REINSURANCE PRICING USING EXPOSURE CURVE OF TWO DEPENDENT RISKS

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ABSTRACT

REINSURANCE PRICING USING EXPOSURE CURVE OF TWO DEPENDENT RISKS

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It is known that experience rating and exposure rating are used for insurance and reinsurance pricing by many practitioners. One of the main tools of exposure rating which is commonly used is exposure curves. It evaluates the percents of pure risk premium shared by insurer and reinsurer. In practice, the exposure curves which depend solely on claim history are widely utilized as an important indicator to determine the price and retention level based on the preferences and strategies of the insurer and reinsurer. The aim of this study is to present exposure curves, their use and statistical distribution for single risk and then develop a theoretical distribution for bivariate case when two risk sources generating claims behave under dependent assumption. Bivariate Pareto distribution constructed is employed to generate exposure curves performing simulation with respect to their parameters. Sensitivity of curves to the different values of parameters are investigated and compared for better estimations. The findings of this thesis show that the joint statistical distribution for dependent risks yield meaningful exposure curves which may be easily implemented to practical use.

Keywords : Reinsurance, reinsurance pricing, exposure rating, exposure curve, Bivariate Pareto distribution, dependent risks

ÖZ

BAĞIMLI İKİ RİSKİN MARUZ KALMA EĞRİSİNİ KULLANARAK REASÜRANS FİYATLANDIRMASI

Akarsu, Gülçin
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Sigorta ve reasürans fiyatlandırması için deneyim oranı ve maruz kalma derecesinin, bir çok uygulayıcı tarafından kullanıldığı bilinmektedir. Yaygın olarak kullanılan ana maruz kalma derecesi araçlarından biri maruz kalma eğrileridir. Sigortacı ve reasürör tarafından paylaşılan saf risk primi yüzdesini hesaplar. Uygulamada yalnızca hasar geçmişine dayanan maruz kalma eğrileri, sigorta şirketinin ve reasürörün tercihleri ve stratejileri temelinde, fiyat ve muafiyet seviyesini belirleyen önemli bir gösterge olarak yaygın bir şekilde kullanılmaktadır. Bu çalışmanın amacı, mevcut maruz kalma eğrilerini, kullanımlarını ve tek bir risk için istatistiksel dağılımlarını, daha sonra, hasarları üreten iki risk kaynağının bağımlı olması varsayımı altında, kuramsal bir dağılım oluşturmasını sunmaktır. İki değişkenli Pareto dağılımı, parametrelerine göre simülasyon yapılarak maruz kalma eğrileri oluşturmak için kullanılır. Eğrilerin farklı parametrelere duyarlılığı araştırılmış ve daha iyi tahminler için karşılaştırılmıştır. Bu tezin bulguları, bağımlı riskler için ortak istatistiksel dağılımın, pratik kullanıma kolaylıkla uygulanabilecek anlamlı maruz kalma eğrileri verdiğini göstermektedir.

Anahtar Kelimeler : Reasürans, reasürans fiyatlandırması, deneyim oranı, maruz kalma eğrisi, iki değişkenli Pareto dağılımı, bağımlı riskler

To My Mother

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LIST OF ABBREVIATIONS

XL	Excess of loss
SI	Sum insured
MPL	Maximum probable loss
EML	Estimated maximum loss
MBBEFD	Maxwell-Boltzmann, Bose-Einstein and Fermi-Dirac
ISO	Insurance Service Office
PSOLD	Property Size of Loss Database

CHAPTER 1

INTRODUCTION

Every company has main target to guarantee itself and its customers under the risk of insolvency. For this reason, they implement reduction techniques to decrease their risks. If all risks are incurred at the same time, insurance companies may not be able to pay all of the incurred claims. Therefore, they need a risk transfer mechanism called reinsurance. Reinsurance is a method of transferring a part of the risk which is undertaken by an insurance company to the third parties.

The main goal of reinsurance is to reduce the risk on ceding company by sharing or transferring the aggregate or individual claims. It provides companies avoiding from catastrophic risks, remaining steady when extraordinary events occur. Insurer's portfolios are more balanced and homogenized, which enable to predict the expected losses more likely. It regulates the income and makes the portfolios more productive and manageable. Also, it affects the capital management in an efficient way and empower the financial position. After reinsurance coverage, insurance companies will be able to increase their capacity to accept more risks and write more business. This allows them to increase underwriting risk capacity both in size and number.

Sharing risk premium with a predetermined loadings is called pricing in reinsurance market. One of the parties, sometimes both of them, determines the price for the reinsurance agreement and the other party chooses either to buy or decline the agreement based on the price set by reinsurer. The price which is evaluated by underwriters depends on the type of reinsurance contract and its characteristics. It is so obvious that the premium determination is simple in proportional reinsurance agreements such as quota share and surplus. The more cumbersome ones are non-proportional reinsurance agreements which are excess of loss and stop loss. For these types of reinsurance, the premiums are calculated by more exposure rating including different models and techniques than experience rating.

Common methods for exposure rating for non-proportional reinsurance is exposure curves. It gives a ratio so that insurer knows how much he\she should retain the part of the premium over the total premium and how much of it to be paid for reinsurance protection. It is apparent that, this ratio tends to change for different size of retention level. Whereas, some reinsurance contracts offer a coverage from high level retention, some of them offer low ones. For example, in the case of high retention level, the insurer has to pay higher claims than reinsurer. Therefore, the insurer deserves higher premium and he\she should determine sharing level of the premium. In that point, exposure curves has a significant role to give a ratio for settling this level. Retention level is not just the one factor that affects the sharing ratio. The factors like type of risks, size of risks and type of perils have impact on exposure curve. Therefore, different kind of mathematical and statistical methods and models are used to construct the exposure curve for incorporating the influence of different factors.

1.1 Literature Survey

There are vast amount of literature in reinsurance. Gerber [13] presents the optimal level of retention for excess of loss reinsurance. He maximizes the adjustment coefficient to minimize the probability of ruin. Waters [26] points out setting an optimal deductible level under the condition that the ruin probability is minimum for the excess of loss reinsurance. He uses insurer's adjustment coefficient as a function of deductible level to minimize the probability of ruin.

Dickson and Waters [9] study on the loading factor of reinsurance when it depends on the deductible level. They analyze the effect of the reinsurance on the ruin probabilities of both infinite and finite time. As further study, Dickson and Waters [10] examine the minimizing the ruin probability in finite time for both discrete and continuous cases to set optimal level of insurance. Thus, they introduce a condition that the expected gain is greater than or equal to zero. They propose a formula for the ruin probability in finite time for both discrete and continuous cases using the Bellman optimality principle [11].

Centeno [2] carries out the combination of excess of loss and quota share reinsurance. She also describes the adjustment coefficient of insurer as a function of the level of deductible [6]. She reveals this level to the combination of excess of loss and quota share reinsurance under Sparre Anderson model. In addition, Centeno [3] determines the optimal deductible level to minimize the upper bound of the probability of ruin. Centeno and Guerra [14] indicate the relation between maximizing both the adjustment coefficient and the expected utility. They use the exponential utility as a utility function.

De Finetti [5] shows that reinsurance decreases the some risks of insurer. These risks that he focuses on are the probability of ruin and the variance of the gain. He indicates the optimal deductible level by reinsurance principles in non-life insurance under the conditions of minimizing the variance of the gain. He presents how this level should be evaluated for non-proportional reinsurance.

Ignatov et al. [16] obtain a formula for the survival probability of cedant for the expected gain. They use the difference between the survival probability of the cedant and the reinsurer for determination of the optimal level of reinsurance. They derive the formula for the expected gain of both parties using joint survival probabilities.

Clark [4] introduces the reinsurance pricing. He explains experience rating and exposure rating to calculate the premiums for both insurer and reinsurer. He mentions the pricing for different types of reinsurance. Deng and Zhang [7] approach to pricing in more theoretical way. They use linear backward stochastic differential equations for reinsurance premiums. Kreps [17] obtains the algorithm that applies investment criterion relating to return to association of reinsurance agreement terms and economic techniques. The algorithm detects the minimum price for an uncommon event, and generally recommends a reduction in risk.

Meyer [20] focuses on the experience rating in his paper. He analyzes the effect of features which does not have any compeer in the literature on Bayesian credibility. He uses collective risk model to examine the effect of these features. He updates Paul Dorweiler's approach of testing experience rating models. The outputs generate the parameters of experience rating models. Wang and Williamson [25] and Topel [24] study on experience rating in unemployment line of business. Sloan [23] discusses the applicability of experience rating in medical malpratice insurance. Ruser [21] analyzes how an increase in the indemnity of workers' compensation insurance influences the claim ratio when the companies use different degrees of experience rating.

The first exposure rating technique and its application is developed by Salzmann [22]. She considers that both the claims transferred and the claims retained should be represent as a percentage. Furthermore, these percentages of claims which belong to each layer of protection of insurance should be determined by a method that performs maximum affordability and flexibility. Firstly, the claim of homeowners fire line of business are used and claim distributions as percents of insured volumes are developed. The relation between fire claims and corresponding volume of insurance is examined. Mack and Fackler [19] studies on liability insurance. They use Riebesell's formula for rating the premium. Ludwig [18] updates the study of Salzmann with homeowners fire claims experience using the data for the accident years between 1984 and 1988. He applies also the same approach to wind claims, property caused of claims and 1989 Hurricane Hugo claims to observe whether the distributions of these claims as percents of insured volumes are different from the fire claims.

Nevertheless, the theoretical studies on exposure curves are limited in literature. The experts of firms or societies examine and develop the curves for reinsurance markets. They use the data of companies to create the exposure curves. Gasser [12] develop a new method for exposure rating to be used in reinsurance. The exposure curves by using the fire loss data of Swiss Re for years 1959 and 1967 are plotted. Bernegger [1] studies on a new parametrisation of Gasser's exposure curves by using Maxwell-Boltzman, Bose-Einstein, Fermi-Dirac (MBBEFD) distribution class. He presents some of Swiss Re curves as an analytical function with two parameters. Guggisberg [15] points out changing Swiss Re exposure curves under the effect of different factors such as size of risk, type of risk and type of peril.

All these studies, which are not many in number, consider the claims with respect to single risk. However, there exist no literature on exposure curves combining two separate risks which may pose a certain dependence to each other.

1.2 The Aim of Study

Exposure curves vary in different lines of business and it is known that some line of businesses sometimes can not be considered as separately. In case of two dependent line of businesses, the availability of exposure curves and their utilization in ratemaking gain importance. Therefore, the question on finding the type of theoretical joint distribution of exposure curves under dependence assumption and its behavior under certain parameter values initiates the motivation for this thesis. In this frame, the aim of this study is to derive the bivariate distribution of two dependent risks and their marginals. Based on this joint distribution, exposure curves are plotted with respect to certain parameter values.

This thesis is composed of six chapters. After the introduction part, Chapter 2 gives the preliminaries on reinsurance and its pricing. The definition and the basis for constructing an exposure curve is given in Chapter 3 with illustrative examples, like Swiss Re exposure curves. Chapter 4 presents the theoretical derivations of Bivariate Pareto distribution yielding exposure curves for dependent risks. Simulations to illustrate the outcome of proposed distribution are done with respect to different parameter values in Chapter 5. The last chapter presents the comments and conclusion of the proposed study.

CHAPTER 2

REINSURANCE

The legal parties of reinsurance process are insurance company and reinsurer company. Insurance company is called ceding company or cedant and it is the party that transfers the risk. Reinsurer is the party that accepts the transferred risk. For uncertain catastrophic risks, even reinsurance company may need a financial protection. In this case, reinsurer becomes a cedant and transfers a part of transferred risk to another reinsurance company. This process is called as retrocession. The reinsurance agreement is basically similar to the insurance agreement with the only difference that the clients of cedant are people in insurance company whereas reinsurer have the cedants as clients.

2.1 Classification of Reinsurance Types

The types of reinsurance are classified as facultative and treaty and both of them can be agreed for proportional or non-proportional forms. Facultative reinsurance contract consists of a specified risk about large losses such as life or catastrophe insurances. Facultative is barely preferred by insurers because: (i) Underwriting and personnel expenses are higher since all calculations are redone for every individual risk. (ii) The insurer has to retain all risks until the agreement is signed with reinsurers. (iii) The reinsurer has an option to reject the facultative reinsurance agreement or to reprice it to cover the combining of all expenses and premium.

On the other hand, treaty reinsurance contract is a group of several policies covered as a lump for an explicit period. It offers stationary obligations and rules for continuity of business. Treaty is more preferred type of reinsurance for insurers because: (i) The reinsurer has to accept all the incurred risks under the treaty reinsurance coverage. This implies that one has no right to pick the risk or policy from the treaty to reject the payment. (ii) The underwriting cost of cedants is lower than facultative reinsurance's underwriting cost, which gives a rise to capability of writing more business. (iii) It provides to cedant to manage the portfolio easily and properly.

In **proportional reinsurance**, the parties share the risk in specified percent. This percent is changed according to the type of risk or the size of risk. The upper bound for the agreement should also be indicated in the contract. Otherwise, the policies would not be balanced and reinsurer would not determine the retained maximum loss. Quota share and surplus are known as proportional reinsurance types. Quota share reinsurance runs with stable percentage for every policy. Furthermore, the shared premium is calculated again according to this percentage. It is usually preferred by small companies or new companies which have just involved to business. Surplus reinsurance also runs with a specified percentage per policy. This percentage is determined by the relation between the limit and the retention level of the policy. In surplus reinsurance, the percentage and the rules of the contract are applied separately for every policy while the rules and the ratios of the quota share are applied in the same percent for all policies.

Non-proportional reinsurance is defined as that the claims are transferred when they exceed the certain level of the claim. This level is determined as an amount or a ratio. Insurer pays the part of the claims which are less than the prescribed level and transfers the claims above the level to reinsurer. Excess of loss and stop loss are subbranches of non-proportional reinsurance. Cedant accepts the risk under the determined level as an amount called deductible, and agrees with reinsurer to transfer the risk that exceeds the deductible in excess of loss reinsurance agreement. The agreement can be signed for each policy or line of business. Whether the agreement may have the upper limit or not depends on the decision of the parties. Stop loss reinsurance agreement has some similarities with excess of loss type. It has also predetermined deductible level but this level is represented only as a ratio. The reinsurer has a liability to pay above the predetermined loss ratio in the agreement. The upper limit as a percentage can be also included in the agreement in this reinsurance type. Besides of similarities, these two types of non-proportional reinsurance have some differences. The main differences are the payment time and deductible level. The parties agree for an option to pay the claim or group of claims when they occur in excess of loss contracts. On the other hand, in stop loss reinsurance, they pay the aggregated losses for the given time period which is generally at the end of the year. Deductible level can change in both ratio or amount according to characteristics of agreement in excess of loss. Unlikely, stop loss uses the given percentage, not an amount, as a deductible level for the accumulation loss of the year.

2.2 Reinsurance Pricing

Risk premium is the part of the premium without commission, costs and reinstatements. It is the complicated process which involves statistical data, mathematical methods, reinsurance market conditions and advanced actuarial calculations. Any mistake on the process may lead to price the reinsurance incorrectly and enormous claim payments which may result in insolvency.

2.2.1 Pricing of Proportional Reinsurance

Reinsurance pricing for proportional type is easier than non-proportional one. We assume that cedant and reinsurer share the premium under the given percentage $k\%$ (such as 70%) in quota share reinsurance contract. It means that reinsurer pays $(1-k)\%$

(30%) of losses occurred under reinsured policies in return to gain $(1-k)\%$ (30%) of the premium.

Surplus reinsurance pricing is more complicated than quota share pricing. The agreement is built on the layers, named lines. There is a retention level for each line, and these two entities determine the upper bound or the limit of the contract. Reinsurer pays the loss exceeding the retention level up to the end of these lines. To make it more clear, Example 1 is given.

Example 1: We assume that the retention level is set to $\$100000$. The contract has 9 lines, which implies the limit of the protection is 9 times of retention and equals to $\$1000000$. If there exists a risk with $\$1500000$ indemnity under the protection of the reinsurance, cedant pays $\$100000$ and reinsurer pays $\$900000$. The rest of the indemnity is out of the contract. Cedant might pay the remaining risk or go for another reinsurance agreement. Suppose that cedant chooses to pay the remaining part $\$500000$. The payment of $\$600000$ is left to cedant in total. So, the percentages are follows: cedant pays $\$600000/\$1500000 = 40\%$ and reinsurer pays $\$900000/\$1500000 = 60\%$. They share the premiums and the incurred losses according to these percents in that policy.

Here, the main question is how the ratios or amounts above should be determined. One of the foundation of the calculations is the historical experience. The data of written premiums or claims incurred for a large time period or previous treaties information should be collected. Extreme losses, which are low frequency with high severity, and catastrophe like earthquake or hurricane are excluded from this part of the evaluation because these kind of risks may lead to the underestimation of the price. To estimate the future claims and premiums is also important for determining the retentions and limits. If all required data is available and estimating models are sufficient, the ratio of historical claims will be approximately equal to the ratio of expected claims for the future claims. After selecting the ratio, the catastrophe ratio and expenses such as commissions, personnel cost and broker free are loaded onto the risk premium.

2.2.2 Pricing of Non-Proportional Reinsurance

Excess of loss and stop loss are the types of non-proportional reinsurance. The claims which exceed the retention level are paid by the reinsurer in both types of reinsurance. They also have some similarities in pricing. Both experience rating and exposure rating are used to price these reinsurance types. The main goal of the experience rating is to obtain the best estimation of claims and premium for next business period. Historical experience and properly ratemaking are the tools of the best estimation. The data of historical claims and earned premiums that belong to past years are required. The more years of data are available, the better is analyzing the experience rating. Like proportional reinsurance, future projection for expected premiums, claims and also coverage levels are determined by the rates. In addition, some analyses for experience rating are evaluated by credibility [20]. Catastrophe rate should be computed separately again like in proportional. Cat-rate and expenses should be added to the selected rate after the calibrating all historical experience and ratemakings. There are several kinds of methods and models in literature for evaluating the cat-rates and weight given to each expense. However, in this study, we only focus on the pure premiums, which are the part of the rating before adding loadings such as commissions, expenses, reinstatements and the remain expenses.

Exposure rating is another method that is more theoretical for approaching to pricing non-proportional than the historical experience. The main characteristic difference between exposure rating and historical experience is modeling the risk profiles in portfolios. Loss distribution is required for modeling and it derives the exposure curves which exposure rating concentrates on. To attain the part of the premium transferred to reinsurer, the rating is obtained as the percentage of claims. So, the method uses percents for all claims, deductibles and limits.

CHAPTER 3

UNIVARIATE EXPOSURE CURVE

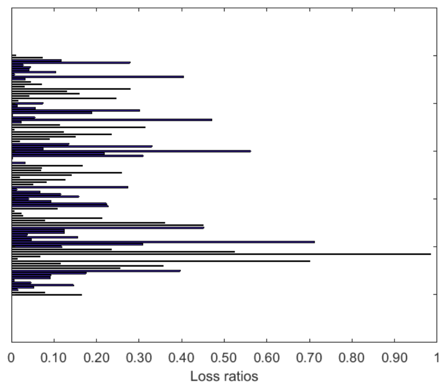
The exposure curves are based on risk profiles which consist of risk bands. All risks have the same volume in terms of sum insured (SI), maximum probable loss (MPL) or estimated maximum loss (EMP) within these risk bands. To be able to apply this method, claim distribution is required. However, the problem is to detect the suitable function for each risk band. Therefore, historical data helps us to define correct distribution function. Incurred claims in the same kind of risk portfolios allow us to predict the type of the claim distribution. These distributions are utilized as the main tool to construct exposure curves.

Exposure curve is a function of the retention level as known as deductible. It gives the information about sharing risk premium between cedant and reinsurer based on this level. Exposure curves need to be used individually for each risk type and risk band. Nevertheless, this leads to rise some questions such that which exposure curve should be selected for a specific risk type and which conditions and assumptions should be attained, if it is necessary. Before defining the construction of exposure curves, the motivation of the basis of exposure curve is given basically.

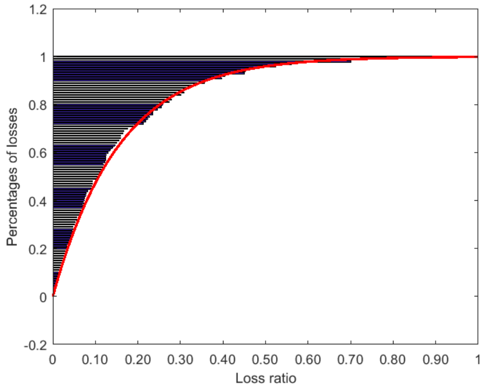
3.1 The Basis of Exposure Curve

Portfolios consist of different line of business. Each policy has different amount of loss in these portfolios. The amount of loss is converted into a percentage based on indemnity so-called total exposure. Thus, loss ratios is used instead of loss amounts, which makes the comparison of different volumed risk easier and standardized. An example on the definition of univariate exposure curves is given to point out the function of these curves under certain statistical distribution assumption.

Example 2: We assume that there are three different portfolios in different line of business. Claims ratio data that are required for construction of curve is available for a known period and assumed to follow Exponential distribution with parameter value 0.3. We first generate claim ratio using random variate generation techniques and claims generated are shown in Figure 3.1a. They are sorted from the smallest to the greatest as shown in graph Figure 3.1b. It is easily noticed that loss ratios create a cumulative distribution and shows the percentage of the loss ratios in the portfolio. It



(a) Exponentially distributed claims ratio



(b) The sorted claims ratio given in part (a)

Figure 3.1: Generated loss ratios and sorted situation.

implies that the probability that the loss ratio is less than or equal to 20% is equivalent to 70% approximately. It means that the policies, which have loss amount less than or equal to 20% of their indemnities, consist of 70% of the portfolio.

If we divide the graph into two parts with a vertical line at any point defined as deductible in Figure 3.2, the part on left of the line represents the loss ratios up to that point. If this area is divided by the area corresponding the entire risk premium, the ratio of deductible level measuring to that point is attained. If this division is applied for every different point, the resulting graph are plotted and shows the exposure curve corresponding to risk portfolio [15].

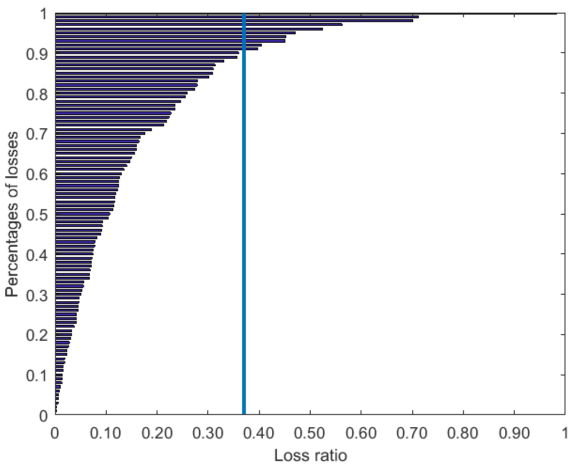
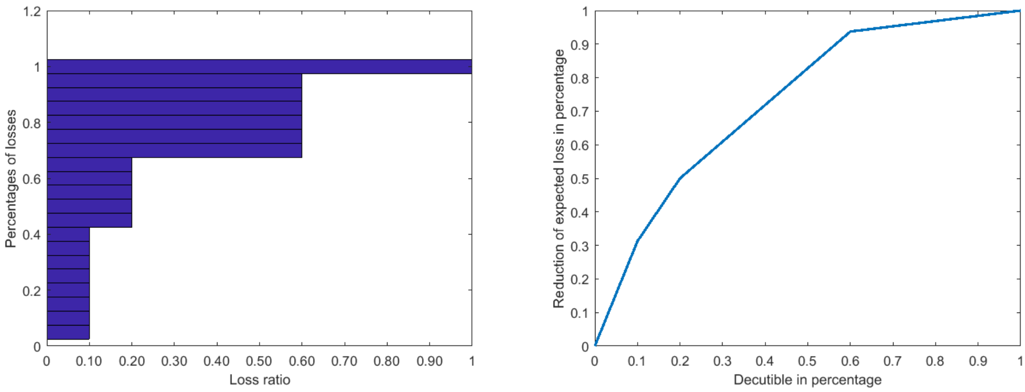


Figure 3.2: Loss ratios and the impact of deductible level.

Loss distribution is the important part to determine exposure curve. Even if loss distribution function does not exist, exposure curve can be found using the loss distribution of portfolio. Some simple examples are given to show how exposure curves occur without loss distribution function.

Example 3: Assume that we have a portfolio A which has total losses about 5% of total portfolio, i.e, 5% of policies have losses amounting indemnity in Figure 3.3a. Assume also that it has loss ratio of 60% which are 30% of portfolio. This means that 30% of policies have losses amounting 60% of their indemnities. The rest of the policies become then as follows: 25% of policies have loss ratios 20% and 40% of policies have loss ratios 10%. Since the portfolio has four different loss levels, the corresponding exposure curve function is a partial function of four components in Figure 3.3b.



(a) Loss distribution

(b) Exposure curve

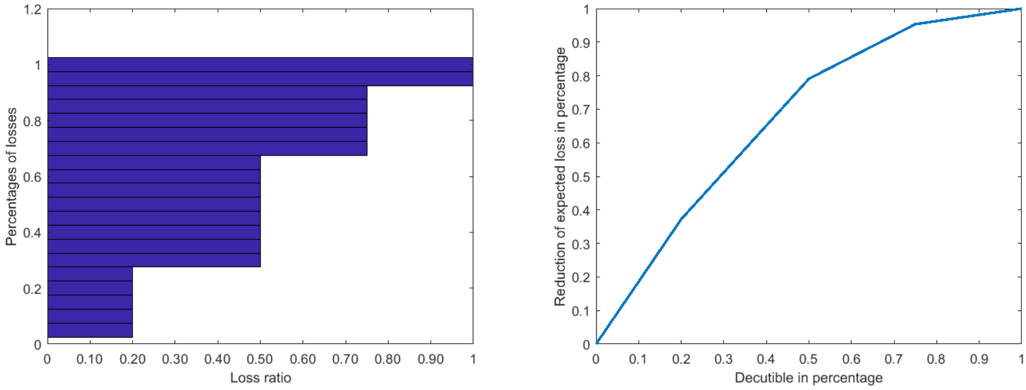
Figure 3.3: Determination of exposure curve for portfolio A.

We obtain the exposure curve for portfolio A, $G_A(d)$, as a partial function. Each component of function is evaluated separately. The expected value of policies equals to 0.32. Deductible level is considered to place in four different intervals since the portfolio has four different loss levels. The area which is on the left of deductible level is divided by its expected value. Finally, the exposure curve is obtained as following:

$$G_A(d) = \frac{1}{0.32} \left\{ \begin{array}{ll} d & d \leq 0.10 \\ 0.60 \cdot d + 0.04 & 0.10 \leq d \leq 0.20 \\ 0.35 \cdot d + 0.05 + 0.04 & 0.20 \leq d \leq 0.60 \\ 0.05 \cdot d + 0.18 + 0.05 + 0.04 & 0.60 \leq d \leq 1 \end{array} \right\} .$$

Example 4: We assume that another portfolio, B, consisting of 10% total losses, losses with ratios 75% of 25% of policies, losses with ratio 50% of 40% of policies and losses with ratio 20% of 25% of policies as shown in Figure 3.4a. Let $G_B(d)$ be the exposure curve function for portfolio B. Following the same steps for $G_A(d)$, we plot the exposure curve for portfolio B as given in Figure 3.4b and Equation (3.1).

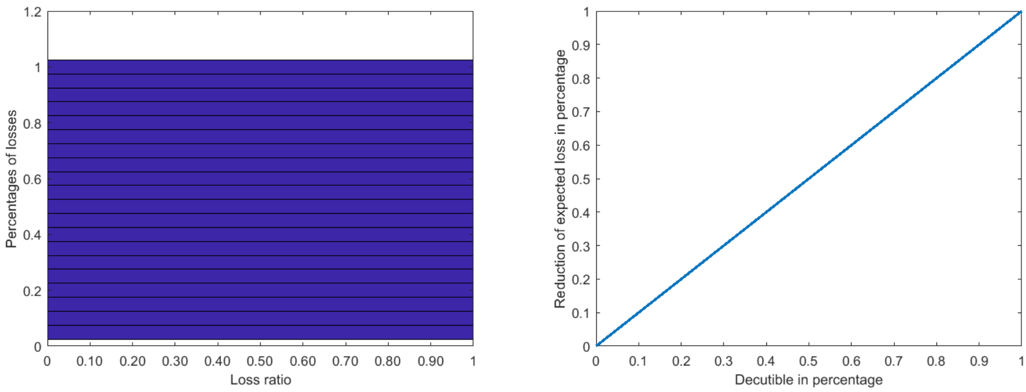
Example 5: In this example we assume portfolio C represents a full damaged portfolio, i.e, all policies have the loss ratio 100% in Figure 3.5a. Therefore, exposure curve function for portfolio C should be linear, i.e. $G_C(d) = d$.



(a) Loss distribution (b) Exposure curve

Figure 3.4: Determination of exposure curve for portfolio B.

$$G_B(d) = \frac{1}{0.5375} \left\{ \begin{array}{ll} d & d \leq 0.20 \\ 0.75 \cdot d + 0.05 & 0.20 \leq d \leq 0.50 \\ 0.35 \cdot d + 0.20 + 0.05 & 0.50 \leq d \leq 0.75 \\ 0.10 \cdot d + 0.1875 + 0.2 + 0.05 & 0.75 \leq d \leq 1 \end{array} \right\}. \quad (3.1)$$



(a) Loss distribution (b) Exposure curve

Figure 3.5: Determination of exposure curve for portfolio C.

We clearly observe that the shapes of the graphs change in every portfolio and all of them except portfolio C are concave with beginning point $(0, 0)$ and end point $(1, 1)$. These graphs imply that exposure curve of portfolios which have high ratio is close to the diagonal. On the other hand, exposure curve is far away from the diagonal when the portfolio has less losses. This concludes that the concavity depends on different type of risk and may also depend on the class of risk, the type of peril and the size of risk.

3.2 The Construction of Exposure Curves

It has been mentioned earlier that on the stage of exposure curves, historical data is also used for exposure rating. There are some exposure curves which are created using both experience and exposure rating in the market. Some of them are used currently but some of them are not preferred due to its timespan to be outdated. Lloyd's [1] curve is one of them and it is not used anymore. Salzmann's [22] curves are made by using homeowners data for 1960 fire losses, thereof the data is also old dated. Reinsurers such as Munich Re and Skandia American Reinsurance Corporation have their curves for mostly corresponding commercial insurance. They do not diversify the insurance by its volume. Ludwig's [18] curves vary by protection and structure class for homeowners and insurance class for personal and commercial and they embrace property insurance with all perils. Insurance Service Office (ISO) uses the current data based on Property Size of Loss Database (PSOLD) and update used data every two years. The curves diversify by volume of insurance, peril, coverage and state. Swiss Re curves developed by Gasser [12] are based on fire statistics of the Swiss Association of Cantonal Fire Institutions. The curves are used generally for fire insurance class in European reinsurance market daily. Bernegger [1] brings out some curves mentioned above with new parametrisation, which is called Maxwell-Boltzmann, Bose-Einstein and Fermi-Dirac (MBBEFD) curves which offers a statistical modeling based on historical information.

It should be noted that exposure curves are partial linear functions, not a curve in the examples given in this chapter. They are given to create basic idea for constructing the exposure curves. If the portfolio has policies with different loss ratios, the lines turn into a curve.

Bernegger [1] suggests a theoretical form for constructing exposure curves.

Definition 3.1. Let X denotes a random variable representing the claim amounts which are normalized by the probable maximum loss, denoted by M. So the loss amount turns into a loss ratio where $0 \leq x \leq 1$. Thus, both the distribution function $F_X(x)$ and probability density function $f_X(x)$ are distributed on the interval [0,1].

Definition 3.2. Let D be the retention level for non-proportional reinsurance. To convert D to a ratio d, D is normalized and it is divided by M.

The mean value of cedant is $E[\min(d,x)]$ and the mean for reinsurer is $E[\min(1,x)] - E[\min(d,x)]$. Therefore, the ratio of $E[\min(d,x)]/E[\min(1,x)]$ derives the exposure curve, denoted by G. G gives the proportion of premium that cedant should have under the retention d [1].

$$G(d) = \frac{E[min(d,x)]}{E[min(1,x)]} = \frac{\int_0^d x f_X(x)dx + \int_d^1 d f_X(x)dx}{\int_0^1 x f_X(x)dx}$$
$$= \frac{\big[-x\big(1-F_X(x)\big)\big]\big|_0^d + \int_0^d \big(1-F_X(x)\big)dx + d\int_d^1 f_X(x)dx}{\int_0^1 x f_X(x)dx}$$

$$= \frac{-d\big(1-F_X(d)\big) + \int_0^d \big(1-F_X(x)\big)dx + d\big(1-F_X(d)\big)}{\int_0^1 x f_X(x)dx}$$
$$= \frac{\int_0^d \big(1-F_X(x)\big)dx}{\int_0^1 \big(1-F_X(x)\big)dx}$$
$$= \frac{\int_0^d \big(1-F_X(x)\big)dx}{E[x]}.$$

Using integration by parts and the properties of probability we prove that $G(d)$ satisfies the requirements of an exposure curve. That is, the first derivative of G with respect to d is $1 - F_X(d) \geq 0$ (by fundamental theorem of calculus), so G is increasing on the interval (0,1). The second derivative of G is $-f_X(d) \leq 0$ and so G is concave on the interval (0,1). It is obvious that $G(0) = 0$ and $G(1) = 1$.

Bernegger[1] uses the Maxwell-Boltzmann, Bose-Einstein and Fermi-Dirac (MBBEFD) distribution to find a new expressions for exposure curves which are still in use currently in reinsurance market. The parameters g and b are the factors that determine the form of the curves. The condition $b \cdot g = 1$ represents the Maxwell-Boltzmann and similarly $b \cdot g > 1$ and $b \cdot g < 1$ represent Bose-Einstein and Fermi-Dirac, respectively. The curve with parameters b and g and variable d is given as:

$$G(d) = \left\{ \begin{array}{ll} d & g = 1 \lor b = 0 \\ \dfrac{ln\big(1+(g-1)d\big)}{ln(g)} & g > 1 \land b = 1 \\ \dfrac{1-b^d}{1-b} & bg = 1 \land g > 1 \\ \dfrac{ln\left(\frac{(g-1)b+(1-gb)b^d}{1-b}\right)}{ln(gb)} & b > 0 \land b \neq 1 \land bg \neq 1 \land g > 1 \end{array} \right\}.$$

Determination of parameters depends on the calculation of the first and the second moment and estimation of some end point probabilities. These curves can approximate to Swiss Re curves at good level. After the calculation of parameters for every Swiss Re curve, it is observed by plotting that the results come up to a smooth continuous curves. These curves are formatted to functions to a single parameter c as follows:

$$b(c) = e^{3.1-0.15(1+c)c},$$
$$g(c) = e^{(0.78+0.12c)c}.$$

MBBEFD curves now depend just the parameter c and when $c = 0$, the curve is a linear function. In other words, the portfolio consists of total losses since $g(0) = 1$. The concavity changes for every value of c. Increasing of c rises the concavity of curve.

It means that the curves which yield a higher value of c have portfolio that consists of less risky, i.e. lower loss ratio, policies. For the values of 1.5, 2, 3 and 4 of c, the curves fit the Swiss Re curves (called Y_1, Y_2, Y_3 and Y_4) very well. The reason why all Swiss Re curves are different from each other depends on several factors. These can be the scope of application, the volume of risk size, the type of total exposure (SI, PML, EML) and the perils covered. When c is equal to 5, the curve coincides with Lloyd's curve which is used for pricing the industrial risk [1].

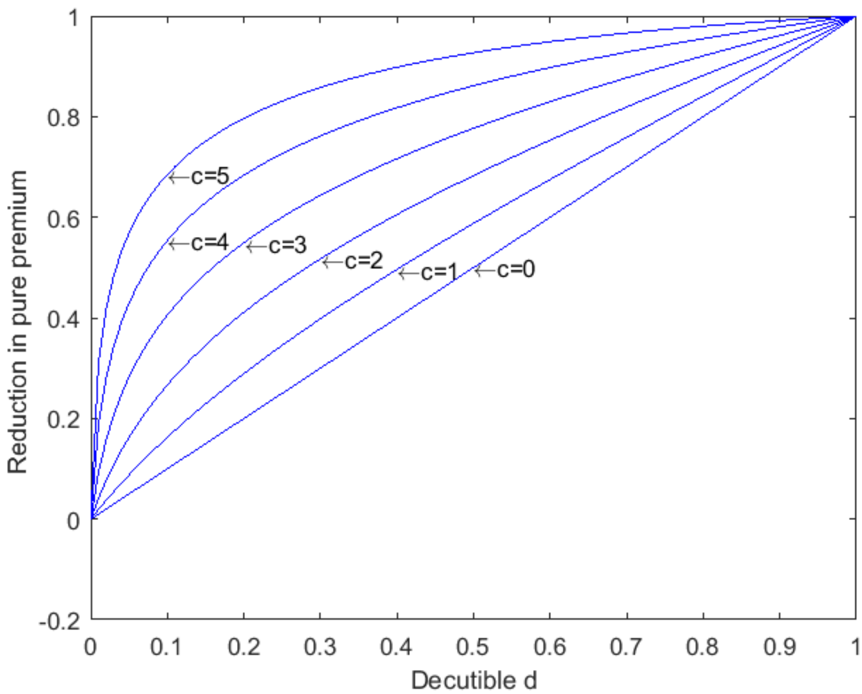


Figure 3.6: Exposure curves based MBBEFD distributions for c = 0,1,2,3,4,5.

3.3 Pricing Using Exposure Curve

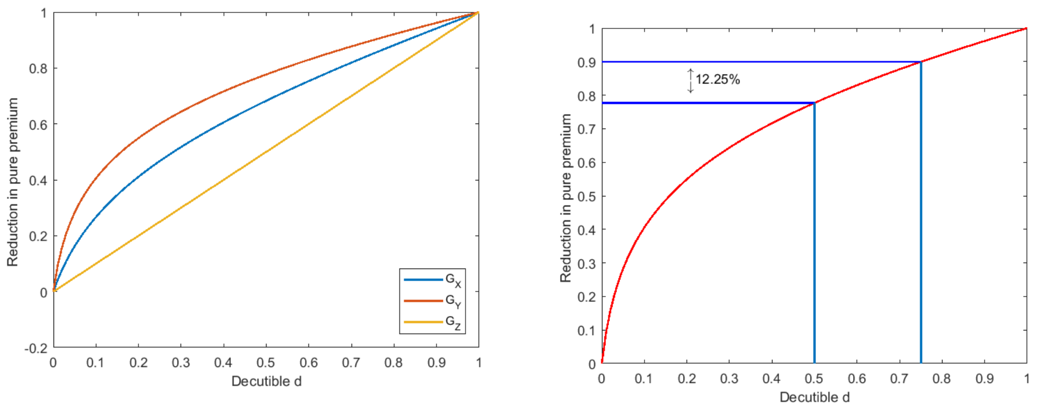
Exposure curves are more preferred tool to price non-proportional reinsurance contracts. The method can be used for both agreements with limit or without limit. It should be kept in mind that sharing the premium is calculated over the pure premium. The other loadings like commissions and expenses should add after sharing the pure premium.

We present some numerical examples using Swiss Re curves. Figure 3.7a implies the curves of three different portfolios under the non-proportional reinsurance agreement without limit.

To present the utilization of Swiss Re exposure curves, an example of three portfolios whose information given in Table 3.1 is introduced. For portfolio X, insurer pays $\%0.4826$ of total premiums to reinsurer for transferring the risk exceeding $\$30000$ (deductible amount of the agreement). On the other side, Figure 3.7b shows the ratio of premium that cedant pays to reinsurer between the deductible and the limit. We assume that there exists a facultative policy with sum insured $\$2000000$ and pure premium $\$50000$. Cedant is provided a reinsurance protection against to this risk by a contract. According to this contract, the risk is transferred when it exceeds $\$1000000$ and up to $\$1500000$ ($\$1000000$ XS $\$500000$). In terms of ratio, the protection is from $\%50$ to $\%75$ and insurer pays $G(\%75) - G(\%50) = \%12.25$ of pure premium, which corresponds to the amount $\$6125$.

Table 3.1: Required entities for portfolios.

Portfolio	D	SI	Total Prem.	d	G(d)	Ins. Prem.	Re. Prem.
Portf. X	\$30K	\$100K	\$10K	0.30	0.5174	\$5174	\$4826
Portf. Y	\$750K	\$1000K	\$30K	0.75	0.8994	\$26981	\$3079
Portf. Z	\$1300K	\$5000K	\$150K	0.26	0.26	\$39K	\$111K



(a) Swiss Re curves of three portfolios without limits.

(b) Swiss RE curve of policy with limit.

Figure 3.7: Swiss Re curves for reinsurance pricing.

CHAPTER 4

BIVARIATE EXPOSURE CURVES UNDER DEPENDENT RISKS

There are different exposure curves that change under the effect of the several factors like type or risk or size of risk in the reinsurance market. Therefore, we expect that to depend two risks under the same policy might rise a different exposure curve with recpect to their individual curves.

Under the assumption of collective risk model, let X_1 and X_2 denote the claim process of two separate risks. Assume that the loss distribution functions of X_1 and X_2 are given as $F_{X_1}(x_1)$ and $F_{X_2}(x_2)$ with probability density functions $f_{X_1}(x_1)$ and $f_{X_2}(x_2)$, respectively.

Assume also that the risks are dependent under the same total exposure as chosen MPL, denoted by B, by collective risk model S such that:

$$S = \sum_{n=1}^{N} X_{1n} + X_{2n},$$

where the random variable N is the number of incurred risks.

Proposition 4.1. $S_I = \sum\limits_{n=1}^{N} \min\big(D, (X_{1n}+X_{2n})\big)$ *is the insurer's expected value under the reinsurance agreement with deductible amount* D*. Therefore, the exposure curve of these two dependent risks is:*

$$G(D) = \frac{\iint\limits_{x_1+x_2<D} (x_1+x_2) f_{X_1,X_2}(x_1,x_2)dx_1dx_2 + \iint\limits_{D<x_1+x_2<B} D \cdot f_{X_1,X_2}(x_1,x_2)dx_1dx_2}{\iint\limits_{x_1+x_2<B} (x_1+x_2) f_{X_1,X_2}(x_1,x_2)dx_1dx_2}.$$

Proof.

$$
\begin{aligned}
G(D) = \frac{E[S_I]}{E[S]} = &\quad \frac{E[N] \cdot E[min(D, X_1 + X_2)]}{E[N] \cdot E[X_1 + X_2]} \\
= &\quad \frac{E[min(D, X_1 + X_2)]}{E[X_1 + X_2]} \\
= &\quad \frac{\iint\limits_{x_1+x_2<D} (x_1 + x_2) f_{X_1,X_2}(x_1, x_2) dx_1 dx_2}{\iint\limits_{x_1+x_2<B} (x_1 + x_2) f_{X_1,X_2}(x_1, x_2) dx_1 dx_2} \\
&\quad + \frac{\iint\limits_{D<x_1+x_2<B} D \cdot f_{X_1,X_2}(x_1, x_2) dx_1 dx_2}{\iint\limits_{x_1+x_2<B} (x_1 + x_2) f_{X_1,X_2}(x_1, x_2) dx_1 dx_2}.
\end{aligned}
$$

□

We need to derive joint probability function $f_{X_1,X_2}(x_1, x_2)$ to calculate the curve above. In order to determine the joint probability, we choose Bivariate Pareto distribution as suggested in literature [8]. We examine the intermediary steps to derive the joint cumulative function.

4.1 Bivariate Pareto Distribution

Let X be a random variable which has Exponential distribution $Exp(\theta)$ with the distribution parameter $\Theta = \theta$ where Θ is also a random variable having $\Theta \sim Gamma(\alpha, \lambda)$.

$$
\begin{aligned}
f_{X|\Theta}(x|\theta) &= \theta e^{-\theta x} && \text{where} \quad x > 0, \theta > 0, \\
f_{\Theta}(\theta) &= \frac{\lambda^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\lambda\theta} && \text{where} \quad \alpha > 0, \lambda > 0.
\end{aligned}
$$

Posterior distribution of $X|\Theta$ is known to have Pareto distributions $X \sim Par(\alpha, \lambda)$ as shown in Equation (4.1).

$$
\begin{aligned}
f_X(x|\alpha, \lambda) &= \int_0^\infty f_{X|\Theta}(x|\theta) f_\Theta(\theta) d\theta \\
&= \int_0^\infty \theta e^{-\theta x} \frac{\lambda^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\lambda\theta} d\theta \\
&= \int_0^\infty \frac{\lambda^\alpha}{\Gamma(\alpha)} \theta^\alpha e^{-\theta(x+\lambda)} d\theta \\
&= \frac{\lambda^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha+1)}{(x+\lambda)^{\alpha+1}} \int_0^\infty \frac{(x+\lambda)^{\alpha+1}}{\Gamma(\alpha+1)} \theta^{(\alpha+1)-1} e^{-\theta(x+\lambda)} d\theta \\
&= \alpha\lambda^\alpha \big(x + \lambda\big)^{-\alpha-1} \qquad\qquad x > 0, \alpha > 0, \lambda > 0,
\end{aligned}
\tag{4.1}
$$

since $\int_0^\infty \frac{(x+\lambda)^{\alpha+1}}{\Gamma(\alpha+1)}\theta^{(\alpha+1)-1}e^{-\theta(x+\lambda)}d\theta = 1.$

The cumulative distribution function of Pareto distribution with parameters α and λ is attained as follows:

$$\begin{aligned}
F_X(x|\alpha,\lambda) &= \int_0^x f_Z(z|\alpha,\lambda)dz \\
&= \int_0^x \alpha\lambda^\alpha(z+\lambda)^{-\alpha-1}dz \\
&= \alpha\lambda^\alpha\Big(\frac{(z+\lambda)^{-\alpha}}{-\alpha}\Big)\Big|_0^x \\
&= -\lambda^\alpha\big[(x+\lambda)^{-\alpha} - \lambda^{-\alpha}\big] \\
&= 1 - \Big(\frac{x+\lambda}{\lambda}\Big)^{-\alpha} \\
&= 1 - \Big(1+\frac{x}{\lambda}\Big)^{-\alpha} \qquad x>0, \alpha>0, \lambda>0.
\end{aligned}$$

Definition 4.1. Suppose that two random variables $X_1 \sim Exp(\theta)$ and $X_2 \sim Exp(\theta)$ share the same random affect $\Theta = \theta$. Therefore, they transform to dependent random variables and their probability functions are given [8]:

$$\begin{aligned}
f_{X_1|\Theta}(x_1|\theta) &= \theta e^{-\theta x_1}, \\
f_{X_2|\Theta}(x_2|\theta) &= \theta e^{-\theta x_2},
\end{aligned}$$

$$\begin{aligned}
f_{X_1}(x_1|\alpha,\lambda) &= \alpha\lambda^\alpha(x_1+\lambda)^{-\alpha-1}, \\
f_{X_2}(x_2|\alpha,\lambda) &= \alpha\lambda^\alpha(x_2+\lambda)^{-\alpha-1},
\end{aligned}$$

$$\begin{aligned}
F_{X_1}(x_1|\alpha,\lambda) &= 1 - \Big(1+\frac{x_1}{\lambda}\Big)^{-\alpha}, \\
F_{X_2}(x_2|\alpha,\lambda) &= 1 - \Big(1+\frac{x_2}{\lambda}\Big)^{-\alpha}.
\end{aligned}$$

The joint cumulative distribution function is then [8]:

$$\begin{aligned}
F_{X_1,X_2}(x_1,x_2|\alpha,\lambda) = 1 - \overline{F}_{x_1}(x_1|\alpha,\lambda) - \overline{F}_{x_2}(x_2|\alpha,\lambda) \\
+ \int_0^\infty e^{-\theta(x_1+x_2)}\Big(\frac{\lambda^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\lambda\theta}\Big)d\theta,
\end{aligned}$$

where $\overline{F}_{x_1}(x_1|\alpha,\lambda)$ is a survival function such that

$$\overline{F}_{x_1}(x_1|\alpha,\lambda) = 1 - F_{x_1}(x_1|\alpha,\lambda).$$

$$
\begin{aligned}
F_{X_1,X_2}(x_1,x_2|\alpha,\lambda) &= 1-\overline{F}_{x_1}(x_1|\alpha,\lambda)-\overline{F}_{x_2}(x_2|\alpha,\lambda) \\
&\quad + \int_0^\infty e^{-\theta(x_1+x_2)}\Big(\frac{\lambda^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\lambda\theta}\Big)d\theta \\
&= 1-\Big(1+\frac{x_1}{\lambda}\Big)^{-\alpha}-\Big(1+\frac{x_2}{\lambda}\Big)^{-\alpha} \\
&\quad + \int_0^\infty e^{-\theta(x_1+x_2)}\Big(\frac{\lambda^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\lambda\theta}\Big)d\theta \\
&= 1-\Big(1+\frac{x_1}{\lambda}\Big)^{-\alpha}-\Big(1+\frac{x_2}{\lambda}\Big)^{-\alpha} \\
&\quad + \lambda^\alpha\int_0^\infty \frac{1}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\theta(x_1+x_2+\lambda)}d\theta \\
&= 1-\Big(1+\frac{x_1}{\lambda}\Big)^{-\alpha}-\Big(1+\frac{x_2}{\lambda}\Big)^{-\alpha} \\
&\quad + \frac{\lambda^\alpha}{(x_1+x_2+\lambda)^\alpha}\int_0^\infty \frac{(x_1+x_2+\lambda)^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\theta(x_1+x_2+\lambda)}d\theta \\
&= 1-\Big(1+\frac{x_1}{\lambda}\Big)^{-\alpha}-\Big(1+\frac{x_2}{\lambda}\Big)^{-\alpha}+\frac{\lambda^\alpha}{(x_1+x_2+\lambda)^\alpha} \\
&= 1-\Big(1+\frac{x_1}{\lambda}\Big)^{-\alpha}-\Big(1+\frac{x_2}{\lambda}\Big)^{-\alpha}+\Big(1+\frac{x_1+x_2}{\lambda}\Big)^{-\alpha}.
\end{aligned}
\tag{4.2}
$$

Since the inner part of integral is the probability density function of θ which is distributed Gamma $\theta \sim Gam(\alpha,\lambda)$, the term $\int_0^\infty \frac{(x_1+x_2+\lambda)^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\theta(x_1+x_2+\lambda)}d\theta$ equals to 1. We obtain the given Equation (4.2) in the form of survival and density functions for simplicity. Hence, the joint cumulative distribution is found to be

$$
\begin{aligned}
F_{X_1,X_2}(x_1,x_2|\alpha,\lambda) &= 1-\overline{F}_{x_1}(x_1|\alpha,\lambda)-\overline{F}_{x_2}(x_2|\alpha,\lambda) \\
&\quad + \int_0^\infty e^{-\theta(x_1+x_2)}\Big(\frac{\lambda^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\lambda\theta}\Big)d\theta \\
&= 1-\overline{F}_{x_1}(x_1|\alpha,\lambda)-\overline{F}_{x_2}(x_2|\alpha,\lambda) \\
&\quad + \int_0^\infty e^{-\theta x_1}e^{-\theta x_2}\Big(\frac{\lambda^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\lambda\theta}\Big)d\theta \\
&= 1-\overline{F}_{x_1}(x_1|\alpha,\lambda)-\overline{F}_{x_2}(x_2|\alpha,\lambda) \\
&\quad + \int_0^\infty \frac{f_{X_1}(x_1|\theta)}{\theta}\frac{f_{X_2}(x_2|\theta)}{\theta}f_\Theta(\theta)d\theta.
\end{aligned}
\tag{4.3}
$$

4.2 Distributions of Dependent Risks

Definition 4.2. Let the risk X has a mixture distribution $F_X(x)$ combining of two distributions $H_X(x)$ and $G_X(x)$ such that

$$F_X(x) = p\cdot H_X(x) + (1-p)\cdot G_{X|\Theta}(x|\theta),$$

where

$$H_X(x) = \left\{ \begin{array}{ll} 0 & X<0 \\ 1 & X \geq 0 \end{array} \right\},$$

$$G_{X|\Theta}(x|\theta) = \left\{ \begin{array}{ll} 0 & X<0 \\ 1 - exp(-\theta x) & X \geq 0 \end{array} \right\},$$

and $G_{X|\Theta}(x|\theta)$ is loss distribution having $Exp(\Theta)$, $p \in$ [0,1].

We obtain the cumulative and density distribution functions of X, using Definition 4.2 as

$$F_{X|\Theta}(x|\theta) = \left\{ \begin{array}{ll} 0 & X<0 \\ 1 - (1-p)e^{-\theta x} & X \geq 0 \end{array} \right\},$$

$$f_{X|\Theta}(x|\theta) = \left\{ \begin{array}{ll} p & X = 0 \\ (1-p)\theta e^{-\theta x} & X > 0 \end{array} \right\},$$

respectively. In more detail, we define F as a mixture distribution to satisfy the condition that the probability of the loss being 0 to be $F_X(0) = p$. If the loss is greater than 0, it is distributed as $Exp(\Theta)$ with $\Theta \sim Gamma(\alpha, \lambda)$. Then, $G_{X|\Theta}(x|\theta)$ has a new distribution with new parameters α and λ, that is Pareto distribution.

$$G_X(x|\alpha, \lambda) = \left\{ \begin{array}{ll} 0 & X<0 \\ 1 - \Big(1 + \dfrac{x}{\lambda}\Big)^{-\alpha} & X \geq 0 \end{array} \right\}.$$

Based on this set up, F becomes mixture Pareto distributions with the probability $F_X(0|\alpha, \lambda) = p$. The function F is obtained:

$$\begin{aligned} F_X(x|\alpha, \lambda) &= p \cdot H_X(x) + (1-p) \cdot G_X(x|\alpha, \lambda) \\ &= p + (1-p)\Big(1 - (1 + \frac{x}{\lambda})^{-\alpha}\Big) \\ &= 1 - (1-p)\big(1 + \frac{x}{\lambda}\big)^{-\alpha} \qquad\qquad x \geq 0. \end{aligned} \tag{4.4}$$

Definition 4.3. Let two risks X_1 and X_2 have mixture Pareto distributions with the probabilities $F_{x_1}(0|\alpha, \lambda) = p_1$ and $F_{X_2}(0|\alpha, \lambda) = p_2$, respectively.

$$F_{X_1}(x_1|\alpha, \lambda) = \left\{ \begin{array}{ll} 0 & x_1<0 \\ 1 - (1-p_1)\Big(1 + \dfrac{x_1}{\lambda}\Big)^{-\alpha} & x_1 \geq 0 \end{array} \right\},$$

$$F_{X_2}(x_2|\alpha, \lambda) = \left\{ \begin{array}{ll} 0 & x_2<0 \\ 1 - (1-p_2)\Big(1 + \dfrac{x_2}{\lambda}\Big)^{-\alpha} & x_2 \geq 0 \end{array} \right\},$$

with probability density distributions:

$$\begin{aligned} f_{X_1}(x_1|\alpha, \lambda) &= \left\{ \begin{array}{ll} p_1 & x_1 = 0 \\ (1-p_1)\dfrac{\alpha}{\lambda}\Big(1 + \dfrac{x_1}{\lambda}\Big)^{-\alpha-1} & x_1 > 0 \end{array} \right\}, \\ f_{X_2}(x_2|\alpha, \lambda) &= \left\{ \begin{array}{ll} p_2 & x_2 = 0 \\ (1-p_2)\dfrac{\alpha}{\lambda}\Big(1 + \dfrac{x_2}{\lambda}\Big)^{-\alpha-1} & x_2 > 0 \end{array} \right\}. \end{aligned} \tag{4.5}$$

It makes sense to define the functions of cumulative distribution and probability density as partial functions. For this reason, we define joint distribution such that probability of the losses are equal to 0 is more likely event. Furthermore, this event's probability is pretty high actually for some lines of business in insurance portfolios.

When X_1 and X_2 are influenced by the same random effect, they may become dependent to each other. The joint probability function of X_1 and X_2 is derived from their bivariate distribution by substituting the terms into Equation (4.3).

$$\begin{aligned}
F_{X_1,X_2}(x_1,x_2) &= 1-\overline{F}_{X_1}(x_1|\alpha,\lambda)-\overline{F}_{X_2}(x_2|\alpha,\lambda)\\
&\quad+\int_0^\infty \frac{f_{X_1}(x_1|\theta)}{\theta}\frac{f_{X_2}(x_2|\theta)}{\theta}f_\Theta(\theta)d\theta\\
&=1-(1-p_1)\Big(1+\frac{x_1}{\lambda}\Big)^{-\alpha}-(1-p_2)\Big(1+\frac{x_2}{\lambda}\Big)^{-\alpha}\\
&\quad+\int_0^\infty (1-p_1)e^{-\theta x_1}(1-p_2)e^{-\theta x_2}\frac{\lambda^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\lambda\theta}d\theta\\
&=1-(1-p_1)\Big(1+\frac{x_1}{\lambda}\Big)^{-\alpha}-(1-p_2)\Big(1+\frac{x_2}{\lambda}\Big)^{-\alpha}\\
&\quad+(1-p_1)(1-p_2)\int_0^\infty e^{-\theta(x_1+x_2)}\frac{\lambda^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\lambda\theta}d\theta\\
&=1-(1-p_1)\Big(1+\frac{x_1}{\lambda}\Big)^{-\alpha}-(1-p_2)\Big(1+\frac{x_2}{\lambda}\Big)^{-\alpha}\\
&\quad+(1-p_1)(1-p_2)\Big(1+\frac{x_1+x_2}{\lambda}\Big)^{-\alpha}.
\end{aligned}$$

Definition 4.4. Let $F_{X_1,X_2}(x_1,x_2)$ denote the joint cumulative distribution function of variables X_1 and X_2 defined in Definition 4.3 with parameters λ and α whose form is given as:

$$F_{X_1,X_2}(x_1,x_2)=\left\{\begin{array}{ll}
0 & x_1<0, x_2<0\\
1-(1-p_1)\Big(1+\dfrac{x_1}{\lambda}\Big)^{-\alpha} & \\
-(1-p_2)\Big(1+\dfrac{x_2}{\lambda}\Big)^{-\alpha} & \\
+(1-p_1)(1-p_2)\Big(1+\dfrac{x_1+x_2}{\lambda}\Big)^{-\alpha} & 0\le x_1, 0\le x_2
\end{array}\right\}.$$

It is shown that $F_{X_1,X_2}(x_1,x_2)$ satisfies the requirements of joint cumulative distribution.

Theorem 4.2. *If $F_{X_1,X_2}(x_1,x_2)$ is a joint cumulative distribution function function, then* $\lim_{x_1,x_2\to\infty}F_{X_1,X_2}(x_1,x_2)=1$.

Proof.

$$\lim_{x_1,x_2\to\infty} F_{X_1,X_2}(x_1,x_2) = \lim_{x_1,x_2\to\infty}\Bigg(1-(1-p_1)\Big(1+\frac{x_1}{\lambda}\Big)^{-\alpha}-(1-p_2)\Big(1+\frac{x_2}{\lambda}\Big)^{-\alpha}$$
$$+(1-p_1)(1-p_2)\Big(1+\frac{x_1+x_2}{\lambda}\Big)^{-\alpha}\Bigg)$$
$$=1-(1-p_1)\cdot 0-(1-p_2)\cdot 0+(1-p_1)(1-p_2)\cdot 0$$
$$=1.$$

□

Theorem 4.3. *If $F_{X_1,X_2}(x_1,x_2)$ is a joint cumulative distribution function, then* $\lim_{x_1,x_2\to-\infty} F_{X_1,X_2}(x_1,x_2)=0$.

Proof. It is proved by Definition 4.4. □

Theorem 4.4. *If $F_{X_1,X_2}(x_1,x_2)$ is a joint cumulative distribution function, then* $\lim_{x_1\to\infty} F_{X_1,X_2}(x_1,x_2)=F_{X_2}(x_2)$ *and* $\lim_{x_2\to\infty} F_{X_1,X_2}(x_1,x_2)=F_{X_1}(x_1)$.

Proof. We give the proof only for the one variable; the other is the same.

$$\lim_{x_1\to\infty} F_{X_1,X_2}(x_1,x_2) = \lim_{x_1\to\infty}\Bigg(1-(1-p_1)\Big(1+\frac{x_1}{\lambda}\Big)^{-\alpha}-(1-p_2)\Big(1+\frac{x_2}{\lambda}\Big)^{-\alpha}$$
$$+(1-p_1)(1-p_2)\Big(1+\frac{x_1+x_2}{\lambda}\Big)^{-\alpha}\Bigg)$$
$$=1-(1-p_1)\cdot 0-(1-p_2)\Big(1+\frac{x_2}{\lambda}\Big)^{-\alpha}+(1-p_1)(1-p_2)\cdot 0$$
$$=1-(1-p_2)\Big(1+\frac{x_2}{\lambda}\Big)^{-\alpha}$$
$$=F_{X_2}(x_2).$$

□

Theorem 4.5. *If $F_{X_1,X_2}(x_1,x_2)$ is a joint cumulative distribution function, then $F_{X_1,X_2}(a_1,a_2)\le F_{X_1,X_2}(b_1,b_2)$ for $a_1\le b_1$ and $a_2\le b_2$, i.e, it is non-decreasing.*

Proof. Take derivative of F with respect to x_1.

$$\frac{\partial}{\partial x_1}F_{X_1,X_2}(x_1,x_2)=(1-p_1)\frac{\alpha}{\lambda}\Big(1+\frac{x_1}{\lambda}\Big)^{-\alpha-1}-(1-p_1)(1-p_2)\frac{\alpha}{\lambda}\Big(1+\frac{x_1+x_2}{\lambda}\Big)^{-\alpha-1}.$$

Consider that the other variable is fixed.

$$\begin{aligned} x_1 \le & x_1 + x_2 \\ \Big(1 + \frac{x_1}{\lambda}\Big)^{\alpha+1} \le & \Big(1 + \frac{x_1 + x_2}{\lambda}\Big)^{\alpha+1} \\ \Big(1 + \frac{x_1}{\lambda}\Big)^{-\alpha-1} \ge & \Big(1 + \frac{x_1 + x_2}{\lambda}\Big)^{-\alpha-1} \\ (1 - p_1)\Big(1 + \frac{x_1}{\lambda}\Big)^{-\alpha-1} \ge & (1 - p_1)\Big(1 + \frac{x_1 + x_2}{\lambda}\Big)^{-\alpha-1}. \end{aligned}$$

Since p_2 is probability and $1 - p_2 \le 1$,

$$(1 - p_1)\Big(1 + \frac{x_1 + x_2}{\lambda}\Big)^{-\alpha-1} \ge (1 - p_1)(1 - p_2)\Big(1 + \frac{x_1 + x_2}{\lambda}\Big)^{-\alpha-1}.$$

Therefore,

$$\begin{aligned} (1 - p_1)\Big(1 + \frac{x_1}{\lambda}\Big)^{-\alpha-1} \ge & (1 - p_1)(1 - p_2)\Big(1 + \frac{x_1 + x_2}{\lambda}\Big)^{-\alpha-1} \\ (1 - p_1)\frac{\alpha}{\lambda}\Big(1 + \frac{x_1}{\lambda}\Big)^{-\alpha-1} \ge & (1 - p_1)(1 - p_2)\frac{\alpha}{\lambda}\Big(1 + \frac{x_1 + x_2}{\lambda}\Big)^{-\alpha-1} \\ \Longrightarrow (1 - p_1)\frac{\alpha}{\lambda}\Big(1 + \frac{x_1}{\lambda}\Big)^{-\alpha-1} & - (1 - p_1)(1 - p_2)\frac{\alpha}{\lambda}\Big(1 + \frac{x_1 + x_2}{\lambda}\Big)^{-\alpha-1} \ge 0. \end{aligned}$$

If we fix x_1 and take partial derivative with respect to x_2, we attain the same result for x_2. Since $\frac{\partial}{\partial x_1} F_{X_1,X_2}(x_1, x_2) \ge 0$ and $\frac{\partial}{\partial x_2} F_{X_1,X_2}(x_1, x_2) \ge 0$, $F_{X_1,X_2}(x_1, x_2)$ is non-decreasing. $\square$

Definition 4.5. The joint probability density function $f_{X_1,X_2}(x_1, x_2)$ has to be considered also at the points where $x_1 = x_2 = 0$,

$$f_{X_1,X_2}(x_1, x_2) = \left\{ \begin{array}{ll} F_{X_1,X_2}(0, 0) & x_1 = 0, x_2 = 0 \\ \frac{\partial}{\partial x_1} F_{X_1,X_2}(x_1, 0) & 0 < x_1, x_2 = 0 \\ \frac{\partial}{\partial x_2} F_{X_1,X_2}(0, x_2) & x_1 = 0, 0 < x_2 \\ \frac{\partial^2}{\partial x_2 \partial x_1} F_{X_1,X_2}(x_1, x_2) & 0 < x_1, 0 < x_2 \end{array} \right\}.$$

$$f_{X_1,X_2}(x_1,x_2)=\left\{\begin{array}{ll} p_1p_2 & x_1=0, x_2=0 \\ (1-p_1)p_2\dfrac{\alpha}{\lambda}\Big(1+\dfrac{x_1}{\lambda}\Big)^{-\alpha-1} & 0<x_1, x_2=0 \\ (1-p_2)p_1\dfrac{\alpha}{\lambda}\Big(1+\dfrac{x_2}{\lambda}\Big)^{-\alpha-1} & x_1=0, 0<x_2 \\ (1-p_1)(1-p_2)\dfrac{\alpha(\alpha+1)}{\lambda^2} & \\ \qquad \cdot\Big(1+\dfrac{x_1+x_2}{\lambda}\Big)^{-\alpha-2} & 0<x_1, 0<x_2 \end{array}\right\}.$$

$f_{X_1,X_2}(x_1,x_2)$ satisfies the characteristics of joint probability density distribution.

Theorem 4.6. *If $f_{X_1,X_2}(x_1,x_2)$ is a joint probability density distribution function, then $\int\limits_0^\infty\int\limits_0^\infty f_{X_1,X_2}(x_1,x_2)dx_1dx_2=1$ for $x_1>0$ and $x_2>0$.*

Proof.

$$\begin{aligned}
\int\limits_0^\infty\int\limits_0^\infty f_{X_1,X_2}(x_1,x_2)dx_1dx_2 &= p_1p_2+\int\limits_0^\infty(1-p_2)p_1\frac{\alpha}{\lambda}\Big(1+\frac{x_2}{\lambda}\Big)^{-\alpha-1}dx_2 \\
&\quad+\int\limits_0^\infty(1-p_1)p_2\frac{\alpha}{\lambda}\Big(1+\frac{x_1}{\lambda}\Big)^{-\alpha-1}dx_1 \\
&\quad+\int\limits_0^\infty\int\limits_0^\infty(1-p_1)(1-p_2)\frac{\alpha(\alpha+1)}{\lambda^2}\Big(1+\frac{x_1+x_2}{\lambda}\Big)^{-\alpha-2}dx_1dx_2 \\
&= p_1p_2+(1-p_2)p_1\frac{\lambda}{-\alpha}\frac{\alpha}{\lambda}\left[\Big(1+\frac{x_2}{\lambda}\Big)^{-\alpha}\Big|_0^\infty\right] \\
&\quad+(1-p_1)p_2\frac{\lambda}{-\alpha}\frac{\alpha}{\lambda}\left[\Big(1+\frac{x_1}{\lambda}\Big)^{-\alpha}\Big|_0^\infty\right] \\
&\quad+(1-p_1)(1-p_2)\frac{\alpha(\alpha+1)}{\lambda^2}\frac{\lambda}{-\alpha-1}\int\limits_0^\infty\left[\Big(1+\frac{x_1+x_2}{\lambda}\Big)^{-\alpha-1}\Big|_0^\infty\right]dx_2 \\
&= p_1p_2+(1-p_2)p_1(-1)(0-1)+(1-p_1)p_2(-1)(0-1) \\
&\quad+(1-p_1)(1-p_2)\frac{-\alpha}{\lambda}\int\limits_0^\infty\left[0-\Big(1+\frac{x_2}{\lambda}\Big)^{-\alpha-1}\right]dx_2 \\
&= p_1p_2+p_1-p_1p_2+p_2-p_1p_2+(1-p_1)(1-p_2)\frac{\alpha}{\lambda}\left[\frac{\lambda}{-\alpha}\Big(1+\frac{x_2}{\lambda}\Big)^{-\alpha}\Big|_0^\infty\right]
\end{aligned}$$

$$
\begin{aligned}
&= p_1 + p_2 - p_1p_2 + (1-p_1)(1-p_2)(-1)(0-1) \\
&= p_1 + p_2 - p_1p_2 + 1 - p_1 - p_2 + p_1p_2 \\
&= 1.
\end{aligned}
$$

□

For mathematical practicality and simplicity in double integration, we employ change of variables to X_1 and X_2.

Definition 4.6. Let μ and η be the functions of U and Y such that

$$
\begin{aligned}
X_1 &= \mu(U, Y) = U, \\
X_2 &= \eta(U, Y) = Y - U.
\end{aligned}
$$

We choose new variables as above since the term $X_1 + X_2$ will turn to Y and this provides to evaluate Proposition in 4.1 using only one integral. In addition, the Jacobian, $J_{U,Y}$, of the transformation $X_1 = \mu(U, Y)$ and $X_2 = \eta(U, Y)$

$$
J_{U,Y} = \frac{\partial(x_1, x_2)}{\partial(u, y)} = \begin{vmatrix} \dfrac{\partial\mu(u,y)}{\partial u} & \dfrac{\partial\mu(u,y)}{\partial y} \\[2ex] \dfrac{\partial\eta(u,y)}{\partial u} & \dfrac{\partial\eta(u,y)}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1.
$$

The integral part of Proposition 4.1 turns into a form as in the right hand side of Equation (4.6).

$$
\begin{aligned}
\iint\limits_{R} (x_1 + x_2) \cdot f_{X_1,X_2}(x_1, x_2) \cdot dx_1 dx_2 &= \iint\limits_{S} y \cdot f_{X_1,X_2}\big(\mu(u,y), \eta(u,y)\big) \cdot J_{U,Y} \cdot dudy \\
&= \iint\limits_{S} y \cdot f_{U,Y}(u, y) \cdot dudy \\
&= \int\limits_{T} y \cdot f_Y(y) \cdot dy.
\end{aligned}
\tag{4.6}
$$

4.3 Joint and Marginal Distributions

We obtain joint probability density function of U and Y with Jacobian transform as follows:

$$
f_{U,Y}(u, y) = \left\{ \begin{array}{ll} f_{X_1,X_2}\big(0, 0\big) \cdot J_{U,Y} & u = 0, y = 0 \\[2ex] f_{X_1,X_2}\big(u, 0\big) \cdot J_{U,Y} & 0 < u, y = u \\[2ex] f_{X_1,X_2}\big(0, y\big) \cdot J_{U,Y} & u = 0, 0 < y \\[2ex] f_{X_1,X_2}\big(u, y - u\big) \cdot J_{U,Y} & 0 < u, 0 < y \end{array} \right\}.
$$

The joint probability distribution function based on the parameters α and λ and variables U and Y is given in Equation (4.7) which also satisfies the rule that the sum of the probabilities equals to 1.

$$f_{U,Y}(u,y) = \left\{ \begin{array}{ll} p_1 p_2 & u=0, y=0 \\ (1-p_1)p_2 \frac{\alpha}{\lambda}\Big(1+\frac{u}{\lambda}\Big)^{-\alpha-1} & 0<u, y=u \\ (1-p_2)p_1 \frac{\alpha}{\lambda}\Big(1+\frac{y}{\lambda}\Big)^{-\alpha-1} & u=0, 0<y \\ (1-p_1)(1-p_2)\frac{\alpha(\alpha+1)}{\lambda^2}\Big(1+\frac{y}{\lambda}\Big)^{-\alpha-2} & 0<u, 0<y \end{array} \right\}. \quad (4.7)$$

Theorem 4.7. *If $f_{U,Y}(u,y)$ is joint probability density function, then* $\int\limits_0^\infty \int\limits_0^\infty f_{U,Y}(u,y)dudy = 1$ *in the defined interval for u and y.*

Proof.

$$\begin{aligned}
\int\limits_0^\infty \int\limits_0^\infty f_{U,Y}(u,y)dudy &= p_1p_2 + \int\limits_0^\infty (1-p_2)p_1\frac{\alpha}{\lambda}\Big(1+\frac{y}{\lambda}\Big)^{-\alpha-1}dy \\
&\quad + \int\limits_0^\infty (1-p_1)p_2\frac{\alpha}{\lambda}\Big(1+\frac{u}{\lambda}\Big)^{-\alpha-1}du \\
&\quad + \int\limits_0^\infty \int\limits_u^\infty (1-p_1)(1-p_2)\frac{\alpha(\alpha+1)}{\lambda^2}\Big(1+\frac{y}{\lambda}\Big)^{-\alpha-2}dydu \\
&= p_1p_2 + (1-p_2)p_1\frac{\lambda}{-\alpha}\frac{\alpha}{\lambda}\left[\Big(1+\frac{y}{\lambda}\Big)^{-\alpha}\Big|_0^\infty\right] \\
&\quad + (1-p_1)p_2\frac{\lambda}{-\alpha}\frac{\alpha}{\lambda}\left[\Big(1+\frac{u}{\lambda}\Big)^{-\alpha}\Big|_0^\infty\right] \\
&\quad + (1-p_1)(1-p_2)\frac{\alpha(\alpha+1)}{\lambda^2}\frac{\lambda}{-\alpha-1}\int\limits_0^\infty \left[\Big(1+\frac{y}{\lambda}\Big)^{-\alpha-1}\Big|_u^\infty\right]du \\
&= p_1p_2 + (1-p_2)p_1(-1)(0-1) + (1-p_1)p_2(-1)(0-1) \\
&\quad + (1-p_1)(1-p_2)\frac{-\alpha}{\lambda}\int\limits_0^\infty \left[0-\Big(1+\frac{u}{\lambda}\Big)^{-\alpha-1}\right]du \\
&= p_1p_2 + p_1 - p_1p_2 + p_2 - p_1p_2 + (1-p_1)(1-p_2)\frac{\alpha}{\lambda}\left[\frac{\lambda}{-\alpha}\Big(1+\frac{u}{\lambda}\Big)^{-\alpha}\Big|_0^\infty\right]
\end{aligned}$$

$$= p_1 + p_2 - p_1p_2 + (1-p_1)(1-p_2)(-1)(0-1)$$
$$= p_1 + p_2 - p_1p_2 + 1 - p_1 - p_2 + p_1p_2$$
$$= 1.$$

□

The marginal distribution functions $f_U(u)$ and $f_Y(y)$ are derived from joint distribution function by taking integral with respect to variables u and y, respectively.

$$f_Y(y) = \int_0^y f_{U,Y}(u,y)du,$$

$$f_U(u) = \int_0^u f_{U,Y}(u,y)dy.$$

We consider for two intervals $y = 0$ and $y > 0$. For the first interval, we obtain the part $f_{U,Y}(0,0)$. For the second interval, we obtain the sum of the rest of Equation (4.7). We integrate the term $(1-p_1)p_2\frac{\alpha}{\lambda}\Big(1+\frac{u}{\lambda}\Big)^{-\alpha-1}$ itself directly since the condition is $y = u$. The term $(1-p_2)p_1\frac{\alpha}{\lambda}\Big(1+\frac{y}{\lambda}\Big)^{-\alpha-1}$ is obtained as the same as itself because to sum of the probabilities of u is not necessary since $u = 0$. We have to take integral just the term $(1-p_1)(1-p_2)\frac{\alpha(\alpha+1)}{\lambda^2}\Big(1+\frac{y}{\lambda}\Big)^{-\alpha-2}$ since $u > 0$.

$$f_Y(y) = \left\{ \begin{array}{ll} p_1p_2 & y=0 \\ & \\ (1-p_1)p_2\frac{\alpha}{\lambda}\Big(1+\frac{y}{\lambda}\Big)^{-\alpha-1} & \\ +(1-p_2)p_1\frac{\alpha}{\lambda}\Big(1+\frac{y}{\lambda}\Big)^{-\alpha-1} & \\ +\int_0^y (1-p_1)(1-p_2)\frac{\alpha(\alpha+1)}{\lambda^2}\Big(1+\frac{y}{\lambda}\Big)^{-\alpha-2}du & y>0 \end{array} \right\} \tag{4.8}$$

$$= \left\{ \begin{array}{ll} p_1p_2 & y=0 \\ & \\ (p_1+p_2-2p_1p_2)\frac{\alpha}{\lambda}\Big(1+\frac{y}{\lambda}\Big)^{-\alpha-1} & \\ +y(1-p_1)(1-p_2)\frac{\alpha(\alpha+1)}{\lambda^2}\Big(1+\frac{y}{\lambda}\Big)^{-\alpha-2} & y>0 \end{array} \right\}.$$

Theorem 4.8. *If $f_Y(y)$ is probability density function, then $\int_0^\infty f_Y(y)dy = 1$ in the defined interval for y.*

Proof.

$$\int_0^\infty f_Y(y)dy = p_1p_2 + \int_0^\infty (p_1+p_2-2p_1p_2)\frac{\alpha}{\lambda}\Big(1+\frac{y}{\lambda}\Big)^{-\alpha-1}dy$$

$$
\begin{aligned}
&+\int_0^\infty (1-p_1)(1-p_2)\frac{\alpha(\alpha+1)}{\lambda^2}y\Big(1+\frac{y}{\lambda}\Big)^{-\alpha-2}dy \\
&= p_1p_2 + (p_1+p_2-2p_1p_2)\frac{\alpha}{\lambda}\frac{\lambda}{-\alpha}\left[\Big(1+\frac{y}{\lambda}\Big)^{-\alpha-1}\Big|_0^\infty\right] \\
&\quad + (1-p_1)(1-p_2)\frac{\alpha(\alpha+1)}{\lambda^2}\left[y\frac{\lambda}{-\alpha-1}\Big(1+\frac{y}{\lambda}\Big)^{-\alpha-1}\Big|_0^\infty \right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad \left. - \int_0^\infty \frac{\lambda}{-\alpha-1}\Big(1+\frac{y}{\lambda}\Big)^{-\alpha-1}dy\right] \\
&= p_1p_2 + (p_1+p_2-2p_1p_2)(-1)(0-1) \\
&\quad + (1-p_1)(1-p_2)\frac{\alpha}{\lambda}\int_0^\infty \Big(1+\frac{y}{\lambda}\Big)^{-\alpha-1}dy \\
&= p_1p_2 + p_1 + p_2 - 2p_1p_2 + (1-p_1)(1-p_2)\frac{\alpha}{\lambda}\left[\frac{\lambda}{-\alpha}\Big(1+\frac{y}{\lambda}\Big)^{-\alpha}\Big|_0^\infty\right] \\
&= p_1+p_2-p_1p_2+1-p_1-p_2+p_1p_2 \\
&= 1.
\end{aligned}
$$

$\square$

We apply the same process to obtain marginal distribution function of U. For the case $u=0$, we obtain sum of the first term p_1p_2 and the third term $(1-p_2)p_1\frac{\alpha}{\lambda}\Big(1+\frac{y}{\lambda}\Big)^{-\alpha-1}$ of Equation (4.7). We should take integral of third term since y changes between 0 and ∞ when $u=0$. For the case $u>0$, we obtain the sum of the left hand side of Equation (4.7). We just take integral of the fourth term $(1-p_1)(1-p_2)\frac{\alpha(\alpha+1)}{\lambda^2}\Big(1+\frac{y}{\lambda}\Big)^{-\alpha-2}$ since y changes from u to ∞ which yields

$$
f_U(u) = \left\{
\begin{array}{ll}
p_1p_2 + \int_0^\infty (1-p_2)p_1\frac{\alpha}{\lambda}\Big(1+\frac{y}{\lambda}\Big)^{-\alpha-1}du & u=0 \\
(1-p_1)p_2\frac{\alpha}{\lambda}\Big(1+\frac{u}{\lambda}\Big)^{-\alpha-1} & \\
+\int_u^\infty (1-p_1)(1-p_2)\frac{\alpha(\alpha+1)}{\lambda^2}\Big(1+\frac{y}{\lambda}\Big)^{-\alpha-2}dy & u>0
\end{array}
\right\}
$$

$$
= \left\{ \begin{array}{ll} p_1p_2 + (1-p_2)p_1\dfrac{\alpha}{\lambda}\dfrac{\lambda}{-\alpha}\left[\left(1+\dfrac{y}{\lambda}\right)^{-\alpha-1}\Big|_0^\infty\right] & u=0 \\ & \\ (1-p_1)p_2\dfrac{\alpha}{\lambda}\left(1+\dfrac{u}{\lambda}\right)^{-\alpha-1} & \\ +(1-p_1)(1-p_2)\dfrac{\alpha(\alpha+1)}{\lambda^2}\dfrac{\lambda}{-\alpha-1}\left[\left(1+\dfrac{y}{\lambda}\right)^{-\alpha-1}\Big|_u^\infty\right] & u>0 \end{array} \right\}
$$

$$
= \left\{ \begin{array}{ll} p_1p_2 + (1-p_2)p_1(-1)(0-1) & u=0 \\ & \\ (1-p_1)p_2\dfrac{\alpha}{\lambda}\left(1+\dfrac{u}{\lambda}\right)^{-\alpha-1} & \\ +(1-p_1)(1-p_2)\dfrac{\alpha}{\lambda}(-1)\left[0-(1+\dfrac{u}{\lambda}\right)^{-\alpha-1}\Big] & u>0 \end{array} \right\}
$$

$$
= \left\{ \begin{array}{ll} p_1p_2 + p_1 - p_1p_2 & u=0 \\ & \\ (p_2+1-p_2)(1-p_1)\dfrac{\alpha}{\lambda}\left(1+\dfrac{u}{\lambda}\right)^{-\alpha-1} & u>0 \end{array} \right\}
$$

$$
= \left\{ \begin{array}{ll} p_1 & u=0 \\ & \\ (1-p_1)\dfrac{\alpha}{\lambda}\left(1+\dfrac{u}{\lambda}\right)^{-\alpha-1} & u>0 \end{array} \right\}.
$$

It is observed that $f_U(u)$ equals to $f_{X_1}(x_1|\alpha,\lambda)$ in Equation (4.5).

Theorem 4.9. *If $f_U(u)$ is probability density function, then $\int\limits_0^\infty f_U(u)du = 1$ in the defined interval for u.*

Proof.

$$
\begin{aligned}
\int\limits_0^\infty f_U(u)du &= p_1 + \int\limits_0^\infty (1-p_1)\frac{\alpha}{\lambda}\left(1+\frac{u}{\lambda}\right)^{-\alpha-1}du \\
&= p_1 + (1-p_1)\frac{\alpha}{\lambda}\left[\frac{\lambda}{-\alpha}\left(1+\frac{u}{\lambda}\right)^{-\alpha}\Big|_0^\infty\right] \\
&= p_1 + (1-p_1) \\
&= 1.
\end{aligned}
$$

□

The **exposure curve of two risks** is derived for the variable Y and using by single integral. We use the marginal distribution $f_Y(y)$ for calculation of the expected values.

$$\begin{aligned} G(D) &= \frac{E[min(D, X_1 + X_2)]}{E[X_1 + X_2]} \\ &= \frac{E[min(D, Y)]}{E[Y]} \\ &= \frac{\int_0^D y \cdot f_Y(y) dy + \int_D^B D \cdot f_Y(y) \cdot dy}{\int_0^B y \cdot f_Y(y) \cdot dy}. \end{aligned}$$

4.4 Distribution Function of Loss Ratios

Up to here, the distributions of claim amounts with respect to certain D are derived. The same analytical derivations are presented for loss ratios which constitute the variable of consideration in exposure curves. We propose that the losses in the portfolio are distributed Pareto with parameters α and λ and all policies in the portfolio have the same total exposure, which is maximum probable loss denoted by B. This means the insurance company has a liability to pay a loss up to amount B for each policy and in the case of the loss is exceeding the amount B is out off the insurance agreement.

Theorem 4.10. *Let B denote an arbitrary number such that $B > 0$. Define X and Z be positive random variables such that X=Z/B, where Z is a loss amount. If Z has a mixture Pareto probability distribution $Par(\alpha, \beta)$ obtained in Equation* (4.4)*, then X has a mixture Pareto probability distribution $Par(\alpha, \beta/B)$.*

Proof. Suppose that $F_Z(z) = 1 - (1-p)\Big(1 + \dfrac{z}{\beta}\Big)^{-\alpha}$ for $\alpha, \beta > 0, Z \geq 0$. Define X such that $X = Z/B$, then

$$\begin{aligned} F_X(x) = \Pr(X \leq x) = \Pr(X \leq \frac{Z}{B}) &= \Pr(XB \leq Z) \\ &= F_Z(XB) = 1 - (1-p)\Big(1 + \frac{XB}{\beta}\Big)^{-\alpha} \\ &= 1 - (1-p)\Big(1 + \frac{X}{\frac{\beta}{B}}\Big)^{-\alpha} \qquad \alpha, \frac{\beta}{B} > 0, X \geq 0. \end{aligned}$$

□

Theorem 4.10 shows that if we consider B as MPL, all losses have to be smaller than or equal to B. If we divide each loss by B, all results are obtained as the form of ratio and can take a value at most 1. We prove that dividing claims to a constant does not affect the distribution itself, but causes change in parameters.

If we change the notation $\dfrac{\beta}{B}$ with λ, we obtain exactly the same function that we obtain in Equation (4.4). The only difference is that X's are **loss ratios** on the interval [0,1].

Similarly, if D is divided by B, it also gives a ratio $d = D/M$ which is called **retention ratio**.

Joint and marginal distributions remains the same except the variable domains which are reduced to the interval [0,1]. Note that the risks X_1 and X_2 are together under the one policy, so the sum of them Y is less than or equal to B. Thus, the ratio of their sum should be less than or equal to 1.

After dividing by MPL, X_1, X_2 and Y have the same domains, which is the interval [0,1]. Therefore, the sum of all probabilities has to be 1 when they take every value from 0 to 1. Therefore, the endpoints of the variables need to been considered separately since the equation $\int_0^1 f_.(.) = 1$ has to been satisfied. Bernegger [1] considers that endpoint by subtracting the sum of all probabilities from 1.

Theorem 4.11. *The density of the ratio Y, at $y = 1$ given as follows:*

$$f_Y(1) = (1 - p_1p_2)\Big(1 + \frac{1}{\lambda}\Big)^{-\alpha} + (1 - p_1)(1 - p_2)\frac{\alpha}{\lambda}\Big(1 + \frac{1}{\lambda}\Big)^{-\alpha-1}.$$

Proof. Given $f_Y(y)$ in Equation (4.8)

$$\begin{aligned}
1 &= \int_0^1 f_Y(y)dy \\
1 &= \int_0^{1^-} f_Y(y)dy + f_Y(1) \\
f_Y(1) &= 1 - \int_0^{1^-} f_Y(y)dy \\
&= 1 - \Bigg[p_1p_2 + \int_0^{1^-} \Bigg((p_1 + p_2 - 2p_1p_2)\frac{\alpha}{\lambda}\Big(1 + \frac{y}{\lambda}\Big)^{-\alpha-1} \\
&\qquad + y(1 - p_1)(1 - p_2)\frac{\alpha(\alpha+1)}{\lambda^2}\Big(1 + \frac{y}{\lambda}\Big)^{-\alpha-2}\Bigg)dy\Bigg] \\
&= 1 - \Bigg[p_1p_2 + (p_1 + p_2 - 2p_1p_2)\frac{\alpha}{\lambda}\frac{\lambda}{-\alpha}\Big(1 + \frac{y}{\lambda}\Big)^{-\alpha}\Big|_0^{1^-} \\
&\qquad + (1 - p_1)(1 - p_2)\frac{\alpha(\alpha+1)}{\lambda^2}\Bigg(y\frac{\lambda}{-\alpha-1}\Big(1 + \frac{y}{\lambda}\Big)^{-\alpha-1}\Big|_0^{1^-} \\
&\qquad - \int_0^{1^-} \frac{\lambda}{-\alpha-1}\Big(1 + \frac{y}{\lambda}\Big)^{-\alpha-1}dy\Bigg)\Bigg] \\
&= 1 - \Bigg[p_1p_2 + (p_1 + p_2 - 2p_1p_2)(-1)\Big[\Big(1 + \frac{1}{\lambda}\Big)^{-\alpha} - 1\Big]
\end{aligned}$$

$$
\begin{aligned}
&\quad + (1-p_1)(1-p_2)\frac{\alpha(\alpha+1)}{\lambda^2}\Bigg(\frac{\lambda}{-\alpha-1}\Big(1+\frac{1}{\lambda}\Big)^{-\alpha-1} \\
&\quad + \frac{\lambda}{\alpha+1}\frac{\lambda}{-\alpha}\Big(1+\frac{y}{\lambda}\Big)^{-\alpha}\Big|_0^{1^-}\Bigg)\Bigg] \\
&= 1 - \Bigg[p_1p_2 - (p_1+p_2-2p_1p_2)\Big[\Big(1+\frac{1}{\lambda}\Big)^{-\alpha} - 1\Big] \\
&\quad + (1-p_1)(1-p_2)\frac{-\alpha}{\lambda}\Big(1+\frac{1}{\lambda}\Big)^{-\alpha-1} \\
&\quad - (1-p_1)(1-p_2)\Big[\Big(1+\frac{1}{\lambda}\Big)^{-\alpha} - 1\Big]\Bigg] \\
&= 1 - \Bigg[p_1p_2 - (1-p_1)(1-p_2)\frac{\alpha}{\lambda}\Big(1+\frac{1}{\lambda}\Big)^{-\alpha-1} \\
&\quad - \big((1-p_1)(1-p_2) + (p_1+p_2-2p_1p_2)\big)\Big[\Big(1+\frac{1}{\lambda}\Big)^{-\alpha} - 1\Big]\Bigg] \\
&= 1 - \Bigg[p_1p_2 - (1-p_1)(1-p_2)\frac{\alpha}{\lambda}\Big(1+\frac{1}{\lambda}\Big)^{-\alpha-1} \\
&\quad - (1-p_2-p_1+p_1p_2+p_1+p_2-2p_1p_2)\Big[\Big(1+\frac{1}{\lambda}\Big)^{-\alpha} - 1\Big]\Bigg] \\
&= 1 - \Bigg[p_1p_2 - (1-p_1)(1-p_2)\frac{\alpha}{\lambda}\Big(1+\frac{1}{\lambda}\Big)^{-\alpha-1} \\
&\quad - (1-p_1p_2)\Big[\Big(1+\frac{1}{\lambda}\Big)^{-\alpha} - 1\Big]\Bigg] \\
&= 1 - \Bigg[p_1p_2 - (1-p_1)(1-p_2)\frac{\alpha}{\lambda}\Big(1+\frac{1}{\lambda}\Big)^{-\alpha-1} \\
&\quad - (1-p_1p_2)\Big(1+\frac{1}{\lambda}\Big)^{-\alpha} + (1-p_1p_2)\Bigg] \\
&= 1 - p_1p_2 + (1-p_1)(1-p_2)\frac{\alpha}{\lambda}\Big(1+\frac{1}{\lambda}\Big)^{-\alpha-1} \\
&\quad + (1-p_1p_2)\Big(1+\frac{1}{\lambda}\Big)^{-\alpha} - 1 + p_1p_2 \\
&= (1-p_1p_2)\Big(1+\frac{1}{\lambda}\Big)^{-\alpha} + (1-p_1)(1-p_2)\frac{\alpha}{\lambda}\Big(1+\frac{1}{\lambda}\Big)^{-\alpha-1}.
\end{aligned}
$$

$\square$

Based on this, the new form of the probability density function and intervals for Y are given as.

$$f_Y(y) = \left\{ \begin{array}{ll} p_1p_2 & y = 0 \\ (p_1 + p_2 - 2p_1p_2)\dfrac{\alpha}{\lambda}\Big(1 + \dfrac{y}{\lambda}\Big)^{-\alpha-1} & \\ +y(1 - p_1)(1 - p_2)\dfrac{\alpha(\alpha+1)}{\lambda^2}\Big(1 + \dfrac{y}{\lambda}\Big)^{-\alpha-2} & 0 < y < 1 \\ (1 - p_1p_2)\Big(1 + \dfrac{1}{\lambda}\Big)^{-\alpha} + (1 - p_1)(1 - p_2)\dfrac{\alpha}{\lambda}\Big(1 + \dfrac{1}{\lambda}\Big)^{-\alpha-1} & y = 1 \end{array} \right\}. \tag{4.9}$$

4.5 Exposure Curve for Two Dependent Risks

Theorem 4.12. *Let Y be the sum of two dependent risks defined in Definition 4.6 and let G be the exposure curve of Y. Given the probability density function of Y in Equation* (4.9). *The function of exposure curve is*

$$G(d) = \frac{(2 - p_1 - p_2)\dfrac{\lambda}{1-\alpha}\Big[\Big(1 + \dfrac{d}{\lambda}\Big)^{-\alpha+1} - 1\Big] - d(1 - p_1)(1 - p_2)\Big(1 + \dfrac{d}{\lambda}\Big)^{-\alpha}}{(2 - p_1 - p_2)\dfrac{\lambda}{1-\alpha}\Big[\Big(1 + \dfrac{1}{\lambda}\Big)^{-\alpha+1} - 1\Big] - (1 - p_1)(1 - p_2)\Big(1 + \dfrac{1}{\lambda}\Big)^{-\alpha}},$$

where $a > 1$.

Proof. We substitute Equation (4.6) to Proposition (4.1).

$$\begin{aligned} G(d) &= \frac{E[\min(d, X_1 + X_2)]}{E[X_1 + X_2]} \\ &= \frac{\iint\limits_{x_1+x_2<d} (x_1 + x_2) f_{X_1,X_2}(x_1, x_2) dx_1 dx_2 + \iint\limits_{d<x_1+x_2<1} d \cdot f_{X_1,X_2}(x_1, x_2) dx_1 dx_2}{\iint\limits_{x_1+x_2<1} (x_1 + x_2) f_{X_1,X_2}(x_1, x_2) dx_1 dx_2} \\ &= \frac{\int_0^d y \cdot f_Y(y) \cdot dy + \int_d^1 d \cdot f_Y(y) \cdot dy}{\int_0^1 y \cdot f_Y(y) \cdot dy} \\ &= \frac{\int_0^d y f_Y(y) dy + d \cdot \left(\int_d^{1^-} f_Y(y) dy + f_Y(1) \right)}{\int_0^{1^-} y f_Y(y) dy + f_Y(1)}. \end{aligned} \tag{4.10}$$

Note that if d takes the value 1, the nominator equals to denominator. In order to simplify equation and not to repeat the same terms, we redefine some of them. Let A

and B denote the terms $(p_1+p_2-2p_1p_2)$ and $(1-p_1)(1-p_2)$, respectively. We divide the terms of nominator into two parts to make the evaluation easier such that

$$\int_0^d yf_Y(y)dy + d\cdot\bigg(\int_d^{1^-} f_Y(y)dy + f_Y(1)\bigg) = I_1 + I_2,$$

where $\int_0^d yf_Y(y)dy = I_1$ and $d\cdot\bigg(\int_d^{1^-} f_Y(y)dy + f_Y(1)\bigg) = I_2$. We evaluate these terms using integration by parts.

$$\begin{aligned}
I_1 &= \int_0^d yf_Y(y)dy\\
&= \int_0^d y(p_1+p_2-2p_1p_2)\frac{\alpha}{\lambda}\Big(1+\frac{y}{\lambda}\Big)^{-\alpha-1}dy\\
&\quad + \int_0^d y^2(1-p_1)(1-p_2)\frac{\alpha(\alpha+1)}{\lambda^2}\Big(1+\frac{y}{\lambda}\Big)^{-\alpha-2}dy\\
&= \int_0^d yA\frac{\alpha}{\lambda}\Big(1+\frac{y}{\lambda}\Big)^{-\alpha-1}dy\\
&\quad + \int_0^d y^2B\frac{\alpha(\alpha+1)}{\lambda^2}\Big(1+\frac{y}{\lambda}\Big)^{-\alpha-2}dy\\
&= A\frac{\alpha}{\lambda}\bigg[y\frac{\lambda}{-\alpha}\Big(1+\frac{y}{\lambda}\Big)^{-\alpha}\Big|_0^d + \frac{\lambda}{\alpha}\int_0^d\Big(1+\frac{y}{\lambda}\Big)^{-\alpha}dy\bigg]\\
&\quad + B\frac{\alpha(\alpha+1)}{\lambda^2}\bigg[y^2\frac{\lambda}{-\alpha-1}\Big(1+\frac{y}{\lambda}\Big)^{-\alpha-1}\Big|_0^d + \frac{\lambda}{\alpha+1}\int_0^d 2y\Big(1+\frac{y}{\lambda}\Big)^{-\alpha-1}dy\bigg]\\
&= A\frac{\alpha}{\lambda}\bigg[d\frac{\lambda}{-\alpha}\Big(1+\frac{d}{\lambda}\Big)^{-\alpha} + \frac{\lambda}{\alpha}\frac{\lambda}{-\alpha+1}\Big(1+\frac{y}{\lambda}\Big)^{-\alpha+1}\Big|_0^d\bigg]\\
&\quad + B\frac{\alpha(\alpha+1)}{\lambda^2}\bigg[d^2\frac{\lambda}{-\alpha-1}\Big(1+\frac{d}{\lambda}\Big)^{-\alpha-1}\\
&\quad + \frac{2\lambda}{\alpha+1}\bigg[y\frac{\lambda}{-\alpha}\Big(1+\frac{y}{\lambda}\Big)^{-\alpha}\Big|_0^d + \frac{\lambda}{\alpha}\int_0^d\Big(1+\frac{y}{\lambda}\Big)^{-\alpha}dy\bigg]\bigg]\\
&= A\frac{\alpha}{\lambda}\bigg[d\frac{\lambda}{-\alpha}\Big(1+\frac{d}{\lambda}\Big)^{-\alpha} + \frac{\lambda^2}{\alpha(-\alpha+1)}\Big[\Big(1+\frac{d}{\lambda}\Big)^{-\alpha+1}-1\Big]\bigg]\\
&\quad + B\frac{\alpha(\alpha+1)}{\lambda^2}\bigg[d^2\frac{\lambda}{-\alpha-1}\Big(1+\frac{d}{\lambda}\Big)^{-\alpha-1}\\
&\quad + \frac{2\lambda}{\alpha+1}\bigg[d\frac{\lambda}{-\alpha}\Big(1+\frac{d}{\lambda}\Big)^{-\alpha} + \frac{\lambda}{\alpha}\frac{\lambda}{-\alpha+1}\Big(1+\frac{y}{\lambda}\Big)^{-\alpha+1}\Big|_0^d\bigg]\bigg]\\
&= A\bigg[-d\Big(1+\frac{d}{\lambda}\Big)^{-\alpha} + \frac{\lambda}{-\alpha+1}\Big[\Big(1+\frac{d}{\lambda}\Big)^{-\alpha+1}-1\Big]\bigg]
\end{aligned}$$

$$
\begin{aligned}
&+ B\Bigg[- d^2\frac{\alpha}{\lambda}\Big(1+\frac{d}{\lambda}\Big)^{-\alpha-1} \\
&+ \frac{2\alpha}{\lambda}\Bigg[d\frac{\lambda}{-\alpha}\Big(1+\frac{d}{\lambda}\Big)^{-\alpha} + \frac{\lambda^2}{\alpha(-\alpha+1)}\Big[\Big(1+\frac{d}{\lambda}\Big)^{-\alpha+1} - 1\Big]\Bigg]\Bigg] \\
= {}& A\Bigg[\frac{\lambda}{-\alpha+1}\Big[\Big(1+\frac{d}{\lambda}\Big)^{-\alpha+1} - 1\Big] - d\Big(1+\frac{d}{\lambda}\Big)^{-\alpha}\Bigg] \\
&+ B\Bigg[- d^2\frac{\alpha}{\lambda}\Big(1+\frac{d}{\lambda}\Big)^{-\alpha-1} \\
&- 2d\Big(1+\frac{d}{\lambda}\Big)^{-\alpha} + \frac{2\lambda}{-\alpha+1}\Big[\Big(1+\frac{d}{\lambda}\Big)^{-\alpha+1} - 1\Big]\Bigg] \\
= {}& A\Bigg[\frac{\lambda}{-\alpha+1}\Big(1+\frac{d}{\lambda}\Big)^{-\alpha+1} - d\Big(1+\frac{d}{\lambda}\Big)^{-\alpha} - \frac{\lambda}{-\alpha+1}\Bigg] \\
&+ B\Bigg[- d^2\frac{\alpha}{\lambda}\Big(1+\frac{d}{\lambda}\Big)^{-\alpha-1} - 2d\Big(1+\frac{d}{\lambda}\Big)^{-\alpha} \\
&+ \frac{2\lambda}{-\alpha+1}\Big(1+\frac{d}{\lambda}\Big)^{-\alpha+1} - \frac{2\lambda}{-\alpha+1}\Bigg] \\
= {}& \frac{A\lambda}{-\alpha+1}\Big(1+\frac{d}{\lambda}\Big)^{-\alpha+1} - Ad\Big(1+\frac{d}{\lambda}\Big)^{-\alpha} - \frac{A\lambda}{-\alpha+1} \\
&- d^2B\frac{\alpha}{\lambda}\Big(1+\frac{d}{\lambda}\Big)^{-\alpha-1} - 2dB\Big(1+\frac{d}{\lambda}\Big)^{-\alpha} \\
&+ \frac{2B\lambda}{-\alpha+1}\Big(1+\frac{d}{\lambda}\Big)^{-\alpha+1} - \frac{2B\lambda}{-\alpha+1} \\
= {}& -d^2B\frac{\alpha}{\lambda}\Big(1+\frac{d}{\lambda}\Big)^{-\alpha-1} + \frac{(A+2B)\lambda}{-\alpha+1}\Big(1+\frac{d}{\lambda}\Big)^{-\alpha+1} \\
&- (A+2B)d\Big(1+\frac{d}{\lambda}\Big)^{-\alpha} - \frac{(A+2B)\lambda}{-\alpha+1}.
\end{aligned}
$$

We use same letters A and B while calculation of I_2.

$$
\begin{aligned}
I_2 &= d\cdot\Bigg(\int_d^{1^-} f_Y(y)dy + f_Y(1)\Bigg) \\
&= d\cdot\Bigg[\int_d^{1^-} (p_1+p_2-2p_1p_2)\frac{\alpha}{\lambda}\Big(1+\frac{y}{\lambda}\Big)^{-\alpha-1}dy
\end{aligned}
$$

$$
\begin{aligned}
&+\int_d^{1^-} y(1-p_1)(1-p_2)\frac{\alpha(\alpha+1)}{\lambda^2}\Big(1+\frac{y}{\lambda}\Big)^{-\alpha-2}dy + f_Y(1)\Bigg] \\
&= d\cdot\Bigg[\int_d^{1^-} A\frac{\alpha}{\lambda}\Big(1+\frac{y}{\lambda}\Big)^{-\alpha-1}dy + \int_d^{1^-} B\frac{\alpha(\alpha+1)}{\lambda^2}y\Big(1+\frac{y}{\lambda}\Big)^{-\alpha-2}dy \quad + f_Y(1)\Bigg] \\
&= d\cdot\Bigg[A\frac{\alpha}{\lambda}\frac{\lambda}{-\alpha}\Big(1+\frac{y}{\lambda}\Big)^{-\alpha}\Big|_d^1 + B\frac{\alpha(\alpha+1)}{\lambda^2}\Bigg[y\frac{\lambda}{-\alpha-1}\Big(1+\frac{y}{\lambda}\Big)^{-\alpha-1}\Big|_d^1 \\
&\quad + \frac{\lambda}{\alpha+1}\int_d^{1^-}\Big(1+\frac{y}{\lambda}\Big)^{-\alpha-1}dy\Bigg] + f_Y(1)\Bigg] \\
&= d\cdot\Bigg[A(-1)\Bigg[\Big(1+\frac{1}{\lambda}\Big)^{-\alpha} - \Big(1+\frac{d}{\lambda}\Big)^{-\alpha}\Bigg] + B\frac{\alpha(\alpha+1)}{\lambda^2}\Bigg[\frac{\lambda}{-\alpha-1}\Bigg[\Big(1+\frac{1}{\lambda}\Big)^{-\alpha-1} \\
&\quad - d\Big(1+\frac{d}{\lambda}\Big)^{-\alpha-1}\Bigg] + \frac{\lambda}{\alpha+1}\frac{\lambda}{-\alpha}\Big(1+\frac{y}{\lambda}\Big)^{-\alpha}\Big|_d^1\Bigg] + f_Y(1)\Bigg] \\
&= d\cdot\Bigg[A\Bigg[\Big(1+\frac{d}{\lambda}\Big)^{-\alpha} - \Big(1+\frac{1}{\lambda}\Big)^{-\alpha}\Bigg] + B\frac{\alpha(\alpha+1)}{\lambda^2}\Bigg[\frac{\lambda}{-\alpha-1}\Bigg[\Big(1+\frac{1}{\lambda}\Big)^{-\alpha-1} \\
&\quad - d\Big(1+\frac{d}{\lambda}\Big)^{-\alpha-1}\Bigg] + \frac{\lambda}{\alpha+1}\frac{\lambda}{-\alpha}\Bigg[\Big(1+\frac{1}{\lambda}\Big)^{-\alpha} - \Big(1+\frac{d}{\lambda}\Big)^{-\alpha}\Bigg]\Bigg] + f_Y(1)\Bigg] \\
&= d\cdot\Bigg[A\Bigg[\Big(1+\frac{d}{\lambda}\Big)^{-\alpha} - \Big(1+\frac{1}{\lambda}\Big)^{-\alpha}\Bigg] + B\Bigg[\frac{-\alpha}{\lambda}\Bigg[\Big(1+\frac{1}{\lambda}\Big)^{-\alpha-1} - d\Big(1+\frac{d}{\lambda}\Big)^{-\alpha-1}\Bigg] \\
&\quad - \Bigg[\Big(1+\frac{1}{\lambda}\Big)^{-\alpha} - \Big(1+\frac{d}{\lambda}\Big)^{-\alpha}\Bigg]\Bigg] + f_Y(1)\Bigg] \\
&= d\cdot\Bigg[A\Big(1+\frac{d}{\lambda}\Big)^{-\alpha} - A\Big(1+\frac{1}{\lambda}\Big)^{-\alpha} + B\Bigg[\frac{-\alpha}{\lambda}\Big(1+\frac{1}{\lambda}\Big)^{-\alpha-1} + \frac{d\alpha}{\lambda}\Big(1+\frac{d}{\lambda}\Big)^{-\alpha-1} \\
&\quad + \Big(1+\frac{d}{\lambda}\Big)^{-\alpha} - \Big(1+\frac{1}{\lambda}\Big)^{-\alpha}\Bigg] + f_Y(1)\Bigg] \\
&= d\cdot\Bigg[A\Big(1+\frac{d}{\lambda}\Big)^{-\alpha} - A\Big(1+\frac{1}{\lambda}\Big)^{-\alpha} - \frac{B\alpha}{\lambda}\Big(1+\frac{1}{\lambda}\Big)^{-\alpha-1} + \frac{Bd\alpha}{\lambda}\Big(1+\frac{d}{\lambda}\Big)^{-\alpha-1} \\
&\quad + B\Big(1+\frac{d}{\lambda}\Big)^{-\alpha} - B\Big(1+\frac{1}{\lambda}\Big)^{-\alpha} + f_Y(1)\Bigg] \\
&= Ad\Big(1+\frac{d}{\lambda}\Big)^{-\alpha} - Ad\Big(1+\frac{1}{\lambda}\Big)^{-\alpha} - \frac{Bd\alpha}{\lambda}\Big(1+\frac{1}{\lambda}\Big)^{-\alpha-1} + \frac{Bd^2\alpha}{\lambda}\Big(1+\frac{d}{\lambda}\Big)^{-\alpha-1} \\
&\quad + Bd\Big(1+\frac{d}{\lambda}\Big)^{-\alpha} - Bd\Big(1+\frac{1}{\lambda}\Big)^{-\alpha} + df_Y(1) \\
&= (A+B)d\Big(1+\frac{d}{\lambda}\Big)^{-\alpha} - (A+B)d\Big(1+\frac{1}{\lambda}\Big)^{-\alpha} - \frac{Bd\alpha}{\lambda}\Big(1+\frac{1}{\lambda}\Big)^{-\alpha-1}
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{Bd^2\alpha}{\lambda}\Big(1+\frac{d}{\lambda}\Big)^{-\alpha-1} + df_Y(1)\\
&= (A+B)d\Big(1+\frac{d}{\lambda}\Big)^{-\alpha} - (A+B)d\Big(1+\frac{1}{\lambda}\Big)^{-\alpha} - \frac{Bd\alpha}{\lambda}\Big(1+\frac{1}{\lambda}\Big)^{-\alpha-1}\\
&+ \frac{Bd^2\alpha}{\lambda}\Big(1+\frac{d}{\lambda}\Big)^{-\alpha-1} + d\Big[(1-p_1p_2)\Big(1+\frac{1}{\lambda}\Big)^{-\alpha} + \frac{B\alpha}{\lambda}\Big(1+\frac{1}{\lambda}\Big)^{-\alpha-1}\Big]\\
&= (A+B)d\Big(1+\frac{d}{\lambda}\Big)^{-\alpha} - (A+B)d\Big(1+\frac{1}{\lambda}\Big)^{-\alpha} - \frac{Bd\alpha}{\lambda}\Big(1+\frac{1}{\lambda}\Big)^{-\alpha-1}\\
&+ \frac{Bd^2\alpha}{\lambda}\Big(1+\frac{d}{\lambda}\Big)^{-\alpha-1} + d(1-p_1p_2)\Big(1+\frac{1}{\lambda}\Big)^{-\alpha} + \frac{Bd\alpha}{\lambda}\Big(1+\frac{1}{\lambda}\Big)^{-\alpha-1}\\
&= (A+B)d\Big(1+\frac{d}{\lambda}\Big)^{-\alpha} - (A+B)d\Big(1+\frac{1}{\lambda}\Big)^{-\alpha} + \frac{Bd^2\alpha}{\lambda}\Big(1+\frac{d}{\lambda}\Big)^{-\alpha-1}\\
&+ d(1-p_1p_2)\Big(1+\frac{1}{\lambda}\Big)^{-\alpha}.
\end{aligned}
$$

We sum the obtained parts I_1 and I_2:

$$
\begin{aligned}
I_1 + I_2 &= -d^2B\frac{\alpha}{\lambda}\Big(1+\frac{d}{\lambda}\Big)^{-\alpha-1} + \frac{(A+2B)\lambda}{-\alpha+1}\Big(1+\frac{d}{\lambda}\Big)^{-\alpha+1} - (A+2B)d\Big(1+\frac{d}{\lambda}\Big)^{-\alpha}\\
&- \frac{(A+2B)\lambda}{-\alpha+1} \quad + \quad (A+B)d\Big(1+\frac{d}{\lambda}\Big)^{-\alpha} - (A+B)d\Big(1+\frac{1}{\lambda}\Big)^{-\alpha}\\
&+ \frac{Bd^2\alpha}{\lambda}\Big(1+\frac{d}{\lambda}\Big)^{-\alpha-1} + d(1-p_1p_2)\Big(1+\frac{1}{\lambda}\Big)^{-\alpha}\\
&= \frac{(A+2B)\lambda}{-\alpha+1}\Big[\Big(1+\frac{d}{\lambda}\Big)^{-\alpha+1} - 1\Big] + (-A-2B+A+B)d\Big(1+\frac{d}{\lambda}\Big)^{-\alpha}\\
&+ (1-p_1p_2-A-B)d\Big(1+\frac{1}{\lambda}\Big)^{-\alpha}\\
&= \frac{(A+2B)\lambda}{-\alpha+1}\Big[\Big(1+\frac{d}{\lambda}\Big)^{-\alpha+1} - 1\Big] + (-B)d\Big(1+\frac{d}{\lambda}\Big)^{-\alpha}\\
&+ (1-p_1p_2-A-B)d\Big(1+\frac{1}{\lambda}\Big)^{-\alpha}\\
&= \frac{(p_1+p_2-2p_1p_2+2-2p_1-2p_2+2p_1p_2)\lambda}{-\alpha+1}\Big[\Big(1+\frac{d}{\lambda}\Big)^{-\alpha+1} - 1\Big]\\
&+ (-1+p_1+p_2-p_1p_2)d\Big(1+\frac{d}{\lambda}\Big)^{-\alpha}\\
&+ (1-p_1p_2-p_1-p_2+2p_1p_2-1+p_1+p_2-p_1p_2)d\Big(1+\frac{1}{\lambda}\Big)^{-\alpha}\\
&= \frac{(2-p_1-p_2)\lambda}{-\alpha+1}\Big[\Big(1+\frac{d}{\lambda}\Big)^{-\alpha+1} - 1\Big] - (1-p_1)(1-p_2)d\Big(1+\frac{d}{\lambda}\Big)^{-\alpha}.
\end{aligned}
$$

We mention that if $d = 1$, we obtain denominator of Equation (4.10).Therefore, the exposure curve G is derived as follows:

$$G(d) = \frac{\dfrac{(2-p_1-p_2)\lambda}{-\alpha+1}\left[\left(1+\dfrac{d}{\lambda}\right)^{-\alpha+1}-1\right]-(1-p_1)(1-p_2)d\left(1+\dfrac{d}{\lambda}\right)^{-\alpha}}{\dfrac{(2-p_1-p_2)\lambda}{-\alpha+1}\left[\left(1+\dfrac{1}{\lambda}\right)^{-\alpha+1}-1\right]-(1-p_1)(1-p_2)\left(1+\dfrac{1}{\lambda}\right)^{-\alpha}}.$$

□

The constraints on weights are investigated and analyzed. For p_1 and p_2, we have $0 \leq p_1 \leq 1$ and $0 \leq p_2 \leq 1$. α and λ are greater than 0. We need to define new constraints for α such that $\alpha > 1$ to avoid $E[Y]$ would be infinity, like the expectation of Pareto distribution in literature. d is deductible in percentage, so it is also in the interval [0,1]. Furthermore, the curve satisfies the objectives $G(0) = 0$ and $G(1) = 1$ of definition of exposure curve.

Theorem 4.13. *If G is an exposure curve, it is increasing in the interval (0,1).*

Proof. Take derivative of G:

$$\frac{\partial G(d)}{\partial d} = \frac{1}{E[Y]}\Bigg[(2-p_1-p_2)\frac{\lambda}{1-\alpha}\frac{(-\alpha+1)}{\lambda}\left(1+\frac{d}{\lambda}\right)^{-\alpha}$$

$$-(1-p_1)(1-p_2)\left(1+\frac{d}{\lambda}\right)^{-\alpha}-d(1-p_1)(1-p_2)\frac{-\alpha}{\lambda}\left(1+\frac{d}{\lambda}\right)^{-\alpha-1}\Bigg]$$

$$= \frac{1}{E[Y]}\left[(2-p_1p_2)\left(1+\frac{d}{\lambda}\right)^{-\alpha}+d(1-p_1)(1-p_2)\frac{\alpha}{\lambda}\left(1+\frac{d}{\lambda}\right)^{-\alpha-1}\right].$$

We check that either the term $(2 - p_1p_2)$ is greater than or equal to 0 since all other terms are positive or equal to 0. Recall that p_1 and p_2 are probabilities:

$$\begin{aligned} 0 \leq p_1 \leq 1, \\ 0 \leq p_1p_2 \leq 1, \\ 1 \leq 2 - p_1p_2 \leq 2. \end{aligned}$$

We prove that $0 < (2 - p_1p_2)$ so $\dfrac{\partial G(d)}{\partial d} \geq 0$. Therefore it is increasing function on the interval (0,1). □

Theorem 4.14. *If G is an exposure curve, it is concave in the interval (0,1).*

Proof. Take second derivative of G:

$$\frac{\partial^2 G(d)}{\partial d^2} = \frac{1}{E[Y]}\Bigg[(2-p_1p_2)\frac{-\alpha}{\lambda}\Big(1+\frac{d}{\lambda}\Big)^{-\alpha-1} + (1-p_1)(1-p_2)\frac{\alpha}{\lambda}\Big(1+\frac{d}{\lambda}\Big)^{-\alpha-1}$$

$$+ d(1-p_1)(1-p_2)\frac{\alpha}{\lambda}\frac{(-\alpha-1)}{\lambda}\Big(1+\frac{d}{\lambda}\Big)^{-\alpha-2}\Bigg]$$

$$= \frac{1}{E[Y]}\Bigg[(2p_1p_2-p_1-p_2-1)\frac{\alpha}{\lambda}\Big(1+\frac{d}{\lambda}\Big)^{-\alpha-1}$$

$$- d(1-p_1)(1-p_2)\frac{\alpha(\alpha+1)}{\lambda}\Big(1+\frac{d}{\lambda}\Big)^{-\alpha-2}\Bigg].$$

Since the other term is negative or equal to 0, we check the term $(2p_1p_2-p_1-p_2-1)$:

$$\begin{aligned} 0 \le p_1 &\le 1, \\ 0 \le p_1p_2 &\le p_2, \\ p_1p_2 &\le p_2 + p_1, \\ p_1p_2 - p_2 - p_1 &\le 0, \\ p_1p_2 - p_2 - p_1 - 1 &\le -1. \end{aligned} \tag{4.11}$$

$$\begin{aligned} 0 \le p_1 &\le 1, \\ 0 \le p_2 &\le 1, \\ 0 \le p_1p_2 &\le 1. \end{aligned} \tag{4.12}$$

When the last steps of Equations (4.11) and (4.12) are added, we obtain the result $2p_1p_2 - p_2 - p_1 - 1 \le 0$. So $\dfrac{\partial^2 G(d)}{\partial d^2} \le 0$. Therefore, G is a concave function on the interval (0,1). $\square$

CHAPTER 5

NUMERICAL ILLUSTRATION

We perform the simulations to construct realistic exposure curves based on the proposed models.

5.1 Constrains for Parameters

In the literature the pdf of Pareto distributions is shown to have a form that is decreasing and convex. We recall the equation of $f_Y(y)$ in Equation (4.8) and take derivative of it with respect to y.

$$\begin{aligned} \frac{df_Y(y)}{dy} = (3p_1p_2 - 2p_1 - 2p_2 + 1)\frac{\alpha(\alpha+1)}{\lambda^2}\left(1+\frac{y}{\lambda}\right)^{-\alpha-2} \\ - (1-p_1)(1-p_2)y\frac{\alpha(\alpha+1)}{\lambda^2}\left(1+\frac{y}{\lambda}\right)^{-\alpha-3}. \end{aligned} \tag{5.1}$$

The second term of Equation (5.1) is negative or equal to zero obviously. The term $(3p_1p_2 - 2p_1 - 2p_2 + 1) \leq 0$ has to be checked if the curve is decreasing. If we assume that $(3p_1p_2 - 2p_1 - 2p_2 + 1) \leq 0$, this condition also satisfies the convexity of the pdf and we attain the decreasing and convex function on the interval (0,1).

The constrains are determined for keeping the probabilities under 1. Since $f_Y(y)$ is decreasing on the interval (0,1), $\lim_{y\to 0} f_Y(y)$ must be at most 1 and also at least 0 because of the definition of probability itself.

$$\begin{aligned} 0 \leq \lim_{y\to 0} f_Y(y) \leq 1, \\ 0 \leq (p_1 + p_2 - 2p_1p_2)\frac{\alpha}{\lambda} \leq 1, \\ (p_1 + p_2 - 2p_1p_2)\alpha \leq \lambda. \end{aligned}$$

Additional to those $0 \leq p_1 \leq 1$, $0 \leq p_2 \leq 1$ and $\alpha > 0$, we remind the other constraints $(3p_1p_2 - 2p_1 - 2p_2 + 1) \leq 0$ and $(p_1 + p_2 - 2p_1p_2)\alpha \leq \lambda$. Figure 5.1 shows the probability density function of Y under some parameter value assumptions.

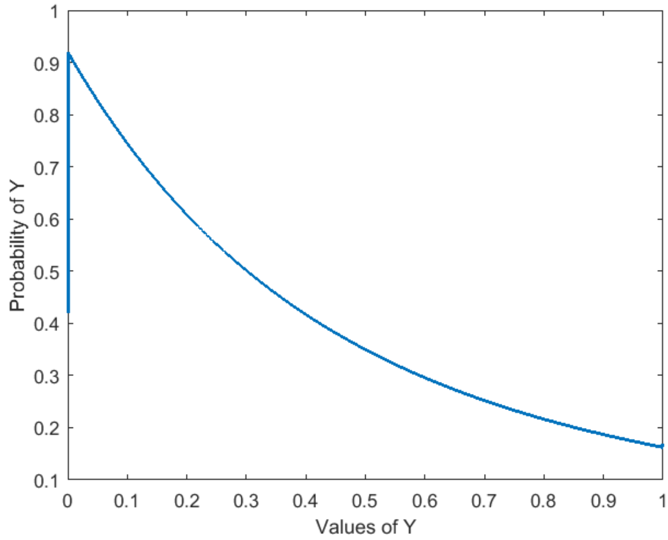


Figure 5.1: Probability density distributions of Y when $p_1 = 0.7, p_2 = 0.6$, $\alpha = 2.2, \lambda = 1.1$.

5.2 The Graph of Exposure Curve

The exposure curve of two dependent risks is obtained in Chapter 4 and the graph of curve is in Figure 5.2. It has four parameters α, λ, p_1 and p_2 and one variable d. They affect to the curve differently. We show that the curve is increasing and concave on the interval (0,1) and $G(0) = 0$ and $G(1) = 1$. They are all observed in Figure 5.2.

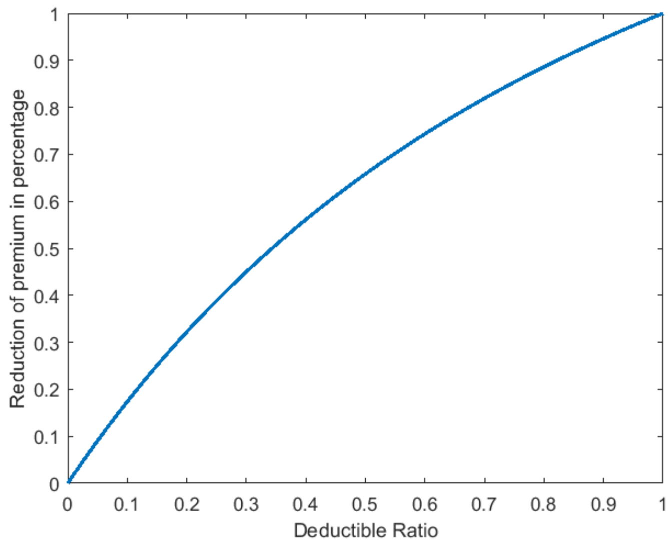


Figure 5.2: Exposure curve of two dependent risks when $p_1 = 0.7, p_2 = 0.6$, $\alpha = 2.5, \lambda = 1.2$.

5.3 Comparison of the Exposure Curves

We derive the exposure curves of the risks X_1 and X_2 from their probability distributions. We compare the exposure curve of Y to individual curves. We realize that even though the expectation values of the risks depend on the probabilities p_1 and p_2, the exposure curves do not depend on them and they are completely the same, as given

$$G_{X_1}(d) = \frac{\left(1+\dfrac{d}{\lambda}\right)^{-\alpha+1} - 1}{\left(1+\dfrac{1}{\lambda}\right)^{-\alpha+1} - 1},$$

$$G_{X_2}(d) = \frac{\left(1+\dfrac{d}{\lambda}\right)^{-\alpha+1} - 1}{\left(1+\dfrac{1}{\lambda}\right)^{-\alpha+1} - 1}.$$

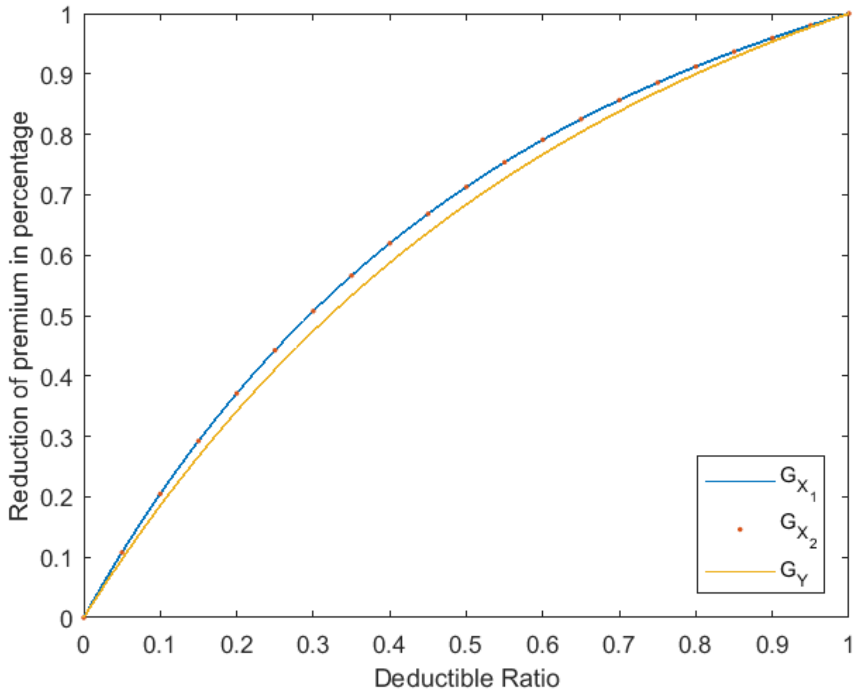
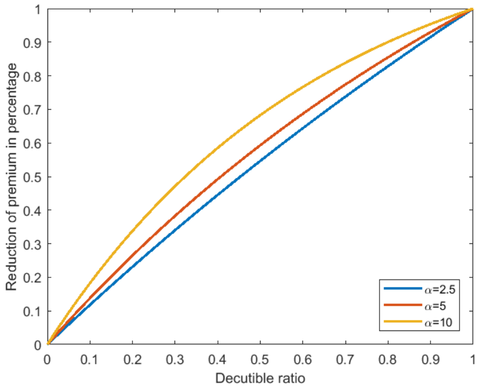


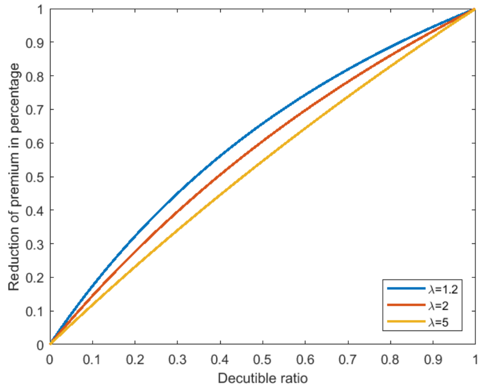
Figure 5.3: Individual and dependent exposure curves of two dependent risks when $p_1 = 0.7, p_2 = 0.6, \alpha = 5, \lambda = 2.3$.

When two risks become dependent, we expect that they create more risky portfolio. Figure 5.3 shows the exposure curve of Y, which is the sum of the dependent risks, is closer to diagonal than individual exposure curves of the risks X_1 and X_2. The location of exposure curve which belongs to more risky portfolio is close to diagonal portfolios, as mentioned in Chapter 3. Thus, the sum Y is more risky portfolio than others and this makes our prediction being justified. Furthermore, more risky portfolio means that almost every policy has a loss ratio up to d or more so the cedant transfers much more claims, and that is why insurer should keep less risk premium for itself and pay more to the reinsurer. Therefore, the fact that exposure curve of Y is placed under the individual exposure curves in Figure 5.3 makes sense.

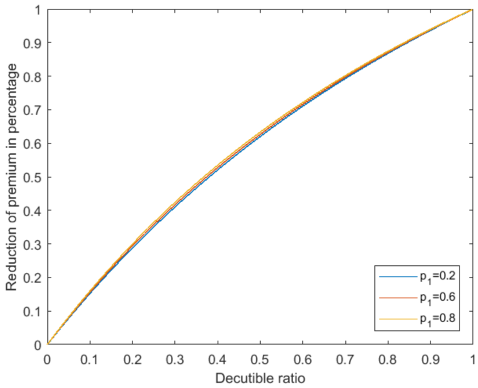
The concavity changes for different values of parameters. For examination this changing, we fix three parameters and give different values to the fourth parameter. Four graphs for four parameters are plotted in Figure 5.4(a-d).



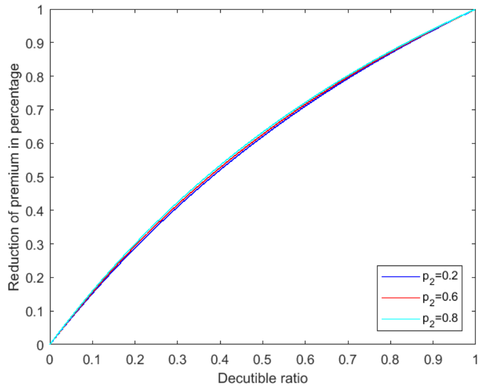
(a) $p_1 = 0.6, p_2 = 0.7, \lambda = 1.2$ for all curves with values of α=2, 2.5, 10.



(b) $p_1 = 0.6, p_2 = 0.7, \alpha = 2.5$ for all curves with values of λ=1.2, 2, 5.



(c) $p_2 = 0.7, \alpha = 2.5, \lambda = 1.6$ for all curves with values of p_1=0.2, 0.6, 0.8.



(d) $p_1 = 0.7, \alpha = 2.5, \lambda = 1.6$ for all curves with values of p_2=0.2, 0.6, 0.8.

Figure 5.4: Exposure curves with different values of parameters.

We observe in Figure 5.4 the parameters α and λ affect the curve more than p_1 and p_2. When α increases, the concavity also increases so $G(d)$ increases in Figure 5.4a. The increment in $G(d)$ means the curve is getting far away from the diagonal. Thus, it can be said that higher value for α brings less risky portfolio and many policies have loss ratios less than the deductible ratio d. For that reason, the cedant pays more claim and tends to keep more risk premium for itself. On the other hand, λ affects the curve inversely. When it increases, the value of the curve decreases and the curve is getting close to diagonal in Figure 5.4b. That means higher value for λ occurs more risky portfolio, and cedant transfers more claim, and it pays more risk premium to the reinsurer. The effect of the parameters p_1 and p_2 to the curve are the identical. $G(d)$ increases very slightly while they increase in Figure 5.4c and Figure 5.4d. We choose these parameters to take into consideration of the probabilities of the claims can be zero. It makes sense since higher p_1 or p_2 shows that the probability of the policy has a claim more than zero is lower and it points the less risky portfolio.

5.4 Sensitivity Analysis

5.4.1 Sensitivity for α

The parameter α affects the curve positively. We realize that the value of G increases when α increases. This is also observed in Table 5.1. For fixed deductible ratio $d = 0.5$ and given parameters $p_1 = 0.6, p_2 = 0.7, \lambda = 1.2$ and some values for α, changes in the values of α and G are obtained as follows:

Table 5.1: The sensitivity of the curves to an increasing value of α.

α	G(0.5)	$\Delta\alpha$	ΔG(0.5)
2.5	65.8257%	-	-
2.6	66.4315%	0.1	60.58x10^{-4}
2.7	67.0331%	0.1	60.16x10^{-4}
2.8	67.6306%	0.1	59.75x10^{-4}
2.9	68.2237%	0.1	59.31x10^{-4}
3.0	68.8119%	0.1	58.82x10^{-4}
5.0	79.3824%	2.0	1057.05x10^{-4}
10.0	94.3607%	5.0	1497.83x10^{-4}

This table shows G increases decreasingly. The increment about 0.1 in α leads to increase the value of curve as less than previous one on the same incremental amount. When it is plotted, this is easily observed this in Figure 5.5

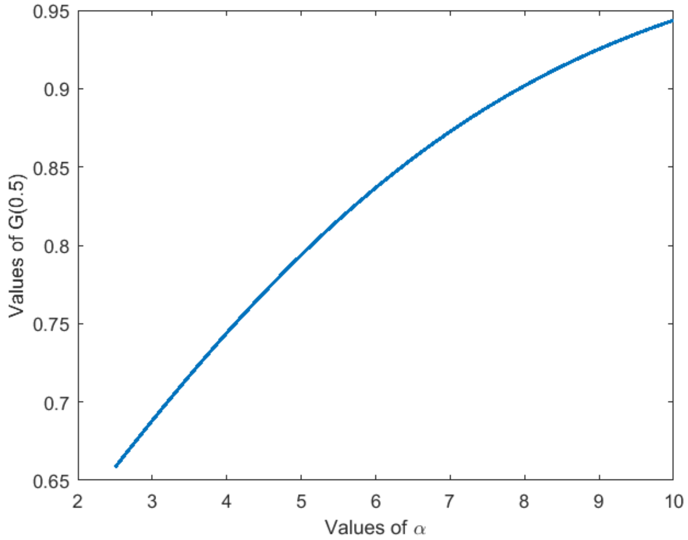


Figure 5.5: Sensitivity of α when $p_1 = 0.6, p_2 = 0.7, \lambda = 1.2$, $d = 0.5$.

5.4.2 Sensitivity for λ

The parameter λ affects the curve negatively. The value of G decreases when λ increases. The values of the result of the changes as amounts 0.01 and 0.1 in λ are given in Table 5.2 when the parameters are $p_1 = 0.6, p_2 = 0.7, \alpha = 1.2$ and fixed $d = 0.5$. According to Table 5.2, we notice that when λ increases, the value of curve decreases decreasingly.

Table 5.2: The sensitivity of the curves to an increasing value of λ.

λ	G(0.5)	$\Delta\lambda$	ΔG(0.5)
0.60	62.5260%	-	-
0.61	62.3947%	0.01	-1.313x10^{-3}
0.62	62.2663%	0.01	-1.284x10^{-3}
0.63	62.1405%	0.01	-1.258x10^{-3}
0.64	62.0175%	0.01	-1.230x10^{-3}
0.65	61.8967%	0.01	-1.208x10^{-3}
0.60	62.5260%	-	-
0.70	61.3286%	0.1	-11.974x10^{-3}
0.80	60.3421%	0.1	-9.865x10^{-3}
0.90	59.5149%	0.1	-8.272x10^{-3}
1.00	58.8110%	0.1	-7.039x10^{-3}
2.00	55.0670%	1.0	-37.44x10^{-3}
5.00	52.2263%	3.0	-28.407x10^{-3}
10.0	51.1504%	5.0	-10.759x10^{-3}
20.0	50.5848%	10.0	-5.656x10^{-3}

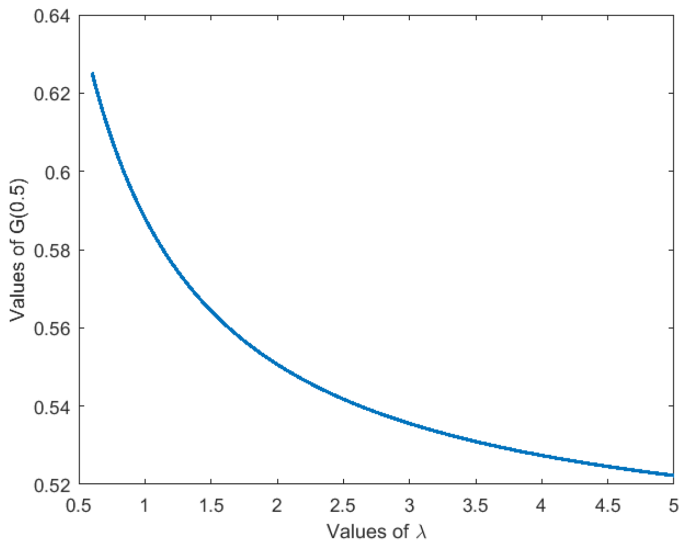


Figure 5.6: Sensitivity of λ when $p_1 = 0.6, p_2 = 0.7, \alpha = 1.2$, $d = 0.5$.

5.4.3 Sensitivity for p_1 and p_2

Both of the effects of the probabilities p_1 and p_2 are the same and positively. Therefore, the Table 5.3 and the Figure 5.7 represent for both parameters p_1 and p_2. For same incremental amounts in probabilities, G rises by increasingly and it is shown in Table 5.3 where one of the probabilities is equal to 0.7 and parameters $\lambda = 0.4, \alpha = 1.2$ and fixed $d = 0.5$.

Table 5.3: The sensitivity of the curves to an increasing values of p_1 or p_2.

p	G(0.5)	Δp	ΔG(0.5)
0.60	65.9143%	-	-
0.61	65.9338%	0.01	1.95x10^{-4}
0.62	65.9538%	0.01	2.00x10^{-4}
0.63	65.9744%	0.01	2.06x10^{-4}
0.64	65.9956%	0.01	2.12x10^{-4}
0.65	66.0172%	0.01	2.16x10^{-4}
0.60	65.9143%	-	-
0.70	66.1358%	0.1	2.215x10^{-3}
0.80	66.4374%	0.1	3.016x10^{-3}
0.90	66.8721%	0.1	4.347x10^{-3}
1.00	67.5527%	0.1	6.806x10^{-3}

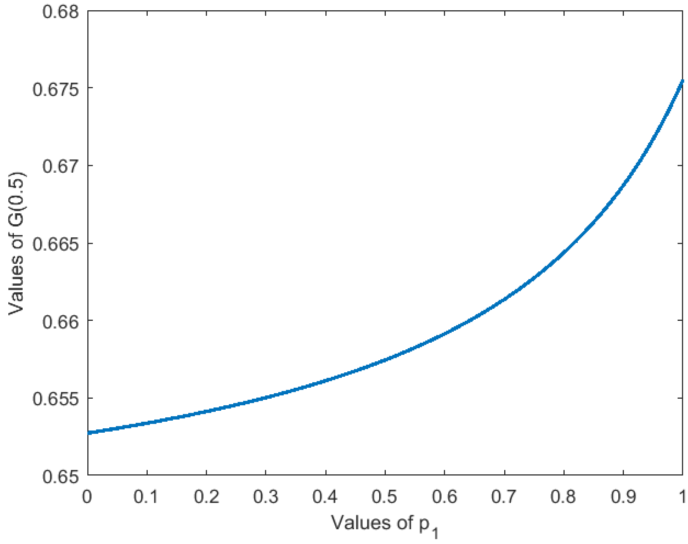


Figure 5.7: Sensitivity of p_1 when $p_2 = 0.7, \lambda = 0.4, \alpha = 1.2$, $d = 0.5$.

CHAPTER 6

CONCLUSION AND FUTURE WORK

6.1 Conclusion

We explain why reinsurance is needed and the benefits. The classification and basic components and rules of them are mentioned. The pricing methods are simplified for both proportional and non-proportional reinsurance. Experience rating and exposure rating techniques are implied and exposure curve is defined as one of a method of exposure rating. The idea behind the exposure curve is explained clearly. Different size of portfolios are given as examples in basically to reflect the the basis of construction of exposure curve. Inferences of Swiss Re curves, their analytical functions and pricing reinsurance by using Swiss Re curves are achieved using numerical examples.

We define the portfolio and distribution functions to approach the thesis's aim. We assume that we have two risk which are dependent in this portfolio. We use Bivariate Pareto Distributions method to create joint distribution of dependent risks. We proved that this distribution function satisfies the conditions of joint distribution function. We change the variables and the intervals of the joint distribution to construct the exposure curve practically and easily. Finally, we obtain the exposure curve of two dependent risks around five parameters α, λ, p_1, p_2 and d. Certain constrains are set to satisfy the existence of the joint probability density function and the curve itself to approach meaningful result. We observe that the value of curve changes when the parameters change. The increasing of α,p_1 and p_2 individually obtain an increment of the value of the curve. However, the increasing of λ leads to decrease the value of the curve.

We incept the idea of the exposure curve of two dependent risks. We show that creation the joint exposure curve under a certain dependency is possible in theoretical and may be easily implemented to practical use. The procedure may be applied for different line of businesses and claim distributions. We expect that the portfolio which has two dependent risks is more risky than each individual portfolio. The joint exposure curve proves that our expectation being justified. This result is observed as the conclusion of the comparison of the curves of dependent risks and individual risks.

6.2 Future work

This work is its first theoretical approach to derive joint distribution of exposure curves in positively correlated risks. The extension of this study is planned on the analytical analysis of loading factor under the same distributional approaches.

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