} B^{\mu\nu}. \tag{2.19}
\end{aligned}$$

This part completes the decomposition of the second part of the action

$$\begin{aligned}
\mathcal{B}^{\mu\nu\rho\sigma} (R_{\mu\nu\rho\sigma} - \mathcal{A}_{\mu\nu\rho\sigma}) &= B^{\mu\nu\rho\sigma} (\mathcal{R}_{\mu\nu\rho\sigma} - A_{\mu\nu\rho\sigma}) - 4B^{\mu\nu\rho} (\mathcal{R}_{\mu\nu\rho} - A_{\mu\nu\rho}) \\
&\quad + 4B^{\mu\nu} (\mathcal{R}_{\mu\nu} - A_{\mu\nu}). \tag{2.20}
\end{aligned}$$

2.5 Decomposition of $f(R_{\alpha\beta\mu\nu})$

Our action involves the functional $f(\mathcal{A}_{\alpha\beta\mu\nu}) = f(R_{\alpha\beta\mu\nu})|_{R_{\alpha\beta\mu\nu} = \mathcal{A}_{\alpha\beta\mu\nu}}$. Now, $f(R_{\alpha\beta\mu\nu})$ involves the Ricci scalar R , the square of the Ricci tensor $R_{\mu\nu} R^{\mu\nu} \equiv (R_{\mu\nu})^2$ and the cube of the Ricci tensor $R_{\mu\alpha} R_\nu^\alpha R^{\mu\nu} \equiv (R_{\mu\nu})^3$ in BINMG. So, we will decompose these elements and then replace $\mathcal{R}_{\alpha\beta\mu\nu} \rightarrow A_{\alpha\beta\mu\nu}$, $\mathcal{R}_{\alpha\mu\nu} \rightarrow A_{\alpha\mu\nu}$, $\mathcal{R}_{\alpha\beta} \rightarrow A_{\alpha\beta}$ to obtain $f(\mathcal{A}_{\alpha\beta\mu\nu})$ decomposition for BINMG in (2.9). Let us decompose the Ricci scalar first.

$$R = g^{\mu\rho} g^{\nu\sigma} R_{\mu\nu\rho\sigma} \tag{2.21}$$

$$\begin{aligned}
&= (\gamma^{\mu\rho} - n^\mu n^\rho) (\gamma^{\nu\sigma} - n^\nu n^\sigma) [\mathcal{R}_{\mu\nu\rho\sigma} - n_\rho \mathcal{R}_{\sigma\nu\mu} - n_\sigma \mathcal{R}_{\rho\mu\nu} - n_\nu \mathcal{R}_{\mu\rho\sigma} \\
&\quad - n_\nu n_\rho \mathcal{R}_{\mu\sigma} + n_\nu n_\sigma \mathcal{R}_{\mu\rho} + n_\mu \mathcal{R}_{\nu\rho\sigma} + n_\mu n_\rho \mathcal{R}_{\nu\sigma} - n_\mu n_\sigma \mathcal{R}_{\nu\rho}] \\
&= \gamma^{\mu\rho} \gamma^{\nu\sigma} \mathcal{R}_{\mu\nu\rho\sigma} - \gamma^{\mu\rho} n^\nu n^\sigma n_\nu n_\sigma \mathcal{R}_{\mu\rho} - \gamma^{\nu\sigma} n^\mu n^\rho n_\mu n_\rho \mathcal{R}_{\nu\sigma} \\
&= \gamma^{\mu\rho} \gamma^{\nu\sigma} \mathcal{R}_{\mu\nu\rho\sigma} - 2\gamma^{\mu\rho} \mathcal{R}_{\mu\rho} \\
&= \mathcal{R}^{\rho\sigma}{}_{\rho\sigma} - 2\mathcal{R}^\rho{}_\rho \tag{2.22}
\end{aligned}$$

$$= \mathcal{R}^i{}_{ij} - 2\mathcal{R}^i{}_i. \quad (2.23)$$

The last expression is the decomposition of the Ricci scalar. We are able to write the last expression with only the spatial indices because we are using coordinates compatible with the foliation and all the involved tensors in the expression are spatial. Let us decompose the Ricci tensor.

$$\begin{aligned} R_{\nu\sigma} &= g^{\mu\rho} R_{\mu\nu\rho\sigma} \quad (2.24) \\ &= (\gamma^{\mu\rho} - n^\mu n^\rho) [\mathcal{R}_{\mu\nu\rho\sigma} - n_\rho \mathcal{R}_{\sigma\nu\mu} - n_\sigma \mathcal{R}_{\rho\mu\nu} - n_\nu \mathcal{R}_{\mu\rho\sigma} \\ &\quad - n_\nu n_\rho \mathcal{R}_{\mu\sigma} + n_\nu n_\sigma \mathcal{R}_{\mu\rho} + n_\mu \mathcal{R}_{\nu\rho\sigma} + n_\mu n_\rho \mathcal{R}_{\nu\sigma} - n_\mu n_\sigma \mathcal{R}_{\nu\rho}] \\ &= \gamma^{\mu\rho} \mathcal{R}_{\mu\nu\rho\sigma} - \mathcal{R}_{\nu\sigma} - n_\nu \gamma^{\mu\rho} \mathcal{R}_{\mu\rho\sigma} - n_\sigma \gamma^{\mu\rho} \mathcal{R}_{\rho\mu\nu} + n_\nu n_\sigma \gamma^{\mu\rho} \mathcal{R}_{\mu\rho} \\ &= \mathcal{R}^\rho{}_{\nu\rho\sigma} - \mathcal{R}_{\nu\sigma} - n_\nu \mathcal{R}^\rho{}_{\rho\sigma} - n_\sigma \mathcal{R}^\mu{}_{\mu\nu} + n_\nu n_\sigma \mathcal{R}^\rho{}_{\rho}. \end{aligned} \quad (2.25)$$

From the decomposition of Ricci tensor, we can find the decomposition of other ingredients, $(R_{\mu\nu}{}^2)$ and $(R_{\mu\nu}{}^3)$. Let us also explicitly show the $(R_{\mu\nu}{}^2)$ decomposition.

$$R^{\nu\sigma} = \gamma^{\mu\rho} \mathcal{R}_{\mu}{}^{\nu}{}_{\rho}{}^{\sigma} - \mathcal{R}^{\nu\sigma} - n^\nu \gamma^{\mu\rho} \mathcal{R}_{\mu\rho}{}^{\sigma} - n^\sigma \gamma^{\mu\rho} \mathcal{R}_{\rho\mu}{}^{\nu} + n^\nu n^\sigma \gamma^{\mu\rho} \mathcal{R}_{\mu\rho} \quad (2.26)$$

$$\begin{aligned} R^{\nu\sigma} R_{\nu\sigma} &= [\gamma^{\mu\rho} \mathcal{R}_{\mu}{}^{\nu}{}_{\rho}{}^{\sigma} - \mathcal{R}^{\nu\sigma} - n^\nu \gamma^{\mu\rho} \mathcal{R}_{\mu\rho}{}^{\sigma} - n^\sigma \gamma^{\mu\rho} \mathcal{R}_{\rho\mu}{}^{\nu} + n^\nu n^\sigma \gamma^{\mu\rho} \mathcal{R}_{\mu\rho}] \\ &\quad \times [\gamma^{\alpha\beta} \mathcal{R}_{\alpha\nu\beta\sigma} - \mathcal{R}_{\nu\sigma} - n_\nu \gamma^{\alpha\beta} \mathcal{R}_{\alpha\beta\sigma} - n_\sigma \gamma^{\alpha\beta} \mathcal{R}_{\beta\alpha\nu} + n_\nu n_\sigma \gamma^{\alpha\beta} \mathcal{R}_{\alpha\beta}], \end{aligned} \quad (2.27)$$

using spatial property of projected parts, some dummy index changes and the identity $n_\alpha n^\alpha = -1$ one finds

$$\begin{aligned} &= \mathcal{R}^\rho{}_{\nu\rho\sigma} \mathcal{R}^{\beta\nu}{}_{\beta}{}^{\sigma} - 2\mathcal{R}^{\nu\sigma} \mathcal{R}^\mu{}_{\nu\mu\sigma} - 2\mathcal{R}^\rho{}_{\rho\nu} \mathcal{R}^{\beta}{}_{\beta}{}^{\nu} + \mathcal{R}^{\nu\sigma} \mathcal{R}_{\nu\sigma} \\ &+ \mathcal{R}^\mu{}_{\mu} \mathcal{R}^\alpha{}_{\alpha}. \end{aligned} \quad (2.28)$$

One can change all the contracted indices into spatial indices since we are using coordinates adapted to the foliation and all the involved tensors are spatial. Now, only $(R_{\mu\nu}{}^3)$ remains. Its calculation is also as straightforward as it was for the others. So, we will only give the result of this multiplication which comprises 18 terms.

$$\begin{aligned} (R_{\mu\nu}{}^3) &= R_1 - R_2 - R_3 - R_4 + R_5 - R_6 + R_7 + R_8 + R_9 - R_{10} \\ &\quad - R_{11} + R_{12} + R_{13} - R_{14} + R_{15} + R_{16} + R_{17} - R_{18}, \end{aligned} \quad (2.29)$$

where

$$R_1 = \mathcal{R}^{\eta\sigma}{}_{\eta}{}^{\alpha} \mathcal{R}^{\rho\nu}{}_{\rho\alpha} \mathcal{R}^{\beta}{}_{\nu\beta\sigma} \quad R_{10} = \mathcal{R}^{\sigma\alpha} \mathcal{R}^\nu{}_{\alpha} \mathcal{R}_{\nu\sigma}$$

$$\begin{aligned}
R_2 &= \mathcal{R}^{\theta\sigma}{}_{\theta}{}^{\alpha} \mathcal{R}^{\rho\nu}{}_{\rho\alpha} \mathcal{R}_{\nu\sigma} & R_{11} &= \mathcal{R}^{\mu}{}_{\mu}{}^{\nu} \mathcal{R}^{\lambda}{}_{\nu\lambda\sigma} \mathcal{R}^{\theta}{}_{\theta}{}^{\sigma} \\
R_3 &= \mathcal{R}^{\theta\sigma}{}_{\theta}{}^{\alpha} \mathcal{R}^{\mu}{}_{\mu\alpha} \mathcal{R}^{\lambda}{}_{\lambda\sigma} & R_{12} &= \mathcal{R}^{\theta}{}_{\theta}{}^{\sigma} \mathcal{R}^{\mu}{}_{\mu\nu} \mathcal{R}^{\nu}{}_{\sigma} \\
R_4 &= \mathcal{R}^{\theta\sigma}{}_{\theta}{}^{\alpha} \mathcal{R}^{\lambda}{}_{\nu\lambda\sigma} \mathcal{R}^{\nu}{}_{\alpha} & R_{13} &= \mathcal{R}^{\theta}{}_{\theta}{}^{\sigma} \mathcal{R}^{\mu}{}_{\mu} \mathcal{R}^{\lambda}{}_{\lambda\sigma} \\
R_5 &= \mathcal{R}^{\theta\sigma}{}_{\theta}{}^{\alpha} \mathcal{R}^{\nu}{}_{\alpha} \mathcal{R}_{\nu\sigma} & R_{14} &= \mathcal{R}^{\theta}{}_{\theta}{}^{\alpha} \mathcal{R}^{\mu\nu}{}_{\mu\alpha} \mathcal{R}^{\lambda}{}_{\lambda\nu} \\
R_6 &= \mathcal{R}^{\sigma\alpha} \mathcal{R}^{\mu\nu}{}_{\mu\alpha} \mathcal{R}^{\lambda}{}_{\nu\lambda\sigma} & R_{15} &= \mathcal{R}^{\theta}{}_{\theta}{}^{\alpha} \mathcal{R}^{\mu}{}_{\mu\alpha} \mathcal{R}^{\lambda}{}_{\lambda} \\
R_7 &= \mathcal{R}^{\sigma\alpha} \mathcal{R}^{\mu\nu}{}_{\mu\alpha} \mathcal{R}_{\nu\sigma} & R_{16} &= \mathcal{R}^{\theta}{}_{\theta\alpha} \mathcal{R}^{\nu}{}_{\alpha} \mathcal{R}^{\lambda}{}_{\lambda\nu} \\
R_8 &= \mathcal{R}^{\mu}{}_{\mu\alpha} \mathcal{R}^{\sigma\alpha} \mathcal{R}^{\lambda}{}_{\lambda\sigma} & R_{17} &= \mathcal{R}^{\mu}{}_{\mu}{}^{\nu} \mathcal{R}^{\lambda}{}_{\lambda\nu} \mathcal{R}^{\theta}{}_{\theta}{}^{\sigma} \\
R_9 &= \mathcal{R}^{\lambda}{}_{\nu\lambda\sigma} \mathcal{R}^{\nu}{}_{\alpha} \mathcal{R}^{\sigma\alpha} & R_{18} &= \mathcal{R}^{\mu}{}_{\mu} \mathcal{R}^{\alpha}{}_{\alpha} \mathcal{R}^{\beta}{}_{\beta}.
\end{aligned}$$

Again, we can change the contracted indices into spatial indices since we are using coordinates compatible with the foliation and the spatial tensors. Throughout this thesis some of the more tedious calculations will involve up to twenty or more terms. We will deal with those calculations term by term so that the calculation proceeds without mistakes. There are only a few quantities that require this scheme (including the first order action and the total Hamiltonian). So, as a notation for the terms we will use Latin letters R, I, H denoting the quantities ($R_{\mu\nu}{}^3$), first order action and the total Hamiltonian, followed by a subscript number indicating the order of the term.

CHAPTER 3

FIRST ORDER LAGRANGIAN, HAMILTONIANS AND PRIMARY CONSTRAINTS

3.1 First Order Lagrangian

Now, if one examines the decomposition of the Riemann tensor, it can be seen that it contains the time derivative of the extrinsic curvature (A.62). Since, the extrinsic curvature is already first order in time derivatives, it is to our advantage that we get rid of the second order time derivatives if we can. Let us separate the action into two parts.

$$\begin{aligned}
 S[g_{\mu\nu}, \mathcal{A}_{\mu\nu\rho\sigma}, \mathcal{B}^{\mu\nu\rho\sigma}] &= \frac{1}{2} \int_{\mathcal{M}} d^D x N \sqrt{\gamma} [f(\mathcal{A}_{\mu\nu\rho\sigma}) + B^{\mu\nu\rho\sigma} \mathcal{R}_{\mu\nu\rho\sigma} \\
 &\quad - 4B^{\mu\alpha\beta} \mathcal{R}_{\mu\alpha\beta} + 4B^{\mu\nu} \mathcal{R}_{\mu\nu} - B^{\mu\nu\rho\sigma} A_{\mu\nu\rho\sigma} \\
 &\quad + 4B^{\mu\alpha\beta} A_{\mu\alpha\beta} - 4B^{\mu\nu} A_{\mu\nu}] \\
 &= S_1 + S_2,
 \end{aligned} \tag{3.1}$$

where

$$\begin{aligned}
 S_1 &= \frac{1}{2} \int_{\mathcal{M}} d^D x N \sqrt{\gamma} [f(\mathcal{A}_{\mu\nu\rho\sigma}) + B^{\mu\nu\rho\sigma} \mathcal{R}_{\mu\nu\rho\sigma} \\
 &\quad - 4B^{\mu\alpha\beta} \mathcal{R}_{\mu\alpha\beta} - B^{\mu\nu\rho\sigma} A_{\mu\nu\rho\sigma} + 4B^{\mu\alpha\beta} A_{\mu\alpha\beta} \\
 &\quad - 4B^{\mu\nu} A_{\mu\nu}],
 \end{aligned} \tag{3.2}$$

$$S_2 = 2 \int_{\mathcal{M}} d^D x N \sqrt{\gamma} B^{\mu\nu} \mathcal{R}_{\mu\nu}. \tag{3.3}$$

We will transfer the time derivative of extrinsic curvature to auxiliary variable $B_{\mu\nu}$ in the integral (3.3) using Leibniz's rule. The integrand in (3.3) is a contraction of spatial tensors. So, we can reduce the dummy index to only spatial indices, since we are using coordinates compatible with the foliation. Then

$$B^{\mu\nu} \mathcal{R}_{\mu\nu} = B^{ij} \mathcal{R}_{ij} \tag{3.4}$$

$$\mathcal{R}_{ij} = \frac{1}{N} {}^{(D)}\mathcal{L}_{\vec{m}} K_{ij} + \frac{1}{N} D_i D_j N + K_{im} K^m{}_j, \quad (3.5)$$

$$\frac{1}{N} B^{ij(D)} \mathcal{L}_{\vec{m}} K_{ij} = \frac{1}{N} {}^{(D)}\mathcal{L}_{\vec{m}} (K_{ij} B^{ij}) - \frac{1}{N} K_{ij} {}^{(D)}\mathcal{L}_{\vec{m}} B^{ij} \quad (3.6)$$

$$\begin{aligned} &= \nabla_{\vec{n}} (K_{ij} B^{ij}) - \frac{K_{ij}}{N} ({}^{(D)}\mathcal{L}_{\vec{\partial}_i} B^{ij} - {}^{(D)}\mathcal{L}_{\vec{\beta}} B^{ij}) \\ &= \nabla_{\mu} (n^{\mu} K_{ij} B^{ij}) - K_{ij} B^{ij} \nabla_{\mu} n^{\mu} \\ &\quad - \frac{K_{ij}}{N} \dot{B}^{ij} + \frac{K_{ij}{}^{(d)}}{N} \mathcal{L}_{\vec{\beta}} B^{ij}, \\ \frac{1}{N} B^{ij(D)} \mathcal{L}_{\vec{m}} K_{ij} &= \nabla_{\mu} (n^{\mu} K_{ij} B^{ij}) + K_{ij} B^{ij} K \\ &\quad - \frac{K_{ij}}{N} \dot{B}^{ij} + \frac{K_{ij}{}^{(d)}}{N} \mathcal{L}_{\vec{\beta}} B^{ij}, \end{aligned} \quad (3.7)$$

where we used the product rule for derivatives and the fact that in coordinates compatible with the foliation

$${}^{(D)}\mathcal{L}_{\vec{\beta}} T_{ij} = {}^{(d)}\mathcal{L}_{\vec{\beta}} T_{ij} \quad (3.8)$$

$${}^{(D)}\mathcal{L}_{\vec{\beta}} T^{ij} = {}^{(d)}\mathcal{L}_{\vec{\beta}} T^{ij} \text{ for a spatial tensor } T. \quad (3.9)$$

By using these equations we transferred the time derivative of the extrinsic curvature to B^{ij} . Now, the integral (3.3) becomes

$$\begin{aligned} S_2 = 2 \int_{\mathcal{M}} d^D x N \sqrt{\gamma} &\left(\frac{D_i D_j N}{N} B^{ij} + K_{im} K^m{}_j B^{ij} + \nabla_{\mu} (n^{\mu} K_{ij} B^{ij}) \right. \\ &\left. + K K_{ij} B^{ij} - \frac{K_{ij}}{N} \dot{B}^{ij} + \frac{K_{ij}{}^{(d)}}{N} \mathcal{L}_{\vec{\beta}} B^{ij} \right), \end{aligned} \quad (3.10)$$

where the third term is a total divergence and since the variations at the boundary vanish, it is fine to drop it out.

$$\begin{aligned} S_{2b} = 2 \int_{\mathcal{M}} d^D x N \sqrt{\gamma} &\left(\frac{D_i D_j N}{N} B^{ij} + K_{im} K^m{}_j B^{ij} \right. \\ &\left. + K K_{ij} B^{ij} - \frac{K_{ij}}{N} \dot{B}^{ij} + \frac{K_{ij}{}^{(d)}}{N} \mathcal{L}_{\vec{\beta}} B^{ij} \right), \end{aligned} \quad (3.11)$$

$$S_b = 2 \int_{\mathcal{M}} d^D x N \sqrt{\gamma} \nabla_{\mu} (n^{\mu} K_{ij} B^{ij}), \quad (3.12)$$

$$S_2 = S_{2b} + S_b, \quad (3.13)$$

$$I = S_1 + S_{2b}. \quad (3.14)$$

Here I denotes the action with the boundary term discarded. The action I is first order in time derivatives and we can start finding the conjugate momentum densities.

3.2 Conjugate Momenta and the Primary Constraints

We have 9 configuration space variables. These are $\{\gamma_{ij}, N, \beta_i, A_{ij}, A_{ijk}, A_{ijkl}, B^{ij}, B^{ijk}, B^{ijkl}\}$. We will write the action I as a functional of these fields. As a short-hand notation, we will use f_A for

$$f_A = f(\mathcal{A}_{\mu\nu\rho\sigma}) = f(R_{\mu\nu\rho\sigma})|_{R_{\mu\nu\rho\sigma}=\mathcal{A}_{\mu\nu\rho\sigma}}. \quad (3.15)$$

To write the action as a functional of these fields, we will need to replace $\mathcal{R}_{\alpha\beta\mu\nu}, \mathcal{R}_{\mu\alpha\beta}$ with the Gauss (A.52) and Codazzi relations (A.56) in the (3.2). After replacing the Gauss and Codazzi relations we have 14 terms. To make the calculations more systematic we will write it term by term separately.

$$S_1 = \frac{1}{2}I_1 + \frac{1}{2}I_2 + \frac{1}{2}I_3 - \frac{1}{2}I_4 - 2I_5 + 2I_6 - \frac{1}{2}I_7 + 2I_8 - 2I_9, \quad (3.16)$$

$$S_{2b} = 2I_{10} + 2I_{11} + 2I_{12} - 2I_{13} + 2I_{14}, \quad (3.17)$$

$$I = S_1 + S_{2b}.$$

$$\begin{aligned} I_1 &= \int_{\mathcal{M}} d^D x N \sqrt{\gamma} f_A & I_8 &= \int_{\mathcal{M}} d^D x N \sqrt{\gamma} B^{ijk} A_{ijk}, \\ I_2 &= \int_{\mathcal{M}} d^D x N \sqrt{\gamma} B^{ijkl(d)} R_{ijkl} & I_9 &= \int_{\mathcal{M}} d^D x N \sqrt{\gamma} B^{ij} A_{ij}, \\ I_3 &= \int_{\mathcal{M}} d^D x N \sqrt{\gamma} B^{ijkl} K_{ik} K_{jl} & I_{10} &= \int_{\mathcal{M}} d^D x \sqrt{\gamma} D_i D_j N B^{ij}, \\ I_4 &= \int_{\mathcal{M}} d^D x N \sqrt{\gamma} B^{ijkl} K_{il} K_{jk} & I_{11} &= \int_{\mathcal{M}} d^D x N \sqrt{\gamma} B^{ij} K_{im} K^m_j, \\ I_5 &= \int_{\mathcal{M}} d^D x N \sqrt{\gamma} B^{ijk} D_k K_{ij} & I_{12} &= \int_{\mathcal{M}} d^D x N \sqrt{\gamma} K K_{ij} B^{ij}, \\ I_6 &= \int_{\mathcal{M}} d^D x N \sqrt{\gamma} B^{ijk} D_j K_{ik} & I_{13} &= \int_{\mathcal{M}} d^D x \sqrt{\gamma} K_{ij} \dot{B}^{ij}, \\ I_7 &= \int_{\mathcal{M}} d^D x N \sqrt{\gamma} B^{ijkl} A_{ijkl} & I_{14} &= \int_{\mathcal{M}} d^D x \sqrt{\gamma} K_{ij}{}^{(d)} \mathcal{L}_{\beta} B^{ij}. \end{aligned}$$

From the 9 configuration space variables 7 of them do not have time derivatives. So, we have 7 primary constraints. First we will introduce a notation for the conjugate momentum densities. We will use the Greek letter π for all the conjugate momentum densities.

$$\begin{aligned} \pi &:= \frac{\delta I}{\delta \dot{N}} & \pi^i &:= \frac{\delta I}{\delta \dot{\beta}_i}, \\ \pi_{\gamma}^{ij} &:= \frac{\delta I}{\delta \dot{\gamma}_{ij}} & \pi_A^{ij} &:= \frac{\delta I}{\delta \dot{A}_{ij}}, \\ \pi_{ij}^B &:= \frac{\delta I}{\delta \dot{B}^{ij}} & \pi_A^{ijk} &:= \frac{\delta I}{\delta \dot{A}_{ijk}}, \end{aligned}$$

$$\pi_{ijk}^B := \frac{\delta I}{\delta \dot{B}^{ijk}} \quad \pi_A^{ijkl} := \frac{\delta I}{\delta \dot{A}_{ijkl}},$$

$$\pi_{ijkl}^B := \frac{\delta I}{\delta \dot{B}^{ijkl}}.$$

These denote the conjugate momentum densities and it is easier to understand than other notations. The indices γ, A, B are only there to distinguish between conjugate momenta that have the same geometric indices. They can be up or down depending on the placement of the first geometric index. We have only 7 primary constraints. Let us list them.

$$\pi = 0 \tag{3.18}$$

$$\pi^i = 0 \tag{3.19}$$

$$\pi_A^{ij} = 0 \tag{3.20}$$

$$\pi_A^{ijk} = 0 \tag{3.21}$$

$$\pi_A^{ijkl} = 0 \tag{3.22}$$

$$\pi_{ijk}^B = 0 \tag{3.23}$$

$$\pi_{ijkl}^B = 0. \tag{3.24}$$

We have only two non-zero conjugate momentum densities. As a side note, we will only show the variational derivatives once, for each unique situation since we have to calculate a lot of variational derivatives. For the other variational derivatives (that had a similar calculation) we will only give the results. Before we do, we will point out two small warnings about the variational derivatives. Consider two tensors one of which, $S_{ij} = S_{ji}$ is a symmetric tensor of rank 2 while the other one, T^{ij} is a generic rank 2 tensor. Then

$$h = S_{ij}T^{ij}$$

$$\frac{\delta h}{\delta S_{ij}} = \frac{\delta(S_{ij}T^{(ij)} + S_{ij}T^{[ij]})}{\delta S_{ij}}$$

$$= T^{(ij)}$$

or incorrectly

$$= T^{ij}.$$

The last result is incorrect because if we accept it as an answer, the answer stops becoming unique. In fact, we can add to the last result any antisymmetric tensor just by adding to the argument of the variational derivative any 0, say

for instance $S^{ij}K_{[ij]}$. So, we will force the symmetries on the results, by hand if necessary. This situation where we get a non-unique answer due to the non-uniqueness of the zero vector also happens when we use projections. Let us give a simple example for this as well. Consider an extended spatial tensor, $A_{\mu\nu}$, in the context of $3 + 1$ split. Then

$$A = A_{\mu\nu}g^{\mu\nu} = A_{\mu\nu}(\gamma^{\mu\nu} + n^\mu n^\nu) = A_{\mu\nu}\gamma^{\mu\nu} \quad (3.25)$$

Then again it seems there are two answers for $\frac{\delta A}{\delta A_{\mu\nu}}$.

$$\frac{\delta A}{\delta A_{\mu\nu}} = \gamma^{\mu\nu} \quad (3.26)$$

or, incorrectly

$$\frac{\delta A}{\delta A_{\mu\nu}} = g^{\mu\nu}. \quad (3.27)$$

Again, the last result is not correct since we can add to it in any expression like $v^\mu n^\nu$ with a generic vector \vec{v} . Here, we must be more careful since the only solution seems to be eliminating the normal vector by hand. This also means that we should be careful in using the chain rule in projections. Let us find the conjugate momentum of B^{ij} .

$$\begin{aligned} \pi_{ij}^B &:= \frac{\delta I}{\delta \dot{B}^{ij}} \\ &= -2 \frac{\delta I_{13}}{\delta \dot{B}^{ij}} \\ &= -2 \int_{\mathcal{M}} d^D y \sqrt{\gamma} K_{ij} \delta \dot{B}^{ij} \delta^{(D)}(\vec{x} - \vec{y}) \end{aligned} \quad (3.28)$$

$$= -2\sqrt{\gamma}K_{ij}. \quad (3.29)$$

Now, we will find the $\pi_{\dot{\gamma}}^{ij}$. $\dot{\gamma}_{ij}$ only exists as part of K_{ij} and there is a simple relation between the two.

$$\begin{aligned} \frac{\delta K_{ij}}{\delta \dot{\gamma}_{ab}} &= -2N \delta_i^a \delta_j^b \\ \frac{\delta I}{\delta \dot{\gamma}_{ij}} &= \frac{\delta I}{\delta K_{ab}} \frac{\delta K_{ab}}{\delta \dot{\gamma}_{ij}} \\ &= \frac{-1}{2N} \frac{\delta I}{\delta K_{ij}} \end{aligned} \quad (3.30)$$

So, if we divide by $-2N$ we can obtain the variations correctly. Another type of trivial variational derivative, like (3.28), we have to calculate many times is

$$\frac{\delta I_3}{\delta K_{ab}} = \int_{\mathcal{M}} d^D y N \sqrt{\gamma} B^{ijkl} (\delta_i^a \delta_k^b K_{jl} + \delta_j^a \delta_l^b K_{ik}) \delta^{(D)}(\vec{x} - \vec{y}) \quad (3.31)$$

$$= 2N\sqrt{\gamma}B^{ajib}K_{ij}, \quad (3.32)$$

where in the last step, we used the symmetries of B^{ijkl} and changed some dummy indices. Another simple variational derivative we have to do is

$$\begin{aligned} \frac{\delta I_6}{\delta K_{ab}} &= \frac{\delta}{\delta K_{ab}} \int_{\mathcal{M}} dt \int d^d x N \sqrt{\gamma} B^{ijk} D_j K_{ik} \\ &= \frac{\delta}{\delta K_{ab}} \left(\int_{\mathcal{M}} dt \int d^d x \sqrt{\gamma} D_j (NK_{ik} B^{ijk}) \right. \\ &\quad \left. - \int_{\mathcal{M}} dt \int d^d x \sqrt{\gamma} K_{ik} D_j (NB^{ijk}) \right) \\ &= \frac{\delta}{\delta K_{ab}} \left(\int_{\mathcal{M}} dt \int d^d x \partial_j (\sqrt{\gamma} K_{ik} NB^{ijk}) \right) \\ &\quad - \frac{\delta}{\delta K_{ab}} \left(\int_{\mathcal{M}} d^D x \sqrt{\gamma} K_{ik} D_j (NB^{ijk}) \right) \\ &= -\frac{1}{2} \left(\sqrt{\gamma} D_j (NB^{ajb}) + \sqrt{\gamma} D_j (NB^{bja}) \right), \end{aligned} \quad (3.33)$$

where at the last step we used the fact that variations vanish at the boundary even if it is only part of the boundary which is the boundary of the space-like slice. Since, theories that do not have any natural boundary conditions include the biggest set of possible solutions there is no generality lost. Now, we can find all the variational derivatives to calculate the conjugate momentum density of spatial metric. The non-zero variations are:

$$\begin{aligned} \frac{\delta I_3}{\delta K_{ij}} &= 2N\sqrt{\gamma}B^{iajb}K_{ab} & \frac{\delta I_4}{\delta K_{ij}} &= 2N\sqrt{\gamma}B^{iabj}K_{ab} \\ \frac{\delta I_5}{\delta K_{ab}} &= -\sqrt{\gamma}D_k(B^{(ab)k}N) & \frac{\delta I_6}{\delta K_{ab}} &= -\sqrt{\gamma}D_j(B^{(a|j|b)}N) \\ \frac{\delta I_{11}}{\delta K_{ij}} &= N\sqrt{\gamma}(K_a^i B^{aj} + K_a^j B^{ai}) & \frac{\delta I_{12}}{\delta K_{ij}} &= N\sqrt{\gamma}(B^{ab}K_{ab}\gamma^{ij} + KB^{ij}) \\ \frac{\delta I_{13}}{\delta K_{ij}} &= \sqrt{\gamma}\dot{B}^{ij} & \frac{\delta I_{14}}{\delta K_{ab}} &= \sqrt{\gamma}^{(d)}\mathcal{L}_{\vec{\beta}}B^{ab}. \end{aligned}$$

Then using (3.14), (3.16), (3.17) and the symmetries one can reach:

$$\begin{aligned} \frac{\delta I}{\delta K_{ij}} &= 2N\sqrt{\gamma}B^{iajb}K_{ab} + 4\sqrt{\gamma}D_k(NB^{ijk}) + 2N\sqrt{\gamma}(K_a^i B^{aj} + K_a^j B^{ai}) \\ &\quad + 2N\sqrt{\gamma}(B^{ab}K_{ab}\gamma^{ij} + B^{ij}K) - 2\sqrt{\gamma}\dot{B}^{ij} + 2\sqrt{\gamma}^d\mathcal{L}_{\vec{\beta}}B^{ij} \\ \pi_{\gamma}^{ij} &= -\sqrt{\gamma}B^{iajb}K_{ab} - 2\frac{\sqrt{\gamma}}{N}D_k(NB^{(ij)k}) - \sqrt{\gamma}(K_a^i B^{aj} + K_a^j B^{ai}) \\ &\quad - \sqrt{\gamma}(B^{ab}K_{ab}\gamma^{ij} + B^{ij}K) + \frac{\sqrt{\gamma}}{N}\dot{B}^{ij} - \frac{\sqrt{\gamma}}{N}^{(d)}\mathcal{L}_{\vec{\beta}}B^{ij}. \end{aligned} \quad (3.34)$$

We have both of the conjugate momenta densities. We will also need the inversions to replace time derivatives of configuration space variables. We can give

them as

$$\dot{\gamma}_{ij} = {}^{(d)}\mathcal{L}_{\tilde{\beta}}\gamma_{ij} + \frac{N}{\gamma}\pi_{ij}^B \quad (3.35)$$

$$\begin{aligned} \dot{B}^{ij} &= \frac{N}{\sqrt{\gamma}}\pi_{ij}^{\dot{\gamma}} - \frac{N}{2\sqrt{\gamma}}B^{iajb}\pi_{ab}^B + 2D_k(NB^{(ij)k}) - \frac{N}{2\sqrt{\gamma}}(\pi_{Ba}^i B^{aj} + \pi_{Ba}^j B^{ai}) \\ &\quad - \frac{N}{2\sqrt{\gamma}}\pi_{ab}^B B^{ab}\gamma^{ij} - \frac{N}{2\sqrt{\gamma}}\pi_{Ba}^a B^{ij} + {}^{(d)}\mathcal{L}_{\tilde{\beta}}B^{ij}. \end{aligned} \quad (3.36)$$

3.3 Canonical Hamiltonian and the Total Hamiltonian

Now, we can start constructing the canonical Hamiltonian, the total Hamiltonian and work out the consistency algorithm of Dirac.

$$\mathcal{H}_c \stackrel{c}{=} (\pi_J\phi^J - \mathcal{L})|_{c_p} \quad (3.37)$$

$$H_c = \int_{\Sigma} d^d x \mathcal{H}_c. \quad (3.38)$$

Using the above relations together with (3.35), (3.36) and inserting the primary constraints we can write the canonical Hamiltonian. Notice that the Lagrangian density is first order in \dot{B}^{ij} so we do not need the longer inversion relation, (3.36), conveniently.

$$\begin{aligned} \mathcal{H}_c \stackrel{c}{=} & \pi\dot{N} + \pi^i\dot{\beta}_i + \pi_A^{ij}\dot{A}_{ij} + \pi_A^{ijk}\dot{A}_{ijk} + \pi_A^{ijkl}\dot{A}_{ijkl} + \pi_{ijk}^B\dot{B}^{ijk} + \pi_{ijkl}^B\dot{B}^{ijkl} \\ & + \pi_{ij}^{\dot{\gamma}}\dot{\gamma}_{ij} + \pi_{ij}^B\dot{B}^{ij} - \mathcal{L}|_{c_p} \end{aligned} \quad (3.39)$$

$$\begin{aligned} \stackrel{c}{=} & \pi_{ij}^{\dot{\gamma}}\dot{\gamma}_{ij} + \pi_{ij}^B\dot{B}^{ij} - \frac{N\sqrt{\gamma}}{2}f_A - \frac{N\sqrt{\gamma}}{2}B^{ijkl}R^{ijkl} - \frac{N\sqrt{\gamma}}{2}B^{ijkl}K_{ik}K_{jl} \\ & + \frac{N\sqrt{\gamma}}{2}B^{ijkl}K_{il}K_{jk} + 2N\sqrt{\gamma}D_kK_{ij}B^{ijk} - 2N\sqrt{\gamma}B^{ijk}D_jK_{ik} \\ & + \frac{N\sqrt{\gamma}}{2}B^{ijkl}A_{ijkl} - 2N\sqrt{\gamma}B^{ijk}A_{ijk} + 2N\sqrt{\gamma}B^{ij}A_{ij} - 2\sqrt{\gamma}D_iD_jNB^{ij} \\ & - 2N\sqrt{\gamma}K_{im}K_j^m B^{ij} - 2N\sqrt{\gamma}KK_{ij}B^{ij} + 2\sqrt{\gamma}K_{ij}\dot{B}^{ij} \\ & - 2\sqrt{\gamma}K_{ij}{}^{(d)}\mathcal{L}_{\tilde{\beta}}B^{ij} \end{aligned} \quad (3.40)$$

$$\begin{aligned} \stackrel{c}{=} & \pi_{ij}^{\dot{\gamma}}{}^{(d)}\mathcal{L}_{\tilde{\beta}}\gamma_{ij} + \frac{N}{\sqrt{\gamma}}\pi_{ij}^B\pi_{ij}^{\dot{\gamma}} - \frac{N\sqrt{\gamma}}{2}f_A - \frac{N\sqrt{\gamma}}{2}B^{ijkl(d)}R^{ijkl} \\ & + \frac{N}{4\sqrt{\gamma}}B^{ijkl}\pi_{il}^B\pi_{jk}^B + 2N\sqrt{\gamma}B^{ijk}D_j\frac{\pi_{ik}^B}{\sqrt{\gamma}} + \frac{N\sqrt{\gamma}}{2}B^{ijkl}A_{ijkl} - 2N\sqrt{\gamma}B^{ijk}A_{ijk} \\ & + 2N\sqrt{\gamma}B^{ij}A_{ij} - 2\sqrt{\gamma}D_iD_jNB^{ij} - \frac{N}{2\sqrt{\gamma}}\pi_{im}^B\pi_{Bj}^m B^{ij} - \frac{N}{2\sqrt{\gamma}}\pi_{Ba}^a\pi_{ij}^B B^{ij} \\ & + \pi_{ij}^B{}^{(d)}\mathcal{L}_{\tilde{\beta}}B^{ij}. \end{aligned} \quad (3.41)$$

The total Hamiltonian will be denoted by H and its density will be denoted by \mathcal{H} . For the Lagrange multipliers that is needed to expand the canonical Hamiltonian off the primary constraint surface we will adopt a notation just like the conjugate momentum densities. That is

$$\begin{aligned} \mathcal{H} = & \mathcal{H}_c + u\pi + u_i\pi^i + u_{ij}^A\pi_A^{ij} + u_{ijk}^A\pi_A^{ijk} + u_{ijkl}^A\pi_A^{ijkl} + u_B^{ijk}\pi_{ijk}^B \\ & + u_B^{ijkl}\pi_{ijkl}^B. \end{aligned} \quad (3.42)$$

$$\begin{aligned} H_1 &= \int_{\Sigma} d^d x \pi_{\gamma}^{ij(d)} \mathcal{L}_{\beta} \gamma_{ij} & H_{11} &= \int_{\Sigma} d^d x \frac{N}{\sqrt{\gamma}} \pi_{im}^B \pi_{Bj}^m B^{ij} \\ H_2 &= \int_{\Sigma} d^d x \frac{N}{\sqrt{\gamma}} \pi_{ij}^B \pi_{\gamma}^{ij} & H_{12} &= \int_{\Sigma} d^d x \frac{N}{\sqrt{\gamma}} \pi_{Ba}^a \pi_{ij}^B B^{ij} \\ H_3 &= \int_{\Sigma} d^d x N \sqrt{\gamma} f_A & H_{13} &= \int_{\Sigma} d^d x \pi_{ij}^{B(d)} \mathcal{L}_{\beta} B^{ij} \\ H_4 &= \int_{\Sigma} d^d x N \sqrt{\gamma} B^{ijkl(d)} R_{ijkl} & H_{14} &= \int_{\Sigma} d^d x u \pi \\ H_5 &= \int_{\Sigma} d^d x \frac{N}{\sqrt{\gamma}} B^{ijkl} \pi_{il}^B \pi_{jk}^B & H_{15} &= \int_{\Sigma} d^d x u_i \pi^i \\ H_6 &= \int_{\Sigma} d^d x N \sqrt{\gamma} B^{ijk} D_j \frac{\pi_{ik}^B}{\sqrt{\gamma}} & H_{16} &= \int_{\Sigma} d^d x u_{ij}^A \pi_A^{ij} \\ H_7 &= \int_{\Sigma} d^d x N \sqrt{\gamma} B^{ijkl} A_{ijkl} & H_{17} &= \int_{\Sigma} d^d x u_{ijk}^A \pi_A^{ijk} \\ H_8 &= \int_{\Sigma} d^d x N \sqrt{\gamma} B^{ijk} A_{ijk} & H_{18} &= \int_{\Sigma} d^d x u_{ijkl}^A \pi_A^{ijkl} \\ H_9 &= \int_{\Sigma} d^d x N \sqrt{\gamma} B^{ij} A_{ij} & H_{19} &= \int_{\Sigma} d^d x u_B^{ijk} \pi_{ijk}^B \\ H_{10} &= \int_{\Sigma} d^d x \sqrt{\gamma} D_i D_j N B^{ij} & H_{20} &= \int_{\Sigma} d^d x u_B^{ijkl} \pi_{ijkl}^B. \end{aligned}$$

$$\begin{aligned} H = & H_1 + H_2 - \frac{H_3}{2} - \frac{H_4}{2} + \frac{H_5}{4} + 2H_6 + \frac{H_7}{2} - 2H_8 + 2H_9 - 2H_{10} \\ & - \frac{H_{11}}{2} - \frac{H_{12}}{2} + H_{13} + H_{14} + H_{15} + H_{16} + H_{17} + H_{18} + H_{19} + H_{20} \end{aligned} \quad (3.43)$$

CHAPTER 4

CONSISTENCY ALGORITHM AND THE SECONDARY CONSTRAINTS

Since, we found the total Hamiltonian we can start working out the consistency algorithm and find the secondary constraints. We will first focus on 4 of the primary constraints $\pi_{ijkl}^B = 0, \pi_{ijk}^B = 0, \pi_A^{ijkl} = 0, \pi_A^{ijk} = 0$ then we will work out the rest.

4.1 Results of $\dot{\pi}_{ijkl}^B = 0$

Let us see the secondary constraint that will come from the time derivative of the $\pi_{ijkl}^B = 0$.

$$\begin{aligned}
 \pi_{ijkl}^B = 0 &\Rightarrow \dot{\pi}_{ijkl}^B = 0 \\
 \{\pi_{ijkl}^B, H\} &= 0 \\
 -\frac{\delta H}{\delta B^{ijkl}} &= 0 \\
 &= \frac{1}{2} \frac{\delta H_4}{\delta B^{ijkl}} - \frac{1}{4} \frac{\delta H_5}{\delta B^{ijkl}} - \frac{1}{2} \frac{\delta H_7}{\delta B^{ijkl}} = 0 \\
 \frac{\delta H_4}{\delta B^{ijkl}} &= N \sqrt{\gamma}^{(d)} R_{ijkl} \\
 \frac{\delta H_5}{\delta B^{ijkl}} &= \frac{N}{2\sqrt{\gamma}} (\pi_{il}^B \pi_{jk}^B - \pi_{ik}^B \pi_{jl}^B) \\
 \frac{\delta H_7}{\delta B^{ijkl}} &= N \sqrt{\gamma} A_{ijkl} \\
 A_{ijkl} &= {}^{(d)} R_{ijkl} - \frac{(\pi_{il}^B \pi_{jk}^B - \pi_{ik}^B \pi_{jl}^B)}{4\gamma}. \tag{4.1}
 \end{aligned}$$

This set of constraints is necessarily second class due to the set of constraints $\pi_A^{ijkl} = 0$. Also notice that taking the derivative of (4.1) we would at most find a restriction on the Lagrange multiplier u_{ijkl}^A , however we do not need this restriction. The reason is simple, since $\pi_A^{ijkl} = 0$ gets multiplied by the Lagrange

multiplier this Lagrange multiplier cannot enter into equations of motion. Also we necessarily have to solve second class constraints (whether we use Dirac brackets or not) so the restriction on the Lagrange multiplier u_{ijkl}^A will not be needed. This is a situation we can see in other cases as well, that is to say, we do not need the restriction on the Lagrange multiplier that belongs to a secondary primary constraint. For instance, we can check the case III of B.3.

4.2 Results of $\dot{\pi}_{ijk}^B = 0$

Let us examine the time derivative of $\pi_{ijk}^B = 0$.

$$\begin{aligned}
\pi_{ijk}^B = 0 &\Rightarrow \dot{\pi}_{ijk}^B = 0 \\
\{\pi_{ijk}^B, H\} &= 0 \\
-\frac{\delta H}{\delta B^{ijk}} &= 0 \\
&= -2\frac{\delta H_6}{\delta B^{ijk}} + 2\frac{\delta H_8}{\delta B^{ijk}} = 0 \\
\frac{\delta H_6}{\delta B^{ijk}} &= N\sqrt{\gamma}A_{ijk} \\
\frac{\delta H_8}{\delta B^{ijk}} &= \frac{N\sqrt{\gamma}}{2}\left(D_j\frac{\pi_{ik}^B}{\sqrt{\gamma}} - D_k\frac{\pi_{ij}^B}{\sqrt{\gamma}}\right) \\
A_{ijk} &= \frac{1}{2}\left(D_j\frac{\pi_{ik}^B}{\sqrt{\gamma}} - D_k\frac{\pi_{ij}^B}{\sqrt{\gamma}}\right). \tag{4.2}
\end{aligned}$$

This is another set of second class constraints due to the constraints $\pi_A^{ijk} = 0$. Again, u_{ijk}^A does not need to be restricted due to the same reasons mentioned in section 4.1.

4.3 Results of $\dot{\pi}_A^{ijkl} = 0$

Let us examine the time derivative of $\pi_A^{ijkl} = 0$.

$$\begin{aligned}
\pi_A^{ijkl} = 0 &\Rightarrow \dot{\pi}_A^{ijkl} = 0 \\
\{\pi_A^{ijkl}, H\} &= 0 \\
-\frac{\delta H}{\delta A_{ijkl}} &= 0 \\
&= \frac{1}{2}\frac{\delta H_3}{\delta A_{ijkl}} - \frac{1}{2}\frac{\delta H_7}{\delta A_{ijkl}} \\
\frac{\delta H_3}{\delta A_{ijkl}} &= N\sqrt{\gamma}\frac{\delta f}{\delta A_{ijkl}}
\end{aligned}$$

$$\begin{aligned}\frac{\delta H_7}{\delta A_{ijkl}} &= N\sqrt{\gamma}B^{ijkl} \\ B^{ijkl} &= \frac{\delta f}{\delta A_{ijkl}}.\end{aligned}\tag{4.3}$$

This is also a set of second class constraints since we have $\pi_B^{ijkl} = 0$. Also, we do not need the restriction on the u_B^{ijkl} due to the same reasons mentioned in section 4.1.

4.4 Results of $\dot{\pi}_A^{ijk} = 0$

Let us check the time derivative of $\pi_A^{ijk} = 0$.

$$\begin{aligned}\pi_A^{ijk} = 0 &\Rightarrow \dot{\pi}_A^{ijk} = 0 \\ \{\pi_A^{ijk}, H\} &= 0 \\ -\frac{\delta H}{\delta A_{ijk}} &= 0 \\ &= \frac{1}{2}\frac{\delta H_3}{\delta A_{ijk}} + 2\frac{\delta H_8}{\delta A_{ijk}} = 0 \\ \frac{\delta H_3}{\delta A_{ijk}} &= N\sqrt{\gamma}\frac{\delta f}{\delta A_{ijk}} \\ \frac{\delta H_8}{\delta A_{ijk}} &= N\sqrt{\gamma}B^{ijk} \\ 4B^{ijk} + \frac{\delta f}{\delta A_{ijk}} &= 0.\end{aligned}\tag{4.4}$$

This set of constraints are also necessarily second class since we have $\pi_{ijk}^B = 0$. We also do not need the restriction on the Lagrange multipliers because of the reasons mentioned in 4.1.

The set of constraints we have found thus far are common in any $f(R_{\mu\nu\alpha\beta})$ theory. They are trivial since they are just the auxiliary variables that represent the actual fields. These constraints are always second class and when they get treated, which is described in the subsection B.3.1.1, we just do what is suggested in [6]. This is what is done in the first part of [1]. The actual constraints that can change between different theories are the other constraints that we will find shortly. Before we continue the consistency algorithm we will list these trivial constraints all of which are second class.

$$\pi_{ijkl}^B = 0 \quad \Rightarrow \quad A_{ijkl} \stackrel{(d)}{=} R_{ijkl} - \frac{(\pi_{il}^B \pi_{jk}^B - \pi_{ik}^B \pi_{jl}^B)}{4\gamma}$$

$$\begin{aligned}
\pi_{ijkl}^B = 0 &\Rightarrow A_{ijk} = \frac{1}{2} \left(D_j \frac{\pi_{ik}^B}{\sqrt{\gamma}} - D_k \frac{\pi_{ij}^B}{\sqrt{\gamma}} \right) \\
\pi_A^{ijkl} = 0 &\Rightarrow B^{ijkl} = \frac{\delta f}{\delta A_{ijkl}} \\
\pi_A^{ijk} = 0 &\Rightarrow 4B^{ijk} = -\frac{\delta f}{\delta A_{ijk}}.
\end{aligned}$$

4.5 Results of $\dot{\pi} = 0$

This constraint $\pi = 0$ is also common in $f(R_{\mu\nu\alpha\beta})$ type theories but the secondary constraint that comes from it is not the same in these types of theories. Let us examine $\dot{\pi} = 0$.

$$\begin{aligned}
\pi = 0 &\Rightarrow \dot{\pi} = 0 \\
\{\pi, H\} = 0 &\Rightarrow \frac{\delta H}{\delta N} = C = 0 \\
C &= \frac{\delta H_2}{\delta N} - \frac{1}{2} \frac{\delta H_3}{\delta N} - \frac{1}{2} \frac{\delta H_4}{\delta N} + \frac{1}{4} \frac{\delta H_5}{\delta N} + 2 \frac{\delta H_6}{\delta N} + \frac{1}{2} \frac{\delta H_7}{\delta N} - 2 \frac{\delta H_8}{\delta N} \\
&\quad + 2 \frac{\delta H_9}{\delta N} - 2 \frac{\delta H_{10}}{\delta N} - \frac{1}{2} \frac{\delta H_{11}}{\delta N} - \frac{1}{2} \frac{\delta H_{12}}{\delta N}. \tag{4.5} \\
\frac{\delta H_2}{\delta N} &= \frac{\pi_{ij}^B \pi_{jk}^B}{\sqrt{\gamma}} & \frac{\delta H_3}{\delta N} &= \sqrt{\gamma} f_A \\
\frac{\delta H_4}{\delta N} &= \sqrt{\gamma} B^{ijkl(d)} R_{ijkl} & \frac{\delta H_5}{\delta N} &= \frac{B^{ijkl} \pi_{il}^B \pi_{jk}^B}{\sqrt{\gamma}} \\
\frac{\delta H_6}{\delta N} &= \sqrt{\gamma} B^{ijk} D_j \frac{\pi_{ik}^B}{\sqrt{\gamma}} & \frac{\delta H_7}{\delta N} &= \sqrt{\gamma} B^{ijkl} A_{ijkl} \\
\frac{\delta H_8}{\delta N} &= \sqrt{\gamma} B^{ijk} A_{ijk} & \frac{\delta H_9}{\delta N} &= \sqrt{\gamma} B^{ij} A_{ij} \\
\frac{\delta H_{10}}{\delta N} &= \sqrt{\gamma} D_j D_i B^{ij} & \frac{\delta H_{11}}{\delta N} &= \frac{\pi_{im}^B \pi_{Bj}^m B^{ij}}{\sqrt{\gamma}} \\
\frac{\delta H_{12}}{\delta N} &= \frac{\pi_a^{Ba} \pi_{ij}^B B^{ij}}{\sqrt{\gamma}}.
\end{aligned}$$

If we examine this constraint, (4.5), we see that it is contaminated by the second class constraints (4.2), (4.1) and we should take those parts out since the contamination is due to second class constraints. The terms $\frac{\delta H_4}{\delta N}, \frac{\delta H_5}{\delta N}, \frac{\delta H_7}{\delta N}$ inside the constraint (4.5), constitute the constraint (4.1) multiplied by B^{ijkl} . The terms $\frac{\delta H_6}{\delta N}, \frac{\delta H_8}{\delta N}$ inside the constraint (4.5), constitute the constraint (4.2) multiplied by B^{ijk} . After taking these out we redefine the constraint as:

$$C = \frac{\delta H_2}{\delta N} - \frac{1}{2} \frac{\delta H_3}{\delta N} + 2 \frac{\delta H_9}{\delta N} - 2 \frac{\delta H_{10}}{\delta N} - \frac{1}{2} \frac{\delta H_{11}}{\delta N} - \frac{1}{2} \frac{\delta H_{12}}{\delta N}, \tag{4.6}$$

where the variations are as they were given before.

4.6 Results of $\pi^i = 0$

Let us examine the results of $\pi^i = 0$.

$$\begin{aligned}
\pi^i = 0 &\Rightarrow \dot{\pi}^i = 0 \\
\{\pi^i, H\} = 0 &\Rightarrow \frac{\delta H}{\delta \beta_i} = 0 \\
\frac{\delta H_1}{\delta \beta_i} &= -\frac{\delta H_{13}}{\delta \beta_i} \\
\frac{\delta H_1}{\delta \beta_i} &= -2\sqrt{\gamma}D_j \frac{\pi_\gamma^{ij}}{\sqrt{\gamma}} \\
\frac{\delta H_{13}}{\delta \beta_a} &= \pi_{ij}^B D^a B^{ij} - 2\sqrt{\gamma}D_k \frac{B^{ik}\pi_i^{Ba}}{\sqrt{\gamma}} \\
C^i &= \frac{\delta H_1}{\delta \beta_i} + \frac{\delta H_{13}}{\delta \beta_i}. \tag{4.7}
\end{aligned}$$

4.7 Results of $\pi_A^{ij} = 0$

This is the last primary constraint but this does not mean the consistency algorithm is over. We will come back to this issue later. Now we will deal with the time derivative of $\pi_A^{ij} = 0$.

$$\begin{aligned}
\pi_A^{ij} = 0 &\Rightarrow \dot{\pi}_A^{ij} = 0 \\
\{\pi_A^{ij}, H\} = 0 &\Rightarrow \frac{\delta H}{\delta A_{ij}} = 0 \\
\frac{\delta H}{\delta A_{ij}} &= -\frac{1}{2} \frac{\delta H_3}{\delta A_{ij}} + 2 \frac{\delta H_9}{\delta A_{ij}} = 0 \\
\frac{\delta H_3}{\delta A_{ij}} &= N\sqrt{\gamma} \frac{\delta f}{\delta A_{ij}} \\
\frac{\delta H_9}{\delta A_{ij}} &= N\sqrt{\gamma} B^{ij} \\
C^{ij} &= 4B^{ij} - \frac{\delta f}{\delta A_{ij}} = 0. \tag{4.8}
\end{aligned}$$

At these stage we have worked the consistency algorithm once for each primary constraint and we should check their Poisson brackets and see whether we get more constraints or restrictions on the Lagrange multipliers.

4.8 Determination of The Classes of The Constraints

We already determined the classes of the trivial constraints (3.21), (3.22), (3.23), (3.24), (4.2), (4.1), (4.4) and (4.3). Now, the difficulties start since the BINMG action is really complicated when decomposed. In fact, I could not determine the classes of any of the constraints (4.6), (4.7), (4.8), (3.18), (3.19) and (3.20). I can only give roundabout reasons on why some of them should be first class and some should be second class. The first thing we must be careful about is that we can only go about finding the classes case by case for the theories which are given with a general form. For instance, if we are given a potential as $V(\vec{x})$ then constraints can change form depending on the potential which can be seen in the example of section B.3 of appendix B. Only for exceptionally simple constraints such as the ones we called trivial, classes can be determined in a general formulation. So, the determination of the classes must be done case by case and it can be really simple as in GR or it can be really complicated as in BINMG.

To determine the classes of the constraints, we must check the Poisson brackets of the constraints with each other and see if the result is consistent, that is, reduces to zero on the constraint surface. Now, the field theory version of this procedure can be thought as taking out the functionals of the constraints and their translations (spatial derivatives) out of the Poisson brackets to reduce the result to zero. Another way to see if the Poisson brackets reduce to zero on the constraint surface is if the Poisson brackets of the smeared quantities is a smeared quantity of the constraints and functionals of the test functions. There are a couple of reasons why $C, C^i, \pi = 0$ and $\pi^i = 0$ should be first class.

- i) The total Hamiltonian can be written as a sum of the first class constraints and a first class Hamiltonian. If we use the product rule and take out the surface terms out of the H_1, H_{10} and H_{13} and also solve the trivial second class constraints, we can write the total Hamiltonian as

$$H = \int_{\Sigma} d^d x (NC + \beta_i C^i + u\pi + u_i \pi^i + u_{ij}^A \pi_A^{ij}). \quad (4.9)$$

- ii) The constraints $C, C^i, \pi = 0$ and $\pi^i = 0$ are supposed to represent diffeomorphism invariance of the action [3], [11].

iii) The original article we review, [1], claims to have calculated the Poisson brackets of the smeared quantities and that they are the same as in GR for the general case. Suppose S, Q are scalar fields and $\tilde{\mathbf{T}}, \tilde{\mathbf{K}}$ are one-forms and that they have corresponding vector fields thanks to the metric. Then, according to the original article, one has

$$\{C[S], C[Q]\} \stackrel{c}{=} C^i[S\partial_i Q - Q\partial_i S] \quad (4.10)$$

$$\{C^i[T_i], C^j[K_j]\} \stackrel{c}{=} C^k[(^d)\mathcal{L}_{\tilde{\mathbf{T}}}\tilde{\mathbf{K}}]^l\gamma_{kl} \quad (4.11)$$

$$\{C^i[T_i], C[S]\} \stackrel{c}{=} C[T^i\partial_i S], \quad (4.12)$$

for any theory whose action contains only $f(R_{\mu\nu\alpha\beta})$. Here we must emphasize that the calculated Poisson brackets above are not enough to prove these constraints are first class. The Poisson brackets with the constraints $C^{ij}, \pi_A^{ij} = 0$ must also be calculated and it must be shown that it reduces to zero on the constraint surface.

Because of all these reasons, we will not concern ourselves with the constraints $C, C^i, \pi = 0$ and $\pi^i = 0$ and assume that they are first class constraints. We will assume that they represent the diffeomorphism invariance. Also notice that if our assumption is correct and the Hamiltonian constraint C and the momentum constraints C^i are first class they do not lead to any more constraints or restrictions on Lagrange multipliers since the total Hamiltonian is a sum of the first class constraints plus the $\pi_A^{ij} = 0$ constraint. We will focus on the remaining constraints $C^{ij}, \pi_A^{ij} = 0$.

4.8.1 Classification of the C^{ij}

We first notice that since the action of the BINMG is infinite order in Riemann tensor the Poisson brackets of C^{ij}, π_A^{ij} is not zero. Let us give our results on this first. From (2.8) and (2.6)

$$f(R_{\mu\nu\alpha\beta}) = -\frac{8m^2}{\kappa^2}F(R, K, S). \quad (4.13)$$

Here we will define some quantities

$$R_A = R|_{R_{\mu\nu\alpha\beta} \rightarrow \mathcal{A}_{\mu\nu\alpha\beta}} \quad (4.14)$$

$$K_A = K \Big|_{R_{\mu\nu\alpha\beta} \rightarrow \mathcal{A}_{\mu\nu\alpha\beta}} \quad (4.15)$$

$$S_A = S \Big|_{R_{\mu\nu\alpha\beta} \rightarrow \mathcal{A}_{\mu\nu\alpha\beta}} \quad (4.16)$$

$$F(R, K, S) = \sqrt{T} - 1 + \frac{\lambda}{2} \quad (4.17)$$

$$T = 1 + \frac{R}{2m^2} - \frac{K}{2m^4} - \frac{S}{24m^6} \quad (4.18)$$

$$T_A = 1 + \frac{R_A}{2m^2} - \frac{K_A}{2m^4} - \frac{S_A}{24m^6} \quad (4.19)$$

$$F_A = \sqrt{T_A} - 1 + \frac{\lambda}{2}. \quad (4.20)$$

Now, the constraint C^{ij} for BINMG action becomes when divided by four

$$\begin{aligned} C^{ij} &= B^{ij} + \frac{2m^2}{\kappa^2} \frac{\delta F_A}{\delta A_{ij}} \\ &= B^{ij} + \frac{2m^2}{\kappa^2} \frac{1}{2\sqrt{T_A}} \frac{\delta T_A}{\delta A_{ij}} \\ &= B^{ij} + \frac{2m^2}{\kappa^2} \frac{1}{2\sqrt{T_A}} \left(\frac{1}{2m^2} \frac{\delta R_A}{\delta A_{ij}} - \frac{1}{2m^4} \frac{\delta K_A}{\delta A_{ij}} - \frac{1}{24m^6} \frac{\delta S_A}{\delta A_{ij}} \right). \end{aligned} \quad (4.21)$$

Let us give the variations

$$\frac{\delta R_A}{\delta A_{ij}} = -2\gamma^{ij} \quad (4.22)$$

$$\frac{\delta K_A}{\delta A_{ij}} = -2A_k^{ikj} + 2A^{ij} - 2\gamma^{ij} A^k_k + 2\gamma^{ij} A^{kl}_{kl} \quad (4.23)$$

$$\begin{aligned} \frac{\delta S_A}{\delta A_{ij}} &= 24A_k^{ikm} A^j_m + 24A^{kj}_{km} A^{im} - 24A^{kj}_{km} A_l^{ilm} + 24A_k^{ki} A_m^{mj} - 24A^{ki} A^j_k \\ &\quad + 12\gamma^{ij} A^k_{mkn} A_l^{mln} - 24\gamma^{ij} A^{kl} A^m_{kml} - 24\gamma^{ij} A^k_{km} A_l^{lm} + 12\gamma^{ij} A^{kl} A_{kl} \\ &\quad - 12\gamma^{ij} A^k_k A^l_l + 12A^{lm}_{lm} A_k^{ikj} - 12A^{lm}_{lm} A^{ij} + 12\gamma^{ij} A^k_k A^{lm}_{lm} \\ &\quad - 24A^l_l A_k^{ikj} + 24A^l_l A^{ij} - 6\gamma^{ij} A^{lm}_{lm} A^{ks}_{ks}. \end{aligned} \quad (4.24)$$

We must determine whether C^{ij} is first class or second class or if it leads to tertiary constraints. For this reason if the assumptions made in the previous section are true, we must check two Poisson brackets, the critical one being $\{\pi_A^{ij}, C^{ab}\}$.

$$\begin{aligned} \{C^{ij}, \pi_A^{ab}\} &= \frac{\delta C^{ij}}{\delta A_{ab}} \\ &= \frac{m^2}{\kappa^2} \frac{\delta}{\delta A_{ab}} \left(\frac{1}{\sqrt{T_A}} \frac{\delta T_A}{\delta A_{ij}} \right) \\ &= \frac{m^2}{2\kappa^2 T_A^{3/2}} \left[2T_A \frac{\delta^2 T_A}{\delta A_{ab} \delta A_{ij}} - \frac{\delta T_A}{\delta A_{ab}} \frac{\delta T_A}{\delta A_{ij}} \right]. \end{aligned} \quad (4.25)$$

This is the result of the Poisson brackets and since there are no spatial derivatives of the auxiliary variable tensors in the BINMG action there is no boundary

terms that needs to be removed while taking the Poisson brackets. The only tool needed is the partial derivative for this calculation. The bad news is that the last expression even with the most optimistic view is about 70 – 80 terms. We can see this easily by considering that T_A itself contains about 20 terms. The problem becomes a simplification problem. The functionals of the constraints and its translations (spatial derivatives) multiplied by some functionals of the phase space variables should be equal to the last expression. That is

$$\frac{m^2}{2\kappa^2 T_A^{3/2}} \left[2T_A \frac{\delta^2 T_A}{\delta A_{ab} \delta A_{ij}} - \frac{\delta T_A}{\delta A_{ab}} \frac{\delta T_A}{\delta A_{ij}} \right] = \sum \left\{ F_1[\phi, \pi] G_1[C, C^i, \dots, C^{ij}, \pi_A^{ij}] + \dots \right\}. \quad (4.26)$$

At this stage there are a few possibilities. Let us write them so that we can guide ourselves to an answer.

- I) Let us suppose that the last expression cannot be written for any of the 9 index combinations of $ijab$, where there is symmetry as $i \leftrightarrow j, a \leftrightarrow b, ij \leftrightarrow ab$. Then both sets of constraints $\pi_A^{ij} = 0, C^{ij} = 0$ become second class and the number of the degrees of freedom are reduced accordingly. Suppose this is true then consistency algorithm is not over and we must check the time derivative of C^{ij} . Luckily, it contains A_{ij} and its time derivative which is a Lagrange multiplier. Then what we will get at most is a restriction on the Lagrange multiplier u_{ij}^A which is not needed since we have to solve the second class constraints. The number of degrees of freedom in this case is 3 which is calculated as in below. We have C, C^i, π, π^i as sets of first class constraints; trivial constraints as second class and π_A^{ij}, C^{ij} as second class constraints. Then the trivial constraints eliminate $A_{ijk}, A_{ijkl}, B^{ijk}, B^{ijkl}$ in all cases. Then we are left with $\gamma_{ij}, N, \beta_i, A_{ij}, B^{ij}$ and the rest of the constraints. Then the number of phase space variables is $2(3+1+2+3+3) = 24$. C, C^i, π, π^i eliminate $2(2 + 1 + 2 + 1) = 12$ and C^{ij}, π_A^{ij} eliminate $(3 + 3) = 6$ of them and we are left with 6 phase space variables. So, we would have 3 degrees of freedom.
- II) The linear analysis shows that BINMG has 2 degrees of freedom [8], [5]. It also seems there are some heuristic arguments as to why BINMG should not contain a ghost degree of freedom [12]. If we accept that BINMG

contain two degrees of freedom then there are only a few possibilities (here we also exclude the possibility of a single degree of freedom.) What is common among these possibilities is that there are constraints among the C^{ij}, π_A^{ij} that have Poisson brackets which reduce to zero on the constraint surface. A combinatorial reasoning shows that the only possibility for this situation is if the constraints C^{12}, π_A^{12} have a Poisson brackets that reduce to zero on the constraint surface. Any other index combination necessitates that some other Poisson brackets also reduce to zero. That is, if the index combinations 1111, 1112, 1122, 1211 have Poisson brackets that reduce to zero, then it must also be true that the index combinations 2222, 2221, 2211, 2122 also have Poisson brackets that reduce to zero. Then there are simply too many constraints and degrees of freedom are lower than two. So, if there is a Poisson brackets that vanish on the constraint surface then it must be the index combination 1212. I had hoped that there would be simplification in the resulting expressions when this particular index combination is inserted in the (4.25). There is simplification but not as much as I hoped. Let us write about the possibilities:

i) Both C^{12}, π_A^{12} are first class and time derivative of C^{12} only leads to zero equal to zero. Then we have to eliminate one more degree of freedom since first class constraints eliminate twice as many degrees of freedom and we have two degrees of freedom.

ii) Time derivatives of C^{12} lead to another constraint but only once. So, together with that we get three constraints. Then since we exclude the possibility BINMG contains a single degree of freedom, these three constraints should contain two second class and one first class constraint. Then we again get two degrees of freedom.

iii) Time derivatives of C^{12} lead to two more constraints. Then, all four of the constraints must be second class and we get two degrees of freedom.

So, then there are a few things that must happen for BINMG to contain two degrees of freedom. The first thing that must happen is that $\{C_{12}, \pi_A^{12}\}$ must vanish on the constraint surface. I could not show this. The expression (4.25) is fourth order in the tensor \mathcal{A} (without the $T_A^{3/2}$ part in front). It is a really big

expression as we have indicated and I only calculated the order zero and order one terms before I gave up on proving that the Poisson brackets vanish. I will give the order one and zero terms of the expression

$$Q^{ijab} := 2T_A \frac{\delta^2 T_A}{\delta A_{ab} \delta A_{ij}} - \frac{\delta T_A}{\delta A_{ab}} \frac{\delta T_A}{\delta A_{ij}} \quad (4.27)$$

$$\mathcal{O}_T^n(S) := \text{the order } n \text{ terms (in tensor } T) \text{ in } S \quad (4.28)$$

$$\mathcal{O}_A^0(Q^{ijab}) = -\frac{1}{m^4} (\gamma^{jb} \gamma^{ia} + \gamma^{ib} \gamma^{ja} - \gamma^{ij} \gamma^{ab}) \quad (4.29)$$

$$\begin{aligned} \mathcal{O}_A^1(Q^{ijab}) = & -\frac{1}{m^6} (A_k^{ikb} \gamma^{ja} + A_k^{ika} \gamma^{jb} + A_k^{jkb} \gamma^{ia} + A_k^{jka} \gamma^{ib} - A^{ai} \gamma^{jb} \\ & - A^{aj} \gamma^{ib} - A^{ib} \gamma^{ja} - A^{bj} \gamma^{ia} - 3A_k^{akb} \gamma^{ij} - 3A_k^{ikj} \gamma^{ab} + 3A^{ab} \gamma^{ij} \\ & + 3A^{ij} \gamma^{ab} - 2\gamma^{ab} \gamma^{ij} A^k_k + 2\gamma^{ab} \gamma^{ij} A^{kl}_{kl}). \end{aligned} \quad (4.30)$$

I gave up here because I could not see a way that I could show that the Poisson brackets reduce to zero on the constraint surface for expressions as complicated as these. There could be symbolic computation software that might be able perform this sort of calculation. Though, I doubt that there is a symbolic computation software available that is powerful enough to simplify the resulting Poisson brackets. We already gave the first partial derivatives according to A_{ij} of the expressions R_A, K_A, S_A . We will also give the second partial derivatives if one wishes to continue this calculation.

$$\frac{\delta^2 R_A}{\delta A_{ab} \delta A_{ij}} = 0 \quad (4.31)$$

$$\frac{\delta^2 K_A}{\delta A_{ab} \delta A_{ij}} = \gamma^{ia} \gamma^{jb} + \gamma^{ib} \gamma^{ja} - 2\gamma^{ij} \gamma^{ab} \quad (4.32)$$

$$\begin{aligned} \frac{\delta^2 S_A}{\delta A_{ab} \delta A_{ij}} = & 12(A_k^{ikb} \gamma^{ja} + A_k^{ika} \gamma^{jb} + A_k^{jkb} \gamma^{ia} + A_k^{jka} \gamma^{ib}) \\ & - 12(A^{ai} \gamma^{jb} + A^{aj} \gamma^{ib} + A^{ib} \gamma^{ja} + A^{bj} \gamma^{ai}) - 24(A_m^{amb} \gamma^{ij} + A_m^{imj} \gamma^{ab}) \\ & + 24(A^{ab} \gamma^{ij} + A^{ij} \gamma^{ab}) + 12A^l_l (\gamma^{ia} \gamma^{jb} + \gamma^{ib} \gamma^{ja}) - 24\gamma^{ij} \gamma^{ab} A^k_k \\ & - 6A^{lm}_{lm} (\gamma^{ia} \gamma^{jb} + \gamma^{ib} \gamma^{ja}) + 12\gamma^{ij} \gamma^{ab} A^{lm}_{lm}. \end{aligned} \quad (4.33)$$

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APPENDIX A

GEOMETRICAL PART OF ADM FORMULATION

This appendix is mostly a summary of the lecture notes of Éricourgoulhon given in [10]. Some calculations that do not exist in the notes are performed which are needed in modified theories of gravity. Some comments are made on some of the confusing parts of the 3+1 formalism.

A.1 Framework and Notations

Spacetime is considered to be a real smooth manifold \mathcal{M} of dimension 4, endowed with a Lorentzian metric \mathbf{g} with signature $(-, +, +, +)$. Also $(\mathcal{M}, \mathbf{g})$ is assumed to be time-orientable. This means that it is possible to divide the light cone at each event into a future and a past part continuously. ∇ is the Levi-Civita connection associated with \mathbf{g} . $\mathcal{T}_p\mathcal{M}$ denotes the tangent space of the manifold \mathcal{M} at point p while $\mathcal{T}_p^*\mathcal{M}$ denotes the cotangent space at p and is dual to the vector space of $\mathcal{T}_p\mathcal{M}$. All Greek indices take values in $(0, 1, 2, 3)$ while Latin indices only take values in $(1, 2, 3)$. Vectors are denoted by $\vec{\mathbf{v}}$, while one forms are denoted by $\tilde{\omega}$ when we do the calculations independently of any basis. $\Omega^p(\mathcal{M})$ denotes the space of p -forms. We use $[\vec{\mathbf{v}}, \vec{\mathbf{u}}]$ to indicate the commutator of two vector fields and $\mathcal{L}_{\vec{\mathbf{v}}}\mathbf{T}$ to indicate the Lie derivatives of any rank tensor with respect to the vector field $\vec{\mathbf{v}}$. $\Gamma(\mathcal{T}_q^p\mathcal{M})$ is used to indicate the smooth sections on the (p, q) tensor bundle. For coordinate basis vectors and the dual basis one-forms, we will use the notation $\vec{\partial}_\alpha, \tilde{d}x^\alpha$ respectively. For non-coordinate basis, we will use $\mathbf{e}_\alpha, \boldsymbol{\theta}^\alpha$ which are only needed in the last section of this appendix. The action of a vector on a function $\vec{\mathbf{v}}\{f\} = v^\mu f_{,\mu}$ is denoted like this. The inner product of two vectors and contraction of a vector and a one-form are denoted as $(\vec{\mathbf{v}} \cdot \vec{\mathbf{u}}), \langle \tilde{\omega}, \vec{\mathbf{u}} \rangle$ respectively.

A.2 Hypersurface Embedded in Spacetime

A hypersurface Σ of \mathcal{M} is the image of a 3-dim manifold $\hat{\Sigma}$ defined by an embedding

$$\begin{aligned}\Phi : \hat{\Sigma} &\longrightarrow \mathcal{M} \\ p &\longmapsto \mathcal{P}\end{aligned}\tag{A.1}$$

where p is a point in $\hat{\Sigma}$ and \mathcal{P} is a point in the \mathcal{M} . If $\Phi : \hat{\Sigma} \longrightarrow \Phi(\Sigma)$ is a diffeomorphism then we have a smooth embedding. There are three ways to define a submanifold. We can set a collection of scalar functions to a constant or we can give a collection of vector fields which form a Lie algebra. We can also give a collection of one-forms which satisfy some conditions [13]. The easiest way to define a submanifold is by setting functions on the manifold to a constant. One of the results of Frobenius's theorem is that the rank of the pushforward matrix (sometimes also called differential map or the Jacobian matrix) is equal to the dimension of the submanifold. In the case of a hypersurface, a codimension one surface, we set only a single function to a constant. To be able to create a Hamiltonian formulation for relativistic theories, which do not assign any specific meaning to coordinates, we choose a function for which the vector form of the gradient of the function is time-like. In the case of 3+1 splitting of gravity theories we have

$$\mathcal{P} \in \mathcal{M}, \mathcal{P} \in \Sigma \iff t(\mathcal{P}) = t_0, t_0 \in \mathbb{R},\tag{A.2}$$

where t is the time coordinate. The theory is still covariant but it is not manifestly covariant. We will use ${}^{(3)}\mathbf{T}$ to denote that the tensor \mathbf{T} is defined on the manifold Σ . We will also use ${}^{(4)}\mathbf{T}$ to denote that the tensor \mathbf{T} is defined on the manifold \mathcal{M} . These notations will only be used when the tensor in question might be confused with another tensor.

A.3 Normal vector to The Hypersurface Σ

The gradient of a function is orthogonal to its level sets so if we have a scalar field t its normal vector is simply $\mathbf{g}^{-1}(d\tilde{t})$. We have $\Phi : \hat{\Sigma} \longrightarrow \mathcal{M}$ and $t \in C^\infty(\mathcal{M})$ where $C^\infty(\mathcal{M})$ denotes the set of smooth functions on \mathcal{M} . Also if we have χ, x

as any coordinates on \mathcal{M} , Σ respectively, then we can see

$$\begin{aligned}
\Phi(p) = \mathcal{P} \in \Phi(\hat{\Sigma}) &\iff t(\mathcal{P}) = t_0 \\
\vec{X} \in \mathcal{T}_p \hat{\Sigma} &\Rightarrow \Phi_* \vec{X} \in \mathcal{T}_{\mathcal{P}} \mathcal{M} \\
\langle \tilde{dt}, \Phi_* \vec{X} \rangle &= \Phi_* \vec{X} \{t\} \\
&= (\Phi_* \vec{X})^\mu \frac{\partial(t \circ \chi^{-1})}{\partial \chi^\mu} \Big|_{\chi(\Phi(p))} \\
&= X^i \frac{\partial(t \circ \Phi \circ x^{-1})}{\partial x^i} \Big|_{x(p)} = 0
\end{aligned} \tag{A.3}$$

since t is constant on the particular hypersurface. We have some conventions that are used in the choice of unit normal vector $\vec{n} \propto \mathbf{g}^{-1}(\tilde{dt})$. Now, the normal vector can be multiplied with any scalar nonvanishing field and it would still be a normal vector. To make it a unit normal vector we normalize it. It is also chosen for unit normal to be future pointing, that is the time component of the vector is positive, $\langle \tilde{dt}, \vec{n} \rangle > 0$. Let us adopt these conventions while remembering that the normal vector is timelike.

$$\begin{aligned}
\vec{n} &= h \mathbf{g}^{-1}(\tilde{dt}), \quad h \in C^\infty(\mathcal{M}) \\
(h \mathbf{g}^{-1}(\tilde{dt}) \cdot h \mathbf{g}^{-1}(\tilde{dt})) &= -1, \text{ and } \langle \tilde{dt}, \vec{n} \rangle > 0 \\
\langle \tilde{dt}, \vec{n} \rangle &= \langle \tilde{dt}, h \mathbf{g}^{-1}(\tilde{dt}) \rangle = h \left(\mathbf{g}^{-1}(\tilde{dt}) \cdot \mathbf{g}^{-1}(\tilde{dt}) \right) > 0 \\
h \frac{-1}{h^2} > 0 &\iff h < 0 \\
h &= - \left(-\mathbf{g}^{-1}(\tilde{dt}) \cdot \mathbf{g}^{-1}(\tilde{dt}) \right)^{-\frac{1}{2}}, \quad N := -h \text{ and } \vec{n} = -N \mathbf{g}^{-1}(\tilde{dt}).
\end{aligned} \tag{A.4}$$

A.4 Moving Tensors Between Manifolds

There are five operations that we will use extensively to be able to move tensors between vector spaces.

A.4.1 Pushforward and Pullback Maps

Since we have a map $\Phi : \hat{\Sigma} \rightarrow \mathcal{M}$ we can uniquely move vectors from $\mathcal{T}_p \hat{\Sigma}$ to $\mathcal{T}_{\mathcal{P}} \mathcal{M}$ and also we can move one-forms from $\mathcal{T}_{\mathcal{P}}^* \mathcal{M}$ to $\mathcal{T}_p^* \hat{\Sigma}$. Also Φ is considered to be identity so that pullback and pushforward maps are trivial. Let's make this part clear. Given a coordinate chart (x, U) on $\hat{\Sigma}$ and another coordinate

chart (χ, V) on \mathcal{M} , $f \in C^\infty(\mathcal{M})$, $\vec{X} \in \mathcal{T}_p \hat{\Sigma}$ and $\tilde{\omega} \in \mathcal{T}_p^* \mathcal{M}$ we have

$$\begin{aligned}
(\Phi_* \vec{X})\{f\}\Big|_{\mathcal{P}} &= \vec{X}\{f \circ \Phi\}\Big|_p, \Phi(p) = \mathcal{P} \\
(\Phi_* \vec{X})^\mu \frac{\partial(f \circ \chi^{-1})}{\partial \chi^\mu} \Big|_{\chi(\mathcal{P})} &= X^i \frac{\partial(f \circ \Phi \circ x^{-1})}{\partial x^i} \Big|_{x(p)} \\
&= X^i \frac{\partial(f \circ \chi^{-1} \circ \chi \circ \Phi \circ x^{-1})}{\partial x^i} \Big|_{x(p)} \\
&= X^i \frac{\partial(f \circ \chi^{-1})}{\partial \chi^\alpha} \Big|_{\chi(\mathcal{P})} \frac{\partial(\chi^\alpha \circ \Phi \circ x^{-1})}{\partial x^i} \Big|_{x(p)} \\
(\Phi_* \vec{X})^\mu &= \frac{\partial(\chi^\mu \circ \Phi \circ x^{-1})}{\partial x^i} \Big|_{x(p)} X^i. \tag{A.5}
\end{aligned}$$

The identity sense and trivial pushforward map is realized when

$$\begin{aligned}
(\chi^\mu \circ \Phi \circ x^{-1})\Big|_{x(p)} &= \chi^\mu\Big|_{\mathcal{P}} = (t_0, x^1(p), x^2(p), x^3(p))^\mu, \\
\frac{\partial(\chi^\mu \circ \Phi \circ x^{-1})}{\partial x^i} \Big|_{x(p)} &= \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^\mu_i = (\Phi_*)^\mu_i. \tag{A.6}
\end{aligned}$$

This last matrix is the pushforward matrix and we also have a pullback matrix.

$$\begin{aligned}
\langle \Phi^* \tilde{\omega}, \vec{X} \rangle &= \langle \tilde{\omega}, \Phi_* \vec{X} \rangle, \\
(\Phi^* \omega)_i X^i &= \omega_\mu (\Phi_* \vec{X})^\mu, \\
(\Phi^*)^\alpha_j \omega_\alpha X^j &= \omega_\mu (\Phi_*)^\mu_i X^i. \tag{A.7}
\end{aligned}$$

Since the last equation is true for all \vec{X} and $\tilde{\omega}$ we have $(\Phi^*)^\mu_i = (\Phi_*)^\mu_i$. With these maps, we also have an induced metric on the hypersurface. Since the pushforward and pullback matrices are trivial, we can quickly do the above transitions without the need to work them out every time. Let's collect the results:

$$\begin{aligned}
(\omega_0, \omega_1, \omega_3, \omega_3) \in \mathcal{T}_p^* \mathcal{M} &\xrightarrow{\Phi^*} (\omega_1, \omega_2, \omega_3) \in \mathcal{T}_p^* \hat{\Sigma} \\
(X^1, X^2, X^3) \in \mathcal{T}_p \hat{\Sigma} &\xrightarrow{\Phi_*} (X^0 = 0, X^1, X^2, X^3) \in \mathcal{T}_p \mathcal{M} \\
\gamma &= \Phi^* g \\
\gamma_{ij} &= g_{ij}. \tag{A.8}
\end{aligned}$$

Since the Φ, Φ^*, Φ_* maps are now trivial we will drop them. That is $p = \mathcal{P}$, $\hat{\Sigma} = \Sigma$, $\Phi_* \vec{X} = \vec{X}$ and so on. We need to be able to also move between other vector

spaces, however these maps cannot be provided by Φ since it is not surjective thus not invertible.

A.4.2 Orthogonal Projection and Orthogonal Extension

At all points of spacetime, we can decompose the vectors in $\mathcal{T}_p\mathcal{M}$ into two parts, one part in $\mathcal{T}_p\Sigma$ other part in the one dimensional vector space spanned by \vec{n} .

$$\mathcal{T}_p\mathcal{M} = \mathcal{T}_p\Sigma \oplus \text{Vect}(\vec{n}). \quad (\text{A.9})$$

The orthogonal projector is given as

$$\begin{aligned} P_{\perp*} : \mathcal{T}_p\mathcal{M} &\longrightarrow \mathcal{T}_p\Sigma \\ \vec{v} &\longmapsto \vec{v} + \vec{n} (\vec{n} \cdot \vec{v}). \end{aligned} \quad (\text{A.10})$$

In components:

$$\begin{aligned} (P_{\perp*}\vec{v})^\alpha &= P_{\perp*}{}^\alpha{}_\beta v^\beta = v^\alpha + n^\alpha n^\mu v^\nu g_{\mu\nu} \\ P_{\perp*}{}^\alpha{}_\beta &= \delta^\alpha_\beta + n^\alpha n_\beta. \end{aligned} \quad (\text{A.11})$$

As a result, we have $P_{\perp*}\vec{v} \perp \vec{n}$. Also, for $\vec{v} \in \mathcal{T}_p\Sigma$, we have $P_{\perp*}\vec{v} = \vec{v}$. We can orthogonally extend one forms from $\mathcal{T}_p^*\Sigma$ to $\mathcal{T}_p^*\mathcal{M}$.

$$\begin{aligned} P_{\perp*} : \mathcal{T}_p^*\Sigma &\longrightarrow \mathcal{T}_p^*\mathcal{M} \\ \langle P_{\perp*}\tilde{\omega}, \vec{v} \rangle &= \langle \tilde{\omega}, P_{\perp*}\vec{v} \rangle. \end{aligned} \quad (\text{A.12})$$

Let's extend the induced metric.

$$\begin{aligned} (P_{\perp*}\gamma)(\vec{v}, \vec{u}) &= \gamma(P_{\perp*}\vec{v}, P_{\perp*}\vec{u}) \\ &= (P_{\perp*}\vec{v} \cdot P_{\perp*}\vec{u}) \\ &= \vec{v} \cdot \vec{u} + (\vec{v} \cdot \vec{n})(\vec{u} \cdot \vec{n}) + (\vec{u} \cdot \vec{n})(\vec{v} \cdot \vec{n}) \\ &\quad + (\vec{n} \cdot \vec{n})(\vec{v} \cdot \vec{n})(\vec{u} \cdot \vec{n}) \\ &= \vec{v} \cdot \vec{u} + (\vec{u} \cdot \vec{n})(\vec{v} \cdot \vec{n}) \\ (P_{\perp*}\gamma)_{\alpha\beta} &= g_{\alpha\beta} + n_\alpha n_\beta. \end{aligned} \quad (\text{A.13})$$

We can see that this is just what we get if we lower one index of $P_{\perp*}$. Also for $\vec{u}, \vec{v} \in \mathcal{T}_p\Sigma$ we have $(P_{\perp*}\gamma)(\vec{u}, \vec{v}) = \gamma(\vec{u}, \vec{v})$. For any $\vec{X} \in \mathcal{T}_p\mathcal{M}$ we have $(P_{\perp*}\gamma)(\vec{X}, \vec{n}) = 0$ and $(P_{\perp*}\gamma)(\vec{n}, \vec{n}) = 0$. So $P_{\perp*}\gamma_{\alpha\beta}$ can be called the extended induced metric, since it reduces to the metric for spatial vectors and is zero if it gets a vector parallel to \vec{n} . We will also drop $P_{\perp*}{}^\alpha{}_\beta$ from expressions and just

use $\gamma^\alpha_\beta = \delta^\alpha_\beta + n^\alpha n_\beta$. We will also use $\gamma^{\alpha\beta}$ and $\gamma_{\alpha\beta}$. There is one more map we need to define. This one is:

$$\begin{aligned} \perp : \mathcal{T}_p\mathcal{M} \otimes \mathcal{T}_p^*\mathcal{M} &\longrightarrow \mathcal{T}_p\mathcal{M} \otimes \mathcal{T}_p^*\mathcal{M} \\ (\perp T)^\alpha_\beta &= T^\mu_\nu \gamma^\alpha_\mu \gamma^\nu_\beta. \end{aligned} \quad (\text{A.14})$$

Extension to higher rank tensors is clear. The expression on the right hand side is orthogonal to \vec{n} in the sense that if it contracts with the normal vector or its one-form dual then it is zero. So the right hand side is tangent to Σ .

A.4.3 Decomposing Tensors

This part is best understood with an example. So let's decompose a second rank tensor. We will just change the identity tensor for the extended induced metric.

$$\begin{aligned} A_{\alpha\beta} &= A_{\mu\nu} \delta^\mu_\alpha \delta^\nu_\beta \\ &= A_{\mu\nu} (\gamma^\mu_\alpha - n^\mu n_\alpha) (\gamma^\nu_\beta - n^\nu n_\beta) \\ &= (\gamma^\mu_\alpha \gamma^\nu_\beta A_{\mu\nu}) - (A_{\mu\nu} \gamma^\mu_\alpha n^\nu) n_\beta - (\gamma^\nu_\beta A_{\mu\nu} n^\mu) n_\alpha \\ &\quad + (A_{\mu\nu} n^\mu n^\nu) n_\beta n_\alpha \\ &= a_{\alpha\beta} - b_\beta n_\alpha - c_\alpha n_\beta + d n_\alpha n_\beta. \end{aligned} \quad (\text{A.15})$$

From this we can see that a second rank tensor is separated into four parts. One part is a completely spatial tensor ($a_{\alpha\beta} = \gamma^\mu_\alpha \gamma^\nu_\beta A_{\mu\nu}$), one part is once projected onto normal direction in the first index and once projected onto space ($b_\beta = \gamma^\nu_\beta n^\mu A_{\mu\nu}$), another part is once projected onto normal direction in the second index and once projected onto space ($c_\alpha = \gamma^\mu_\alpha n^\nu A_{\mu\nu}$) and a part that is completely projected onto normal direction ($d = n^\mu n^\nu A_{\mu\nu}$). When the tensor to be decomposed have symmetries the number of independent parts will decrease. For instance if A was symmetric in its indices then we would have $b_\alpha = c_\alpha$. Also for spatial tensors we can change the contraction range to only spatial indices, that is, $S^{\alpha\dots\beta\dots} T_{\alpha\dots\beta\dots} = S^{i\dots j\dots} T_{i\dots j\dots}$ for S, T spatial tensors. The reason is $\tilde{d}x^0 = \tilde{d}t \propto \vec{n}$ and so upper zero indices are zero for spatial tensors.

A.5 The Link between the ∇ and D connections

Given a tensor field on Σ its covariant derivative according to D is expressible in terms of the covariant derivative on \mathcal{M} according to:

$$P_{\perp}^*(D\mathbf{T}) = \perp(\nabla P_{\perp}^*\mathbf{T}), \quad (\text{A.16})$$

$$D_{\rho}T^{\alpha}_{\beta} = \gamma^{\sigma}_{\rho}\gamma^{\alpha}_{\mu}\gamma^{\nu}_{\beta}\nabla_{\sigma}T^{\mu}_{\nu}. \quad (\text{A.17})$$

Normally, we would need to extend the tensor field \mathbf{T} on Σ to \mathcal{M} in some way since the support (domain of definition) of \mathbf{T} is not an open set and we need open sets to be able to take derivatives. In the end however the extension does not enter into the formula since we are projecting orthogonally.

A.6 Extrinsic Curvature of The Hypersurface

Intrinsic curvature of Σ is determined fully by the induced metric, γ , in the usual way. We can define another curvature called the extrinsic curvature for embeddings. For example, we know all one dimensional manifolds are Riemann-flat. However, a circle looks curved. This is because it is embedded in a higher dimensional manifold. We define the Weingarten map (or the shape operator) as the endomorphism

$$\begin{aligned} \chi : \mathcal{T}_p\Sigma &\longrightarrow \mathcal{T}_p\Sigma \\ \vec{v} &\longmapsto \nabla_{\vec{v}}\vec{n} = \chi(\vec{v}), \end{aligned}$$

which satisfies

$$\vec{n} \cdot \nabla_{\vec{v}}\vec{n} = \frac{1}{2}\nabla_{\vec{v}}(\vec{n} \cdot \vec{n}) = 0. \quad (\text{A.18})$$

The last equation indicates that $\nabla_{\vec{v}}\vec{n} \in \mathcal{T}_p\Sigma$ so the map is well-defined. Weingarten map is also self adjoint according to the inner product. That is

$$\vec{u} \cdot \chi(\vec{v}) = \vec{v} \cdot \chi(\vec{u}) \quad (\text{A.19})$$

for $\forall(\vec{u}, \vec{v}) \in \mathcal{T}_p\Sigma$. We can show this as follows:

$$\begin{aligned} \vec{u} \cdot \chi(\vec{v}) &= \vec{u} \cdot \nabla_{\vec{v}}\vec{n} \\ &= \nabla_{\vec{v}}(\vec{u} \cdot \vec{n}) - \vec{n} \cdot \nabla_{\vec{v}}\vec{u} \\ &= \vec{n} \cdot (-\nabla_{\vec{u}}\vec{v} - [\vec{v}, \vec{u}]) \end{aligned}$$

$$\begin{aligned}
&= -\vec{n} \cdot \nabla_{\vec{u}} \vec{v} = -\nabla_{\vec{u}}(\vec{v} \cdot \vec{n}) + \vec{v} \cdot \nabla_{\vec{u}} \vec{n} \\
&= \vec{v} \cdot \nabla_{\vec{u}} \vec{n} = \vec{v} \cdot \chi(\vec{u}),
\end{aligned} \tag{A.20}$$

where we used the fact that the commutator of two spatial vectors is again a spatial vector according to the Frobenius's theorem. Since the Weingarten map is an endomorphism we can find its eigenvalues and eigenvectors by solving the equation $\chi(\vec{v}) = \kappa \vec{v}$. The eigenvalues κ are called principal curvatures and the eigenvectors \vec{v} are called principal directions. The mean curvature of Σ is the average of the principal curvatures. We define the bilinear form \mathbf{K} as

$$\begin{aligned}
\mathbf{K} &: \mathcal{T}_p \Sigma \otimes \mathcal{T}_p \Sigma \longrightarrow \mathbb{R}, \\
(\vec{u}, \vec{v}) &\longmapsto -\vec{u} \cdot \chi(\vec{v}),
\end{aligned} \tag{A.21}$$

which is called the extrinsic curvature of Σ . It is also called the second fundamental form of Σ . It is symmetric in its arguments. The minus sign in the definition is a convention. Numerical relativity community [10] and [9] use it while [13],[14] and [15] use the opposite convention. We will extend $\mathbf{K} \in \mathcal{T}_p^* \Sigma \otimes \mathcal{T}_p^* \Sigma$ to $\mathcal{T}_p^* \mathcal{M} \otimes \mathcal{T}_p^* \mathcal{M}$ with the help of the orthogonal projector. First let's define a new quantity. Normally we can't take the gradient of \vec{n} since it does not lie on an open subset of \mathcal{M} . Consider any extension of \vec{n} . Then $\vec{a} = \nabla_{\vec{n}} \vec{n}$. Since $(\vec{n} \cdot \vec{n}) = -1$ which is a constant, we have $\vec{a} \in \mathcal{T}_p \Sigma$. So we don't even need to project it. It is called the four-acceleration of \vec{n} since \vec{n} is timelike and has unit magnitude.

$$\begin{aligned}
\mathbf{K}(\vec{u}, \vec{v}) &= (P_{\perp}^* \mathbf{K})(\vec{u}, \vec{v}) \quad \forall (\vec{u}, \vec{v}) \in \mathcal{T}_p \mathcal{M} \\
\mathbf{K}(P_{\perp}^* \vec{u}, P_{\perp}^* \vec{v}) &= -P_{\perp}^* \vec{u} \cdot \nabla_{P_{\perp}^* \vec{v}} \vec{n} \\
&= -\left(\vec{u} + \vec{n} (\vec{n} \cdot \vec{u}) \right) \cdot \nabla_{\vec{v} + \vec{n} (\vec{n} \cdot \vec{v})} \vec{n} \\
&= \left(\vec{u} + \vec{n} (\vec{n} \cdot \vec{u}) \right) \cdot \left(\nabla_{\vec{v}} \vec{n} + (\vec{v} \cdot \vec{n}) \nabla_{\vec{n}} \vec{n} \right) \\
&= -\vec{u} \cdot \nabla_{\vec{v}} \vec{n} - (\vec{n} \cdot \vec{v}) (\vec{u} \cdot \nabla_{\vec{n}} \vec{n}) \\
&= -\nabla \vec{n}(\vec{u}, \vec{v}) - (\vec{u} \cdot \vec{a}) (\vec{n} \cdot \vec{v}) \\
K_{\alpha\beta} &= -\nabla_{\beta} n_{\alpha} - n_{\beta} a_{\alpha},
\end{aligned} \tag{A.22}$$

where we used the fact that $\vec{n} \cdot \nabla_{\vec{x}} \vec{n} = 0$. It can be shown that

$$\mathbf{K} = -\perp(\nabla \vec{n}) \tag{A.23}$$

and $K^\alpha{}_\alpha = -\nabla_\alpha n^\alpha = K^i{}_i$. We have an interesting equation which is called the Gauss-Weingarten equation. Let us calculate the spatial covariant derivative of a spatial vector along another spatial vector.

$$\begin{aligned}
\forall(\vec{u}, \vec{v}) \in \mathcal{T}_p\Sigma, \quad (D_{\vec{u}}\vec{v})^\alpha &= u^\sigma D_\sigma v^\alpha = u^\sigma \gamma^\nu{}_\sigma \gamma^\alpha{}_\beta \nabla_\nu v^\beta \\
&= u^\nu (\delta_\beta^\alpha + n^\alpha n_\beta) \nabla_\nu v^\beta \\
&= u^\nu \nabla_\nu v^\alpha + u^\nu n^\alpha n_\beta \nabla_\nu v^\beta \\
&= u^\nu \nabla_\nu v^\alpha + u^\nu n^\alpha n_\beta \nabla_\nu v^\beta \\
&\quad + u^\nu n^\alpha v_\beta \nabla_\nu n^\beta - u^\nu n^\alpha v^\beta \nabla_\nu n_\beta \\
&= u^\nu \nabla_\nu v^\alpha - u^\nu n^\alpha v^\beta \nabla_\nu n_\beta \\
&= (\nabla_{\vec{u}}\vec{v})^\alpha - n^\alpha (\vec{v} \cdot \nabla_{\vec{u}}\vec{n}) \\
&= (\nabla_{\vec{u}}\vec{v})^\alpha + n^\alpha \mathbf{K}(\vec{u}, \vec{v}) \\
D_{\vec{u}}\vec{v} &= \nabla_{\vec{u}}\vec{v} + \vec{n}\mathbf{K}(\vec{u}, \vec{v}), \quad \forall(\vec{u}, \vec{v}) \in \mathcal{T}_p\Sigma. \quad (\text{A.24})
\end{aligned}$$

When $\mathbf{K} = 0$ we have what is called a totally geodesic hypersurface, since the geodesics of Σ are also geodesics of \mathcal{M} . So, in a sense \mathbf{K} measures the deviation between covariant derivatives ∇ and D , as is clear from $D_{\vec{u}}\vec{v} - \nabla_{\vec{u}}\vec{v} = \vec{n}\mathbf{K}(\vec{u}, \vec{v})$.

A.7 Globally hyperbolic spacetimes and foliations

We will assume that the spacetime can be foliated. This assumption is equivalent to accepting that the spacetime is globally hyperbolic that is it admits a Cauchy surface. A Cauchy surface is a spacelike hypersurface in the spacetime that is intersected by causal curves (worldline of real particles i.e. photons and electrons and such, not tachyons) only once. Its domain of dependence (the events that are affected by it and the events that affect it) is the whole spacetime. The topology of a globally hyperbolic spacetime is necessarily homeomorphic to $\Sigma \otimes \mathbb{R}$.

A.8 Foliation Kinematics

A.8.1 The Lapse Function

When the coordinates are chosen so that the shift vector is zero, (described in the section A.9.1) the lapse function gives the amount of proper time between

the two points which has the same spatial coordinates [9].

$$\begin{aligned}\vec{\nabla}t &:= \mathbf{g}^{-1}(\tilde{dt}), \quad N = \sqrt{-\left(\vec{\nabla}t \cdot \vec{\nabla}t\right)} \\ \vec{n} &= -N\vec{\nabla}t.\end{aligned}\tag{A.25}$$

N is called the lapse function and due to the future pointing convention and the regularity of time scalar field, it is positive definite.

A.8.2 Eulerian Observers

Since \vec{n} is a unit timelike vector pointing towards future it can be considered as the four velocity of some observer. This observer is called a Eulerian observer. Such an observer has the four-acceleration $\vec{a} = \nabla_{\vec{n}}\vec{n}$. Let us find a nice relation on this expression.

$$\begin{aligned}a_\alpha &= n^\mu \nabla_\mu n_\alpha = n^\mu \nabla_\mu (-N \nabla_\alpha t) \\ &= -N n^\mu \nabla_\mu \nabla_\alpha t - \nabla_\alpha t (n^\mu \nabla_\mu N) \\ &= -N n^\mu \nabla_\alpha \nabla_\mu t + \frac{(n_\alpha)}{N} (n^\mu \nabla_\mu N) \\ &= -n^\mu \nabla_\alpha (N \nabla_\mu t) + n^\mu \nabla_\mu t \nabla_\alpha N + \frac{(n_\alpha)}{N} (n^\mu \nabla_\mu N) \\ &= n^\mu \nabla_\alpha n_\mu + \frac{\nabla_\alpha N}{N} + \frac{(n_\alpha)}{N} (n^\mu \nabla_\mu N) \\ &= \frac{1}{N} \nabla_\mu N (\delta_\alpha^\mu + n_\alpha n^\mu) = \gamma^\mu_\alpha \frac{\nabla_\mu N}{N} \\ &= \frac{D_\alpha N}{N} = D_\alpha (\ln N),\end{aligned}\tag{A.26}$$

where we have used $\nabla_{\vec{n}}t = 1/N$.

A.8.3 Normal evolution vector

Suppose we have a tensor field $\mathbf{A} \in \Gamma(\mathcal{T}_1^1\mathcal{M})$ and two infinitesimally close points $p \in \Sigma_{t_0}$ and $\acute{p} \in \Sigma_{t_0+\delta t}$. What is the first order correction to the tensor between these two points? Let us find it.

$$x^\mu|_{\acute{p}} = x^\mu + \delta t \xi^\mu|_p\tag{A.27}$$

One might suspect that this looks like an infinitesimal diffeomorphism and expect that there will be Lie derivatives in the first order correction. Let us make this part clear. The basis vectors act on coordinate functions as in $\vec{\partial}_\alpha \{x^\beta\}|_p = \delta_\alpha^\beta$

and dual coordinate basis is defined as $\langle \tilde{d}x^\alpha, \vec{\partial}_\beta \rangle = \delta_\beta^\alpha$. The first order correction to the bases can be found from these relations.

$$\begin{aligned}\vec{\partial}_\alpha \Big|_{\tilde{p}} &= (\vec{\partial}_\alpha + \delta\vec{\partial}_\alpha) \Big|_p, \quad \delta\vec{\partial}_\alpha = \epsilon^\lambda{}_\alpha \vec{\partial}_\lambda \\ \vec{\partial}_\alpha \{x^\mu\} \Big|_{\tilde{p}} &= \delta_\alpha^\mu \\ (\vec{\partial}_\alpha + \epsilon^\lambda{}_\alpha \vec{\partial}_\lambda) \{x^\mu + \delta t \xi^\mu\} \Big|_p &= \delta_\beta^\alpha \\ \delta_\alpha^\mu + \delta t \xi^\mu{}_{,\alpha} + \epsilon^\mu{}_\alpha &= \delta_\alpha^\mu \Rightarrow \epsilon^\mu{}_\alpha = -\delta t \xi^\mu{}_{,\alpha}\end{aligned}$$

and the correction to the dual basis is calculated as

$$\begin{aligned}\langle \tilde{d}x^\alpha, \vec{\partial}_\beta \rangle &= \delta_\beta^\alpha, \quad \tilde{d}x^\alpha \Big|_{\tilde{p}} = \tilde{d}x^\alpha \Big|_p + \delta \tilde{d}x^\alpha \Big|_p \\ \langle \tilde{d}x^\alpha + \delta \tilde{d}x^\alpha, \vec{\partial}_\beta - \delta t \xi^\mu{}_{,\beta} \vec{\partial}_\mu \rangle &= \delta_\beta^\alpha \\ \delta \tilde{d}x^\alpha &= \delta t \xi^\alpha{}_{,\beta} \tilde{d}x^\beta.\end{aligned}\tag{A.28}$$

Now we can find the correction a tensor gets.

$$\begin{aligned}\delta \mathbf{A} \Big|_p &= \mathbf{A} \Big|_{\tilde{p}} - \mathbf{A} \Big|_p \\ &= A^\mu{}_\nu \vec{\partial}_\mu \tilde{d}x^\nu \Big|_{\tilde{p}} - A^\mu{}_\nu \vec{\partial}_\mu \tilde{d}x^\nu \Big|_p \\ &= (A^\mu{}_\nu (x^\alpha + \delta t \xi^\alpha)) (\vec{\partial}_\mu - \delta t \xi^\lambda{}_{,\mu} \vec{\partial}_\lambda) (\tilde{d}x^\nu + \delta t \xi^\nu{}_{,\lambda} \tilde{d}x^\lambda) \Big|_p \\ &\quad - A^\mu{}_\nu (x^\alpha) \vec{\partial}_\mu \tilde{d}x^\nu \Big|_p \\ \delta \mathbf{A} \Big|_p &= \delta t \left(\xi^\alpha \frac{\partial A^\mu{}_\nu}{\partial x^\alpha} - \xi^\mu{}_{,\lambda} A^\lambda{}_\nu + \xi^\lambda{}_{,\nu} A^\mu{}_\lambda \right) \vec{\partial}_\mu \tilde{d}x^\nu \Big|_p.\end{aligned}\tag{A.29}$$

If we define a vector field $\vec{\xi} = \xi^\mu \vec{\partial}_\mu$ then we can see that the first order correction is going to be the Lie derivative of this tensor along $\vec{\xi}$. Not every vector field is acceptable. We have the restriction that

$$t \Big|_{\tilde{p}} = t_0 + \delta t = t_0 + \delta t \xi^0 \Big|_p \Rightarrow \xi^0 = 1.\tag{A.30}$$

We might also impose other restrictions on $\vec{\xi}$. The most obvious restriction one might want is that a spatial tensor remains spatial. This brings about another restriction on $\vec{\xi}$. Let us see what it is. First we rewrite $\vec{\xi} = N \vec{n} + \vec{s}$, $\vec{s} \in \Gamma(\mathcal{T}\Sigma)$ since ξ^0 is fixed and every spatial vector has its zero component as zero. Let us first check for spatial one-forms.

$$\begin{aligned}\langle \tilde{\omega}, \vec{n} \rangle &= 0, \quad \langle \tilde{\omega} + \delta \tilde{\omega}, \vec{n} + \delta \vec{n} \rangle = 0, \\ \delta t (\langle \mathcal{L}_{\vec{\xi}} \tilde{\omega}, \vec{n} \rangle + \langle \tilde{\omega}, \mathcal{L}_{\vec{\xi}} \vec{n} \rangle) &= 0,\end{aligned}$$

$$\delta t \mathcal{L}_{\vec{\xi}} \langle \tilde{\omega}, \vec{n} \rangle = 0. \quad (\text{A.31})$$

For spatial one forms we do not get a further restriction on $\vec{\xi}$. Now, let us look at spatial vectors.

$$\begin{aligned} \vec{v} \cdot \vec{n} &= 0, \quad (\vec{v} + \delta \vec{v}) \cdot (\vec{n} + \delta \vec{n}) = 0, \\ \delta t \vec{v} \cdot [N \vec{n} + \vec{s}, \vec{n}] + \delta t \vec{n} \cdot [N \vec{n} + \vec{s}, \vec{v}] &= 0. \end{aligned} \quad (\text{A.32})$$

After using $[\vec{v}, \vec{u}] = \nabla_{\vec{v}} \vec{u} - \nabla_{\vec{u}} \vec{v}$ and the product rule for covariant derivative one reaches

$$\begin{aligned} N \vec{v} \cdot \nabla_{\vec{n}} \vec{n} + \vec{v} \cdot \nabla_{\vec{s}} \vec{n} - (\vec{v} \cdot \vec{n}) \vec{n} \{N\} - N \vec{v} \cdot \nabla_{\vec{n}} \vec{n} - \vec{v} \cdot \nabla_{\vec{n}} \vec{s} \\ + N \vec{n} \cdot \nabla_{\vec{n}} \vec{v} + \vec{n} \cdot \nabla_{\vec{s}} \vec{v} - \vec{v} \{N\} (\vec{n} \cdot \vec{n}) - N \vec{n} \cdot \nabla_{\vec{v}} \vec{n} - \vec{n} \cdot \nabla_{\vec{v}} \vec{s} &= 0. \end{aligned} \quad (\text{A.33})$$

After using $\vec{n} \cdot \vec{n} = -1$, $\vec{v} \cdot \vec{n} = 0$, $\vec{s} \cdot \vec{n} = 0$, $\vec{v} \cdot \nabla_{\vec{n}} \vec{n} = \frac{\vec{v} \{N\}}{N}$ one gets

$$\vec{v} \cdot \nabla_{\vec{n}} \vec{s} + \vec{n} \cdot \nabla_{\vec{v}} \vec{s} = 0, \quad (\text{A.34})$$

or in component form

$$v^\alpha n^\beta (\nabla_\beta s_\alpha + \nabla_\alpha s_\beta) = 0. \quad (\text{A.35})$$

The last equation is like an equation for Killing vectors and unless the spacetime has an isometry it will not be satisfied for non-trivial solutions. So, in the end for vectors to remain spatial we get a restriction that is, $\vec{s} = 0$. So, as long as we are evolving the spatial tensors with $\vec{\xi} = N \vec{n}$ spatial tensors will remain spatial. We will rename $\vec{\xi}$ to $\vec{m} = N \vec{n}$ and call it the normal evolution vector.

A.8.4 Gradients of the unit normal and the normal evolution vector

These are calculated from the equation (A.22) and the definition of the normal evolution vector easily.

$$\nabla_\beta n_\alpha = -K_{\alpha\beta} - \frac{D_\alpha N}{N} n_\beta, \quad (\text{A.36})$$

$$\nabla_\beta m^\alpha = -N K^\alpha_\beta - D^\alpha N n_\beta + n^\alpha \nabla_\beta N. \quad (\text{A.37})$$

A.8.5 Evolution of the induced metric

We can calculate this by substituting the equations (A.36), (A.37) and (A.13).

$$\begin{aligned} \mathcal{L}_{\vec{m}} \gamma_{\alpha\beta} &= m^\sigma \nabla_\sigma \gamma_{\alpha\beta} + \gamma_{\sigma\beta} \nabla_\alpha m^\sigma + \gamma_{\alpha\sigma} \nabla_\beta m^\sigma, \\ \mathcal{L}_{\vec{m}} \gamma_{\alpha\beta} &= -2N K_{\alpha\beta}. \end{aligned} \quad (\text{A.38})$$

A.8.6 Evolution of the Orthogonal Projector

We have $P_{\perp*}^{\alpha\beta} = P_{\perp}^{*\alpha\beta} = \perp^{\alpha\beta} = \gamma^{\alpha\beta}$. Again, by substituting the equations (A.37), (A.36) and (A.13) one can show

$$\mathcal{L}_{\vec{m}}\gamma^{\alpha\beta} = m^{\sigma}\nabla_{\sigma}\gamma^{\alpha\beta} - \gamma^{\sigma\beta}\nabla_{\sigma}m^{\alpha} + \gamma^{\alpha\sigma}\nabla_{\beta}m^{\sigma} = 0. \quad (\text{A.39})$$

This has the implication that a spatial tensor remains spatial while we are moving between slices of the foliation since for spatial tensors one has $A^{\alpha}_{\beta} = \gamma^{\alpha}_{\mu}A^{\mu}_{\nu}\gamma^{\nu}_{\beta}$.

A.9 Coordinates adapted to the foliation

On each hypersurface, one can introduce a coordinate system $x^i = \{x^1, x^2, x^3\}$. If this coordinate system changes smoothly between slices of the foliation then $x^{\alpha} = \{x^0 = t, x^1, x^2, x^3\}$ constitutes a well-behaved coordinate system.

A.9.1 The Shift vector

The difference between $\vec{\partial}_t$ and \vec{m} is called the shift vector $\vec{\beta} := \vec{\partial}_t - \vec{m}$. It is a spatial vector. Then we can write \vec{n} , \tilde{n} as below.

$$n^{\alpha} = \{1/N, -\beta^i/N\}, \quad (\text{A.40})$$

$$n_{\alpha} = \{-N, 0, 0, 0\}. \quad (\text{A.41})$$

With this knowledge we can write the metric tensor and the extended induced metric explicitly.

$$\begin{aligned} g_{\alpha\beta} &= \vec{\partial}_{\alpha} \cdot \vec{\partial}_{\beta}, \\ &= \begin{bmatrix} -N^2 + \beta_k\beta^k & \beta_j \\ \beta_i & \gamma_{ij} \end{bmatrix}_{\alpha\beta}. \end{aligned} \quad (\text{A.42})$$

$$\gamma_{\alpha\beta} = g_{\alpha\beta} + n_{\alpha}n_{\beta}, \quad (\text{A.43})$$

$$= \begin{bmatrix} \beta_k\beta^k & \beta_j \\ \beta_i & \gamma_{ij} \end{bmatrix}_{\alpha\beta}. \quad (\text{A.44})$$

By taking the matrix inverse of $g_{\alpha\beta}$ we get

$$g^{\alpha\beta} = \begin{bmatrix} -1/N^2 & \beta^j/N^2 \\ \beta^i/N^2 & \gamma^{ij} - (\beta^i\beta^j)/N^2 \end{bmatrix}^{\alpha\beta}, \quad (\text{A.45})$$

$$\gamma^{\alpha\beta} = g^{\alpha\beta} + n^\alpha n^\beta \quad (\text{A.46})$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & \gamma^{ij} \end{bmatrix}^{\alpha\beta}. \quad (\text{A.47})$$

We can also express the γ^α_β explicitly:

$$\gamma^\alpha_\beta = \delta^\alpha_\beta + n^\alpha n_\beta \quad (\text{A.48})$$

$$= \begin{bmatrix} 0 & 0 \\ \beta^i/N^2 & \delta_j^i \end{bmatrix}^\alpha_\beta. \quad (\text{A.49})$$

These coordinates are called the ADM coordinates.

A.10 Decomposition of the Riemann tensor

We are just going to list the results here and give the main points in the derivation of the decomposition. They are done very explicitly in Éricourgoulhon's lecture notes [10].

A.10.1 Gauss Relation

Let us fully project the Riemann tensor onto Σ . It is much easier to start with the Ricci identity for the connection D .

$$\begin{aligned} D_\alpha D_\beta v^\gamma - D_\beta D_\alpha v^\gamma &= R^\gamma_{\mu\alpha\beta} v^\mu; \quad \vec{v} \in \mathcal{T}_p \Sigma \\ D_\alpha D_\beta v^\gamma &= \gamma^\mu_\alpha \gamma^\nu_\beta \gamma^\gamma_\rho \nabla_\alpha (D_\nu v^\rho) \\ &= \gamma^\mu_\alpha \gamma^\nu_\beta \gamma^\gamma_\rho \nabla_\alpha (\gamma^\sigma_\nu \gamma^\rho_\lambda \nabla_\sigma v^\lambda). \end{aligned} \quad (\text{A.50})$$

With the help of equations (A.36) and (A.22) and also using the idempotent property of projection operators one can show

$$D_\alpha D_\beta v^\gamma = -K_{\alpha\beta} \gamma^\gamma_\lambda n^\sigma \nabla_\sigma v^\lambda - K^\gamma_\alpha K_{\beta\lambda} v^\lambda + \gamma^\mu_\alpha \gamma^\sigma_\beta \gamma^\gamma_\lambda \nabla_\mu \nabla_\sigma v^\lambda. \quad (\text{A.51})$$

Then subtracting and using the symmetry of extrinsic curvature one gets

$$\begin{aligned} D_\alpha D_\beta v^\gamma - D_\beta D_\alpha v^\gamma &= (K_{\alpha\mu} K^\gamma_\beta - K_{\beta\mu} K^\gamma_\alpha) v^\mu \\ &\quad + \gamma^\rho_\alpha \gamma^\sigma_\beta \gamma^\gamma_\lambda (\nabla_\rho \nabla_\sigma v^\lambda - \nabla_\sigma \nabla_\rho v^\lambda) \\ \gamma^\mu_\alpha \gamma^\nu_\beta \gamma^\gamma_\rho \gamma^\sigma_\lambda {}^{(4)}R^\rho_{\sigma\mu\nu} &= R^\gamma_{\delta\alpha\beta} + K_{\delta\beta} K^\gamma_\alpha - K_{\alpha\sigma} K^\gamma_\beta. \end{aligned} \quad (\text{A.52})$$

By using $\gamma^\alpha_\mu v^\mu = v^\alpha$, $\vec{v} \in \mathcal{T}_p \Sigma$ one can reach the last equation. The equation (A.52) is called the Gauss relation. By contracting on γ , α indices one gets the

contracted Gauss relation

$$\gamma^\mu{}_\alpha \gamma^\nu{}_\beta {}^{(4)}R_{\mu\nu} + \gamma_{\mu\alpha} n^\nu \gamma^\rho{}_\beta n^\sigma {}^{(4)}R^\mu{}_{\nu\rho\sigma} = R_{\alpha\beta} + KK_{\alpha\beta} - K_{\alpha\mu} K^\mu{}_\beta. \quad (\text{A.53})$$

Contracting once more one gets

$${}^{(4)}R + 2{}^{(4)}R_{\mu\nu} n^\mu n^\nu = R + K^2 - K^{ij} K_{ij}. \quad (\text{A.54})$$

A.10.2 Codazzi Relation

Apply the Ricci identity to unit normal vector and then project it three times onto Σ .

$$\begin{aligned} (\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha) n^\gamma &= {}^{(4)}R^\gamma{}_{\mu\alpha\beta} n^\mu \\ \gamma^\mu{}_\alpha \gamma^\nu{}_\beta \gamma^\gamma{}_\rho (\nabla_\mu \nabla_\nu n^\rho - \nabla_\nu \nabla_\mu n^\rho) &= \gamma^\mu{}_\alpha \gamma^\nu{}_\beta \gamma^\gamma{}_\rho {}^{(4)}R^\rho{}_{\sigma\mu\nu} n^\sigma. \end{aligned} \quad (\text{A.55})$$

By using (A.36), (A.17) and (A.23) one can show

$$\gamma^\gamma{}_\rho n^\sigma \gamma^\mu{}_\alpha \gamma^\nu{}_\beta {}^4R^\rho{}_{\sigma\mu\nu} = D_\beta K^\gamma{}_\alpha - D_\alpha K^\gamma{}_\beta \quad (\text{A.56})$$

called Codazzi relation or Codazzi-Mainardi relation. Contracting it on γ and α one gets

$$\gamma^\mu{}_\alpha n^\nu {}^{(4)}R_{\mu\nu} = D_\alpha K - D_\mu K^\mu{}_\alpha. \quad (\text{A.57})$$

A.10.3 Ricci Equation

First we are going to need a quantity that will be substituted later. Let us show it.

$$\mathcal{L}_{\vec{m}} K_{\alpha\beta} = N n^\mu \nabla_\mu K_{\alpha\beta} - 2N K_{\alpha\mu} K^\mu{}_\beta - K_{\alpha\mu} D^\mu N n_\beta - K_{\beta\mu} D^\mu N n_\alpha. \quad (\text{A.58})$$

which is obtained by just substituting (A.37) in the Lie derivative. Since extrinsic curvature is spatial we can project its Lie derivative according to \vec{m} , by \perp and not lose any information.

$$\mathcal{L}_{\vec{m}} K_{\alpha\beta} = N \gamma^\mu{}_\alpha \gamma^\nu{}_\beta n^\sigma \nabla_\sigma K_{\mu\nu} - 2N K_{\alpha\mu} K^\mu{}_\beta. \quad (\text{A.59})$$

Now we can find the part of Riemann tensor that is twice projected onto normal direction. Start by applying Ricci identity to unit normal and project the tensor twice onto Σ and once onto normal direction.

$$\gamma_{\alpha\mu} n^\sigma \gamma^\nu{}_\beta (\nabla_\nu \nabla_\sigma - \nabla_\sigma \nabla_\nu) n^\mu = \gamma_{\alpha\mu} n^\sigma \gamma^\nu{}_\beta {}^{(4)}R^\mu{}_{\rho\nu\sigma} n^\rho \quad (\text{A.60})$$

$$\begin{aligned} \gamma_{\alpha\mu}n^\rho\gamma^\nu_\beta n^{\sigma(4)}R^\mu_{\rho\nu\sigma} &= -K_{\alpha\sigma}K^\sigma_\beta + \frac{1}{N}D_\beta D_\alpha N \\ &+ \gamma^\mu_\alpha\gamma^\nu_\beta n^\sigma\nabla_\sigma K_{\mu\nu}. \end{aligned} \quad (\text{A.61})$$

The way to get this equation is by using the equation (A.36) successively. After replacing the last term which also exists in equation (A.59) we reach to the result:

$$\gamma_{\alpha\mu}n^\rho\gamma^\nu_\beta n^{\sigma(4)}R^\mu_{\rho\nu\sigma} = \frac{1}{N}\mathcal{L}_{\vec{m}}K_{\alpha\beta} + K_{\alpha\sigma}K^\sigma_\beta + \frac{1}{N}D_\beta D_\alpha N. \quad (\text{A.62})$$

This equation is called the Ricci equation. With these three equations we have the full extent of Riemann tensor expressed entirely in terms projected parts. We can't project the Riemann tensor three times onto normal direction because of the antisymmetry of Riemann tensor.

A.11 Calculation of Christoffel Symbols

For some of the modified theories of gravity we need the Christoffel symbols because we have not been able to calculate some of the terms like $\nabla_{\vec{n}}T_{ij}$ $\mathbf{T} \in \mathcal{T}_p^*\Sigma \otimes \mathcal{T}_p^*\Sigma$ without the Christoffel symbols. We will not use the usual formula for Christoffel symbols since it will be hard to pull out spatial tensor quantities from the partial derivatives. This section is entirely based on the knowledge gathered from chapter 14 of [9]. We can even calculate the decomposition of Riemann tensor with the method that will be described shortly but we will be content with Christoffel symbols. We will try to give the calculations concisely so some of the trivial steps will not be shown explicitly.

A.11.1 Extended exterior derivative

One can extend the definition of exterior derivative. First we will make a change of point of view and look at some tensors as tensor valued forms. This will be clear shortly. For the usual exterior derivative we have

$$\langle d\tilde{f}, \vec{u} \rangle := \nabla_{\vec{u}}f = \vec{u}\{f\} \quad (\text{A.63})$$

$$(d\tilde{\omega})(\vec{u}, \vec{v}) := \vec{u}\{\langle \tilde{\omega}, \vec{v} \rangle\} - \vec{v}\{\langle \tilde{\omega}, \vec{u} \rangle\} - \langle \tilde{\omega}, [\vec{u}, \vec{v}] \rangle \quad (\text{A.64})$$

These two equations together with the product rule for the exterior derivative that is,

$$d\alpha \wedge \beta = d\alpha \wedge \beta + (-1)^p\alpha \wedge d\beta \quad (\text{A.65})$$

$\alpha \in \Omega^p(\mathcal{M})$, $\beta \in \Omega^q(\mathcal{M})$ allow us to find the exterior derivative for all p-forms. To extend the definition of exterior derivative to tensor valued forms we will just add another defining equation and modify the product rule.

$$\langle d\vec{v}, \vec{u} \rangle := \nabla_{\vec{u}} \vec{v} \Rightarrow d\vec{v} := \nabla \vec{v} \Rightarrow (d\vec{v})^\alpha{}_\beta \vec{\partial}_\alpha \tilde{d}x^\beta = \nabla_\beta v^\alpha \vec{\partial}_\alpha \tilde{d}x^\beta \quad (\text{A.66})$$

$$d\mathbf{S} \wedge \beta = d\mathbf{S} \wedge \beta + (-1)^p \mathbf{S} \wedge d\beta \quad (\text{A.67})$$

where \mathbf{S} is tensor valued p-form and β is an ordinary q-form. We can treat vectors as vector valued 0-forms so treating it like this we expect it to be a vector valued one-form after taking its exterior derivative which can be seen more clearly in the component form from equation (A.66). Now let us show how we can use the equation (A.67) to define higher rank tensor valued forms. A general vector valued one form is a linear combination of objects $\vec{u} \otimes \tilde{\sigma}$.

$$\vec{u} \otimes \tilde{\sigma} = \vec{u} \wedge \tilde{\sigma}$$

$$d\vec{u} \wedge \tilde{\sigma} = d\vec{u} \wedge \tilde{\sigma} + (-1)^0 \vec{u} \wedge d\tilde{\sigma}$$

$$d\vec{u} = \nabla_\beta u^\alpha \vec{\partial}_\alpha \tilde{d}x^\beta \Rightarrow d\vec{u} \wedge \tilde{\sigma} = (\sigma_\mu \nabla_\beta u^\alpha \vec{\partial}_\alpha) \tilde{d}x^\beta \wedge \tilde{d}x^\mu$$

$$d\tilde{\sigma} = \sigma_{\mu,\beta} \tilde{d}x^\beta \wedge \tilde{d}x^\mu \Rightarrow \vec{u} \wedge d\tilde{\sigma} = (u^\alpha \sigma_{\mu,\beta} \vec{\partial}_\alpha) \tilde{d}x^\beta \wedge \tilde{d}x^\mu$$

$$d\vec{u} \wedge \tilde{\sigma} = (\sigma_\mu \nabla_\beta u^\alpha \vec{\partial}_\alpha + u^\alpha \sigma_{\mu,\beta} \vec{\partial}_\alpha) \tilde{d}x^\beta \wedge \tilde{d}x^\mu. \quad (\text{A.68})$$

For the first equation we used the definition of wedge product of forms with a zero form. Notice that the vector valued two-form has components which themselves are vectors.

A.11.2 Connection one-form and curvature two-form

Now choose any maximally independent minimally spanning set of vectors as a non-coordinate basis e_α for the tangent space $\mathcal{T}_p\mathcal{M}$. These are not necessarily the tetrads or vielbeins that are used in literature which are chosen so as to make the metric tensor look like Minkowski metric at all points. By using $\langle \theta^\alpha, e_\beta \rangle = \delta^\alpha_\beta$ one can find a dual non-coordinate basis for the dual space. One defines the curvature one-form as $\omega^\alpha{}_\beta$ as in below equation.

$$de_\mu := e_\nu \omega^\nu{}_\mu \quad (\text{A.69})$$

From these we can also give the connection coefficients a general definition. First write the connection one-forms as $\omega_{\mu\alpha} = C^\mu{}_{\gamma\alpha} \theta^\gamma$ then we find

$$\langle de_\alpha, e_\beta \rangle = \nabla_{\underline{\beta}} e_\alpha = \langle e_\mu \omega^\mu{}_\alpha, e_\beta \rangle = e_\mu C^\mu{}_{\gamma\alpha} \delta^\gamma_\beta = e_\mu C^\mu{}_{\beta\alpha} \quad (\text{A.70})$$

$$C^\mu_{\beta\alpha} = \langle \theta^\mu, \nabla_\beta e_\alpha \rangle \quad (\text{A.71})$$

(A.70) or (A.71) as the definition for connection coefficients. They are not symmetric in lower indices for non-coordinate bases so index placement is important. To make the formulas easier let us define $\mathbf{e} := \begin{bmatrix} e_0 & e_1 & \dots & e_n \end{bmatrix}$ as a row vector whose elements are basis vectors, also $\boldsymbol{\theta} := \begin{bmatrix} \theta^0 & \theta^1 & \dots & \theta^n \end{bmatrix}^T$ as a column vector whose elements are dual basis one-forms where $(\cdot)^T$ denotes the transpose operation. We also define a matrix Ω as

$$\Omega := \begin{bmatrix} \omega^0_0 & \omega^0_1 & \dots & \omega^0_n \\ \omega^1_0 & \omega^1_1 & \dots & \omega^1_n \\ \vdots & & & \\ \omega^n_0 & \omega^n_1 & \dots & \omega^n_n \end{bmatrix} \quad (\text{A.72})$$

then we can write the equation (A.69) as

$$d\mathbf{e} = \mathbf{e}\Omega \quad (\text{A.73})$$

Now we can write any vector as $\vec{v} = \mathbf{v}\mathbf{e}$ where $\mathbf{v} = \begin{bmatrix} v^0 & v^1 & \dots & v^n \end{bmatrix}^T$ is a column vector whose elements are the components of the vector. Let's calculate $d\vec{v}$ and $d^2\vec{v} = d(d\vec{v})$.

$$\begin{aligned} d\vec{v} &= d(\mathbf{v}\mathbf{e}) = d(v^\mu e_\mu) \\ &= dv^\mu \wedge e_\mu + (-1)^0 v^\mu \wedge de_\mu \\ d^2\vec{v} &= d(dv^\mu \wedge e_\mu) + dv^\mu \wedge de_\mu \\ &= -dv^\mu \wedge de_\mu + dv^\mu \wedge e_\mu + v^\mu d(de_\mu) \\ &= v^\mu d(e_\nu \Omega^\nu_\mu) \\ &= v^\mu ((de_\nu) \wedge \Omega^\nu_\mu + (-1)^0 e_\nu \wedge d\Omega^\nu_\mu) \\ &= v^\mu (d\Omega^\nu_\mu + \Omega^\nu_\alpha \wedge \Omega^\alpha_\mu) e_\nu \end{aligned} \quad (\text{A.74})$$

where we have used the fact that $d^2 = 0$ for ordinary scalars. It is not zero for tensor valued p-forms, $p > 1$. The last equation can also be written as $d^2\vec{v} = \mathbf{e}\mathcal{R}\mathbf{v}$, where $\mathcal{R}^\nu_\mu = d\Omega^\nu_\mu + \Omega^\nu_\alpha \wedge \Omega^\alpha_\mu$ is a (1,1) valued 2-form. We have the equation

$$(d\tilde{\omega})(\vec{u}, \vec{v}) := \vec{u}\{\langle \tilde{\omega}, \vec{v} \rangle\} - \vec{v}\{\langle \tilde{\omega}, \vec{u} \rangle\} - \langle \tilde{\omega}, [\vec{u}, \vec{v}] \rangle \quad (\text{A.75})$$

We can generalize this equation to tensor valued one-forms as follows.

$$(dS)(\vec{u}, \vec{v}) = \vec{u}\{\langle S, \vec{v} \rangle\} - \vec{v}\{\langle S, \vec{u} \rangle\} - \langle S, [\vec{u}, \vec{v}] \rangle \quad (\text{A.76})$$

Taking $S = d\vec{w}$ one can write

$$\begin{aligned} (d(d\vec{w}))(\vec{u}, \vec{v}) &= \vec{u}\{\langle d\vec{w}, \vec{v} \rangle\} - \vec{v}\{\langle d\vec{w}, \vec{u} \rangle\} - \langle d\vec{w}, [\vec{u}, \vec{v}] \rangle \\ d^2\vec{w}(\vec{u}, \vec{v}) &= \nabla_{\vec{u}}\nabla_{\vec{v}}\vec{w} - \nabla_{\vec{v}}\nabla_{\vec{u}}\vec{w} - \nabla_{[\vec{u}, \vec{v}]} \vec{w} \end{aligned} \quad (\text{A.77})$$

where we have used $\langle d\vec{v}, \vec{u} \rangle := \nabla_{\vec{u}}\vec{v}$, and also we changed vector action on function to covariant derivative so that we have a covariant expression for vector action on vector valued forms. The last equation is the definition of Riemann tensor. So what we have as \mathcal{R} is just the Riemann tensor. Now suppose we make a change of basis. Then

$$\acute{e} = eA \quad (\text{A.78})$$

$$\acute{\theta} = A^{(-1)}\theta \quad (\text{A.79})$$

$$d\acute{e} = \acute{e}\acute{\Omega} \quad (\text{A.80})$$

$$\acute{\Omega} = A^{(-1)}\Omega A + A^{(-1)}dA \quad (\text{A.81})$$

where the first equation defines the basis change, second equation can be obtained from the requirement of leaving the identity tensor, $\delta := e_{\mu}\theta^{\mu}$, invariant and last equation is easily calculated from the definition of connection one-form which is the third equation. The Levi-Civita connection is the torsion-free (the covariant derivative of identity tensor is zero) metric compatible (the covariant derivative of metric tensor is zero) connection. These two restrictions are given by

$$\begin{aligned} d\delta = 0 &\Rightarrow d(e\theta) = 0 \\ &= de \wedge \theta + (-1)^0 e \wedge d\theta = 0 \\ &= e\Omega \wedge \theta + ed\theta = 0 \\ &= \Omega \wedge \theta + d\theta = 0 \end{aligned} \quad (\text{A.82})$$

$$0 = \omega^{\alpha}_{\mu} \wedge \theta^{\mu} + d\theta^{\alpha}$$

$$d(\vec{u} \cdot \vec{v}) = (d\vec{u} \cdot \vec{v}) + (\vec{u} \cdot d\vec{v}) \quad (\text{A.83})$$

$$d(g_{\mu\nu}) = \omega_{\mu\nu} + \omega_{\nu\mu} \quad (\text{A.84})$$

where $\omega_{\alpha\beta} = g_{\alpha\lambda}\omega^{\lambda}_{\beta}$ and $d(g_{\mu\nu}) = e_{\alpha}\{g_{\mu\nu}\}\theta^{\alpha}$ since the metric is a (0,2) valued zero-form. The equations (A.82) and (A.83) are called the torsion-free condition and metric compatibility conditions respectively. Before finding the connection one-forms for the metric in ADM coordinates let us do a simple example and

analyze the simplest case, that is, polar coordinates on Euclidean plane.

$$g_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix} \quad (\text{A.85})$$

As a general tactic we will choose non-coordinate bases to simplify the metric as much as possible. We can choose $\mathbf{e}_r = \vec{\partial}_r$, $\mathbf{e}_\theta = r\vec{\partial}_\theta$ and $\boldsymbol{\theta}^r = \tilde{d}x^r$, $\boldsymbol{\theta}^\theta = \tilde{d}x^\theta/r$ as a non-coordinate basis in which the metric is simply the identity matrix.

$$g_{\underline{i}\underline{j}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (\text{A.86})$$

In this case $dg_{\underline{i}\underline{j}} = 0$ and the metric compatibility condition is $\boldsymbol{\omega}^{\underline{i}}_{\underline{j}} + \boldsymbol{\omega}^{\underline{j}}_{\underline{i}} = 0$ here we use $\boldsymbol{\omega}_{\underline{i}\underline{j}} = g_{ik}\boldsymbol{\omega}^k_{\underline{j}}$. Now let us find the torsion free conditions.

$$\boldsymbol{\theta}^r = \tilde{d}x^r \Rightarrow d\boldsymbol{\theta}^r = 0 \quad (\text{A.87})$$

$$\boldsymbol{\theta}^\theta = \tilde{d}x^\theta/r \Rightarrow d\boldsymbol{\theta}^\theta = -\frac{\boldsymbol{\theta}^r \wedge \boldsymbol{\theta}^\theta}{r} \quad (\text{A.88})$$

So $0 = d\boldsymbol{\omega}^\alpha_\mu \wedge \boldsymbol{\theta}^\mu + d\boldsymbol{\theta}^\alpha$ becomes

$$\begin{aligned} \boldsymbol{\omega}^r_\theta \wedge \boldsymbol{\theta}^\theta + \boldsymbol{\omega}^r_r \wedge \boldsymbol{\theta}^r &= 0 \\ -\frac{\boldsymbol{\theta}^r \wedge \boldsymbol{\theta}^\theta}{r} + \boldsymbol{\omega}^\theta_\theta \wedge \boldsymbol{\theta}^\theta + \boldsymbol{\omega}^\theta_r \wedge \boldsymbol{\theta}^r &= 0 \end{aligned} \quad (\text{A.89})$$

$$\boldsymbol{\omega}^r_r = 0 \quad (\text{A.89})$$

$$\boldsymbol{\omega}^\theta_\theta = 0 \quad (\text{A.90})$$

$$\boldsymbol{\omega}^r_\theta + \boldsymbol{\omega}^\theta_r = 0 \quad (\text{A.91})$$

together with the metric compatible conditions. The solutions are very easy and they are given below.

$$\boldsymbol{\omega}^\theta_r = -\boldsymbol{\omega}^r_\theta = -\boldsymbol{\theta}^\theta/r \quad (\text{A.92})$$

$$\boldsymbol{\omega}^r_r = \boldsymbol{\omega}^\theta_\theta = 0 \quad (\text{A.93})$$

Then from these we can find the connection coefficients with the equation (A.70). We will apply these same steps to find the connection coefficients for the metric in ADM coordinates. The equations are a bit long but since they are linear they always have a solution.

A.11.3 Calculation of Christoffel Symbols in ADM coordinates

First we will calculate the extrinsic curvature in ADM coordinates since it will come as a connection coefficient later and we will need to recognize it. From the

definition of extrinsic curvature

$$\begin{aligned}
K_{ij} &= -\vec{\partial}_j \cdot \nabla_i \vec{n} \\
&= -\left(\nabla_i (\vec{\partial}_j \cdot \vec{n}) + \vec{n} \cdot \nabla_i \vec{\partial}_j \right) \\
&= \vec{n} \cdot \vec{\partial}_\alpha^{(4)} \Gamma^\alpha_{ij} \\
&= n_0^{(4)} \Gamma^0_{ij} \\
{}^{(4)}\Gamma^0_{ij} &= \frac{1}{2} g^{0\lambda} (g_{i\lambda,j} + g_{j\lambda,i} - g_{ij,\lambda}) \\
K_{ij} &= \frac{1}{2N} (g_{i0,j} + g_{j0,i} - g_{ij,0}) - \frac{\beta^k}{N} \gamma_{mk} {}^{(3)}\Gamma^m_{ij} \\
&= (D_j \beta_i + D_j \beta_i - \gamma_{ij,0}) / 2N \\
&= \frac{{}^{(3)}\mathcal{L}_{\vec{\beta}} \gamma_{ij} - \gamma_{ij,0}}{2N}
\end{aligned} \tag{A.94}$$

Here the ${}^{(3)}\mathcal{L}_{\vec{v}}\mathbf{A}$ denotes that the Lie derivative is taken with respect to the covariant derivative D of the hypersurface. To calculate the connection coefficients in ADM coordinates we follow the same steps described in the previous example. First we adopt a non-coordinate basis that simplifies the metric.

$$\mathbf{e}_0 = \vec{n} = (1/N)\vec{\partial}_0 - (\beta^i/N)\vec{\partial}_i, \quad \mathbf{e}_i = \vec{\partial}_i \tag{A.95}$$

$$\boldsymbol{\theta}^0 = -\tilde{n} = N\tilde{d}x^0, \quad \boldsymbol{\theta}^i = \beta^i \tilde{d}x^0 + \tilde{d}x^i \tag{A.96}$$

$$\vec{\partial}_j = \mathbf{e}_j, \quad \vec{\partial}_0 = N\mathbf{e}_0 + \beta^i \mathbf{e}_i \tag{A.97}$$

$$\tilde{d}x^0 = \frac{\boldsymbol{\theta}^0}{N}, \quad \tilde{d}x^i = -\frac{\beta^i}{N}\boldsymbol{\theta}^0 + \boldsymbol{\theta}^i \tag{A.98}$$

These equations describe the transformation between the coordinate basis and non-coordinate basis. In the non-coordinate basis the metric becomes

$$g_{\underline{\mu}\underline{\nu}} = \begin{bmatrix} -1 & 0 \\ 0 & \gamma_{\underline{i}\underline{j}} \end{bmatrix}_{\underline{\mu}\underline{\nu}} \tag{A.99}$$

$$g^{\underline{\mu}\underline{\nu}} = \begin{bmatrix} -1 & 0 \\ 0 & \gamma^{\underline{i}\underline{j}} \end{bmatrix}^{\underline{\mu}\underline{\nu}}. \tag{A.100}$$

For the exterior derivative of dual non-coordinate basis we have

$$d\boldsymbol{\theta}^0 = \frac{N_{,i}}{N} (\boldsymbol{\theta}^i \wedge \boldsymbol{\theta}^0) \tag{A.101}$$

$$d\boldsymbol{\theta}^i = \frac{\beta^i_{,k}}{N} (\boldsymbol{\theta}^k \wedge \boldsymbol{\theta}^0). \tag{A.102}$$

For the exterior derivative of the metric a (0,2) valued 0-form we have

$$dg_{\underline{\mu}\underline{\nu}} = \mathbf{e}_\alpha \{g_{\underline{\mu}\underline{\nu}}\} \boldsymbol{\theta}^\alpha \tag{A.103}$$

$$= e_k \{\gamma_{ij}\} \theta^k + e_0 \{\gamma_{ij}\} \theta^0 \quad (\text{A.104})$$

$$= \gamma_{ij,k} \theta^k + (\gamma_{ij,0} - \gamma_{ij,k} \frac{\beta^k}{N}) \theta^0. \quad (\text{A.105})$$

We can also write for $\omega_{\alpha\beta}$

$$\omega_{00} = g_{\alpha 0} \omega^\alpha_0 = -\omega^0_0 \quad (\text{A.106})$$

$$\omega_{0i} = g_{\alpha 0} \omega^\alpha_i = -\omega^0_i \quad (\text{A.107})$$

$$\omega_{i0} = g_{\alpha i} \omega^\alpha_0 = \gamma_{ij} \omega^j_0 \quad (\text{A.108})$$

$$\omega_{ij} = g_{\alpha i} \omega^\alpha_j = \gamma_{ik} \omega^k_j. \quad (\text{A.109})$$

Now let us write the torsion-free and metric compatible conditions explicitly for ADM coordinates.

$$0 = \omega^\alpha_\mu \wedge \theta^\mu + d\theta^\alpha$$

$$0 = \frac{N_{,i}}{N} (\theta^i \wedge \theta^0) + (\omega^0_0 \wedge \theta^0) + (\omega^0_k \wedge \theta^k) \quad (\text{A.110})$$

$$0 = \frac{\beta^i_{,k}}{N} (\theta^k \wedge \theta^0) + (\omega^i_0 \wedge \theta^0) + (\omega^i_k \wedge \theta^k) \quad (\text{A.111})$$

These are the torsion-free conditions.

$$d(g_{\mu\nu}) = \omega_{\mu\nu} + \omega_{\nu\mu}$$

$$0 = \omega^0_0 \quad (\text{A.112})$$

$$0 = -\omega^0_i + \gamma_{ji} \omega^j_0 \quad (\text{A.113})$$

$$\left(\frac{\gamma_{ij,0}}{N} - \frac{\beta^k \gamma_{ij,k}}{N} \right) \theta^0 + \gamma_{ij,k} \theta^k = \gamma_{ki} \omega^k_j + \gamma_{kj} \omega^k_i \quad (\text{A.114})$$

These are the conditions for metric compatibility. By using $\omega^\mu_\alpha = C^\mu_{\beta\alpha} \theta^\beta$ we can turn all of these equations into equations for the connection coefficients. Normally we would directly solve for the connection one-form but here we will just use the equations for the connection coefficients.

$$0 = C^0_{jk} - C^0_{kj} \quad (\text{A.115})$$

$$0 = C^i_{jk} - C^i_{kj} \quad (\text{A.116})$$

$$\frac{N_{,i}}{N} = -C^0_{i0} + C^0_{0i} \quad (\text{A.117})$$

$$\frac{\beta^i_{,k}}{N} = C^i_{0k} - C^i_{k0} \quad (\text{A.118})$$

$$0 = C^0_{\alpha 0} \quad (\text{A.119})$$

$$0 = -C^0_{\alpha i} + \gamma_{ji} C^j_{\alpha 0} \quad (\text{A.120})$$

$$\frac{\gamma_{ij,0}}{N} - \frac{\beta^k \gamma_{ij,k}}{N} = \gamma_{ki} C_{0j}^k + \gamma_{kj} C_{0i}^k \quad (\text{A.121})$$

$$\gamma_{ij,m} = \gamma_{ki} C_{mj}^k + \gamma_{kj} C_{mi}^k \quad (\text{A.122})$$

Let us solve these equations. All the equations carrying information about the C_{ij}^m are from equations (A.122) and (A.116).

$$C_{ij}^m - C_{ji}^m = 0 \quad (\text{A.123})$$

$$\gamma_{ij,m} - \gamma_{ki} C_{mj}^k - \gamma_{kj} C_{mi}^k = 0 \quad (\text{A.124})$$

These equations are exactly the equations a coordinate basis connection coefficient will satisfy for the spacelike hypersurface. So we have the solution $C_{ij}^m = {}^{(3)}\Gamma_{ij}^m$. Now we try to find C_{jk}^0 . By substituting C_{k0}^j from equation (A.120) into equation (A.118) we get

$$\begin{aligned} -C_{ki}^0 + \gamma_{ij} (C_{0k}^j - \frac{\beta^j{}_{,k}}{N}) &= 0 \\ -C_{ik}^0 + \gamma_{kj} (C_{0i}^j - \frac{\beta^j{}_{,i}}{N}) - C_{ki}^0 + \gamma_{ij} (C_{0k}^j - \frac{\beta^j{}_{,k}}{N}) &= 0. \end{aligned}$$

The last equation is just obtained by symmetrizing in the i, k indices. Now we can use equation (A.115) to get

$$-2C_{ik}^0 + \gamma_{ij} C_{0k}^j + \gamma_{kj} C_{0i}^j - \gamma_{ij} \frac{\beta^j{}_{,k}}{N} - \gamma_{kj} \frac{\beta^j{}_{,i}}{N} = 0.$$

At this step we use the equation (A.121) to get

$$\begin{aligned} 0 &= -2C_{ik}^0 + \frac{\gamma_{ik,0}}{N} - \frac{\gamma_{ik,j} \beta^j}{N} - \frac{\gamma_{ji} \beta^j{}_{,k}}{N} - \frac{\gamma_{jk} \beta^j{}_{,i}}{N} \\ 2NC_{ik}^0 &= \gamma_{ik,0} - \gamma_{ik,j} \beta^j - \gamma_{ji} \beta^j{}_{,k} - \gamma_{jk} \beta^j{}_{,i} \\ &= \gamma_{ik,0} - \gamma_{ik,j} \beta^j - \beta_{i,k} + \gamma_{ji,k} \beta^j - \beta_{k,i} + \gamma_{jk,i} \beta^j \\ &= \gamma_{ik,0} - \beta^j (\gamma_{ik,j} - \gamma_{ji,k} - \gamma_{jk,i}) - \beta_{k,i} - \beta_{i,k} \\ &= \gamma_{ik,0} + 2\beta^j \gamma_{jm} {}^{(3)}\Gamma_{ik}^m - \beta_{k,i} - \beta_{i,k} \\ 2NC_{ik}^0 &= \gamma_{ik,0} - D_k \beta_i - D_i \beta_k \\ C_{ik}^0 &= -K_{ik} \end{aligned} \quad (\text{A.125})$$

Now other equations for the coefficients can be solved easily. From (A.120)

$$\begin{aligned} C_{ji}^0 &= \gamma_{ki} C_{j0}^k \\ C_{j0}^k &= -K^k{}_j. \end{aligned} \quad (\text{A.126})$$

From equation (A.118)

$$C_{0k}^i = K_{ik} + \frac{\beta^i{}_{,k}}{N}. \quad (\text{A.127})$$

From equations (A.120), (A.119) and (A.117)

$$0 = -C^0_{\underline{0}i} + \gamma_{ij}C^j_{\underline{0}0}$$

$$C^0_{\underline{j}0} = 0 \tag{A.128}$$

$$C^0_{\underline{0}i} - C^0_{i0} = \frac{N_{,i}}{N}$$

$$C^0_{\underline{0}i} = \frac{N_{,i}}{N} \tag{A.129}$$

$$C^j_{\underline{0}0} = \gamma^{ij} \frac{N_{,i}}{N} \tag{A.130}$$

Now we can list all of the connection coefficients.

$$C^m_{\underline{i}j} = {}^{(3)}\Gamma^m_{ij} \tag{A.131}$$

$$C^0_{\underline{i}j} = -K_{ij} \tag{A.132}$$

$$C^k_{\underline{j}0} = -K^k_j \tag{A.133}$$

$$C^i_{\underline{0}k} = -K^i_k + \frac{\beta^i_{,k}}{N} \tag{A.134}$$

$$C^0_{\underline{\alpha}0} = 0 \tag{A.135}$$

$$C^0_{\underline{0}i} = \frac{N_{,i}}{N} \tag{A.136}$$

$$C^j_{\underline{0}0} = \gamma^{ij} \frac{N_{,i}}{N}. \tag{A.137}$$

Now these equations are useful in non-coordinate basis. But we know how to convert them into coordinate basis. So when we need them we will do the conversion there.

APPENDIX B

CONSTRAINED HAMILTONIAN SYSTEMS

This appendix is a basic review of the constrained Hamiltonian systems. We will give a non-geometrical summary since it is good enough for our purposes and the geometrical considerations might make one get lost in the details. We will use the notations for finite degree of freedom systems but the methods developed here can be used in field theory as well, if one is careful. There are of course problems which arise due to field theory aspects, but the methods described here are still the best ones for dealing with constraints. Also, there are a lot of open questions yet, so there are disagreements about *the best* method [16]. So, it is better to be more flexible in the treatment of this subject until a rigorous treatment is available. The standard reference on this subject is [3]. For geometrical aspects of this subject one can examine [17], [18], [19] and [20]. For pedagogical treatment the lecture notes of Bernard Whiting [21] and Dirac's Yeshiva lectures [11] is very helpful.

B.1 Motivation - Gauge Invariance and Constraints

A gauge theory emerges when the equations of motion involve arbitrary functions of coordinates. If we want to make a quantum theory then we need to be able to upgrade this symmetry of the solutions into a local (in terms of spacetime) symmetry. For instance in electromagnetism the four-potential is not completely determined by equations of motion. We can add the gradient of an arbitrary smooth scalar to it and it would still be a solution. Now if we change the wave function with a local $U(1)$ symmetry, that is, $\Psi \mapsto \exp(\alpha(x))\Psi$, which does not change the probabilities, to get the same equation of motion for the Ψ we would need to vary the four-potential in the Hamiltonian with the gradient of $\alpha(x)$. So,

one is inclined to seek the symmetries in the Hamiltonian equations of motion. This arbitrariness in equations of motion is also encountered in constrained Hamiltonian systems. So, it is important to understand constraints really well. This is the basic motivation to study constrained Hamiltonian systems. Another motivation is to be able to deal with the dynamics of singular Lagrangians (the ones with non-invertible Legendre maps) consistently and efficiently.

B.2 Dynamics in terms of Lagrangian or Various Hamiltonians

In this section we will not be interested in quantization of the dynamical system but only try to get the correct dynamical equations. We will also classify the constraints but not treat them. They will be treated in the next section with examples.

B.2.1 Primary Constraints

We will suppose that we are given a Lagrangian $L = L(q, \dot{q})$. The action is given as

$$S = \int dt L(q, \dot{q}) \quad (\text{B.1})$$

where $q = \{q^i(t)\}_{i=1}^N$ are the configuration space variables. The equations of motion are given by

$$\begin{aligned} \frac{\delta S}{\delta q^i} &= 0, \quad \forall q^i(t) \\ \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i} \right) - \frac{\partial L}{\partial q^i} &= 0 \end{aligned} \quad (\text{B.2})$$

or

$$\ddot{q}^j \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} = \frac{\partial L}{\partial q^i} - \dot{q}^j \frac{\partial^2 L}{\partial q^j \partial \dot{q}^i} \quad (\text{B.3})$$

where the matrix

$$W_{ij} := \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \quad (\text{B.4})$$

is given as the Hessian matrix of the Lagrangian map. If W_{ij} is invertible then for some initial conditions we can find the solution to the equations of motion uniquely. If it is not invertible then equations of motion will not be uniquely determined and the arbitrariness in the equations of motion will be interpreted

as gauge freedom. We will not assume that W_{ij} is invertible but we will assume that it has constant rank everywhere. When W_{ij} is not invertible we will have equations $\phi_m(q, p) = 0$, $m = 1, \dots, M$ satisfied for any physical (satisfying the equations of motion) solution of the system at all times. Here p represent the conjugate momenta and are defined as usual.

$$p_i = \frac{\partial L}{\partial \dot{q}^i}. \quad (\text{B.5})$$

The equations $\phi_m = 0$ are called primary constraints. They are called primary since we did not use any time evolution of the system (equations of motion). The submanifold defined by the primary constraints is called the primary constraint surface. There are some conditions that these constraints (and any other constraint we will come to) must satisfy so that some theorems are satisfied.

B.2.2 Regularity Conditions

Not all descriptions of the constraint surface is acceptable. There are four equivalent descriptions of the conditions which must be satisfied by the constraint equations so that they are admissible. The basic idea is to be able to define a submanifold according to the conditions of the Frobenius theorem [22].

- i) The $2N - M$ dimensional primary constraint surface should be coverable with open regions, on each of which the constraint functions ϕ can be separated into independent constraints ϕ_m and dependent constraints $\tilde{\phi} = 0$ which hold as consequences of the independent ones. Furthermore the rank of the Jacobian matrix $\frac{\partial \phi_m}{\partial (q^i, p_i)}$ should be M .
- ii) There must exist Frobenius charts, that is, charts for which the coordinates on the submanifold, say $\{\chi^A\}$, are a part of the coordinates on the manifold together with the constraint functions $\{\phi_m, \chi^A\}$ where the coordinate description of the submanifold reduces to $\phi_m = 0$.
- iii) The gradients $d\phi_1, \dots, d\phi_M$ should be linearly independent or $d\phi_1 \wedge \dots \wedge d\phi_M \neq 0$.
- iv) The variations of ϕ_m are of order ϵ for arbitrary variations of (q, p) of order ϵ . This means that ϕ_m are first order in (q, p) but this is not the most

comprehensive way to define a submanifold.

Let us examine an example from [3]. Suppose we are given the below equivalent representations of the same constraint surface.

$$p_1 = 0 \tag{B.6}$$

$$(p_1)^2 = 0 \tag{B.7}$$

$$p_1 = 0, (p_1)^2 = 0 \tag{B.8}$$

$$\sqrt{p_1} = 0 \tag{B.9}$$

The first description is admissible. Second and the fourth descriptions are inadmissible and should be replaced with the first one. The third description is redundant. We will always assume that the regularity conditions hold. If they do not hold then we can make them hold by taking a subset of the constraint equations. There are two theorems which are correct when the regularity conditions hold.

Theorem B.2.1 *If a smooth phase space function disappears on the primary constraint surface then it can be written as*

$$G(q, p) = g^m(q, p)\phi_m(q, p) \text{ for some functions } g^m. \tag{B.10}$$

Theorem B.2.2 *If $\lambda_i\delta q^i + \mu^i\delta p_i = 0$ for arbitrary variations $\delta q, \delta p$ then on the constraint surface we can write*

$$\lambda_i = u^m \frac{\partial \phi_m}{\partial q^i} \tag{B.11}$$

$$\mu^i = u^m \frac{\partial \phi_m}{\partial p_i} \tag{B.12}$$

where u^m is any function of (q, p) .

The first theorem (B.2.1) can be inferred from the existence of Frobenius charts which are charts compatible with both the submanifold and the manifold itself. The second theorem is an exercise in nonlinear functional analysis. If the regularity conditions hold and no redundant constraints appear then the variations $\delta\phi_m$ form a basis for the null space for the functions defined on the constraint surface. However, if there happens to be redundant constraints (squares of the constraints etc.) then u^m will not be unique. For details, one can look at [3].

B.2.3 The Canonical Hamiltonian

The canonical Hamiltonian is defined as usual, that is,

$$H(q, p) = \dot{q}^i p_i - L. \quad (\text{B.13})$$

The equations of motion can be found from the canonical Hamiltonian as

$$\delta S = \delta \int dt (\dot{q}^i p_i - H) = 0, \quad (\text{B.14})$$

subject to the conditions

$$\phi_m(q, p) = 0, \quad (\text{B.15})$$

$$\delta \phi_m(q, p) = 0 \quad m = 1, \dots, M \quad (\text{B.16})$$

where S is the action integral. Let us derive the equations of motion by using the canonical Hamiltonian. We will suppress the coordinate indices. $\delta S = 0$ implies the below equations only.

$$\begin{aligned} \delta S &= \int dt \left(\dot{q} \delta p + \frac{d(p \delta q)}{dt} - \dot{p} \delta q - \frac{\delta H}{\delta q} \delta q - \frac{\delta H}{\delta p} \delta p \right) \\ 0 &= (p \delta q) \Big|_{t_1}^{t_2} + \int dt \left(\delta p \left\{ \dot{q} - \frac{\delta H}{\delta p} \right\} + \delta q \left\{ -\dot{p} - \frac{\delta H}{\delta q} \right\} \right). \end{aligned} \quad (\text{B.17})$$

Now the first term in the last equation simply defines the conjugate momenta (by equating to the boundary term of the variation of the Lagrangian in the case of variations which do not vanish at the boundary). We also have the constraints so, we can not directly write the equations of motion. As in any singular linear equation system we can have many solutions or no solutions. No solution systems are not physically interesting (they are inconsistent), indicating a poor choice of Lagrangian like $L = q$, which has the equations of motion $1 = 0$ [11]. Now, from $\delta \phi_m = 0$ one gets

$$0 = \frac{\delta \phi_m}{\delta q} \delta q + \frac{\delta \phi_m}{\delta p} \delta p. \quad (\text{B.18})$$

If there are no inconsistencies the best we can deduce is

$$\left(\delta p \left\{ \dot{q} - \frac{\delta H}{\delta p} \right\} + \delta q \left\{ -\dot{p} - \frac{\delta H}{\delta q} \right\} \right) = u^m(t) \left(\frac{\delta \phi_m}{\delta q} \delta q + \frac{\delta \phi_m}{\delta p} \delta p \right). \quad (\text{B.19})$$

Here u_m is arbitrary and by also including the constraint

$$\phi_m = 0, \quad (\text{B.20})$$

we can write all of the equations of motion.

$$\dot{q}^i = \frac{\partial H}{\partial p_i} + u^m(t) \frac{\partial \phi_m}{\partial p_i}, \quad (\text{B.21})$$

$$\dot{p}_i = -\frac{\partial H}{\partial q^i} - u^m(t) \frac{\partial \phi_m}{\partial q^i}, \quad (\text{B.22})$$

$$\phi_m = 0, \quad (\text{B.23})$$

for all i and m . The regularity conditions must be satisfied for these equations of motion to be the correct ones [3].

B.2.4 The Total Hamiltonian

Another way to obtain the correct dynamical equations (ones that are equivalent to the Euler-Lagrange equations) is through the total Hamiltonian. We derived the equations of motion by restricting the possible motions to the primary constraint surface in the case of canonical Hamiltonian. This means however that we can extend the canonical Hamiltonian off the primary constraint surface arbitrarily. This way we can obtain the total Hamiltonian.

$$H_T = H + u^m(p, q) \phi_m \quad (\text{B.24})$$

and the action becomes

$$S_T[q, p, u] = \int dt (p\dot{q} - H_T) \quad (\text{B.25})$$

where the $[q, p, u]$ means the action is a functional of q^i, p_i, u^m . We have the equations of motion

$$\dot{q}^i = \frac{\partial H}{\partial p_i} + u^m(t) \frac{\partial \phi_m}{\partial p_i} \quad (\text{B.26})$$

$$\dot{p}_i = -\frac{\partial H}{\partial q^i} - u^m(t) \frac{\partial \phi_m}{\partial q^i} \quad (\text{B.27})$$

$$\phi_m = 0 \quad (\text{B.28})$$

for all i and m . Here we changed u^m to a function of time only since starting from an initial condition $q(0), p(0)$, u^m can be written as a function of time. As a side note, in general, satisfying $\phi_m = 0$ does not satisfy $\dot{\phi}_m = 0$ for all times. So we might have to impose some restrictions on u 's or the (q, p) 's at all times. They will be clear with examples. Now, if the Legendre map is not invertible the transformation from conjugate momenta to the generalized coordinates is multivalued, that is to say we have multiple generalized velocities corresponding to the same conjugate momentum. Here the variables u^m can be considered as the extra parameters in the phase space needed to make the Legendre map

invertible. So then it is possible to define the Legendre map as

$$\mathcal{FL} : \mathcal{T}Q \longrightarrow \mathcal{T}^*Q$$

$$q^i \longmapsto q^i \tag{B.29}$$

$$\dot{q}^i \longmapsto p_i \tag{B.30}$$

$$\phi^m \longmapsto u^m \tag{B.31}$$

Here \mathcal{FL} denotes the fibre derivative with respect to function L and is properly defined in [17]. It is a bit hard so we will be content with the local version $p_i = \frac{\partial L}{\partial \dot{q}^i}$. We can write equations of motion more briefly with the help of Poisson brackets as

$$\dot{\Omega}(q, p) = \{\Omega, H_T\} \tag{B.32}$$

$$\dot{\phi}_m = 0. \tag{B.33}$$

B.2.5 Secondary Constraints

The constraint equations must be satisfied at all times. This leads to some consistency equations requiring

$$\dot{\phi}_m = \{\phi_m, H_T\} = 0 \tag{B.34}$$

$$= \{\phi_m, H\} + u^{\dot{m}} \{\phi_m, \phi_{\dot{m}}\} \tag{B.35}$$

Here we are free to enforce $\phi_m = 0$ after all the Poisson brackets are calculated. Then, these equations if independent from the Lagrange multipliers u will lead to either $0 = 0$ or more restrictions on (q, p) say $X(q, p) = 0$. Then we have to check the time evolution of X with the total Hamiltonian again and see what we get. We must repeat this process for all the constraints we obtained until all the time evolution of constraints lead only to $0 = 0$ or an equation on u 's. Suppose we obtained K new equations then in total we will have $M + K = J$ constraints.

$$\phi_m = 0 \quad \forall m \in 1, \dots, M \tag{B.36}$$

$$\phi_k = 0 \quad \forall k \in M + 1, \dots, M + K \tag{B.37}$$

$$\phi_j = 0 \quad \forall j \in 1, \dots, J \tag{B.38}$$

The difference between primary and secondary constraints will not be extremely important in the final form of the theory so the notation is a bit loose but is in

accord with [3]. The constraints $\phi_k = 0$ will be called secondary constraints since we used the time evolution (equations of motion). We will assume $\phi_j = 0$ obey the regularity conditions as well. We will also assume the rank of the matrix $\{\phi_j, \phi_j\}$ is constant (throughout the surface $\phi_j = 0$, $j = 1, \dots, J$ describes).

B.2.6 Weak and Strong Equations

We will denote the surface the constraints $\phi_j = 0$ describes as \mathcal{C} and call it the constraint surface. The surface described by only primary constraints $\phi_m = 0$ will be denoted as \mathcal{C}_p . If two functions $F(q, p)$, $G(q, p)$ are equal on the \mathcal{C} we will denote it by $F \stackrel{c}{=} G$. This equality will be called weakly equal. If two functions are only equal on primary constraint surface we will denote it by $F \stackrel{c_p}{=} G$. This equality won't be used much. Notice that it is appropriate to use this equality in the consistency equations. We also enforce the primary constraints in the definition of canonical Hamiltonian. Now if two smooth functions F , G and their gradients are weakly equal on the constraint surface then they are called strongly equal which means they are equal in an open set around the constraint surface but we will take this open set to be the entire phase space. Strong equality will be denoted by the ordinary equal sign. Due to theorem B.2.1 we can write

$$F \stackrel{c}{=} G \iff F - G = c^j(q, p)\phi_j \quad (\text{B.39})$$

We can rewrite the equations of motion as $\hat{\Omega}(q, p) \stackrel{c}{=} \{\Omega, H_T\}$.

B.2.7 Restrictions on Lagrange multipliers

When the consistency algorithm is playing out we might have some equations on the Lagrange multipliers u^m . We will not get a solution for all of the Lagrange multipliers and while some of them will be completely solvable some of them will be partially solved and will remain arbitrary. This will be clear in the next section when we do some simple examples, but we can still get an expression for the solutions of the equations on the Lagrange multipliers.

$$\{\phi_j, H\} + u^m \{\phi_j, \phi_m\} = 0. \quad (\text{B.40})$$

Some equations in (B.40) are not trivial, that is, $0 = 0$. Now consider the linear homogeneous equation

$$V^m \{\phi_j, \phi_m\} = 0 \quad (\text{B.41})$$

which will have some non-trivial solutions V^m_a , $a = 1, \dots, A$ and the general solution of the inhomogeneous equation on u^m can be written as

$$u^m = U^m + v^a V^m_a \quad (\text{B.42})$$

where v^a is arbitrary functions of time and U^m is any specific solution of the inhomogeneous equation. Notice that we can now rewrite the total Hamiltonian as $H_T = \dot{H} + v^a \phi_a$ where $\dot{H} = H + U^m \phi_m$ and $\phi_a = V^m_a \phi_m$.

B.2.8 First and Second Class Functions

A function $F(q, p)$ is said to be first class if it satisfies

$$\{F, \phi_j\} \stackrel{c}{=} 0, \quad j = 1, \dots, J \quad (\text{B.43})$$

or equivalently

$$\{F, \phi_j\} = f^j_j \phi_j \quad (\text{B.44})$$

If it does not satisfy the above equations then the function is said to be second class. With this classification we can separate the constraints into two sets: first class ones and second class ones. Obviously, we will have to set the constraints to zero after the expression in Poisson brackets explicitly calculated. However the separation into first and second class might not be trivial to perform. Namely, the constraints might be given impurely that is as a sum of second class and first class constraints. The complete separation is achieved by satisfying irreducibility conditions which will be described later at the end of this section B.2. First class functions satisfy a nice property. The Poisson bracket of two first class functions is again a first class function which can be demonstrated with the help of the Jacobi identity. Also it can be checked that \dot{H} and ϕ_a are first class. Since the v^a are arbitrary, there will be an equivalence class of (q, p) that correspond to a single physical state. This can be shown as follows. Suppose we have chosen two different v 's v^a and \acute{v}^a . We also have some initial conditions at t_0 . After some small amount of time δt a phase space function F will evolve to $F + \delta t \dot{F}$.

The difference between the two values of F , with respect to the choice of v 's, at $t + \delta t$, denoted δF , will be given by

$$\delta F = \delta t(v^a - \dot{v}^a) \{F, \phi_a\}. \quad (\text{B.45})$$

Now the only way we can have a deterministic system classically (without considering the tougher issues associated with the quantum picture) is if the two descriptions of the F are equivalent. Also the change in F is proportional to $\delta v^a = \delta t(v^a - \dot{v}^a)$. So, the two systems described by different choices of v 's show the same dynamical picture, meaning, they are physically equivalent. Then by extending a terminology used in the theory of gauge fields one says that the primary first class constraints generate the gauge transformations [3]. However there are good reasons to consider that secondary first class constraints generate the gauge transformations as well. These reasons are due to these two following facts. The first is that $\{\phi_a, \phi_{\dot{a}}\}$ also a gauge generator since it leads to the same dynamics for F . The second is that $\{\dot{H}, \phi_a\}$ is also a gauge generator. So one might consider that some of the secondary first class constraints generate gauge transformations as well. This is not entirely true unless we also restrict the variations of some of the Lagrange multipliers to show the invariance explicitly [3]. The basic idea in [3] is that all first class constraints generate the gauge transformations of a dynamical system and we should use an extended Hamiltonian (to be described in the next section) to count for all gauge freedom. This is so that a method of quantization can be achieved. So, it is postulated that all first class constraints generate gauge transformations, even if the equations of motion does not contain arbitrary functions. There are also papers which create the gauge transformations only using the total Hamiltonian [19]. As we said in the beginning of this appendix a fully rigorous treatment has not been performed. Nevertheless, in either approach, one has to use both primary and secondary constraints in the generating function for a gauge transformation. That is why the distinction between primary and secondary constraints is not extremely important. However the distinction between first and second class constraints is important because second class constraints do not generate gauge transformations or any other transformation of significance. However the treatment of second class constraints is also much easier than first class constraints, at least with some methods. We will use the notation γ_a to denote a constraint

which is first class and χ_α to denote a constraint which is second class.

B.2.9 Reducible First and Second Class Constraints

In the irreducible case we can separate the constraints by following the method which will be described now. To separate the constraints into first and second class sets we first check the rank of $C_{j\dot{j}} := \{\phi_j, \phi_{\dot{j}}\}$. If the determinant of the matrix $C_{j\dot{j}}$ vanishes on the constraint surface then there must be at least one first class constraint among ϕ_j . By solving the linear equation $c^j C_{j\dot{j}} \stackrel{c}{=} 0$ for c^j , we can redefine a constraint $\gamma_1 = c^j \phi_j$. A solution for c^j must exist since the matrix is not full rank so its null space is not trivial. This new constraint $\gamma_1 = 0$ is automatically first class since $\{c^j \phi_j, \phi_{\dot{j}}\} = c^j \{\phi_j, \phi_{\dot{j}}\} + \phi_j \{c^j, \phi_{\dot{j}}\} \stackrel{c}{=} 0$. Here c^j might depend on the (q, p) . Then by repeatedly applying this procedure one can take linear combinations of the constraints to reach to a separation such that

$$0 \stackrel{c}{=} \begin{matrix} & \gamma_b & \chi_\beta \\ \gamma_a & \begin{bmatrix} 0 & 0 \\ 0 & C_{\alpha\beta} \end{bmatrix} \\ \chi_\alpha & \end{matrix}. \quad (\text{B.46})$$

In the reducible case constraints might be expressed with extra parts, which are unnecessary. For instance take this set of three constraints.

$$\phi_1 = \chi_1 \equiv q^1 + p_1 = 0 \quad (\text{B.47})$$

$$\phi_2 = \chi_2 \equiv q^1 = 0 \quad (\text{B.48})$$

$$\phi_3 = \chi_3 \equiv p_1 = 0 \quad (\text{B.49})$$

Now the constraints cannot be separated simply by checking their Poisson brackets since the first one is contaminated. The determinant of the $C_{j\dot{j}}$ vanishes but we do not have any first class constraints. A set of constraints $\{\phi_j = 0\}$ are completely separated into first and second class constraints $\{\gamma_a = 0, \chi_\alpha = 0\}$ when the following conditions hold [3].

- i) The reducibility conditions split the constraints into pure first-class and pure second class sets as

$$Z_{\dot{a}}^a \gamma_a = 0 \quad (a = 1, \dots, A; \dot{a} = 1, \dots, \dot{A}) \quad (\text{B.50})$$

$$Z_{\acute{\alpha}}^{\alpha} \chi_{\alpha} = 0 \quad (\alpha = 1, \dots, B; \acute{\alpha} = 1, \dots, \acute{B}) \quad (\text{B.51})$$

where the reducibility functions Z 's may depend on the q, p .

ii) The following brackets weakly vanish.

$$\{\gamma_a, \gamma_b\} \stackrel{c}{=} 0 \quad (\text{B.52})$$

$$\{\gamma_a, \chi_{\alpha}\} \stackrel{c}{=} 0. \quad (\text{B.53})$$

iii) The matrix $\{\chi_{\alpha}, \chi_{\beta}\}$ is of maximal rank.

$$\text{rank}(\{\chi_{\alpha}, \chi_{\beta}\}) = \acute{B} - B \quad (\text{B.54})$$

B.2.10 The Extended Hamiltonian

The extended Hamiltonian is defined as in below.

$$H_E = \acute{H} + u^j \phi_j = \acute{H} + u^a \gamma_a \quad (\text{B.55})$$

$$S_E[q, p, u] = \int dt (p\dot{q} - H_E) \quad (\text{B.56})$$

where $u^j \phi_j = u^a A^j_a \phi_j$ and $\gamma_a = A^j_a \phi_j$. The extended Hamiltonian does involve more arbitrary functions than the total Hamiltonian. This is to account for a more general description of gauge freedom which might be required if one has quantization in mind. The subject of quantization is also not entirely settled and there are many methods with different caveats.

B.3 Treatment of The Constraints in Constrained Hamiltonian Systems

We will first examine a system with a singular Lagrangian and classify the constraints. Then we will list the ways one can deal with the constraints. In many problems the method of attack will require a case by case analysis. Let us take the below hypothetical Lagrangian as our starting point.

$$L = \frac{1}{2} \dot{y}^2 + \dot{x}x - V(x, y). \quad (\text{B.57})$$

This Lagrangian has the primary constraint:

$$p_x = \frac{\partial L}{\partial \dot{x}} = x. \quad (\text{B.58})$$

The other conjugate momentum is defined as

$$p_x = \frac{\partial L}{\partial \dot{x}} = \dot{y}. \quad (\text{B.59})$$

The momenta defined through the equation (B.58) is a primary constraint since we can not invert the \dot{x} from the conjugate momentum. The Hessian matrix of the Legendre map is

$$W_{ij} := \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}_{ij}. \quad (\text{B.60})$$

The canonical Hamiltonian is defined as

$$H \stackrel{c_p}{=} p_i \dot{q}^i - L(q, \dot{q}). \quad (\text{B.61})$$

Here, as we have said before, unless we enforce the primary constraints we cannot get a canonical Hamiltonian since not all generalized velocities will be invertible due to the singular Lagrangian. The canonical Hamiltonian for (B.57) is given by

$$H \stackrel{c_p}{=} p_x \dot{x} + p_y \dot{y} - L(x, y, \dot{x}, \dot{y}) \quad (\text{B.62})$$

$$H \stackrel{c_p}{=} \dot{x}(p_x - x) + \frac{p_y^2}{2} + V(x, y) \quad (\text{B.63})$$

$$H \stackrel{c_p}{=} \frac{p_y^2}{2} + V(x, y) \quad (\text{B.64})$$

So now our system can be described as

$$H \stackrel{c_p}{=} \frac{p_y^2}{2} + V(x, y) \quad (\text{B.65})$$

$$\phi_1 \equiv p_x - x = 0 \quad (\text{B.66})$$

where $\phi_1 = 0$ is a primary constraint. We can also write the total Hamiltonian as

$$H_T = H + \lambda(t)\phi_1 = \frac{p_y^2}{2} + V(x, y) + \lambda(p_x - x) \quad (\text{B.67})$$

where λ is a Lagrange multiplier. We can also write the equations of motion:

$$\dot{x} = \lambda \quad (\text{B.68})$$

$$\dot{p}_x = -\frac{\partial V}{\partial x} + \lambda \quad (\text{B.69})$$

$$\dot{y} = p_y \quad (\text{B.70})$$

$$\dot{p}_y = -\frac{\partial V}{\partial y} \quad (\text{B.71})$$

$$\phi_1 = p_x - x = 0. \quad (\text{B.72})$$

To be able to satisfy the primary constraints at all times we need to work out the consistency algorithm and see what we get.

$$\dot{\phi}_1 = \dot{p}_x - \dot{x} = 0 \quad (\text{B.73})$$

$$= \frac{\partial V}{\partial x} \equiv \xi = 0 \quad (\text{B.74})$$

$$\dot{\xi} = \zeta \equiv \frac{\partial^2 V}{\partial x^2} \lambda + \frac{\partial^2 V}{\partial x \partial y} p_y = 0. \quad (\text{B.75})$$

As one can see we can not determine what the consistency algorithm will bring in a general situation. The equations for ξ and ζ might be another constraint or might be a restriction on the Lagrange multiplier, depending on the form of the function $V(x, y)$. Let us take the function V as the below three cases.

Case I)

$$V \rightsquigarrow 0 \quad (\text{B.76})$$

$$\xi = 0 \rightsquigarrow 0 = 0 \quad (\text{B.77})$$

$$\zeta = 0 \rightsquigarrow 0 = 0 \quad (\text{B.78})$$

We only have a single primary constraint and no restrictions on λ exist. Since there is only a single constraint its Poisson bracket commutes with itself so it is a first class constraint. We will rename $\phi_1 \equiv \gamma_1$.

Case II)

$$V \rightsquigarrow xy \quad (\text{B.79})$$

$$\xi = 0 \rightsquigarrow y = 0 \quad (\text{B.80})$$

$$\zeta = 0 \rightsquigarrow p_y = 0 \quad (\text{B.81})$$

Here, we have no restrictions on the Lagrange multiplier and we have three constraints: $\gamma_1 \equiv \phi_1 = p_x - x = 0$ a first class constraint, $\chi_1 \equiv \xi = y = 0$ a second class constraint and lastly $\chi_2 \equiv \zeta = p_y = 0$ another second class constraint.

Case III)

$$V \rightsquigarrow x^2 \quad (\text{B.82})$$

$$\xi = 0 \rightsquigarrow x = 0 \quad (\text{B.83})$$

$$\zeta = 0 \rightsquigarrow \lambda = 0 \quad (\text{B.84})$$

Here, we have two constraint equations and a single restriction on the Lagrange multiplier: $\chi_1 \equiv \phi_1 = p_x - x = 0$ a second class constraint, $\chi_2 \equiv x = 0$ another second class constraint and $\lambda = 0$ a restriction on the Lagrange multiplier.

Now we will try to see what we can do about the constraints. First we will deal with second class constraints, since they are easier to handle.

B.3.1 Systems with Only Second Class Constraints

Here we will assume the system contains only second class constraints. This assumption is harmless since one can usually convert the first and second class constraints into one another, which is a method for dealing with the constraints. So, the order of dealing with the constraints does not matter. This will be clear when we finish the section B.3.

B.3.1.1 The First Method: Solve the Constraints First

The simplest method of dealing with second class constraints is directly solving them. By solving the second class constraints we can get rid of some degrees of freedom. For instance for the example we presented at the beginning of this section (B.57), the total Hamiltonian involved two degree of freedom and a Lagrange multiplier to enforce the primary constraint, namely, (x, y, p_x, p_y) and λ . Here we are considering the case presented in (B.82). By solving the constraints $\chi_2 = x = 0$ and $\chi_1 = p_x - x = 0$ we can get rid of x, p_x and λ and treat the total Hamiltonian as if it is an ordinary Hamiltonian with a single degree of freedom, (y, p_y) . Of course the phase space functions must also be constrained to the constraint surface. More symbolically we can write:

$$\dot{F}|_c = \{F|_c, H_T|_c\}. \quad (\text{B.85})$$

Here we should be careful to take the Poisson brackets according to the remaining variables. Let us perform this method for our example.

$$H_T|_c = \frac{p_y^2}{2} \quad (\text{B.86})$$

$$\dot{F}|_c = p_y \frac{\partial F}{\partial y} \Big|_{x=p_x=0} \quad (\text{B.87})$$

Notice that by solving the second class constraints we did not have to solve for Lagrange multiplier λ . We will need constraints to be irreducible to be able to use this method however, otherwise we might set some of the first class constraints to zero.

B.3.1.2 The Second Method: Dirac Brackets in the case of Irreducible Constraints

Sometimes eliminating the constraint equations might be difficult or undesirable. In this case we can define a new bracket in terms of the ordinary Poisson bracket to define a new bracket structure, called Dirac bracket, to obtain the correct dynamics. Here we impose the second class constraints after we have taken the Dirac bracket. Dirac bracket will be denoted by $\{F, G\}^*$. The Dirac bracket is defined as

$$\{F, G\}^* := (\{F, G\} - \{F, \chi_\alpha\} C^{\alpha\beta} \{\chi_\beta, G\})|_c \quad (\text{B.88})$$

$$C_{\alpha\beta} := \{\chi_\alpha, \chi_\beta\} \quad (\text{B.89})$$

$$C^{\alpha\beta} C_{\beta\rho} = \delta_\rho^\alpha. \quad (\text{B.90})$$

Let us apply the method for our example (B.82). First we will apply the first method described in (B.3.1.1), to show how Poisson brackets are computed in the case we solve the constraints first.

$$\{F|_c, G|_c\} = \left(\frac{\partial F}{\partial y} \frac{\partial G}{\partial p_y} - \frac{\partial F}{\partial p_y} \frac{\partial G}{\partial y} \right) \Big|_{x=p_x=0}. \quad (\text{B.91})$$

Now we will compute the Dirac brackets for the same system.

$$C_{\alpha\beta} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}_{\alpha\beta} \quad (\text{B.92})$$

$$C^{\alpha\beta} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{\alpha\beta} \quad (\text{B.93})$$

$$\begin{aligned} \{F, G\}^* = & \left[\{F, G\} - \left(\frac{\partial F}{\partial x} + \frac{\partial F}{\partial p_x} \right) \left(\frac{\partial G}{\partial p_x} \right) \right. \\ & \left. + \left(-\frac{\partial F}{\partial p_x} \right) \left(-\frac{\partial G}{\partial p_x} - \frac{\partial G}{\partial x} \right) \right] \Big|_{x=p_x=0} \end{aligned} \quad (\text{B.94})$$

$$= \left(\frac{\partial F}{\partial y} \frac{\partial G}{\partial p_y} - \frac{\partial F}{\partial p_y} \frac{\partial G}{\partial y} \right) \Big|_{x=p_x=0} \quad (\text{B.95})$$

As we can see the Dirac brackets allow us to calculate without the need to solve the second class constraints. The Dirac brackets satisfy some good properties:

i) The Dirac bracket is antisymmetric,

$$\{F, G\}^* = -\{G, F\}^*. \quad (\text{B.96})$$

ii) The Dirac bracket obeys the Leibniz rule,

$$\{F, GR\}^* = \{F, G\}^* R + \{F, R\}^* G. \quad (\text{B.97})$$

iii) It obeys the Jacobi identity,

$$\{\{F, G\}^*, R\}^* + \{\{G, R\}^*, F\}^* + \{\{R, F\}^*, G\}^* = 0. \quad (\text{B.98})$$

iv) Second class constraints Dirac bracket commute with all phase space functions,

$$\{\chi_\alpha, F\}^* = 0 \quad (\text{B.99})$$

v) For a first class function G and an arbitrary F ,

$$\{F, G\}^* \stackrel{c}{=} \{F, G\}. \quad (\text{B.100})$$

vi) For F, G first class functions and an arbitrary R ,

$$\{R, \{F, G\}^*\}^* \stackrel{c}{=} \{R, \{F, G\}\}. \quad (\text{B.101})$$

This method can also be used in the reducible case if we manage to reduce the constraints.

B.3.1.3 The Third Method: Dirac Brackets in the case of Reducible Constraints

It is also possible to write the Dirac bracket in the case of reducible constraints. The method is to write the Dirac bracket more generally which is equivalent to the Dirac bracket in the case we make the constraints irreducible. It is defined like this:

$$\{A, B\}^* := \{A, B\} - \{A, \chi_\alpha\} D^{\alpha\beta} \{\chi_\beta, B\} \quad (\text{B.102})$$

where

$$D^{\alpha\beta} = -D^{\beta\alpha} \quad (\text{B.103})$$

obeys on $\chi_\alpha = 0$

$$D^{\alpha\beta} \{\chi_\beta, \chi_\rho\} = \delta_\rho^\alpha + Z^\alpha_{\dot{\alpha}} \lambda^{\dot{\alpha}}_\rho \text{ for some } \lambda^{\dot{\alpha}}_\rho. \quad (\text{B.104})$$

Let us show the equivalence of this method with the first reduce then create the Dirac bracket method. Suppose we have the below constraints, $\chi_1 = q^1 = 0$, $\chi_2 = p_1 = 0$ and $\chi_3 = q^1 + p_1 = 0$. Clearly we can take out any one of the χ 's to pass to the irreducible case. Let us assume we eliminated χ_2 . Then the Dirac bracket can be calculated as below.

$$C_{\alpha\beta} = \{\chi_\alpha, \chi_\beta\} = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}_{\alpha\beta} \quad (\alpha, \beta) \in \{1, 2, 3\} \quad (\text{B.105})$$

$$C_{\Lambda\Gamma} = \{\chi_\Lambda, \chi_\Gamma\} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}_{\Lambda\Gamma} \quad (\Lambda, \Gamma) \in \{1, 3\} \quad (\text{B.106})$$

So, we chose χ_1, χ_3 as the reduced complete set of constraints. Now the Dirac bracket reads:

$$\{A, B\}^* = \left(\{A, B\} - \{A, \chi_\Lambda\} C^{\Lambda\Gamma} \{\chi_\Gamma, B\} \right) \Big|_c \quad (\text{B.107})$$

$$= \{A, B\} + \{A, \chi_1\} \{\chi_3, B\} - \{A, \chi_3\} \{\chi_1, B\} \Big|_{\chi_1=0, \chi_3=0}. \quad (\text{B.108})$$

Now we will try to create the Dirac bracket without reducing the system.

$$Z^\alpha_{\dot{\alpha}} \chi_\alpha = 0 \quad \rightsquigarrow \quad Z^\alpha \chi_\alpha = -\chi_3 + \chi_1 + \chi_2 = 0 \quad (\text{B.109})$$

$$D^{\alpha\beta} \{\chi_\beta, \chi_\rho\} = \delta_\rho^\alpha + Z^\alpha_{\dot{\alpha}} \lambda^{\dot{\alpha}}_\rho \rightsquigarrow D^{\alpha\beta} \{\chi_\beta, \chi_\rho\} = \delta_\rho^\alpha + Z^\alpha \lambda_\rho. \quad (\text{B.110})$$

Here we have to solve the last equation as best as we can for both λ and D matrices, while also accounting the fact that the latter is antisymmetric. It is a linear equation system so it is pretty easy.

$$D^{\alpha\beta} C_{\beta\rho} = \begin{bmatrix} -D^{12} - D^{13} & D^{13} & -D^{12} \\ -D^{23} & D^{23} - D^{12} & -D^{12} \\ D^{23} & -D^{13} & -D^{13} + D^{23} \end{bmatrix}_{\beta}^{\alpha} \quad (\text{B.111})$$

$$= Z^\alpha_{\dot{\alpha}} \lambda^{\dot{\alpha}}_\rho = \begin{bmatrix} 1 + \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1 & 1 + \lambda_2 & \lambda_3 \\ -\lambda_1 & -\lambda_2 & 1 - \lambda_3 \end{bmatrix}_{\beta}^{\alpha} \quad (\text{B.112})$$

From these equations we get a single restriction on the possible λ , that is, $-\lambda_2 - \lambda_1 = 1 - \lambda_3$. We also solve for D in terms of λ , which gives $D^{13} = \lambda_2$, $D^{12} =$

$-\lambda_3$, $D^{23} = -\lambda_1$. Now, we simply insert this in the definition of Dirac bracket in the reducible case.

$$\{A, B\}^* = \{A, \chi_\alpha\} D^{\alpha\beta} \{\chi_\beta, B\} \Big|_{\mathcal{C}}. \quad (\text{B.113})$$

By using the restrictions on D and λ one gets:

$$\{A, B\}^* = \{A, B\} + \{A, \chi_1\} \{\chi_3, B\} - \{A, \chi_3\} \{\chi_1, B\} \Big|_{\chi_1=0, \chi_3=0}. \quad (\text{B.114})$$

There are some more flexibility in the definition of D matrix, but we were not able to understand how those parts in [3] came about.

B.3.1.4 The Fourth Method: Convert the Second Class Constraints into First Class Constraints

This last method is intimately related to a method of dealing with the first class constraints so we will describe it there, in B.3.2.1.

B.3.2 Systems with only First Class Constraints

As far as the classical dynamical picture is concerned we were done in section B.2. There are simply some equations of motion which are not uniquely determined but this is alright since the arbitrariness just means we have an equivalence class of the physical states rather than a single unique description. However, once we wish to create a quantum field theory we might need to get rid of this gauge freedom since we wish to avoid multiple counting of the states, or extend the phase space and treat it accordingly as in BRST quantization [23], [3], [16]. So, the main problem of systems with first class constraints is to be able to quantize the system. Classically, we have the problem that the two-form induced from the phase space onto constraint surface is degenerate and some degrees of freedom are not actual degrees of freedom. That is to say, first class constraints do not just reduce the dimension of the physical phase space by one but also introduces gauge symmetries which are the null eigenvectors of the induced two form. Treating the first class constraints is much harder than the second class constraints. That is why we will focus on a single method B.3.2.1 and only describe the other methods very briefly. There are also other methods

in the literature than the ones we are going to discuss. So we will try to do simple examples and show how the counting of the true degrees of freedom in a system goes. We will assume we are in the irreducible case.

B.3.2.1 The First Method: Gauge Fixation

Now, to avoid multiple counting of the states, which is important in calculating the path integrals [23], we introduce ad-hoc equations restricting the gauge freedom. For instance, in electromagnetism we may introduce the Lorenz gauge or the Coulomb gauge or for ADM formulation of general relativity we can choose the Gaussian normal coordinates. There are two conditions a satisfactory set of gauge conditions $C_b(q, p) \stackrel{c}{=} 0$ must satisfy [3].

- i) The gauge must be accessible. That is, all possible states of the dynamical system should have a representative in the chosen gauge condition and starting from this representative we should be able to reach any other equivalent state by using infinitesimal transformations of the form $\delta u^a \{F, \gamma_a\}$.
- ii) The gauge conditions should fix the gauge completely. That is, any state should have only a single unique representative. Even discrete transformations, the ones which are created in a single finite step rather than infinitely many arbitrarily small steps, are not allowed. This is equivalently stated by the condition, $\delta u^a \{C_b, \gamma_a\} \stackrel{c}{=} 0$ should imply $\delta u^a = 0$.

If this conditions are satisfied by gauge conditions they are called canonical gauges. The Lorenz gauge is not canonical, since we can still use scalar functions with vanishing Laplacians in creating a gauge transformation. Choosing Gaussian normal coordinates is not a canonical gauge condition, since Gaussian normal coordinates can be chosen smoothly in only a part of a manifold and not everywhere in the manifold. By looking at the gauge fixing conditions as a set of constraints we can see that C_b, γ_a form a set of second class constraints. So, after introducing the gauge fixing conditions one can apply the methods described in B.3.1, to deal with the dynamical system. Now, we can also ask the question: if it is possible turn first class constraints into second class constraints

by introducing a set of gauge conditions, then is it also possible to eliminate some constraints (or introduce more hypothetical degrees of freedom) to turn a system with a set of second class constraints into a system with a set of first class constraints and gauge fixing conditions? The answer is yes, although performing the said feat might be difficult. Let us examine two examples from [3]. First example is a system with constraints: $\chi_1 = q^1 = 0$ and $\chi_2 = p_1 = 0$. By eliminating the constraint χ_1 we can turn the system into a system with first class constraints. We can also introduce more degrees of freedom as in this next example. Suppose we have the same set of constraints as before. Now introduce a hypothetical degree of freedom (q^2, p_2) and turn the constraints χ_1, χ_2 into $\gamma_1 \equiv q^1 + q^2 = 0$ and $\gamma_2 \equiv p_1 - p_2$. The Poisson brackets of the constraints vanish and the system can be treated as a first class system. The gauge fixing conditions might be impossible to construct globally [3], this problem is called Gribov obstruction.

B.3.2.2 The Second Method: Reduced Phase Space

Suppose we define a gauge orbit as the equivalence class of states which are related by a gauge transformation. Then it should be possible to quotient the constraint surface by these gauge orbits and get a reduced phase space. This method has the caveat that the reduced phase space may not be a manifold [16].

B.3.2.3 The Third Method: Dirac Constraint Quantization

Dirac's proposition is to upgrade the first class constraint functions into their quantum analogs and take the physical Hilbert space as the smaller set of the states which vanish when taken as an argument for the constraint operators. This method has the caveat that we might need to deal with the operating ordering issues.

B.3.2.4 The Fourth Method: BRST Quantization

The main idea in this quantization is to replace the gauge symmetry by a rigid BRST symmetry which will be present even after gauge fixing. This is achieved

by introducing more fields into the theory [23].

B.4 Counting of The Degrees of Freedom

Now, the strategy we will follow can be summarized as:

- Step 1) Find the canonical Hamiltonian and determine the primary constraints,
- Step 2) Find the total Hamiltonian, work out the consistency algorithm,
- Step 3) Classify the constraints into first and second class constraints,
- Step 4) Solve the second class constraints if it is possible if not write the Dirac bracket,
- Step 5) For dealing with the first class constraints assume a canonical gauge is given and they turn into second class constraints.

APPENDIX C

BORN - INFELD EXTENSION OF A THEORY AND THE PRINCIPLE OF FINITENESS

This is a short summary of the original article [24] which gives a description for creating theories with finite field strength. Now, the article only described how to use the principle of finiteness for the electromagnetic theory of Maxwell. There is also another article describing the same principle for gravitational actions, [25]. I do not understand every part of that article so I will only summarize the [24]. It seems that [25], consider the ghost-freedom as a secondary principle in creating a BI extension of a gravity theory, however they also seem to open the action only up to second order. I am not entirely sure if it can be claimed that a BI extended gravity theory is ghost-free since the action is of infinite order in Ricci tensor. In this appendix, we are working with the assumption of flat space-time.

C.1 Principle of Invariant Action

Since, all field theories must be independent of the possible coordinate systems they must have an invariant action. This is a requirement for all field theories so we will talk just a little bit about this. The requirement simply means that the action must have a determinant form inside a square root since that makes the integral invariant. This is because of the fact that totally antisymmetric tensors, top forms, are all proportional to the totally antisymmetric Levi-Civita tensor and all of them can be written as

$$f\epsilon = f\epsilon_{\alpha\beta\dots\mu}\tilde{d}x^\alpha\tilde{d}x^\beta\dots\tilde{d}x^\mu \quad (\text{C.1})$$

$$= f\sqrt{\det(g_{\rho\lambda})}\tilde{d}x^1\wedge\tilde{d}x^2\wedge\dots\wedge\tilde{d}x^D \quad (\text{C.2})$$

where $f\epsilon$ is any top form. The $\tilde{d}x^\alpha \wedge \tilde{d}x^\beta \wedge \dots \wedge \tilde{d}x^\mu$ only gives the orientation for the infinitesimal volume element. Since the orientation remains the same, due to its definition, what must be the integrand is always a scalar times the square root of determinant of the metric. Since determinants are all proportional to each other by scalars we can use any square root of a determinant to make an invariant action integral.

C.2 Principle of Finiteness

Here, the inspiration is taken from special relativity. If one examines the free particle Lagrangian in special relativity it is found as

$$L_{free,rel} = -m\sqrt{1 - v^2}. \quad (\text{C.3})$$

Also it is interesting that it can be written as a determinantal form

$$L_{free,rel} = -m\sqrt{\det(\delta_j^i - v^i v_j)}. \quad (\text{C.4})$$

As it is well known velocity has a finite limit in special relativity and now we have one more reason to use determinantal actions for theories with finite field strength. This does not mean that this is the only way to create theories with finite field strength but it seems to be common. There could be other methods. We will examine how the original article [24] crates a theory for electromagnetism and then we will write a little bit about the same procedure for gravitational actions which is much more explained in [25]. Since the action should be determinantal there are many possibilities for the integrand. Taking inspiration from (C.4) one claims that the action should be a linear combination of $\sqrt{-\det(g_{\mu\nu} + F_{\mu\nu})}$, $\sqrt{-\det g_{\mu\nu}}$, $\sqrt{\det F_{\mu\nu}}$ in some suitable units. Let us write

$$L = \epsilon\sqrt{-\det(g_{\mu\nu} + F_{\mu\nu})} + A\sqrt{-\det g_{\mu\nu}} + B\sqrt{\det F_{\mu\nu}}. \quad (\text{C.5})$$

The article claims that $\sqrt{\det F_{\mu\nu}}$ can be shown to be a surface integral. Let us try to show this. First, we will give a couple of facts about skew-symmetric matrices (here adapted for geometrical quantities, two forms). Suppose \mathbf{A} is a two-form in a $2n$ dimensional manifold. Then

$$\mathbf{A} = \frac{1}{2}A_{\mu\nu}(\tilde{d}x^\mu \wedge \tilde{d}x^\nu) \quad (\text{C.6})$$

$$\det(A_{\mu\nu}) = \text{pf}(A)^2 \quad (\text{C.7})$$

where $\text{pf}(A)$ denotes the Pfaffian of A . We also have another nice formula about the Pfaffians,

$$A \wedge A \wedge \dots \wedge A = n! \text{pf}(A) \tilde{d}x^1 \wedge \tilde{d}x^2 \wedge \dots \wedge \tilde{d}x^{2n}. \quad (\text{C.8})$$

Using these facts and the fact that $\mathbf{F} = d\tilde{\mathbf{A}}$ we can show that $\sqrt{\det F_{\mu\nu}}$ reduces to a surface integral. Let us show this.

$$\begin{aligned} B\sqrt{\det F_{\mu\nu}} &= B\text{pf}(F_{\mu\nu}) \\ B\text{pf}(F_{\mu\nu})\tilde{d}x^0 \wedge \tilde{d}x^1 \wedge \tilde{d}x^2 \wedge \tilde{d}x^3 &= \frac{B}{2}\mathbf{F} \wedge \mathbf{F} \\ &= \frac{B}{2}d(d\tilde{\mathbf{A}} \wedge \tilde{\mathbf{A}}) = \frac{B}{2}d(\tilde{\mathbf{A}} \wedge d\tilde{\mathbf{A}}) \end{aligned} \quad (\text{C.9})$$

reduces to a surface integral due to Stokes' theorem. Here, B is a constant. So, we will deal with Lagrangians that can be constructed from the other two terms, that is,

$$L = \epsilon\sqrt{-\det(g_{\mu\nu} + F_{\mu\nu})} + A\sqrt{-\det g_{\mu\nu}}. \quad (\text{C.10})$$

Here we might put another obvious condition. If the field strength is low, then it must reduce to Maxwell's theory. This condition fixes A and ϵ . In Cartesian coordinates

$$\begin{aligned} \det(\eta_{\mu\nu} + F_{\mu\nu}) &= -1 + F_{0i}^2 - F_{ij}^2 + \det(F_{\mu\nu}) \quad \text{sum over } i \text{ and } j \\ L_{\text{Maxwell}} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ &= \frac{F_{0i}^2 - F_{ij}^2}{2} \quad \text{sum over } i \text{ and } j \\ \sqrt{1+x} &= 1 + \frac{x}{2} + \mathcal{O}^2(x) \\ L &= \epsilon\sqrt{1 - F_{0i}^2 + F_{ij}^2 + \det(F_{\mu\nu})} + A \quad \text{sum over } i \text{ and } j \\ &= \epsilon\left(1 - \frac{F_{0i}^2 - F_{ij}^2}{2}\right) + A + \mathcal{O}^4(F_{\mu\nu}) \quad \text{sum over } i \text{ and } j. \end{aligned} \quad (\text{C.11})$$

This fixes $\epsilon = -1$ and $A = 1$. This is not the only possible Lagrangian we can construct. We can construct other Lagrangians as well. Let us construct another one. Let us examine this quantity $\sqrt{-\det(g_{\mu\nu} + F_{\mu\nu})}$. It, being a polynomial, can be Taylor expanded in both $g_{\mu\nu}$ and $F_{\mu\nu}$ in any order. We first open the series up to first order in $g_{\mu\nu}$

$$\det(g_{\mu\nu} + F_{\mu\nu}) = \det(g_{\mu\nu}) + \Phi[F_{\mu\nu}, g_{\mu\nu}] \quad (\text{C.12})$$

$$\Phi[F_{\mu\nu}, g_{\mu\nu}] = \det(g_{\mu\nu} + F_{\mu\nu}) - \det(g_{\mu\nu}). \quad (\text{C.13})$$

We also Taylor expand this up to first order in $F_{\mu\nu}$.

$$\begin{aligned} \Phi[F_{\mu\nu}, g_{\mu\nu}] &= \det(F_{\mu\nu}) + \phi[F_{\mu\nu}, g_{\mu\nu}] \\ \det(g_{\mu\nu} + F_{\mu\nu}) &= \det(g_{\mu\nu}) + \det(F_{\mu\nu}) + \phi[F_{\mu\nu}, g_{\mu\nu}]. \end{aligned} \quad (\text{C.14})$$

We can determine the form of $\phi[F_{\mu\nu}, g_{\mu\nu}]$ in a geodesic coordinate system. Remember we are in flat space-time in some general coordinate system so, we have to find the value of $\phi[F_{\mu\nu}, g_{\mu\nu}]$ in a generic coordinate system.

$$\frac{\det(g_{\mu\nu} + F_{\mu\nu})}{\det(g_{\mu\nu})} = \frac{\det(\eta_{\hat{\mu}\hat{\nu}} + F_{\hat{\mu}\hat{\nu}})}{\det(\eta_{\hat{\mu}\hat{\nu}})} \quad (\text{C.15})$$

where the indices with the hat indicate Cartesian coordinates

$$\begin{aligned} &= -1 \det(\eta_{\hat{\mu}\hat{\nu}} + F_{\hat{\mu}\hat{\nu}}) \\ &= -1(-1 - F_{\hat{i}\hat{j}}^2 + F_{\hat{0}\hat{i}}^2 + \det(F_{\hat{\mu}\hat{\nu}})) \\ &= 1 + F_{\hat{i}\hat{j}}^2 - F_{\hat{0}\hat{i}}^2 - \det(F_{\hat{\mu}\hat{\nu}}) \\ \frac{\det(g_{\mu\nu} + F_{\mu\nu})}{\det(g_{\mu\nu})} &= 1 + \frac{1}{2} F_{\hat{\mu}\hat{\nu}} F^{\hat{\mu}\hat{\nu}} + \frac{\det(F_{\hat{\mu}\hat{\nu}})}{\det(g_{\hat{\mu}\hat{\nu}})} \end{aligned} \quad (\text{C.16})$$

where the right hand side can be made into a general coordinate system

$$\begin{aligned} &= 1 + \frac{1}{2} F_{\mu\nu} F^{\mu\nu} + \frac{\det(F_{\mu\nu})}{\det(g_{\mu\nu})} \\ \det(g_{\mu\nu} + F_{\mu\nu}) &= \det(g_{\mu\nu}) + \det(g_{\mu\nu}) \frac{1}{2} F_{\mu\nu} F^{\mu\nu} + \det(F_{\mu\nu}) \\ &= \det(g_{\mu\nu})(1 - G^2 + F) \\ F &= \frac{1}{2} F_{\mu\nu} F^{\mu\nu} \\ G^2 &= \frac{\det(F_{\mu\nu})}{\det(g_{\mu\nu})}. \end{aligned} \quad (\text{C.17})$$

Both F and G^2 are also invariant quantities. So, they can also be used to create Lagrangians. For instance,

$$\begin{aligned} L_{\text{trial}} &= \sqrt{-\det(g_{\mu\nu})} \sqrt{1 + F} \\ &= \sqrt{-\det(g_{\mu\nu} + F_{\mu\nu}) + \det(F_{\mu\nu})}. \end{aligned} \quad (\text{C.18})$$

This is the general method of creating a theory whose field strength is bounded. The rest of the article [24] deals with the physical consequences of the Lagrangians we have constructed and examines which one is more appropriate for the nature.

C.3 Gravitational Actions

What we have described above for the Born- Infeld extension of Maxwell's theory can be applied to Einstein's theory. In fact, such an extension in Einstein's theory preceded Maxwell's theory. [See the PhD thesis [8] and the references therein for more details.] I do not know much about this method but as far as I understand [25] uses a general action

$$I = \int_{\mathcal{M}} d^D x \sqrt{-\det(ag_{\mu\nu} + bR_{\mu\nu} + cX_{\mu\nu}[R])} \quad (\text{C.19})$$

with a fudge tensor, $X_{\mu\nu}$ which is a functional of Ricci tensor, to impose some freedom on the theory and apply whichever extra principle one can fancy. For instance, they use the ghost freedom as an extra principle. BINMG is also created like this [8], [5].