GRAVITATIONAL WAVES AND GRAVITATIONAL MEMORY

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ABSTRACT

GRAVITATIONAL WAVES AND GRAVITATIONAL MEMORY

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We study the gravitational waves produced by compact binary systems in the linear regime of massless general relativity and calculate the gravitational memory produced by these waves on a detector.

Keywords: gravitational waves, gravitational memory

ÖZ

YERÇEKİMSEL DALGALAR VE YERÇEKİMSEL HAFIZA

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Kompakt ikili sistemler tarafından üretilen kütleçekim dalgaları kütlesiz ve teoride doğrusal rejimde araştırılıp, bu dalgalar tarafından bir detektör üzerinde üretilen kütle çekimsel bellek hesaplanacaktır.

Anahtar Kelimeler: yerçekimsel dalgalar, yerçekimsel hafıza

Aileme

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CHAPTER 1

INTRODUCTION

In this thesis, we do not present new ideas or computations, we only follow the references and lay set detailed computations. So it is only a review of earlier published material. Especially the gravitational wave part follows closely the discussion of Maggiore [1], and the memory part based on the Tolish [2].

In the first chapter, we will find the Einstein equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu}$$

by using variation of the gravitational action $S = S_E + S_M$ where

$$S_E = \frac{c^3}{16\pi G}\int d^4x\sqrt{-g}R,$$

and

$$\delta S_M = \frac{1}{2c}\int d^4x\sqrt{-g}T^{\mu\nu}\delta g_{\mu\nu}$$

under metric change $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$. After that we will use the linearized theory by writing the metric such as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad |h_{\mu\nu}| << 1,$$

then we will get the linearized Einstein equations

$$\Box h_{\mu\nu} + \eta_{\mu\nu}\partial^\alpha\partial^\beta h_{\alpha\beta} - \partial^\alpha\partial_\nu h_{\mu\alpha} - \partial^\alpha\partial_\mu h_{\nu\alpha} + \partial_\nu\partial_\mu h - \eta_{\mu\nu}\Box h = \frac{-16\pi G}{c^4}T_{\mu\nu}.$$

Then, using Lorenz gauge

$$\partial^\nu \bar{h}_{\mu\nu} = 0,$$

we will get a simple wave equation

$$\Box\bar{h}_{\mu\nu} = \frac{-16\pi G}{c^4}T_{\mu\nu}$$

where

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h.$$

Next, we will use another gauge

$$h_{0\mu} = 0, \qquad h^i{}_i = 0, \qquad \partial^j h_{ij} = 0$$

which is known as transverse-traceless gauge in order to solve this wave equation. The solution of this wave equations describe the gravitational wave. Also, we will find the geodesic equation

$$\frac{d^2x^\beta}{d\tau^2} + \Gamma^\beta{}_{\alpha\mu}\frac{dx^\mu}{d\tau}\frac{dx^\alpha}{d\tau} = 0,$$

and geodesic deviation

$$\frac{D^2\xi^\mu}{D\tau^2} = -R^\mu{}_{\nu\rho\sigma}u^\nu u^\sigma \xi^\rho$$

which is an important equation to find the gravitational wave effect. Lastly, we will look at some physical effect of the gravitational wave.

In the second chapter, we will describe what memory effect is, and we will give our notation. In the third chapter, we will find the memory effect of some important fields. As a first example, we will find the solution of the scalar wave equation

$$\partial^a\partial_a\varphi = -4\pi S$$

where φ is a scalar field and S is a scalar charge distribution, then we can find the memory effect of the scalar fields. Secondly, we will find the solution of the electromagnetic wave equation

$$\partial^b\partial_b A^a = -4\pi J^a$$

where we have the Lorentz gauge

$$\partial_a A^a = 0,$$

and J^a is the electromagnetic current density in order to get electromagnetic memory.

Finally, we will use the our gravitational wave solution in order to prove that there is non-trivial gravitational memory effect

$$\begin{aligned}\triangle d^i(U) &= \int_{-\infty}^{U} dU' \int_{-\infty}^{U'} dU'' \frac{d^2 d^i}{dU''^2} \\ &= \frac{1}{r}\triangle_k{}^i d^k.\end{aligned}$$

in the 4-dimensional flat Minkowski space. To find this memory expression, we will solve the equation of the geodesic deviation.This result will be important for us since it is a solid proof of the general relativity.

CHAPTER 2

THE GEOMETRIC APPROACH TO GRAVITATIONAL WAVES

In this Chapter, we generally follow Maggiore [1] in order to understand Gravitational Waves.

Notation

Greek indices take the values 0,...,3 while the Latin letters, $i, j, ... = 1, 2, 3$ is used for the spatial indices. The 4-dimensional flat space metric is

$$\eta_{\mu\nu} = diag(-,+,+,+).$$

We also define

$$\begin{aligned} x^\mu &= (x^0, \vec{x}), \quad x^0 = ct, \\ \partial_\mu &= \frac{\partial}{\partial x^\mu} = (\frac{1}{c}\partial_t, \partial_i), \\ d^4x &= dx^0 d^3x = cdtd^3x. \end{aligned}$$

A dot refers to the time derivative such that $\dot{f}(t) = \partial_t f = c\partial_0 f$.

The four-momentum is defined as $p^\mu = (E/c, \vec{p})$, so $p_\mu x^\mu = -Et + \vec{p}.\vec{x}$, and $d^4p = (1/c)dEd^3p$. The Einstein summation rule with repeated upper and lower indices are summed over is used.

2.1 Expansion Around Minkowski Space

The gravitational action is given as $S = S_E + S_M$, where

$$S_E = \frac{c^3}{16\pi G}\int d^4x\sqrt{-g}R. \tag{2.1}$$

It is known as the Einstein-Hilbert action and S_M is called the matter action where g is the determinant of the metric. The energy-momentum tensor of matter, $T^{\mu\nu}$, is defined from the variation of matter action S_M under metric change $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$, according to

$$\delta S_M = \frac{1}{2c}\int d^4x\sqrt{-g}T^{\mu\nu}\delta g_{\mu\nu}. \tag{2.2}$$

In order to get the Einstein equations, we need to find the variation δS_E under the change of the metric which is

$$\delta S_E = \frac{c^3}{16\pi G}\Big[\int d^4x(\delta\sqrt{-g})R + \int d^4x\sqrt{-g}\delta R\Big] \tag{2.3}$$

by the product rule. If we use the definition of the Ricci scalar R which is

$$R = g^{\mu\nu}R_{\mu\nu},$$

then the variation of the Ricci scalar is

$$\delta R = (\delta g^{\mu\nu})R_{\mu\nu} + g^{\mu\nu}(\delta R_{\mu\nu}). \tag{2.4}$$

Using eqn.(2.4) in eqn.(2.3), we can get the variation of Einstein-Hilbert action

$$\delta S_E = \frac{c^3}{16\pi G}\Big[\int d^4x\delta(\sqrt{-g})R + \int d^4x\sqrt{-g}R_{\mu\nu}\delta g^{\mu\nu} + \int d^4x\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu}\Big]. \tag{2.5}$$

Now, let's split eqn.(2.5) into three parts,

$$\delta S_{E1} := \int d^4x\delta(\sqrt{-g})R, \tag{2.6}$$

$$\delta S_{E2} := \int d^4x\sqrt{-g}R_{\mu\nu}\delta g^{\mu\nu}, \tag{2.7}$$

$$\delta S_{E3} := \int d^4x\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu}. \tag{2.8}$$

First of all, let's find eqn.(2.8). By definition, the Ricci tensor is

$$R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu} = \partial_\alpha\Gamma^{\alpha}{}_{\mu\nu} - \partial_\nu\Gamma^{\alpha}{}_{\mu\alpha} + \Gamma^{\alpha}{}_{\lambda\alpha}\Gamma^{\lambda}{}_{\mu\nu} - \Gamma^{\alpha}{}_{\lambda\nu}\Gamma^{\lambda}{}_{\mu\alpha}, \tag{2.9}$$

so the variation of the Ricci tensor is

$$\begin{aligned}\delta R_{\mu\nu} = {} & \partial_\alpha\delta(\Gamma^{\alpha}{}_{\mu\nu}) - \partial_\nu\delta(\Gamma^{\alpha}{}_{\mu\alpha}) + \delta(\Gamma^{\alpha}{}_{\lambda\alpha})\Gamma^{\lambda}{}_{\mu\nu} + \Gamma^{\alpha}{}_{\lambda\alpha}\delta(\Gamma^{\lambda}{}_{\mu\nu}) \\ & - \delta(\Gamma^{\alpha}{}_{\lambda\nu})\Gamma^{\lambda}{}_{\mu\alpha} - \Gamma^{\alpha}{}_{\lambda\nu}\delta(\Gamma^{\lambda}{}_{\mu\alpha}).\end{aligned} \tag{2.10}$$

Let's rewrite eqn.(2.10), adding and subtracting the term $\Gamma^{\lambda}{}_{\nu\alpha}\delta(\Gamma^{\alpha}{}_{\lambda\mu})$,

$$\begin{aligned}\delta R_{\mu\nu} = {} & \big[\partial_\alpha\delta(\Gamma^{\alpha}{}_{\mu\nu}) + \Gamma^{\alpha}{}_{\lambda\alpha}\delta(\Gamma^{\lambda}{}_{\mu\nu}) - \Gamma^{\lambda}{}_{\mu\alpha}\delta(\Gamma^{\alpha}{}_{\lambda\nu}) - \Gamma^{\lambda}{}_{\nu\alpha}\delta(\Gamma^{\alpha}{}_{\lambda\mu})\big] \\ & - \big[\partial_\nu\delta(\Gamma^{\alpha}{}_{\mu\alpha}) + \Gamma^{\alpha}{}_{\lambda\nu}\delta(\Gamma^{\lambda}{}_{\mu\alpha}) - \Gamma^{\lambda}{}_{\mu\nu}\delta(\Gamma^{\alpha}{}_{\lambda\alpha}) - \Gamma^{\lambda}{}_{\nu\alpha}\delta(\Gamma^{\alpha}{}_{\lambda\mu})\big],\end{aligned} \tag{2.11}$$

and using the definition of the covariant derivative

$$\nabla_\alpha V^{\beta}{}_{\mu\nu} = \partial_\alpha V^{\beta}{}_{\mu\nu} + \Gamma^{\beta}{}_{\lambda\alpha}V^{\lambda}{}_{\mu\nu} - \Gamma^{\lambda}{}_{\mu\alpha}V^{\beta}{}_{\lambda\nu} - \Gamma^{\lambda}{}_{\nu\alpha}V^{\beta}{}_{\lambda\mu}, \tag{2.12}$$

one can write the variation of the Ricci tensor

$$\delta R_{\mu\nu} = \nabla_\alpha \delta({\Gamma^\alpha}_{\mu\nu}) - \nabla_\nu \delta({\Gamma^\alpha}_{\mu\alpha}). \tag{2.13}$$

Note that ${\Gamma^\mu}_{\alpha\beta}$ is not a tensor but $\delta({\Gamma^\mu}_{\alpha\beta})$ is a tensor, hence we know how the covariant derivative acts on it. If we use eqn.(2.13) in eqn.(2.8), we get

$$\begin{aligned}\delta S_{E(3)} &= \int d^4x\sqrt{-g}g^{\mu\nu}\left(\nabla_\alpha \delta{\Gamma^\alpha}_{\mu\nu} - \nabla_\nu \delta{\Gamma^\alpha}_{\mu\alpha}\right) \\ &= \int d^4x\sqrt{-g}\left[\nabla_\alpha (g^{\mu\nu}\delta{\Gamma^\alpha}_{\mu\nu}) - \nabla_\nu (g^{\mu\nu}\delta{\Gamma^\alpha}_{\mu\alpha})\right],\end{aligned}$$

where we use the fact that the covariant derivative of the Riemannian metric is zero since we have metric compatibility. If dummy indices α and ν are replaced with each other for the second term of the integral, we get

$$\begin{aligned}\delta S_{E(3)} &= \int d^4x\sqrt{-g}\left[\nabla_\alpha (g^{\mu\nu}\delta{\Gamma^\alpha}_{\mu\nu}) - \nabla_\alpha (g^{\mu\alpha}\delta{\Gamma^\nu}_{\mu\nu})\right] \\ &= \int d^4x\sqrt{-g}\nabla_\alpha J^\alpha. \end{aligned} \tag{2.14}$$

where $J^\alpha := g^{\mu\nu}\delta({\Gamma^\alpha}_{\mu\nu}) - g^{\mu\alpha}\delta({\Gamma^\nu}_{\mu\nu})$. Let J be a vector field over the region V with boundary ∂V, then using the Stokes' theorem:

$$\int_V d^4x\sqrt{-g}\nabla_\alpha J^\alpha = \int_{\partial V} d^3x\sqrt{|\gamma|}n_\alpha J^\alpha, \tag{2.15}$$

where n_α is the normal unit vector on the hyper surface ∂V, and $\sqrt{|\gamma|}$ is the integration measure for ∂V.

$$\delta S_{E(3)} = \int_{\partial V} d^3x\sqrt{|\gamma|}n_\alpha J^\alpha = 0, \tag{2.16}$$

using eqn.(2.15) in eqn.(2.14), $\delta S_{E(3)}$ is equal to a contribution which is zero by vanishing of variation at infinity.

Secondly, we need to know the variation of $\sqrt{-g}$ in order to find eqn.(2.6).

$$\delta(\sqrt{-g}) = -\frac{1}{2\sqrt{-g}}\delta g = -\frac{1}{2\sqrt{-g}}\frac{\partial g}{\partial g_{\lambda\nu}}\delta g_{\lambda\nu}. \tag{2.17}$$

It is known that one can write the inverse of metric $g_{\nu\lambda}$ such that

$$g^{\nu\lambda} = \frac{1}{g}(\tilde{g}^{\nu\lambda})^T = \frac{\tilde{g}^{\lambda\nu}}{g}$$

where $\tilde{g}^{\nu\lambda}$ is the cofactor of metric $g_{\nu\lambda}$.

$$\begin{aligned} gg^{\nu\lambda} &= \tilde{g}^{\lambda\nu}, \\ gg_{\lambda\nu}g^{\nu\lambda} &= g_{\lambda\nu}\tilde{g}^{\lambda\nu}, \\ g &= g_{\lambda\nu}\tilde{g}^{\lambda\nu}, \\ \Rightarrow \frac{\partial g}{\partial g_{\lambda\nu}} &= \tilde{g}^{\lambda\nu} = gg^{\nu\lambda}. \end{aligned} \tag{2.18}$$

Using eqn.(2.18) in eqn.(2.17), we get

$$\begin{aligned}\delta(\sqrt{-g}) &= -\frac{1}{2\sqrt{-g}} g g^{\nu\lambda}\delta g_{\lambda\nu} \\ &= \frac{1}{2}\sqrt{-g} g^{\nu\lambda}\delta g_{\lambda\nu} \\ &= \frac{1}{2}\sqrt{-g} g^{\mu\nu}\delta g_{\mu\nu}, \end{aligned} \tag{2.19}$$

and

$$\begin{aligned} g_{\mu\nu}g^{\mu\nu} &= 4 \\ \delta(g_{\mu\nu})g^{\mu\nu} + g_{\mu\nu}\delta(g^{\mu\nu}) &= 0 \\ g^{\mu\nu}\delta(g_{\mu\nu}) &= -g_{\mu\nu}\delta(g^{\mu\nu}). \end{aligned} \tag{2.20}$$

Also, if we use eqn.(2.20) in eqn.(2.19), then we get

$$\delta(\sqrt{-g}) = \frac{1}{2}\sqrt{-g} g_{\mu\nu}\delta g^{\mu\nu}. \tag{2.21}$$

Hence, eqn.(2.6) takes the following form

$$\delta S_{E(1)} = \int d^4x\big(-\frac{1}{2}\sqrt{-g} g_{\mu\nu}\delta g^{\mu\nu}\big). \tag{2.22}$$

As a result, eqn.(2.5) can be recast as

$$\begin{aligned}\delta S_E &= \frac{c^3}{16\pi G}\Big[-\frac{1}{2}\int d^4x\sqrt{-g} g_{\mu\nu} R\delta g^{\mu\nu} + \int d^4x\sqrt{-g} R_{\mu\nu}\delta g^{\mu\nu} + 0\Big] \\ &= \frac{c^3}{16\pi G}\int d^4x\big(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\big)\sqrt{-g}\delta g^{\mu\nu}. \end{aligned} \tag{2.23}$$

On the other hand, the variation of the matter action is

$$\delta S_M = \frac{1}{2c}\int d^4x\sqrt{-g} T^{\mu\nu}\delta g_{\mu\nu} \tag{2.24}$$

by the definition. We need to arrange eqn.(2.24) to find the variation of the action $S = S_E + S_M$. To do this, we can use the expression

$$\begin{aligned} g_{\alpha\beta} g^{\beta\nu} &= \delta_\alpha{}^\nu, \\ g_{\alpha\beta}\delta g^{\beta\nu} &= -\delta(g_{\alpha\beta}) g^{\beta\nu}, \\ g^{\mu\alpha} g_{\alpha\beta}\delta g^{\beta\nu} &= -g^{\mu\alpha}\delta(g_{\alpha\beta}) g^{\beta\nu}, \\ \delta^\mu{}_\beta\delta g^{\beta\nu} &= -g^{\mu\alpha}\delta(g_{\alpha\beta}) g^{\beta\nu}, \\ \delta g^{\mu\nu} &= -g^{\mu\alpha}\delta(g_{\alpha\beta}) g^{\beta\nu}. \end{aligned}$$

Then, we have

$$\begin{aligned} T_{\mu\nu}\delta g^{\mu\nu} &= -T_{\mu\nu} g^{\mu\alpha}\delta(g_{\alpha\beta}) g^{\beta\nu} \\ &= -T^{\alpha\beta}\delta g_{\alpha\beta} \end{aligned}$$

$$= -T^{\mu\nu}\delta g_{\mu\nu}. \tag{2.25}$$

Plugging eqn.(2.25) in eqn.(2.24), we can write the variation of the matter action as

$$\delta S_M = -\frac{1}{2c}\int d^4x\sqrt{-g}T_{\mu\nu}\delta g^{\mu\nu}. \tag{2.26}$$

The variation of the action $S = S_E + S_M$ vanishes under change of metric,

$$\begin{aligned}\delta S &= \delta S_E + \delta S_M \\ 0 &= \frac{c^3}{16\pi G}\int d^4x\sqrt{-g}\big(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\big)\delta g^{\mu\nu} - \frac{1}{2c}\int d^4x\sqrt{-g}T_{\mu\nu}\delta g^{\mu\nu} \\ &= \frac{c^3}{16\pi G}\int d^4x\sqrt{-g}\big[R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \frac{8\pi G}{c^4}T_{\mu\nu}\big]\delta g^{\mu\nu}.\end{aligned}$$

As a result, the Einstein equation is found by taking the variation of the total action with respect to metric $g_{\mu\nu}$

$$\Rightarrow R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu}. \tag{2.27}$$

Let us discuss the symmetry of general relativity. It is invariant under a big symmetry group. Let

$$x^\mu \to x'^\mu(x) \tag{2.28}$$

refer to all possible coordinate transformations, where x'^μ is arbitrary smooth function of the coordinate x^μ. In other words, it is a diffeomorphism. The metric transforms as

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}g_{\rho\sigma}(x) \tag{2.29}$$

under the coordinate transformations (2.28).

We will expand the the Einstein equation around the flat-space metric. To do this, we can firstly write the metric as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad |h_{\mu\nu}| << 1, \tag{2.30}$$

and then we expand the equation of the motion up to linear order in $h_{\mu\nu}$. This theory is known as *linearized theory*.

Consider coordinate transformations defined as

$$x^\mu \to x'^\mu(x) = x^\mu + \xi^\mu(x), \tag{2.31}$$

where $|\partial_\mu \xi_\nu|$ has the at most same order of smallness as $h_{\mu\nu}$. Let's use this transformations in eqn.(2.29). In order to do this one needs to find the inverse transformation

$$x^\rho = x'^\rho - \xi^\mu(x), \tag{2.32}$$

$$
\begin{aligned}
\frac{\partial x^\rho}{\partial x'^\mu} &= \frac{\partial x'^\rho}{\partial x'^\mu} - \frac{\partial \xi^\rho(x)}{\partial x'^\mu}, \\
&= \delta^\rho{}_\mu - \partial_\alpha \xi^\rho \frac{\partial x^\alpha}{\partial x'^\mu}, \\
&= \delta^\rho{}_\mu - \partial_\alpha \xi^\rho [\delta^\alpha{}_\mu - \partial_\lambda \xi^\alpha \frac{\partial x^\lambda}{\partial x'^\mu}], \\
&= \delta^\rho{}_\mu - \partial_\mu \xi^\rho + O(|\partial_\mu \xi^\rho|^2).
\end{aligned} \tag{2.33}
$$

Now, if we use eqn.(2.33) with appropriate indices in eqn.(2.29), then we have

$$
\begin{aligned}
g'_{\mu\nu} &= (\delta^\rho{}_\mu - \partial_\mu \xi^\rho)(\delta^\sigma{}_\nu - \partial_\nu \xi^\sigma)(\eta_{\rho\sigma} + h_{\rho\sigma}) \\
&= (\delta^\rho{}_\mu - \partial_\mu \xi^\rho)(\eta_{\rho\nu} + h_{\rho\nu} - \partial_\nu \xi_\rho - \partial_\nu \xi^\sigma h_{\rho\sigma}) \\
&= \eta_{\mu\nu} + h_{\mu\nu} - \partial_\nu \xi_\mu - \partial_\mu \xi_\nu.
\end{aligned} \tag{2.34}
$$

Hence, the term of perturbation transforms as

$$
h_{\mu\nu} \to h'_{\mu\nu} = h_{\mu\nu} - (\partial_\nu \xi_\mu + \partial_\mu \xi_\nu), \tag{2.35}
$$

under the transformation (2.31). As a special case, we can consider constant translations, i.e. $x^\mu \to x'^\mu = x^\mu + a^\mu$, where a^μ is not infinitesimal but can be finite. We can see that $h_{\mu\nu}$ is invariant, if we look at eqn.(2.34). Since,

$$
g'_{\mu\nu} = \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} [\eta_{\rho\sigma} + h_{\rho\sigma}] = \delta^\rho_\mu \delta^\sigma_\nu [\eta_{\rho\sigma} + h_{\rho\sigma}] = \eta_{\mu\nu} + h_{\mu\nu}(x). \tag{2.36}
$$

In addition, we can look at the finite, global Lorentz transformations

$$
x^\mu = \Lambda^\mu{}_\nu x^\nu, \tag{2.37}
$$

and by definition

$$
\Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma \eta_{\rho\sigma} = \eta_{\mu\nu}. \tag{2.38}
$$

Lorentz transformation of the metric is

$$
\begin{aligned}
g_{\mu\nu}(x) \to g'_{\mu\nu}(x') &= \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma g_{\rho\sigma}(x) \\
&= \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma [\eta_{\rho\sigma} + h_{\rho\sigma}] \\
&= \eta_{\mu\nu} + \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma h_{\rho\sigma}(x).
\end{aligned} \tag{2.39}
$$

So we have $g'_{\mu\nu} = \eta_{\mu\nu} + h'_{\mu\nu}$, with

$$
h'_{\mu\nu} = \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma h_{\rho\sigma}(x). \tag{2.40}
$$

It means that $h_{\mu\nu}$ is a tensor under global Lorentz transformations.

Now, let's find the linearization of the Riemann tensor at the linear order in $h_{\mu\nu}$

$$
R^\mu{}_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu{}_{\nu\sigma} - \partial_\sigma \Gamma^\mu{}_{\nu\rho} + \Gamma^\mu{}_{\alpha\rho} \Gamma^\alpha{}_{\nu\sigma} - \Gamma^\mu{}_{\alpha\sigma} \Gamma^\alpha{}_{\nu\rho}, \tag{2.41}
$$

and Christoffel symbol

$$\Gamma^{\rho}{}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}[\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}] \tag{2.42}$$

by the definition [11]. First we need to use the linearized metric in eqn.(2.42),

$$\begin{aligned} \Gamma^{\rho}{}_{\mu\nu} &= \frac{1}{2} g^{\rho\sigma}[\partial_\mu(\eta_{\sigma\nu} + h_{\sigma\nu}) + \partial_\nu(\eta_{\sigma\mu} + h_{\sigma\mu}) - \partial_\sigma(\eta_{\mu\nu} + h_{\mu\nu})] \\ &= \frac{1}{2} g^{\rho\sigma}[\partial_\mu h_{\sigma\nu} + \partial_\nu h_{\sigma\mu} - \partial_\sigma h_{\mu\nu}], \end{aligned} \tag{2.43}$$

and we need to find the inverse metric $g^{\rho\sigma}$ using $g_{\mu\rho} g^{\rho\sigma} = \delta_\mu{}^\sigma$. Suppose that $g^{\rho\sigma} = a\eta^{\rho\sigma} + bh^{\rho\sigma}$. Hence,

$$(\eta^{\mu\rho} + h^{\mu\rho})(a\eta^{\rho\sigma} + bh^{\rho\sigma}) = a\delta_\mu{}^\rho + bh_\mu{}^\sigma(x) + ah_\mu{}^\sigma(x) + b|h|^2 = \delta_\mu{}^\rho$$

$$\Rightarrow a = 1 a + b = 0 \Rightarrow b = -a \Rightarrow b = -1,$$

the inverse metric is

$$g^{\rho\sigma} = \eta^{\rho\sigma} - h^{\rho\sigma}(x). \tag{2.44}$$

Using eqn.(2.44) in eqn.(2.43), we get

$$\begin{aligned} \Gamma^{\rho}{}_{\mu\nu} &= \frac{1}{2}(\eta^{\rho\sigma} - h^{\rho\sigma})[\partial_\mu h_{\sigma\nu} + \partial_\nu h_{\sigma\mu} - \partial_\sigma h_{\mu\nu})] \\ &= \frac{1}{2}\eta^{\rho\sigma}[\partial_\mu h_{\sigma\nu} + \partial_\nu h_{\sigma\mu} - \partial_\sigma h_{\mu\nu}] + |h|^2. \end{aligned} \tag{2.45}$$

Using eqn.(2.45) with appropriate indices, one can find the linearized Riemann tensor

$$\begin{aligned} R^{\mu}{}_{\nu\rho\sigma} &= \partial_\rho[\frac{1}{2}\eta^{\mu\alpha}(\partial_\nu h_{\alpha\sigma} + \partial_\sigma h_{\alpha\nu} - \partial_\alpha h_{\nu\sigma})] - \partial_\sigma[\frac{1}{2}\eta^{\mu\alpha}(\partial_\nu h_{\alpha\rho} + \partial_\rho h_{\alpha\nu} - \partial_\alpha h_{\nu\rho})] \\ &+ \frac{1}{2}\eta^{\mu\alpha}[\partial_\lambda h_{\alpha\rho} + \partial_\rho h_{\alpha\lambda} - \partial_\alpha h_{\lambda\rho}]\frac{1}{2}\eta^{\lambda\alpha}[\partial_\nu h_{\alpha\sigma} + \partial_\sigma h_{\alpha\nu} - \partial_\alpha h_{\nu\sigma}] \\ &- \frac{1}{2}\eta^{\mu\alpha}[\partial_\lambda h_{\alpha\sigma} + \partial_\sigma h_{\alpha\lambda} - \partial_\alpha h_{\lambda\rho}]\frac{1}{2}\eta^{\lambda\alpha}[\partial_\nu h_{\alpha\rho} + \partial_\rho h_{\alpha\nu} - \partial_\alpha h_{\nu\rho}] \\ R^{\mu}{}_{\nu\rho\sigma} &= \frac{1}{2}\eta^{\mu\alpha}[\partial_\rho\partial_\nu h_{\alpha\sigma} + \cancel{\partial_\rho\partial_\sigma h_{\alpha\nu}} - \partial_\rho\partial_\alpha h_{\nu\sigma}] \\ &- \frac{1}{2}\eta^{\mu\alpha}[\partial_\sigma\partial_\nu h_{\alpha\rho} + \cancel{\partial_\sigma\partial_\rho h_{\alpha\nu}} - \partial_\sigma\partial_\alpha h_{\nu\rho}] + |h|^2 \\ &= \frac{1}{2}\eta^{\mu\alpha}[\partial_\rho\partial_\nu h_{\alpha\sigma} + \partial_\sigma\partial_\alpha h_{\nu\rho} - \partial_\rho\partial_\alpha h_{\nu\sigma} - \partial_\sigma\partial_\nu h_{\alpha\rho}] \\ g_{\gamma\mu}R^{\mu}{}_{\nu\rho\sigma} &= (\eta_{\gamma\mu} + h_{\gamma\mu})\frac{1}{2}\eta^{\mu\alpha}[\partial_\rho\partial_\nu h_{\alpha\sigma} + \partial_\sigma\partial_\alpha h_{\nu\rho} - \partial_\rho\partial_\alpha h_{\nu\sigma} - \partial_\sigma\partial_\nu h_{\alpha\rho}] \\ R_{\gamma\nu\rho\sigma} &= \frac{1}{2}\delta_\gamma{}^\alpha[\partial_\rho\partial_\nu h_{\alpha\sigma} + \partial_\sigma\partial_\alpha h_{\nu\rho} - \partial_\rho\partial_\alpha h_{\nu\sigma} - \partial_\sigma\partial_\nu h_{\alpha\rho}] + |h|^2 \\ &= \frac{1}{2}[\partial_\rho\partial_\nu h_{\gamma\sigma} + \partial_\sigma\partial_\gamma h_{\nu\rho} - \partial_\rho\partial_\gamma h_{\nu\sigma} - \partial_\sigma\partial_\nu h_{\gamma\rho}] + |h|^2 \\ \Rightarrow R_{\mu\nu\rho\sigma} &= \frac{1}{2}[\partial_\rho\partial_\nu h_{\mu\sigma} + \partial_\sigma\partial_\mu h_{\nu\rho} - \partial_\rho\partial_\mu h_{\nu\sigma} - \partial_\sigma\partial_\nu h_{\mu\rho}]. \end{aligned} \tag{2.46}$$

Plugging eqn.(2.31) into eqn.(2.46), one gets

$$\begin{aligned}
R'_{\mu\nu\rho\sigma} &= \frac{1}{2}\{\partial_\rho\partial_\nu[h_{\mu\sigma}-(\partial_\mu\xi_\sigma+\partial_\sigma\xi_\mu)]+\partial_\sigma\partial_\mu[h_{\nu\rho}-(\partial_\nu\xi_\rho+\partial_\rho\xi_\nu)] \\
&\quad -\partial_\rho\partial_\mu[h_{\nu\sigma}-(\partial_\nu\xi_\sigma+\partial_\sigma\xi_\nu)]-\partial_\sigma\partial_\nu[h_{\mu\rho}-(\partial_\mu\xi_\rho+\partial_\rho\xi_\mu)]\} \\
&= \frac{1}{2}[\partial_\rho\partial_\nu h_{\mu\sigma}+\partial_\sigma\partial_\mu h_{\nu\rho}-\partial_\rho\partial_\mu h_{\nu\sigma}-\partial_\sigma\partial_\nu h_{\mu\rho}] \\
&\quad -\frac{1}{2}[\cancel{\partial_\rho\partial_\nu\partial_\mu\xi_\sigma}+\cancel{\partial_\rho\partial_\nu\partial_\sigma\xi_\mu}+\cancel{\partial_\sigma\partial_\mu\partial_\nu\xi_\rho}+\cancel{\partial_\sigma\partial_\mu\partial_\rho\xi_\nu} \\
&\quad -\cancel{\partial_\rho\partial_\mu\partial_\nu\xi_\sigma}-\cancel{\partial_\rho\partial_\mu\partial_\sigma\xi_\nu}-\cancel{\partial_\sigma\partial_\nu\partial_\rho\xi_\rho}-\cancel{\partial_\sigma\partial_\nu\partial_\rho\xi_\mu}] \\
&= R_{\mu\nu\rho\sigma},
\end{aligned}$$

which means that the linearized Riemann tensor is invariant under infinitesimal gauge transformations (2.31).

Defining,

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h, \tag{2.47}$$

where

$$h = \eta^{\mu\nu}h_{\mu\nu}, \tag{2.48}$$

the linearized Einstein equations can be written more compactly. In addition, it is easy to see that $\bar{h} \equiv \eta^{\mu\nu}\bar{h}_{\mu\nu} = \eta^{\mu\nu}(h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h) = -h$. Therefore,

$$\begin{aligned}
h_{\mu\nu} &= \bar{h}_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}h, \\
&= \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h}.
\end{aligned} \tag{2.49}$$

Now we can compute the Einstein equations. First we will find $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$,

$$\begin{aligned}
R^{\mu}{}_{\nu\rho\sigma} &= \frac{1}{2}\eta^{\mu\alpha}\{\partial_\rho\partial_\nu h_{\alpha\sigma}+\partial_\sigma\partial_\alpha h_{\nu\rho}-\partial_\rho\partial_\alpha h_{\nu\sigma}-\partial_\sigma\partial_\nu h_{\alpha\rho}\}, \\
R_{\nu\sigma} = R^{\mu}{}_{\nu\mu\sigma} &= \frac{1}{2}\eta^{\mu\alpha}\{\partial_\mu\partial_\nu h_{\alpha\sigma}+\partial_\sigma\partial_\alpha h_{\nu\mu}-\partial_\mu\partial_\alpha h_{\nu\sigma}-\partial_\sigma\partial_\nu h_{\alpha\mu}\}, \\
R_{\mu\nu} &= \frac{1}{2}\eta^{\sigma\alpha}\{\partial_\sigma\partial_\mu h_{\alpha\nu}+\partial_\nu\partial_\alpha h_{\mu\sigma}-\partial_\sigma\partial_\alpha h_{\mu\nu}-\partial_\nu\partial_\mu h_{\alpha\sigma}\},
\end{aligned} \tag{2.50}$$

and

$$\begin{aligned}
R = \eta^{\mu\nu}R_{\mu\nu} &= \frac{1}{2}\eta^{\mu\nu}\{\partial^\alpha\partial_\mu h_{\alpha\nu}+\partial_\nu\partial_\alpha h_\mu{}^\alpha-\partial^\alpha\partial_\alpha h_{\mu\nu}-\partial_\nu\partial_\mu h_\alpha{}^\alpha\}, \\
&= \frac{1}{2}\{\partial^\alpha\partial_\mu h_\alpha{}^\mu+\partial^\nu\partial_\alpha h_\mu{}^\alpha-\Box h_\mu{}^\mu-\partial^\mu\partial_\mu h\} \\
&= \frac{1}{2}\{2\partial_\alpha\partial_\mu h^{\mu\alpha}-2\Box h\} \\
&= \partial_\alpha\partial_\mu h^{\mu\alpha}-\Box h.
\end{aligned} \tag{2.51}$$

Then the Einstein tensor becomes

$$
\begin{aligned}
G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \\
&= \frac{1}{2}\eta^{\sigma\alpha}\{\partial_\sigma\partial_\mu h_{\alpha\nu} + \partial_\nu\partial_\alpha h_{\mu\sigma} - \partial_\sigma\partial_\alpha h_{\mu\nu} - \partial_\nu\partial_\mu h_{\alpha\sigma}\} - \frac{1}{2}\eta_{\mu\nu}(\partial_\alpha\partial_\alpha h^{\sigma\alpha} - \Box h) \\
&= \frac{1}{2}\{\partial^\alpha\partial_\mu h_{\alpha\nu} + \partial_\nu\partial_\alpha h_\mu{}^\alpha - \Box h_{\mu\nu} - \partial_\nu\partial_\mu h - \eta_{\mu\nu}\partial_\sigma\partial_\alpha h^{\sigma\alpha} - \eta_{\mu\nu}\Box h\} \\
&= \frac{1}{2}\{-\Box h_{\mu\nu} + \partial^\alpha\partial_\nu h_{\mu\alpha} + \partial^\alpha\partial_\mu h_{\nu\alpha} - \partial_\nu\partial_\mu h - \partial_\alpha\partial_\beta h^{\alpha\beta}\eta_{\mu\nu} + \eta_{\mu\nu}\Box h\}.
\end{aligned}
\tag{2.52}
$$

From $G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$, we have the linearized Einstein equation

$$
\Box h_{\mu\nu} + \eta_{\mu\nu}\partial^\alpha\partial^\beta h_{\alpha\beta} - \partial^\alpha\partial_\nu h_{\mu\alpha} - \partial^\alpha\partial_\mu h_{\nu\alpha} + \partial_\nu\partial_\mu h - \eta_{\mu\nu}\Box h = \frac{-16\pi G}{c^4} T_{\mu\nu}.
\tag{2.53}
$$

To use definition (2.47), we will add and subtract some suitable terms to eqn.(2.53) such that,

$$
\begin{aligned}
\frac{-16\pi G}{c^4} T_{\mu\nu} &= \Box(h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h) - \frac{1}{2}\eta_{\mu\nu}\Box h + \eta_{\mu\nu}\partial^\alpha\partial^\beta(h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}h) \\
&\quad + \frac{1}{2}\eta_{\mu\nu}\partial^\alpha\partial^\beta\eta_{\alpha\beta}h - \partial^\alpha\partial_\nu(h_{\mu\alpha} - \frac{1}{2}\eta_{\mu\alpha}h) - \frac{1}{2}\partial^\alpha\partial_\nu\eta_{\mu\alpha}h \\
&\quad - \partial^\alpha\partial_\mu(h_{\nu\alpha} - \frac{1}{2}\eta_{\nu\alpha}h) - \frac{1}{2}\partial^\alpha\partial_\mu\eta_{\nu\alpha}h + \partial_\nu\partial_\mu h \\
&= \Box\bar{h}_{\mu\nu} + \eta_{\mu\nu}\partial^\alpha\partial^\beta\bar{h}_{\alpha\beta} - \partial^\alpha\partial_\nu\bar{h}_{\mu\alpha} - \partial^\alpha\partial_\mu\bar{h}_{\nu\alpha} \\
&\quad - \cancel{\frac{1}{2}\eta_{\mu\nu}}\Box h + \cancel{\frac{1}{2}\partial_\beta\partial^\beta} h - \cancel{\frac{1}{2}\partial_\mu}\partial_\nu h - \cancel{\frac{1}{2}\partial_\nu}\partial_\mu h + \cancel{\partial_\nu\partial_\mu h},
\end{aligned}
$$

$$
\Rightarrow \Box\bar{h}_{\mu\nu} + \eta_{\mu\nu}\partial^\alpha\partial^\beta\bar{h}_{\alpha\beta} - \partial^\alpha\partial_\nu\bar{h}_{\mu\alpha} - \partial^\alpha\partial_\mu\bar{h}_{\nu\alpha} = \frac{-16\pi G}{c^4} T_{\mu\nu}.
\tag{2.54}
$$

Up to this point, we have not made a choice of gauge. But we can do this to simplify these equations. We will choose the *Lorenz* gauge

$$
\partial^\nu\bar{h}_{\mu\nu} = 0.
\tag{2.55}
$$

Now, let's use the gauge symmetry (2.31),

$$
\begin{aligned}
\bar{h}_{\mu\nu} \to \bar{h}'_{\mu\nu} &= \bar{h}_{\mu\nu} - (\partial_\mu\xi_\nu + \partial_\nu\xi_\mu) - \frac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta}(h_{\alpha\beta} - \partial_\alpha\xi_\beta - \partial_\beta\xi_\alpha) \\
&= (h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h) - (\partial_\mu\xi_\nu + \partial_\nu\xi_\mu - \eta_{\mu\nu}\partial_\beta\xi^\beta) \\
&= \bar{h}_{\mu\nu} - (\partial_\mu\xi_\nu + \partial_\nu\xi_\mu - \eta_{\mu\nu}\partial_\beta\xi^\beta),
\end{aligned}
$$

so

$$
\partial^\nu\bar{h}_{\mu\nu} \to (\partial^\nu\bar{h}_{\mu\nu})' = \partial^\nu\bar{h}_{\mu\nu} - (\cancel{\partial^\nu\partial_\mu\xi_\nu} + \partial^\nu\partial_\nu\xi_\mu - \cancel{\eta_{\mu\nu}\partial^\nu\partial_\beta\xi^\beta})
$$

$$= \partial^{\nu}\bar{h}_{\mu\nu} - \Box\xi_{\mu}, \tag{2.56}$$

to prove the existence of the Lorenz gauge. If we choose the initial field configuration as $\partial^{\nu}\bar{h}_{\mu\nu} = f_{\mu}(x)$, we must choose,

$$\Box\xi_{\mu} = f_{\mu}(x), \tag{2.57}$$

to get $(\partial^{\nu}\bar{h}_{\mu\nu})' = 0$. Then, the solution of eqn.(2.56) is

$$\xi_{\mu}(x) = \int d^4xG(x-y)f_{\mu}(y), \tag{2.58}$$

where $G(x)$ is a Green's function of the d'Alembertian operator such that

$$\Box_x G(x-y) = \delta^4(x-y). \tag{2.59}$$

If we use the Lorenz gauge in our main eqn.(2.54), the last three terms on the left hand side will vanish, then we get the simple wave equation

$$\Box\bar{h}_{\mu\nu} = \frac{-16\pi G}{c^4}T_{\mu\nu}. \tag{2.60}$$

Now, let us note the following observations:

$\boldsymbol{i}$)the gauge condition (2.55) gives four conditions which reduce the ten independent components of the symmetric 4x4 matrix $h_{\mu\nu}$ to six independent components,

$\boldsymbol{ii}$)if we take the derivative of the wave equation, we will get an expression,

$$\begin{aligned} \partial^{\nu}(\Box\bar{h}_{\mu\nu}) &= \partial^{\nu}(\frac{-16\pi G}{c^4}T_{\mu\nu}) \\ \Box\partial^{\nu}\bar{h}_{\mu\nu} &= \frac{-16\pi G}{c^4}\partial^{\nu}T_{\mu\nu} \\ &\Rightarrow \partial^{\nu}T_{\mu\nu} = 0, \end{aligned} \tag{2.61}$$

which is the conservation of the energy-momentum in the linearized theory.

2.2 The transverse-traceless gauge

Firstly, we will analyze eqn.(2.60) outside the source where the energy-momentum tensor is zero:

$$\Box\bar{h}_{\mu\nu} = 0. \tag{2.62}$$

There is an important result which is the fact that Gravitational waves travel at the speed of light, because the definition of d'Alembertian $\Box = -(\frac{1}{c^2})\partial_0^2 + \nabla^2$. Under transformation (2.31), the Lorenz gauge is not spoiled if

$$\Box\xi_{\mu} = 0. \tag{2.63}$$

Then, $\Box\xi_{\mu\nu} = 0$, where

$$\xi_{\mu\nu} \equiv \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu}\partial_\rho \xi^\rho, \tag{2.64}$$

because the flat space d'Alembertian always commutes with partial derivatives. If the $\xi_{\mu\nu}$ which depends on ξ_μ is subtracted from $\bar{h}_{\mu\nu}$, then the result will satisfy the same equation $\Box(\bar{h}_{\mu\nu} - \xi_{\mu\nu}) = 0$. Taking a suitable value of ξ^0, we can make $\bar{h} = 0$, then it is obvious that $\bar{h}_{\mu\nu} = h_{\mu\nu}$. Also, choosing suitable ξ^i we can make $h^{0i} = 0$. Then the Lorentz gauge for $\mu = 0$,

$$\begin{aligned} \partial^0 h_{00} + \partial^i h_{0i} &= 0, \\ \Rightarrow \partial^0 h_{00} &= 0, \\ \Rightarrow h_{00} &= const. \end{aligned} \tag{2.65}$$

The time independent term h_{00} is the static part of gravitational interaction that is the Newtonian potential of the source which generates the gravitational wave. We can take the $h_{00} = 0$. So, $h_{0\mu} = 0$, then the Lorenz gauge becomes $\partial^j h_{ij} = 0$, and $h^i{}_i = 0$. As a result,

$$h_{0\mu} = 0, \qquad h^i{}_i = 0, \qquad \partial^j h_{ij} = 0. \tag{2.66}$$

This is called the transverse-traceless gauge (TT gauge).

Using the Lorenz gauge which gives four conditions, $h_{\mu\nu}$ has 10-4=6 independent components. Lastly, we impose the infinitesimal gauge to eqn.(2.63) which gives us four conditions; therefore, the independent components of the $h_{\mu\nu}$ is reduced from 6 to 2. We will denote the metric in the TT gauge by h_{ij}^{TT}.

The plane wave solution of eqn.(2.62) is $h_{ij}^{TT} = e_{ij}(\vec{k})e^{ikx}$ with $k^\mu = (\frac{\omega}{c}, \vec{k})$ and $\frac{\omega}{c} = |\vec{k}|$. The tensor $e_{ij}(\vec{k})$ is known as the polarization tensor. From eqn.(2.66), the non-zero components of the h_{ij}^{TT} are in the plane transverse to $\hat{n} = \frac{\vec{k}}{|\vec{k}|}$ since on plane wave $\partial^j h_{ij} = 0 \Rightarrow n^j h_{ij} = 0$. Let's choose the $\hat{n} = \hat{z}$, and use the symmetry and traceless conditions of h_{ij}^{TT}, we can write the solution as

$$h_{ij}^{TT}(t,z) = \begin{pmatrix} h_+ & h_x & 0 \\ h_x & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij} \cos[\omega(t - \frac{z}{c})], \tag{2.67}$$

or,

$$h_{ab}^{TT}(t,z) = \begin{pmatrix} h_+ & h_x \\ h_x & -h_+ \end{pmatrix}_{ab} \cos[\omega(t - \frac{z}{c})], \tag{2.68}$$

where $a,b = 1,2$. h_+ and h_x are called the amplitudes of the "plus" and "cross" polarizations of the wave. Then the metric is

$$\begin{aligned} ds^2 = -c^2dt^2 + dz^2 + \{1 + h_+ \cos[\omega(t - \frac{z}{c})]\}dx^2 \\ + \{1 - h_+ \cos[\omega(t - \frac{z}{c})]\}dy^2 + 2h_x \cos[\omega(t - \frac{z}{c})]dxdy. \end{aligned} \tag{2.69}$$

The plane wave solution $h_{\mu\nu}(x)$ propagating in the direction $\hat{n}$, outside the source, follows the Lorenz gauge but it is not suitable for TT gauge, yet. To make it suitable for TT gauge, we will firstly define the tensor

$$P_{ij}(\hat{n}) = \delta_{ij} - n_i n_j, \tag{2.70}$$

which is symmetric, is transverse($n^i P_{ij} = 0$), is a projector ($P_{ik}P_{kj} = P_{ij}$), and its trace is 2. Using the tensor (2.70), we define

$$\Lambda_{ij,kl} = P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl}. \tag{2.71}$$

The question is if this is still a projector. The answer is yes since

$$\begin{aligned} \Lambda_{ij,kl}\Lambda_{kl,mn} &= (P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl})(P_{km}P_{ln} - \frac{1}{2}P_{kl}P_{mn}) \\ &= P_{ik}P_{jl}P_{km}P_{ln} - \frac{1}{2}P_{ik}P_{jl}P_{kl}P_{mn} - \frac{1}{2}P_{ij}P_{kl}P_{km}P_{ln} + \frac{1}{4}P_{ij}P_{kl}P_{kl}P_{mn} \\ &= P_{ik}P_{jl}P_{km}P_{ln} - \frac{1}{2}P_{ik}P_{jk}P_{mn} - \frac{1}{2}P_{ij}P_{lm}P_{ln} + \frac{1}{4}.2P_{ij}P_{mn} \\ &= P_{ik}P_{jl}P_{km}P_{ln} - \frac{1}{2}P_{ij}P_{mn} - \frac{1}{2}P_{ij}P_{mn} + \frac{1}{2}P_{ij}P_{mn} \\ &= P_{ik}P_{jl}P_{km}P_{ln} - \frac{1}{2}P_{ij}P_{mn}, \end{aligned} \tag{2.72}$$

and eqn.(2.70) implies that

$$P_{ik}P_{jl} = (\delta_{ik} - n_i n_k)(\delta_{jl} - n_j n_l) = P_{jl}P_{ik}. \tag{2.73}$$

Using eqn.(2.73) in eqn.(2.72), we get

$$\begin{aligned} \Lambda_{ij,kl}\Lambda_{kl,mn} &= P_{ik}P_{km}P_{jl}P_{ln} - \frac{1}{2}P_{ij}P_{mn} \\ &= P_{im}P_{jn} - \frac{1}{2}P_{ij}P_{mn} \\ &= \Lambda_{ij,mn}. \end{aligned} \tag{2.74}$$

Also, it is transverse for all indices, $n^i\Lambda_{ij,kl} = 0$,etc., and it is traceless for the (i,j) and (k,l) indices,

$$n^i\Lambda_{ij,kl} = n^i P_{ik}P_{jl} - -\frac{1}{2}n^i P_{ij}P_{kl} = 0,$$

$$\Lambda_{ii,kl} = P_{ik}P_{il} - \frac{1}{2}P_{ii}P_{kl} = 0,$$

and it is symmetric under the exchange $(i,j) \leftrightarrow (k,l)$. In terms of $\hat{n}$, its explicit form is

$$\begin{aligned}
\Lambda_{ij,kl} &= P_{ik}P_{jl} - -\frac{1}{2}P_{ij}P_{kl} \\
&= (\delta_{ik} - n_i n_k)(\delta_{jl} - n_j n_l) - \frac{1}{2}(\delta_{ij} - n_i n_j)(\delta_{kl} - n_k n_l) \\
&= \delta_{ik}\delta_{jl} - \delta_{ik}n_j n_l - n_i n_k \delta_{jl} + n_i n_k n_j n_l - \frac{1}{2}\delta_{ij}\delta_{kl} \\
&\quad + \frac{1}{2}n_k n_l \delta_{ij} + \frac{1}{2}n_i n_j \delta_{kl} - \frac{1}{2}n_i n_j n_k n_l \\
&= \delta_{ik}\delta_{jl} - \frac{1}{2}\delta_{ij}\delta_{kl} - n_j n_l \delta_{ik} - n_i n_k \delta_{jl} + \frac{1}{2}n_k n_l \delta_{ij} + \frac{1}{2}n_i n_j \delta_{kl} \\
&\quad + \frac{1}{2}n_i n_j n_k n_l.
\end{aligned} \tag{2.75}$$

It is called the Lambda tensor. In general, any symmetric tensor can be written as

$$S_{ij}^{TT} = \Lambda_{ij,kl}S_{kl}, \tag{2.76}$$

in the TT gauge. Hence, the gravitational wave is given by

$$h_{ij}^{TT} = \Lambda_{ij,kl}h_{kl}. \tag{2.77}$$

We know that in the TT gauge, the equation of motion is $\Box h_{ij}^{TT} = 0$, so

$$h_{ij}^{TT}(x) = \int \frac{d^3k}{(2\pi)^3}(A_{ij}(\vec{k})e^{ikx} + A_{ij}^*(\vec{k})e^{-ikx}). \tag{2.78}$$

Since, $k^\mu = (\frac{\omega}{c}, \vec{k})$ and $|\vec{k}| = \frac{\omega}{c} = \frac{2\pi f}{c}$, then $d^3k = |\vec{k}|^2 d|\vec{k}| d\Omega = (\frac{2\pi}{c})^3 f^2 df d\Omega$, with $f > 0$. If we denote the $d^2\hat{n} = d\cos\theta d\phi$, then

$$h_{ij}^{TT}(x) = \frac{1}{c^3}\int_0^\infty df f^2 \int d^2\hat{n}(A_{ij}(f,\hat{n})e^{-2\pi i f(t-\frac{\hat{n}.\vec{x}}{c})} + c.c.). \tag{2.79}$$

Because of the TT gauge condition,

$$h^i{}_i^{TT}(x) = \frac{1}{c^3}\int_0^\infty df f^2 \int d^2\hat{n}(A^i{}_i(\vec{k})e^{-2\pi i f(t-\frac{\hat{n}.\vec{x}}{c})} + c.c.) = 0$$

which implies that $A^i{}_i(\vec{k}) = 0$, and

$$\partial^i h_{ij}^{TT}(x) = k^i h_{ij}^{TT}(x) = \frac{1}{c^3}\int_0^\infty df f^2 \int d^2\hat{n}(k^i A_{ij}(\vec{k})e^{-2\pi i f(t-\frac{\hat{n}.\vec{x}}{c})} + c.c.) = 0$$

and also this leads to $k^i A_{ij}(\vec{k}) = 0$. For simplicity, we can omit the superscript TT by using the $a,b = 1,2$ indices for the TT gauge metric in the transverse plane. Then

$$h_{ab}(t,\vec{x}) = \int_0^\infty df\Big(\tilde{h}_{ab}(f,\vec{x})e^{-2\pi i f t} + \tilde{h}_{ab}{}^*(f,\vec{x})e^{2\pi i f t}\Big), \tag{2.80}$$

where

$$\tilde{h}_{ab}(f,\vec{x}) = \frac{f^2}{c^3}\int d^2\hat{n}A_{ab}(f,\hat{n})e^{2\pi i f\frac{\hat{n}.\vec{x}}{c}}. \tag{2.81}$$

If we observe on Earth a gravitational wave emitted by a single source, we can define the direction of propagation of the wave $\hat{n_0}$, and we can write

$$A_{ij}(\vec{k}) = A_{ij}(f)\delta^{(2)}(\hat{n} - \hat{n_0}) \tag{2.82}$$

Using eqn.(2.82) in eqn.(2.81) we get

$$\tilde{h}_{ab}(f,\vec{x}) = \frac{f^2}{c^3}A_{ab}(f)e^{2\pi i f\frac{\hat{n_0}.\vec{x}}{c}}. \tag{2.83}$$

For the detectors, we have

$$e^{2\pi i f\frac{\hat{n}.\vec{x}}{c}} = e^{i\frac{\hat{n}.\vec{x}}{\lambda}} \cong 1, \tag{2.84}$$

all over the detector with choosing the origin of the coordinate system centered on the detector, because $\hat{n}.\vec{x} << \lambda$. When we want to look at the gravitational wave at the detector location, we can omit all x-dependences and write

$$h_{ab}(t) = \int_0^\infty df\Big(\tilde{h}_{ab}(f)e^{-2\pi i f t} + \tilde{h}_{ab}{}^*(f)e^{2\pi i f t}\Big), \tag{2.85}$$

with $\tilde{h}_{ab}(f) = \tilde{h}_{ab}(f,\vec{x}=0)$. We don't have to keep x-dependence unless we compare the gravitational wave signal at two different detectors.

From eqn.(2.68)

$$\tilde{h}_{ab}(t) = \begin{pmatrix} \tilde{h}_+ & \tilde{h}_x \\ \tilde{h}_x & -\tilde{h}_+ \end{pmatrix}_{ab}. \tag{2.86}$$

If we rotate by an angle ψ the system of axes for their definition,

$$\begin{pmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{pmatrix}\begin{pmatrix} h_+ & h_x \\ h_x & -h_+ \end{pmatrix}\begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix}$$

$$= \begin{pmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{pmatrix}\begin{pmatrix} h_+\cos\psi - h_x\sin\psi & h_+\sin\psi + h_x\cos\psi \\ h_x\cos\psi + h_+\sin\psi & h_+\cos\psi - h_x\sin\psi \end{pmatrix}$$

$$= \begin{pmatrix} h_+c^2\psi - h_xs\psi c\psi - h_xs\psi c\psi - h_+s^2\psi & h_+c\psi s\psi + h_xc^2\psi - h_xs^2\psi + h_+s\psi c\psi \\ h_+c\psi s\psi + h_xc^2\psi - h_xs^2\psi + h_+s\psi c\psi & h_+s^2\psi + h_xs\psi c\psi + h_xs\psi c\psi - h_+c^2\psi \end{pmatrix}$$

$$= \begin{pmatrix} h_+\cos2\psi - h_x\sin2\psi & h_+\sin2\psi + h_x\cos2\psi \\ h_+\sin2\psi + h_x\cos2\psi & -(h_+\cos2\psi - h_x\sin2\psi) \end{pmatrix},$$

then h_+ and h_x transform as

$$h_+ \to h_+\cos2\psi - h_x\sin2\psi, \tag{2.87}$$

$$h_x \to h_+\sin2\psi + h_x\cos2\psi. \tag{2.88}$$

Until now, we have looked at the only physical frequencies $f > 0$, but now we can rewrite eqn.(2.80) by defining

$$\tilde{h}_{ab}(-f,\vec{x}) = \tilde{h}_{ab}{}^{*}(f,\vec{x}), \tag{2.89}$$

then

$$h_{ab}(t) = \int_{-\infty}^{\infty} df \tilde{h}_{ab}(f) e^{-2\pi i f t}, \tag{2.90}$$

and the inversion of eqn.(2.90) is

$$\tilde{h}_{ab}(f) = \int_{-\infty}^{\infty} dt h_{ab}(t) e^{2\pi i f t}. \tag{2.91}$$

Also, we can use the polarization tensor $e^{A}_{ij}(\hat{n})$ (with $A = +, x$) to write the plane wave expansion, making the definition

$$e^{+}_{ij}(\hat{n}) = \hat{u}_i\hat{u}_j - \hat{v}_i\hat{v}_j, \qquad e^{x}_{ij}(\hat{n}) = \hat{u}_i\hat{v}_j + \hat{v}_i\hat{u}_j \tag{2.92}$$

with $\hat{u}\perp\hat{n}$,$\hat{v}\perp\hat{n}$, and $\hat{u}\perp\hat{v}$ where $\hat{u}$ and $\hat{v}$ are the unit vectors. It is obvious from this definition that

$$e^{A}_{ij}(\hat{n}) e^{A',ij}(\hat{n}) = 2\delta^{AA'}. \tag{2.93}$$

As a special case, in the frame $\hat{n} = \hat{z}$, if $\hat{u} = \hat{x}$ and $\hat{v} = \hat{y}$ then

$$e^{+}_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad e^{x}_{ab} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.94}$$

with $a, b = 1, 2$ spanning the (x, y) plane. In a generic frame, if we define

$$\frac{f^2}{c^3} A_{ij}(f,\hat{n}) = \sum_{A=+,x} \tilde{h}_A(f,\hat{n}) e^{A}_{ij}(\hat{n}), \tag{2.95}$$

then eqn.(2.79) reads

$$h_{ab}(t,\vec{x}) = \sum_{A=+,x} \int_{-\infty}^{\infty} df \int d^2\hat{n} \tilde{h}_A(f,\hat{n}) e^{A}_{ab}(\hat{n}) e^{-2\pi i f (t - \frac{\hat{n}.\vec{x}}{c})}, \tag{2.96}$$

where we used the definition (2.89).

2.3 Interaction of gravitational waves with test masses

2.3.1 Geodesic equation and geodesic deviation

Let $x^{\mu}(\lambda)$ be a curve in some reference frame parametrized by a parameter λ. The interval ds is

$$ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu} = g_{\mu\nu} \frac{dx^{\mu}}{d\lambda} \frac{dx^{\nu}}{d\lambda} d\lambda^2. \tag{2.97}$$

There are some cases such that

$i)$ if $ds^2 > 0$, then it is known as a space-like curve,

$ii)$ if $ds^2 < 0$, then it is known as a time-like curve.

$iii)$ if $ds^2 = 0$, then it is known as a light-like (null) curve.

In the second case, we can define the proper time τ,

$$c^2 d\tau^2 = -ds^2 = -g_{\mu\nu} dx^\mu dx^\nu. \tag{2.98}$$

The proper time τ is the time which is measured by a clock which goes along this trajectory. So we can use τ as a parameter, $x^\mu = x^\mu(\tau)$. From eqn.(2.98), we have

$$g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = -c^2. \tag{2.99}$$

The four-velocity u^μ is defined as

$$u^\mu = \frac{dx^\mu}{d\tau}, \tag{2.100}$$

then using this definition, we can rewrite eqn.(2.99) as

$$g_{\mu\nu} u^\mu u^\nu = -c^2. \tag{2.101}$$

For all time-like curves which have the value on the boundary $x^\mu(\tau_A) = x^\mu{}_A$ and $x^\mu(\tau_B) = x^\mu{}_B$, the action is

$$S = -m \int_{\tau_A}^{\tau_B} d\tau. \tag{2.102}$$

Its variation gives us the trajectory of a point-like test mass m. To show this we will start to use eqn.(2.98) which gives us

$$d\tau = \frac{1}{c} \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda. \tag{2.103}$$

If we use eqn.(2.103) in the action (2.102), we will get

$$S = -\frac{m}{c} \int_{\tau_A}^{\tau_B} \sqrt{-g_{\mu\nu} d\dot{x}^\mu d\dot{x}^\nu} d\lambda, \tag{2.104}$$

where $d\dot{x}^\mu = \frac{dx^\mu}{d\lambda}$. The variation of the action is

$$\begin{aligned} \delta S &= -\frac{m}{c} \int_{\tau_A}^{\tau_B} \delta(\sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}) d\lambda \\ &= -\frac{m}{c} \int_{\tau_A}^{\tau_B} \frac{1}{2\sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}} [(-\delta g_{\mu\nu}) \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} - 2 g_{\mu\nu} \frac{d(\delta x^\mu)}{d\lambda} \frac{dx^\nu}{d\lambda}] d\lambda. \end{aligned}$$

The last term can be rewritten as

$$-2 g_{\mu\nu} \frac{d(\delta x^\mu)}{d\lambda} \frac{dx^\nu}{d\lambda} = \frac{d}{d\lambda} [-2 g_{\mu\nu} \delta x^\mu \frac{dx^\nu}{d\lambda}] + 2 \frac{d g_{\mu\nu}}{d\lambda} \delta x^\mu \frac{dx^\nu}{d\lambda} + 2 g_{\mu\nu} \delta x^\mu \frac{d^2 x^\nu}{d\lambda^2}. \tag{2.105}$$

Then using this, we have

$$\delta S = -\frac{m}{2c}\int_{\tau_A}^{\tau_B} \frac{1}{\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}}\{(-\delta g_{\mu\nu})\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} + \frac{d}{d\lambda}[-2g_{\mu\nu}\delta x^\mu\frac{dx^\nu}{d\lambda}]$$

$$+2\frac{dg_{\mu\nu}}{d\lambda}\delta x^\mu\frac{dx^\nu}{d\lambda} + 2g_{\mu\nu}\delta x^\mu\frac{d^2x^\nu}{d\lambda^2}\}d\lambda.$$

Using the fact that $\frac{d\tau}{d\lambda} = \frac{1}{c}\sqrt{(-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda})}$, the variation of the action boils down to

$$\delta S = -\frac{m}{2c^2}\int_{\tau_A}^{\tau_B} d\lambda\frac{d\lambda}{d\tau^2}d\tau\{(-\delta g_{\mu\nu})\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} + 2\frac{dg_{\mu\nu}}{d\lambda}\delta x^\mu\frac{dx^\nu}{d\lambda} + 2g_{\mu\nu}\delta x^\mu\frac{d^2x^\nu}{d\lambda^2}\}$$

$$= -\frac{m}{2c^2}\int_{\tau_A}^{\tau_B} d\tau\{(-\delta g_{\mu\nu})\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} + 2\frac{dg_{\mu\nu}}{d\tau}\delta x^\mu\frac{dx^\nu}{d\tau} + 2g_{\mu\nu}\delta x^\mu\frac{d^2x^\nu}{d\tau^2}\}$$

$$= -\frac{m}{2c^2}\int_{\tau_A}^{\tau_B} d\tau\{-\frac{\partial g_{\mu\nu}}{\partial x^\alpha}\delta x^\alpha\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} + 2\frac{dg_{\mu\nu}}{dx^\alpha}\frac{dx^\alpha}{d\tau}\frac{dx^\nu}{d\tau}\delta x^\mu + 2g_{\mu\nu}\frac{d^2x^\nu}{d\tau^2}\delta x^\mu\}$$

$$= -\frac{m}{2c^2}\int_{\tau_A}^{\tau_B} d\tau\{-\partial_\alpha g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\delta x^\alpha + \frac{dg_{\mu\nu}}{dx^\alpha}\frac{dx^\alpha}{d\tau}\frac{dx^\nu}{d\tau}\delta x^\mu + \frac{dg_{\mu\nu}}{dx^\alpha}\frac{dx^\alpha}{d\tau}\frac{dx^\nu}{d\tau}\delta x^\mu$$

$$+2g_{\mu\nu}\frac{d^2x^\nu}{d\tau^2}\delta x^\mu\}.$$

If we change some dummy indices such that in the first term $\alpha \leftrightarrow \nu$, in the second term $\mu \leftrightarrow \nu$, in the third term $\alpha \to \mu, \mu \to \nu, \nu \to \alpha$, and in the last term $\mu \leftrightarrow \nu$ then we have

$$\delta S = -\frac{m}{2c^2}\int_{\tau_A}^{\tau_B} d\tau\{-\partial_\nu g_{\mu\alpha}\frac{dx^\mu}{d\tau}\frac{dx^\alpha}{d\tau}\delta x^\nu + \partial_\alpha g_{\nu\mu}\frac{dx^\alpha}{d\tau}\frac{dx^\mu}{d\tau}\delta x^\nu + \partial_\mu g_{\nu\alpha}\frac{dx^\mu}{d\tau}\frac{dx^\alpha}{d\tau}\delta x^\nu$$

$$+2g_{\nu\mu}\frac{d^2x^\mu}{d\tau^2}\delta x^\nu\}$$

$$= -\frac{m}{2c^2}\int_{\tau_A}^{\tau_B} d\tau\{(-\partial_\nu g_{\mu\alpha} + \partial_\alpha g_{\nu\mu} + \partial_\mu g_{\nu\alpha})\frac{dx^\mu}{d\tau}\frac{dx^\alpha}{d\tau} + 2g_{\nu\mu}\frac{d^2x^\mu}{d\tau^2}\}\delta x^\nu$$

$$= -\frac{m}{c^2}\int_{\tau_A}^{\tau_B} d\tau\{\frac{1}{2}(-\partial_\nu g_{\mu\alpha} + \partial_\alpha g_{\nu\mu} + \partial_\mu g_{\nu\alpha})\frac{dx^\mu}{d\tau}\frac{dx^\alpha}{d\tau} + g_{\nu\mu}\frac{d^2x^\mu}{d\tau^2}\}\delta x^\nu.$$

Since the $\delta S = 0$, we get

$$g_{\nu\mu}\frac{d^2x^\mu}{d\tau^2} + \frac{1}{2}(-\partial_\nu g_{\mu\alpha} + \partial_\alpha g_{\nu\mu} + \partial_\mu g_{\nu\alpha})\frac{dx^\mu}{d\tau}\frac{dx^\alpha}{d\tau} = 0. \tag{2.106}$$

Multiplying with the inverse metric $g^{\beta\nu}$,

$$\delta^\beta{}_\mu\frac{d^2x^\mu}{d\tau^2} + \frac{1}{2}g^{\beta\nu}(-\partial_\nu g_{\mu\alpha} + \partial_\alpha g_{\nu\mu} + \partial_\mu g_{\nu\alpha})\frac{dx^\mu}{d\tau}\frac{dx^\alpha}{d\tau} = 0, \tag{2.107}$$

we get

$$\frac{d^2x^\beta}{d\tau^2} + \Gamma^\beta{}_{\alpha\mu}\frac{dx^\mu}{d\tau}\frac{dx^\alpha}{d\tau} = 0 \tag{2.108}$$

where we use the definition of the Christoffel symbol $\Gamma^\beta{}_{\alpha\mu}$ [5]. This equation is known as the geodesic equation, and it is the equation of motion of a test mass in the space-time which has the metric $g_{\mu\nu}$ when there is no external forces. In terms of four-velocity which has been defined in eqn.(2.100), geodesic equation takes the form

$$\frac{du^{\beta}}{d\tau} + \Gamma^{\beta}{}_{\alpha\mu} u^{\mu} u^{\alpha} = 0. \tag{2.109}$$

Let's look at the two nearby geodesics, one parametrized by $x^{\mu}(\tau)$ and the other by $x^{\mu}(\tau) + \xi^{\mu}(\tau)$ where $|\xi^{\mu}|$ is much smaller than the scale of the variation of gravitational field. We know $x^{\mu}(\tau)$ satisfies the geodesic eqn.(2.108), and $x^{\mu}(\tau) + \xi^{\mu}(\tau)$ satisfies the "nearby" geodesic equation [5]

$$\frac{d^2(x^{\mu} + \xi^{\mu})}{d\tau^2} + \Gamma^{\mu}{}_{\nu\rho}(x+\xi) \frac{d(x^{\nu} + \xi^{\nu})}{d\tau} \frac{d(x^{\rho} + \xi^{\rho})}{d\tau} = 0. \tag{2.110}$$

Let's use the Taylor series expansion for the $\Gamma^{\mu}{}_{\nu\rho}(x+\xi)$ which is

$$\Gamma^{\mu}{}_{\nu\rho}(x+\xi) = \Gamma^{\mu}{}_{\nu\rho}(x) + \xi^{\sigma} \partial_{\sigma} \Gamma^{\mu}{}_{\nu\rho}(x) + O(|\xi|^2)$$

then using this in eqn.(2.110)

$$\begin{aligned}
0 &= \frac{d^2x^{\mu}}{d\tau^2} + \frac{d^2\xi^{\mu}}{d\tau^2} + [\Gamma^{\mu}{}_{\nu\rho}(x) + \xi^{\sigma}\partial_{\sigma}\Gamma^{\mu}{}_{\nu\rho}(x)][\frac{dx^{\nu}}{d\tau}\frac{dx^{\rho}}{d\tau} + \frac{dx^{\nu}}{d\tau}\frac{d\xi^{\rho}}{d\tau} + \frac{d\xi^{\nu}}{d\tau}\frac{dx^{\rho}}{d\tau} + \cancel{\frac{d\xi^{\nu}}{d\tau}\frac{d\xi^{\rho}}{d\tau}}] \\
&= \frac{d^2x^{\mu}}{d\tau^2} + \frac{d^2\xi^{\mu}}{d\tau^2} + \Gamma^{\mu}{}_{\nu\rho}(x)\frac{dx^{\nu}}{d\tau}\frac{dx^{\rho}}{d\tau} + \Gamma^{\mu}{}_{\nu\rho}(x)\frac{dx^{\nu}}{d\tau}\frac{d\xi^{\rho}}{d\tau} + \Gamma^{\mu}{}_{\nu\rho}(x)\frac{d\xi^{\nu}}{d\tau}\frac{dx^{\rho}}{d\tau} \\
&\quad + \xi^{\sigma}\partial_{\sigma}\Gamma^{\mu}{}_{\nu\rho}(x)[\frac{dx^{\nu}}{d\tau}\frac{dx^{\rho}}{d\tau} + \cancel{\frac{dx^{\nu}}{d\tau}\frac{d\xi^{\rho}}{d\tau}} + \cancel{\frac{d\xi^{\nu}}{d\tau}\frac{dx^{\rho}}{d\tau}}] \\
&= \frac{d^2x^{\mu}}{d\tau^2} + \Gamma^{\mu}{}_{\nu\rho}(x)\frac{dx^{\nu}}{d\tau}\frac{dx^{\rho}}{d\tau} + \frac{d^2\xi^{\mu}}{d\tau^2} + \Gamma^{\mu}{}_{\nu\rho}(x)\frac{dx^{\nu}}{d\tau}\frac{d\xi^{\rho}}{d\tau} + \Gamma^{\mu}{}_{\nu\rho}(x)\frac{d\xi^{\nu}}{d\tau}\frac{dx^{\rho}}{d\tau} \\
&\quad + \xi^{\sigma}\partial_{\sigma}\Gamma^{\mu}{}_{\nu\rho}(x)\frac{dx^{\nu}}{d\tau}\frac{dx^{\rho}}{d\tau}.
\end{aligned}$$

In the last equation, the first two terms will vanish if we use eqn.(2.108), one has

$$0 = \frac{d^2\xi^{\mu}}{d\tau^2} + 2\Gamma^{\mu}{}_{\nu\rho}(x)\frac{dx^{\nu}}{d\tau}\frac{d\xi^{\rho}}{d\tau} + \xi^{\sigma}\partial_{\sigma}\Gamma^{\mu}{}_{\nu\rho}(x)\frac{dx^{\nu}}{d\tau}\frac{dx^{\rho}}{d\tau}. \tag{2.111}$$

By the definition of the covariant derivative, we know that

$$\frac{DV^{\mu}}{D\tau} = \frac{dV^{\mu}}{d\tau} + \Gamma^{\mu}{}_{\nu\rho}V^{\nu}\frac{dx^{\rho}}{d\tau}. \tag{2.112}$$

Then, we have

$$\begin{aligned}
\frac{D^2\xi^{\mu}}{D\tau^2} &= \frac{D}{D\tau}(\frac{D\xi^{\mu}}{D\tau}) = \frac{D}{D\tau}(\frac{d\xi^{\mu}}{d\tau} + \Gamma^{\mu}{}_{\nu\rho}\xi^{\nu}\frac{dx^{\rho}}{d\tau}) \\
&= \frac{d}{d\tau}(\frac{d\xi^{\mu}}{d\tau} + \Gamma^{\mu}{}_{\nu\rho}\xi^{\nu}\frac{dx^{\rho}}{d\tau}) + +\Gamma^{\mu}{}_{\nu\rho}(\frac{d\xi^{\nu}}{d\tau} + \Gamma^{\nu}{}_{\alpha\beta}\xi^{\alpha}\frac{dx^{\beta}}{d\tau})\frac{dx^{\rho}}{d\tau} \\
&= \frac{d^2\xi^{\mu}}{d\tau^2} + \frac{d\Gamma^{\mu}{}_{\nu\rho}}{d\tau}\xi^{\nu}\frac{dx^{\rho}}{d\tau} + \Gamma^{\mu}{}_{\nu\rho}\frac{d\xi^{\nu}}{d\tau}\frac{dx^{\rho}}{d\tau} + \Gamma^{\mu}{}_{\nu\rho}\xi^{\nu}\frac{d^2x^{\rho}}{d\tau^2} + \Gamma^{\mu}{}_{\nu\rho}\frac{d\xi^{\nu}}{d\tau}\frac{dx^{\rho}}{d\tau} \\
&\quad + \Gamma^{\mu}{}_{\nu\rho}\Gamma^{\nu}{}_{\alpha\beta}\xi^{\alpha}\frac{dx^{\beta}}{d\tau}\frac{dx^{\rho}}{d\tau}.
\end{aligned} \tag{2.113}$$

On the other hand, we can use

$$\frac{d\Gamma^{\mu}{}_{\nu\rho}}{d\tau} = \frac{\partial \Gamma^{\mu}{}_{\nu\rho}}{\partial x^{\sigma}} \frac{dx^{\sigma}}{d\tau} = \partial_{\sigma}\Gamma^{\mu}{}_{\nu\rho} \frac{dx^{\sigma}}{d\tau},$$

and

$$\frac{d^2x^{\rho}}{d\tau^2} + \Gamma^{\rho}{}_{\alpha\beta} \frac{dx^{\alpha}}{d\tau}\frac{dx^{\beta}}{d\tau} = 0 \Rightarrow \frac{d^2x^{\rho}}{d\tau^2} = -\Gamma^{\rho}{}_{\alpha\beta} \frac{dx^{\alpha}}{d\tau}\frac{dx^{\beta}}{d\tau},$$

in eqn.(2.113),

$$\begin{aligned} \frac{D^2\xi^{\mu}}{D\tau^2} = {} & \frac{d^2\xi^{\mu}}{d\tau^2} + \partial_{\sigma}\Gamma^{\mu}{}_{\nu\rho}u^{\sigma}u^{\rho}\xi^{\nu} + \Gamma^{\mu}{}_{\nu\rho}\frac{d\xi^{\nu}}{d\tau}u^{\rho} - \Gamma^{\mu}{}_{\nu\rho}\Gamma^{\rho}{}_{\alpha\beta}u^{\alpha}u^{\beta}\xi^{\nu} + \Gamma^{\mu}{}_{\nu\rho}\frac{d\xi^{\nu}}{d\tau}u^{\rho} \\ & + \Gamma^{\mu}{}_{\nu\rho}\Gamma^{\nu}{}_{\alpha\beta}u^{\beta}u^{\rho}\xi^{\alpha}. \end{aligned} \tag{2.114}$$

From eqn.(2.111) we have

$$\frac{d^2\xi^{\mu}}{d\tau^2} = -2\Gamma^{\mu}{}_{\nu\rho}\frac{dx^{\nu}}{d\tau}\frac{d\xi^{\rho}}{d\tau} - \partial_{\sigma}\Gamma^{\mu}{}_{\nu\rho}u^{\nu}u^{\rho}\xi^{\sigma},$$

using this in eqn.(2.114), we obtain

$$\begin{aligned} \frac{D^2\xi^{\mu}}{D\tau^2} = {} & \cancel{-2\Gamma^{\mu}{}_{\nu\rho}u^{\nu}\frac{d\xi^{\rho}}{d\tau}} - \partial_{\sigma}\Gamma^{\mu}{}_{\nu\rho}u^{\nu}u^{\rho}\xi^{\sigma} + \partial_{\sigma}\Gamma^{\mu}{}_{\nu\rho}u^{\sigma}u^{\rho}\xi^{\nu} + \cancel{2\Gamma^{\mu}{}_{\nu\rho}u^{\rho}\frac{d\xi^{\nu}}{d\tau}} \\ & - \Gamma^{\mu}{}_{\nu\rho}\Gamma^{\rho}{}_{\alpha\beta}u^{\alpha}u^{\beta}\xi^{\nu} + \Gamma^{\mu}{}_{\nu\rho}\Gamma^{\nu}{}_{\alpha\beta}u^{\beta}u^{\rho}\xi^{\alpha} \\ = {} & -\partial_{\sigma}\Gamma^{\mu}{}_{\nu\rho}u^{\nu}u^{\rho}\xi^{\sigma} + \partial_{\sigma}\Gamma^{\mu}{}_{\nu\rho}u^{\sigma}u^{\rho}\xi^{\nu} - \Gamma^{\mu}{}_{\nu\rho}\Gamma^{\rho}{}_{\alpha\beta}u^{\alpha}u^{\beta}\xi^{\nu} + \Gamma^{\mu}{}_{\nu\rho}\Gamma^{\nu}{}_{\alpha\beta}u^{\beta}u^{\rho}\xi^{\alpha}. \end{aligned}$$

In the last equation, if we change the dummy indices such that in the first term $\sigma \leftrightarrow \rho$, in the second and third terms $\nu \leftrightarrow \rho$, in the last term $\alpha \leftrightarrow \rho$, then we have

$$\frac{D^2\xi^{\mu}}{D\tau^2} = -\partial_{\rho}\Gamma^{\mu}{}_{\nu\sigma}u^{\nu}u^{\sigma}\xi^{\rho} + \partial_{\sigma}\Gamma^{\mu}{}_{\nu\rho}u^{\sigma}u^{\nu}\xi^{\rho} - \Gamma^{\mu}{}_{\nu\rho}\Gamma^{\nu}{}_{\alpha\beta}u^{\alpha}u^{\beta}\xi^{\rho} + \Gamma^{\mu}{}_{\nu\alpha}\Gamma^{\nu}{}_{\rho\beta}u^{\beta}u^{\alpha}\xi^{\rho}.$$

Now, in the last equation, if we again change the dummy indices such that in the third term $\alpha \leftrightarrow \nu, \beta \to \sigma$, and in the last term $\beta \leftrightarrow \nu, \alpha \to \sigma, \beta \to \alpha$, then we arrive at

$$\begin{aligned} \frac{D^2\xi^{\mu}}{D\tau^2} = {} & -\partial_{\rho}\Gamma^{\mu}{}_{\nu\sigma}u^{\nu}u^{\sigma}\xi^{\rho} + \partial_{\sigma}\Gamma^{\mu}{}_{\nu\rho}u^{\sigma}u^{\nu}\xi^{\rho} - \Gamma^{\mu}{}_{\alpha\rho}\Gamma^{\alpha}{}_{\nu\sigma}u^{\nu}u^{\sigma}\xi^{\rho} + \Gamma^{\mu}{}_{\alpha\sigma}\Gamma^{\alpha}{}_{\rho\nu}u^{\nu}u^{\sigma}\xi^{\rho} \\ = {} & -(\partial_{\rho}\Gamma^{\mu}{}_{\nu\sigma} - \partial_{\sigma}\Gamma^{\mu}{}_{\nu\rho} + \Gamma^{\mu}{}_{\alpha\rho}\Gamma^{\alpha}{}_{\nu\sigma} + \Gamma^{\mu}{}_{\alpha\sigma}\Gamma^{\alpha}{}_{\rho\nu})u^{\nu}u^{\sigma}\xi^{\rho}. \end{aligned}$$

The term which is in the parenthesis is the Riemann tensor by definition, so we have the so called "geodesic deviation" equation

$$\frac{D^2\xi^{\mu}}{D\tau^2} = -R^{\mu}{}_{\nu\rho\sigma}u^{\nu}u^{\sigma}\xi^{\rho}. \tag{2.115}$$

Hence, we can say that two nearby time-like geodesics experience a tidal gravitational force, which is determined by the Riemann tensor. Also, eqn.(2.115) is very important for us, since we will use it in order to compute the gravitational memory in Chapter 3.

2.3.2 Local inertial frames and freely falling frames

In General Relativity, it is known that we can perform a change of coordinates such that the Christoffel symbol always vanishes at a given point P, ${\Gamma^{\mu}}_{\nu\rho}(P) = 0$. In such a frame, the geodesic equation at point P is

$$\frac{d^2x^\mu}{d\tau^2}\bigg|_P = 0.$$

Hence, we can say that a test mass is freely falling in this frame, but only at one point in space-time. This kind of frame is called a local inertial frame(LIF). This is also important for this thesis because we will imagine a detector which consists of the freely falling particles when we try to compute the gravitational memory.

However, it is possible to construct a reference frame in which a test mass is in free fall all along the geodesic. To do this, we observe that a freely spinning object which goes on along a time-like geodesic $x^\mu(\tau)$ then we have

$$\frac{ds^\mu}{d\tau} + {\Gamma^{\mu}}_{\nu\rho} s^\nu \frac{dx^\rho}{d\tau} = 0,$$

where s^μ is the spin four-vector which is $s^\mu = (0, \vec{s})$ in the rest frame. From the conservation of the angular momentum, we have $\frac{ds^\mu}{d\tau} = 0$. This is true along the entire time-like geodesic, then we can see that ${\Gamma^{\mu}}_{\nu\rho}$ is zero from last expression. Such a reference frame is called a freely falling frame, and its coordinates are known as Fermi normal coordinates.

2.3.3 TT frame and proper detector frame

The TT frame

We have used the TT gauge to give gravitational waves a simple form. We denote the corresponding reference frame as the TT frame.

Let's look at the geodesic eqn.(2.108) in order to understand TT frame. Say a test mass is at rest at $\tau = 0$, then

$$\begin{aligned}\frac{d^2x^i}{d\tau^2}\bigg|_{\tau=0} &= -\left[{\Gamma^{i}}_{\nu\rho}(x)\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}\right]_{\tau=0} \\ &= -\left[{\Gamma^{i}}_{00}\left(\frac{dx^0}{d\tau}\right)^2\right]_{\tau=0},\end{aligned}$$

where we used the fact that $\frac{dx^i}{d\tau}\big|_{\tau=0} = 0$. We have computed the Christoffel symbol ${\Gamma^\mu}_{\nu\rho}$ explicitly for the $g_{\mu\nu} = \eta_{\mu\nu}$ which is

$$
{\Gamma^\mu}_{\nu\rho} = \frac{1}{2}\eta^{\mu\sigma}(\partial_\nu h_{\rho\sigma} + \partial_\rho h_{\nu\sigma} - \partial_\sigma h_{\nu\rho}),
$$

so

$$
\begin{aligned}
{\Gamma^i}_{00} &= \frac{1}{2}\eta^{i\sigma}(\partial_0 h_{0\sigma} + \partial_0 h_{0\sigma} - \partial_\sigma h_{00}) \\
&= \frac{1}{2}(2\partial_0 h_{0i} - \partial_i h_{00}).
\end{aligned}
$$

In the TT gauge from eqn.(2.66), h_{00} and h_{0i} are zero. Hence we can say that if $\frac{dx^i}{d\tau}\big|_{\tau=0} = 0$, then also its derivative $\frac{d^2x^i}{d\tau^2}\big|_{\tau=0} = 0$ vanishes in the TT gauge. This means that $\frac{dx^i}{d\tau}$ is always zero. As a result, particle which is at rest before the wave arrives will be at rest when wave arrives, even after wave is gone. However, this is true only to linear order in $h_{\mu\nu}$. In other words, in the TT frame the position of the test masses initially at rest do not change, this looks like to use the free test masses themselves as the coordinate.

Let's look at coordinate separations of the two nearby test masses. From eqn.(2.111), we have for the $(\mu = i)$

$$
\begin{aligned}
\frac{d^2\xi^i}{d\tau^2}\Big|_{\tau=0} &= -\left[2{\Gamma^i}_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{d\xi^\rho}{d\tau} + \xi^\sigma\partial_\sigma{\Gamma^i}_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}\right]_{\tau=0} \\
&= -\left[2c{\Gamma^i}_{0\rho}\frac{d\xi^\rho}{d\tau} + c^2\xi^\sigma\partial_\sigma{\Gamma^i}_{00}\right]_{\tau=0}
\end{aligned}
$$

where we use the conditions $\frac{dx^0}{d\tau}\Big|_{\tau=0} = c$ and $\frac{dx^i}{d\tau}\Big|_{\tau=0} = 0$ at rest. We saw that before ${\Gamma^i}_{00} = 0$, since $h_{00} = h_{0i} = 0$. From the definition of the Christoffel symbol,

$$
\begin{aligned}
{\Gamma^i}_{0\rho} &= \frac{1}{2}\eta^{i\sigma}\left(\partial_0 h_{\rho\sigma} + \cancel{\partial_\rho h_{0\sigma}} - \cancel{\partial_\sigma h_{0\rho}}\right) \\
&= \frac{1}{2}\partial_0 h_{ij},
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\frac{d^2\xi^i}{d\tau^2}\Big|_{\tau=0} &= -\left[2c\frac{1}{2}\frac{\partial}{\partial x^0}h_{ij}\frac{d\xi^i}{d\tau}\right]_{\tau=0} \\
&= -\left[\dot{h}_{ij}\frac{d\xi^i}{d\tau}\right]_{\tau=0},
\end{aligned}
$$

where we used $\frac{\partial}{\partial x^0} = \frac{\partial\tau}{\partial x^0}\frac{\partial}{\partial\tau} = \frac{1}{c}\frac{\partial}{\partial\tau}$, and denotes $\frac{\partial h_{ij}}{\partial\tau}$ as $\dot{h}_{ij}$.

When we take the condition $\frac{d\xi^i}{d\tau}\Big|_{\tau=0} = 0$ into account we can say that $\frac{d^2\xi^i}{d\tau^2}\Big|_{\tau=0} = 0$

from the last equation. As a result, the separation is constant at all times. Also,

$$
\begin{aligned}
c^2 d\tau^2 &= -ds^2(\tau) \\
&= c^2 dt^2(\tau) - \left(\delta_{ij} + h_{ij}^{TT}\right) dx^i(\tau) dx^j(\tau) \\
&= c^2 dt^2(\tau) - \left(\delta_{ij} + h_{ij}^{TT}\right) \frac{dx^i}{d\tau}(\tau) \frac{dx^j}{d\tau}(\tau) d\tau^2,
\end{aligned}
$$

where $x^0(\tau) = ct(\tau)$. To find the physical effect of the gravitational wave, we can consider two events at $(t, x_1, 0, 0)$ and at $(t, x_2, 0, 0)$. In the TT gauge the coordinate distance $x_2 - x_1 = L$ remains. From eqn.(2.69),

$$
\begin{aligned}
ds^2 = -c^2 dt^2 + dz^2 + \left\{1 + h_+ \cos[w(t - \frac{z}{c})]\right\} dx^2 + \left\{1 - h_+ \cos[w(t - \frac{z}{c})]\right\} dy^2 \\
+ 2h_x \cos[w(t - \frac{z}{c})] dx dy,
\end{aligned}
$$

the proper distance s between these events is given by

$$
s = (x_2 - x_1)\{1 + h_+ \cos wt\}^{\frac{1}{2}} \equiv L\{1 + h_+ \cos wt\}^{\frac{1}{2}},
$$

where $ds = s$, $dx = x_2 - x_1$ and $dy = dx = 0$. Hence, we can say that the proper distance will change periodically in time with existence of the gravitational wave.

The Proper Detector Frame

Let us give our attention to a very small region of space. In such a region, if we choose the coordinates $(t, \vec{x})$, the metric is flat even if there is gravitational waves such that

$$
ds^2 = -c^2 dt^2 + \delta_{ij} dx^i dx^j.
$$

If we look at the this last expression with second order $g_{\mu\nu}$ in terms of Riemann tensor, the result is

$$
ds^2 = -c^2 dt^2 \left[1 + R_{0i0j} x^i x^j\right] - 2c dt dx^i \left(\frac{2}{3} R_{0ijk} x^j x^k\right) + dx^i dx^j \left[\delta_{ij} - \frac{1}{3} R_{ijkl} x^k x^l\right],
$$

where we are around at the point P which implies the Christoffel symbol vanishes.

The detector moves non-relativistically, so we have

$$
\frac{dx^i}{d\tau} << \frac{dx^0}{d\tau}.
$$

Using this last expression in the geodesic deviation equation, we get

$$
\frac{d^2 \xi^i}{d\tau^2} + \xi^\sigma \partial_\sigma \Gamma^i{}_{00} \left(\frac{dx^0}{d\tau}\right)^2 = 0.
$$

If we look at the last expression around the point P, i.e. $x^i = 0$. Because $g_{\mu\nu} = \eta_{\mu\nu} + O(x^i x^j)$, the time derivative of the Christoffel symbol at P gives zero. Also, we know that at the point P, both $\Gamma^{\mu}{}_{\nu\rho} = 0$ and $\partial_0 \Gamma^{i}{}_{0j} = 0$, then we have $R^{i}{}_{0j0} = \partial_j \Gamma^{i}{}_{00} - \partial_0 \Gamma^{i}{}_{0j} = \partial_j \Gamma^{i}{}_{00}$. Using this, we get

$$\frac{d^2 \xi^i}{d\tau^2} = -R^{i}{}_{0j0} \xi^j \left(\frac{dx^0}{d\tau}\right)^2 = 0.$$

We have already found that the Riemann tensor order h. Hence, if we limit ourselves to study at linear order in h, we can write $t = \tau$, so $dx^0/d\tau = c$, then we obtain

$$\ddot{\xi}^i = -c^2 R^{i}{}_{0j0} \xi^j.$$

2.4 The energy of gravitational waves

2.4.1 Separation of gravitational waves from the background

Now, we will expand the Einstein equation around the curved background metric $\bar{g}_{\mu\nu}(x)$, and write

$$g_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x) + h_{\mu\nu}(x), \qquad |h_{\mu\nu}| << 1 \tag{2.116}$$

where the diagonal elements of $\bar{g}_{\mu\nu}(x)$ are $O(0)$ with respect to $h_{\mu\nu}(x)$ on the region of the space. We need to find which part of $g_{\mu\nu}$ is the background and which is the fluctuation. Let $\bar{g}_{\mu\nu}(x)$ has a scale of spatial variation L_B, such that

$$\lambda\!\!\!^{-} << L_B, \tag{2.117}$$

where $\lambda\!\!\!^{-} = \frac{\lambda}{2\pi}$ is the reduced wavelength, and λ is the wavelength of the small perturbations $h_{\mu\nu}$.

Or, equivalently, we can assume that the background metric has frequencies which can be f_B as the maximum value, and let f be the frequency of the perturbation $h_{\mu\nu}$ such that

$$f >> f_B. \tag{2.118}$$

The conditions (2.117) and (2.118) are independent since L_B and f_B are unrelated. However, if one of them is satisfied then we can distinguish the metric as background metric plus perturbation based upon the gravitational wave. In the next section, we will find the answers for the two main questions such that

1. $h_{\mu\nu}$ is called a gravitational wave to answer the question: how does this high-frequency(or short wavelength) perturbation effect the background space-time,

2. $t_{\mu\nu}$ an energy momentum tensor can be defined to answer the question how it effects the background metric itself.

2.4.2 How gravitational waves curve the background

The Einstein equations can be rewritten as

$$R_{\mu\nu} = \frac{8\pi G}{c^4}(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T), \tag{2.119}$$

where $T_{\mu\nu}$ is the energy-momentum tensor of the matter and T is its trace. If we expand the Ricci tensor up to $O(h^2)$, we will get

$$R_{\mu\nu} = \bar{R}_{\mu\nu} + R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu} + ..., \tag{2.120}$$

where $\bar{R}_{\mu\nu}$ is based on the $\bar{g}_{\mu\nu}$ only, $R^{(1)}_{\mu\nu}$ is linear in $h_{\mu\nu}$, and $R^{(2)}_{\mu\nu}$ is quadratic in $h_{\mu\nu}$. Since, the $\bar{R}_{\mu\nu}$ is based on $\bar{g}_{\mu\nu}$, it contains only low-frequency modes. Let $\vec{k}$ separate the low frequency from the high frequency modes, then $\bar{g}_{\mu\nu}$ has only modes up a typical wave-vector $k_B \simeq \frac{2\pi}{L_B}$ with $k_B << k$. The Christoffel symbols for the $\bar{g}_{\mu\nu}$ are quadratic, so they have modes up to $2k_B$. The Ricci tensor is quadratic with respect to the Christoffel symbols, so its modes are up to $4k_B$. Fortunately, $4k_B << k$ implies that $\bar{R}_{\mu\nu}$ has only low-frequency modes. Since, $R^{(1)}_{\mu\nu}$ is linear in $h_{\mu\nu}$ by definition, it contains only high-frequency modes. On the other hand, $R^{(2)}_{\mu\nu}$ is quadratic in $h_{\mu\nu}$ implies that it has both low- and high-frequency modes. When we take these factors into account, we can separate the Einstein equations into two parts which are low- and high-frequency parts, the first one is

$$\bar{R}_{\mu\nu} = -[R^{(2)}_{\mu\nu}]^{Low} + \frac{8\pi G}{c^4}(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T)^{Low}, \tag{2.121}$$

and the other one is

$$R^{(1)}_{\mu\nu} = -[R^{(2)}_{\mu\nu}]^{High} + \frac{8\pi G}{c^4}(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T)^{High}, \tag{2.122}$$

where the superscript "Low" denotes projection on the low momenta or on the low frequencies which depend on if eqn.(2.117) or eqn.(2.118) applies, and the superscript "High" has similar meaning.

Let's start to compute $R^{(1)}_{\mu\nu}$ explicitly. To do this, first of all we need to find the Christoffel symbols. We know that the metric $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$ and its inverse metric is $g^{\mu\nu} = \bar{g}^{\mu\nu} - h^{\mu\nu} + O(h^2)$. By definition, the Christoffel symbol is

$$\Gamma^{\mu}{}_{\nu\rho} = \frac{1}{2}g^{\mu\alpha}(\partial_\nu g_{\alpha\rho} + \partial_\rho g_{\alpha\nu} - \partial_\alpha g_{\nu\rho}), \tag{2.123}$$

and writing the metric explicitly

$$
\begin{aligned}
\Gamma^{\mu}{}_{\nu\rho} &= \frac{1}{2}(\bar{g}^{\mu\alpha} - h^{\mu\alpha})[\partial_\nu(\bar{g}_{\alpha\rho} + h_{\alpha\rho}) + \partial_\rho(\bar{g}_{\alpha\nu} + h_{\alpha\nu}) - \partial_\alpha(\bar{g}_{\nu\rho} + h_{\nu\rho})] \\
&= \frac{1}{2}\bar{g}^{\mu\alpha}(\partial_\nu\bar{g}_{\alpha\rho} + \partial_\rho\bar{g}_{\alpha\nu} - \partial_\alpha\bar{g}_{\nu\rho}) + \frac{1}{2}\bar{g}^{\mu\alpha}(\partial_\nu h_{\alpha\rho} + \partial_\rho h_{\alpha\nu} - \partial_\alpha h_{\nu\rho}) \\
&\quad - \frac{1}{2}h^{\mu\alpha}(\partial_\nu\bar{g}_{\alpha\rho} + \partial_\rho\bar{g}_{\alpha\nu} - \partial_\alpha\bar{g}_{\nu\rho}) + O(h^2).
\end{aligned}
$$

We can change the derivatives in the second term of the last equation to covariant derivatives if we add some suitable terms,

$$
\begin{aligned}
\Gamma^{\mu}{}_{\nu\rho} &= \frac{1}{2}\bar{g}^{\mu\alpha}(\partial_\nu\bar{g}_{\alpha\rho} + \partial_\rho\bar{g}_{\alpha\nu} - \partial_\alpha\bar{g}_{\nu\rho}) + \frac{1}{2}\bar{g}^{\mu\alpha}(\bar{D}_\nu h_{\alpha\rho} + \bar{D}_\rho h_{\alpha\nu} - \bar{D}_\alpha h_{\nu\rho}) \\
&\quad + \frac{1}{2}\bar{g}^{\mu\alpha}(\bar{\Gamma}^{\beta}{}_{\nu\rho}h_{\beta\alpha} + \bar{\Gamma}^{\beta}{}_{\nu\alpha}h_{\rho\beta}) + \frac{1}{2}\bar{g}^{\mu\alpha}(\bar{\Gamma}^{\beta}{}_{\rho\nu}h_{\beta\alpha} + \bar{\Gamma}^{\beta}{}_{\rho\alpha}h_{\nu\beta}) \\
&\quad - \frac{1}{2}h^{\mu\alpha}(\partial_\nu\bar{g}_{\alpha\rho} + \partial_\rho\bar{g}_{\alpha\nu} - \partial_\alpha\bar{g}_{\nu\rho}).
\end{aligned}
$$

Using the definition of the Christoffel symbol, and making suitable cancellations

$$
\begin{aligned}
\Gamma^{\mu}{}_{\nu\rho} &= \bar{\Gamma}^{\mu}{}_{\nu\rho} + \frac{1}{2}\bar{g}^{\mu\alpha}(\bar{D}_\nu h_{\alpha\rho} + \bar{D}_\rho h_{\alpha\nu} - \bar{D}_\alpha h_{\nu\rho}) \\
&\quad + \frac{1}{2}\bar{g}^{\mu\alpha}\{\frac{1}{2}\bar{g}^{\beta\sigma}(\partial_\nu\bar{g}_{\sigma\rho} + \partial_\rho\bar{g}_{\sigma\nu} - \partial_\sigma\bar{g}_{\nu\rho})h_{\beta\alpha} + \frac{1}{2}\bar{g}^{\beta\sigma}(\partial_\nu\bar{g}_{\sigma\alpha} + \partial_\alpha\bar{g}_{\sigma\nu} - \partial_\sigma\bar{g}_{\nu\alpha})h_{\rho\beta} \\
&\quad + \frac{1}{2}\bar{g}^{\beta\sigma}(\partial_\rho\bar{g}_{\sigma\nu} + \partial_\nu\bar{g}_{\sigma\rho} - \partial_\sigma\bar{g}_{\rho\nu})h_{\beta\alpha} + \frac{1}{2}\bar{g}^{\beta\sigma}(\partial_\rho\bar{g}_{\sigma\alpha} + \partial_\nu\bar{g}_{\sigma\rho} - \partial_\sigma\bar{g}_{\rho\alpha})h_{\nu\beta} \\
&\quad + \frac{1}{2}\bar{g}^{\beta\sigma}(\partial_\alpha\bar{g}_{\sigma\rho} + \partial_\rho\bar{g}_{\sigma\alpha} - \partial_\sigma\bar{g}_{\alpha\rho})h_{\beta\nu} + \frac{1}{2}\bar{g}^{\beta\sigma}(\partial_\alpha\bar{g}_{\sigma\nu} + \partial_\nu\bar{g}_{\sigma\alpha} - \partial_\sigma\bar{g}_{\alpha\nu})h_{\rho\beta}\} \\
&\quad - \frac{1}{2}h^{\mu\alpha}(\partial_\nu\bar{g}_{\alpha\rho} + \partial_\rho\bar{g}_{\alpha\nu} - \partial_\alpha\bar{g}_{\nu\rho}) \\
&= \bar{\Gamma}^{\mu}{}_{\nu\rho} + \frac{1}{2}\bar{g}^{\mu\alpha}(\bar{D}_\nu h_{\alpha\rho} + \bar{D}_\rho h_{\alpha\nu} - \bar{D}_\alpha h_{\nu\rho}) \\
&\quad + \frac{1}{4}\bar{g}^{\mu\alpha}\bar{g}^{\beta\sigma}\{\partial_\nu\bar{g}_{\sigma\rho}h_{\beta\alpha} + \partial_\rho\bar{g}_{\sigma\nu}h_{\beta\alpha} - \partial_\sigma\bar{g}_{\nu\rho}h_{\beta\alpha} + \cancel{\partial_\nu\bar{g}_{\sigma\alpha}h_{\rho\beta}} + \cancel{\partial_\alpha\bar{g}_{\sigma\nu}h_{\rho\beta}} \\
&\quad - \cancel{\partial_\sigma\bar{g}_{\nu\alpha}h_{\rho\beta}} + \partial_\rho\bar{g}_{\sigma\nu}h_{\beta\alpha} + \partial_\nu\bar{g}_{\sigma\rho}h_{\beta\alpha} - \partial_\sigma\bar{g}_{\rho\nu}h_{\beta\alpha} + \cancel{\partial_\rho\bar{g}_{\sigma\alpha}h_{\nu\beta}} + \cancel{\partial_\alpha\bar{g}_{\sigma\rho}h_{\nu\beta}} \\
&\quad - \cancel{\partial_\sigma\bar{g}_{\rho\alpha}h_{\nu\beta}} - \cancel{\partial_\alpha\bar{g}_{\sigma\rho}h_{\beta\nu}} - \cancel{\partial_\rho\bar{g}_{\sigma\alpha}h_{\beta\nu}} + \cancel{\partial_\sigma\bar{g}_{\alpha\rho}h_{\beta\nu}} - \cancel{\partial_\alpha\bar{g}_{\sigma\nu}h_{\rho\beta}} - \cancel{\partial_\nu\bar{g}_{\sigma\alpha}h_{\rho\beta}} \\
&\quad + \cancel{\partial_\sigma\bar{g}_{\alpha\nu}h_{\rho\beta}}\} - \frac{1}{2}h^{\mu\alpha}(\partial_\nu\bar{g}_{\alpha\rho} + \partial_\rho\bar{g}_{\alpha\nu} - \partial_\alpha\bar{g}_{\nu\rho}),
\end{aligned}
$$

we get

$$
\begin{aligned}
\Gamma^{\mu}{}_{\nu\rho} &= \bar{\Gamma}^{\mu}{}_{\nu\rho} + \frac{1}{2}\bar{g}^{\mu\alpha}(\bar{D}_\nu h_{\alpha\rho} + \bar{D}_\rho h_{\alpha\nu} - \bar{D}_\alpha h_{\nu\rho}) \\
&\quad + \frac{1}{2}\bar{g}^{\mu\alpha}\bar{g}^{\beta\sigma}(\partial_\nu\bar{g}_{\sigma\rho}h_{\beta\alpha} + \partial_\rho\bar{g}_{\sigma\nu}h_{\beta\alpha} - \partial_\sigma\bar{g}_{\nu\rho}h_{\beta\alpha}) - \frac{1}{2}h^{\mu\alpha}(\partial_\nu\bar{g}_{\alpha\rho} + \partial_\rho\bar{g}_{\alpha\nu} - \partial_\alpha\bar{g}_{\nu\rho}) \\
&= \bar{\Gamma}^{\mu}{}_{\nu\rho} + \frac{1}{2}\bar{g}^{\mu\alpha}(\bar{D}_\nu h_{\alpha\rho} + \bar{D}_\rho h_{\alpha\nu} - \bar{D}_\alpha h_{\nu\rho}) + \frac{1}{2}h^{\mu\sigma}(\partial_\nu\bar{g}_{\sigma\rho} + \partial_\rho\bar{g}_{\sigma\nu} - \partial_\sigma\bar{g}_{\nu\rho}) \\
&\quad - \frac{1}{2}h^{\mu\alpha}(\partial_\nu\bar{g}_{\alpha\rho} + \partial_\rho\bar{g}_{\alpha\nu} - \partial_\alpha\bar{g}_{\nu\rho})
\end{aligned}
$$

where we use $\bar{g}^{\mu\alpha}\bar{g}^{\beta\sigma}h_{\beta\alpha} = h^{\mu\sigma}$. If we change the dummy index in the last term as $\alpha \to \sigma$, we obtain

$$\Gamma^{\mu}{}_{\nu\rho} = \bar{\Gamma}^{\mu}{}_{\nu\rho} + \frac{1}{2}\bar{g}^{\mu\alpha}(\bar{D}_\nu h_{\alpha\rho} + \bar{D}_\rho h_{\alpha\nu} - \bar{D}_\alpha h_{\nu\rho}) + \frac{1}{2}h^{\mu\sigma}(\partial_\nu \bar{g}_{\sigma\rho} + \partial_\rho \bar{g}_{\sigma\nu} - \partial_\sigma \bar{g}_{\nu\rho})$$
$$- \frac{1}{2}h^{\mu\sigma}(\partial_\nu \bar{g}_{\sigma\rho} + \partial_\rho \bar{g}_{\sigma\nu} - \partial_\sigma \bar{g}_{\nu\rho}).$$

As a result,

$$\Gamma^{\mu}{}_{\nu\rho} = \bar{\Gamma}^{\mu}{}_{\nu\rho} + \frac{1}{2}\bar{g}^{\mu\alpha}(\bar{D}_\nu h_{\alpha\rho} + \bar{D}_\rho h_{\alpha\nu} - \bar{D}_\alpha h_{\nu\rho}). \tag{2.124}$$

At any given point x, we can simplify eqn.(2.124) using the $\bar{\Gamma}^{\mu}{}_{\nu\rho}(x) = 0$ with a suitable coordinate system. However, its derivative is not zero. In addition, we see that $\Gamma^{\mu}{}_{\nu\rho} = O(h)$ from eqn.(2.124), so in the Riemann tensor we will omit the terms $\Gamma\Gamma$ because they are order in $O(h^2)$. Then we can say that $R^{\mu}{}_{\nu\rho\sigma} = \partial_\rho \Gamma^{\mu}{}_{\nu\sigma} - \partial_\sigma \Gamma^{\mu}{}_{\nu\rho} + O(h^2)$ by the definition of Riemann tensor.

In such frame where $\bar{\Gamma}^{\mu}{}_{\nu\rho}(x) = 0$, we need to compute the Riemann tensor.

$$\begin{aligned}
R^{\mu}{}_{\nu\rho\sigma} &= \partial_\rho \Gamma^{\mu}{}_{\nu\sigma} - \partial_\sigma \Gamma^{\mu}{}_{\nu\rho} \\
&= \partial_\rho \{\bar{\Gamma}^{\mu}{}_{\nu\sigma} + \frac{1}{2}\bar{g}^{\mu\alpha}(\bar{D}_\nu h_{\alpha\sigma} + \bar{D}_\sigma h_{\alpha\nu} - \bar{D}_\alpha h_{\nu\sigma}) - \partial_\sigma \bar{\Gamma}^{\mu}{}_{\nu\rho} \\
&\quad + \frac{1}{2}\bar{g}^{\mu\alpha}(\bar{D}_\nu h_{\alpha\rho} + \bar{D}_\rho h_{\alpha\nu} - \bar{D}_\alpha h_{\nu\rho})\} \\
&= \{\partial_\rho \bar{\Gamma}^{\mu}{}_{\nu\sigma} - \partial_\sigma \bar{\Gamma}^{\mu}{}_{\nu\rho}\} + \frac{1}{2}\bar{g}^{\mu\alpha}\{\partial_\rho(\bar{D}_\nu h_{\alpha\sigma} + \bar{D}_\sigma h_{\alpha\nu} - \bar{D}_\alpha h_{\nu\sigma}) \\
&\quad - \partial_\sigma(\bar{D}_\nu h_{\alpha\rho} + \bar{D}_\rho h_{\alpha\nu} - \bar{D}_\alpha h_{\nu\rho})\}.
\end{aligned}$$

Since $\bar{\Gamma}^{\mu}{}_{\nu\rho}(x) = 0$, we can write the Riemann tensor for the background metric instead of the first two terms, and covariant derivative instead of the normal derivative. Then we have

$$\begin{aligned}
R^{\mu}{}_{\nu\rho\sigma} &= \bar{R}^{\mu}{}_{\nu\rho\sigma} + \frac{1}{2}\bar{g}^{\mu\alpha}\{\bar{D}_\rho\bar{D}_\nu h_{\alpha\sigma} + \bar{D}_\rho\bar{D}_\sigma h_{\alpha\nu} - \bar{D}_\rho\bar{D}_\alpha h_{\nu\sigma} - \bar{D}_\sigma\bar{D}_\nu h_{\alpha\rho} - \bar{D}_\sigma\bar{D}_\rho h_{\alpha\nu} \\
&\quad + \bar{D}_\sigma\bar{D}_\alpha h_{\nu\rho}\} \\
&= \bar{R}^{\mu}{}_{\nu\rho\sigma} + \frac{1}{2}\{\bar{D}_\rho\bar{D}_\nu h^{\mu}{}_{\sigma} + \bar{D}_\rho\bar{D}_\sigma h^{\mu}{}_{\nu} - \bar{D}_\rho\bar{D}^{\mu} h_{\nu\sigma} - \bar{D}_\sigma\bar{D}_\nu h^{\mu}{}_{\rho} - \bar{D}_\sigma\bar{D}_\rho h^{\mu}{}_{\nu} \\
&\quad + \bar{D}_\sigma\bar{D}^{\mu} h_{\nu\rho}\}.
\end{aligned}$$

Let's rewrite the last equation such that

$$\begin{aligned}
g_{\beta\mu}R^{\mu}{}_{\nu\rho\sigma} &= (\bar{g}_{\beta\mu} + h_{\beta\mu})R^{\mu}{}_{\nu\rho\sigma} \\
R_{\beta\nu\rho\sigma} &= \bar{g}_{\beta\mu}\bar{R}^{\mu}{}_{\nu\rho\sigma} + \frac{1}{2}\bar{g}_{\beta\mu}\bar{g}^{\mu\alpha}\{\bar{D}_\rho\bar{D}_\nu h_{\alpha\sigma} + \bar{D}_\rho\bar{D}_\sigma h_{\alpha\nu} - \bar{D}_\rho\bar{D}_\alpha h_{\nu\sigma} - \bar{D}_\sigma\bar{D}_\nu h_{\alpha\rho}
\end{aligned}$$

$$
\begin{aligned}
&\quad - \bar{D}_\sigma \bar{D}_\rho h_{\alpha\nu} + \bar{D}_\sigma \bar{D}_\alpha h_{\nu\rho}\} + h_{\beta\mu}\bar{R}^\mu{}_{\nu\rho\sigma} + O(h^2) \\
&= \bar{R}_{\beta\nu\rho\sigma} + \frac{1}{2}\{\bar{D}_\rho \bar{D}_\nu h_{\beta\sigma} + \bar{D}_\rho \bar{D}_\sigma h_{\beta\nu} - \bar{D}_\rho \bar{D}_\beta h_{\nu\sigma} - \bar{D}_\sigma \bar{D}_\nu h_{\beta\rho} - \bar{D}_\sigma \bar{D}_\rho h_{\beta\nu} \\
&\quad + \bar{D}_\sigma \bar{D}_\beta h_{\nu\rho}\} + h_{\beta\mu}\bar{R}^\mu{}_{\nu\rho\sigma} + O(h^2).
\end{aligned}
$$

Now we can use the definition of the Ricci tensor,

$$
\begin{aligned}
g^{\beta\rho} R_{\beta\nu\rho\sigma} &= \left(\bar{g}^{\beta\rho} - h^{\beta\rho}\right) R_{\beta\nu\rho\sigma} \\
R_{\nu\sigma} &= \bar{g}^{\beta\rho}\bar{R}_{\beta\nu\rho\sigma} + \frac{1}{2}\bar{g}^{\beta\rho}\{\bar{D}_\rho \bar{D}_\nu h_{\beta\sigma} + \bar{D}_\rho \bar{D}_\sigma h_{\beta\nu} - \bar{D}_\rho \bar{D}_\beta h_{\nu\sigma} - \bar{D}_\sigma \bar{D}_\nu h_{\beta\rho} \\
&\quad - \bar{D}_\sigma \bar{D}_\rho h_{\beta\nu} + \bar{D}_\sigma \bar{D}_\beta h_{\nu\rho}\} + \bar{g}^{\beta\rho} h_{\beta\mu}\bar{R}^\mu{}_{\nu\rho\sigma} - h^{\beta\rho}\bar{R}_{\beta\nu\rho\sigma} + O(h^2) \\
&= \bar{R}_{\nu\sigma} + \frac{1}{2}\{\bar{D}^\beta \bar{D}_\nu h_{\beta\sigma} + \bar{D}^\beta \bar{D}_\sigma h_{\beta\nu} - \bar{D}^\beta \bar{D}_\beta h_{\nu\sigma} - \bar{D}_\sigma \bar{D}_\nu h - \cancel{\bar{D}_\sigma \bar{D}^\beta h_{\beta\nu}} \\
&\quad + \cancel{\bar{D}_\sigma \bar{D}_\beta h_\nu{}^\beta}\} + \cancel{h^\rho{}_\mu \bar{R}^\mu{}_{\nu\rho\sigma}} - \cancel{h^\rho{}_\beta \bar{R}^\beta{}_{\nu\rho\sigma}} + O(h^2) \\
&= \bar{R}_{\nu\sigma} + \frac{1}{2}\{\bar{D}^\beta \bar{D}_\nu h_{\beta\sigma} + \bar{D}^\beta \bar{D}_\sigma h_{\beta\nu} - \bar{D}^\beta \bar{D}_\beta h_{\nu\sigma} - \bar{D}_\sigma \bar{D}_\nu h\}.
\end{aligned}
$$

We can separate the Ricci tensor with respect to its h orders such as

$$
R_{\mu\nu} = \bar{R}_{\mu\nu} + R^{(1)}_{\mu\nu} \tag{2.125}
$$

where the superscript (1) refers to order of h is 1. Hence, we can define $R^{(1)}_{\mu\nu}$ such that

$$
R^{(1)}_{\mu\nu} = \frac{1}{2}\Big(\bar{D}^\alpha \bar{D}_\mu h_{\nu\alpha} + \bar{D}^\alpha \bar{D}_\nu h_{\mu\alpha} - \bar{D}^\alpha \bar{D}_\alpha h_{\mu\nu} - \bar{D}_\nu \bar{D}_\mu h\Big). \tag{2.126}
$$

Next, we will find the Ricci tensor at quadratic order. To do this, we need to find inverse metric at quadratic order. Say $g^{\nu\alpha} = \bar{g}^{\nu\alpha} - h^{\nu\alpha} + x^{\nu\alpha}(h^2)$ where $x^{\nu\alpha}(h^2)$ is the quadratic term. Let's find it to use the definition $g_{\mu\nu}g^{\nu\alpha} = \delta_\mu{}^\alpha$

$$
\begin{aligned}
\delta_\mu{}^\alpha &= \left(\bar{g}_{\mu\nu} + h_{\mu\nu}\right)\left(\bar{g}^{\nu\alpha} - h^{\nu\alpha} + x^{\nu\alpha}\right) \\
&= \bar{g}_{\mu\nu}\bar{g}^{\nu\alpha} - \bar{g}_{\mu\nu}h^{\nu\alpha} + \bar{g}_{\mu\nu}x^{\nu\alpha} + h_{\mu\nu}\bar{g}^{\nu\alpha} - h_{\mu\nu}h^{\nu\alpha} \\
\cancel{\delta_\mu{}^\alpha} &= \cancel{\delta_\mu{}^\alpha} - \cancel{h_\mu{}^\alpha} + x_\mu{}^\alpha + \cancel{h_\mu{}^\alpha} - h_{\mu\nu}h^{\nu\alpha} \\
x_\mu{}^\alpha &= h_{\mu\nu}h^{\nu\alpha} \\
x^{\mu\alpha} &= h^\mu{}_\nu h^{\nu\alpha},
\end{aligned}
$$

so the inverse metric at quadratic order is

$$
g^{\mu\alpha} = \bar{g}^{\mu\alpha} - h^{\mu\alpha} + h^\mu{}_\lambda h^{\lambda\alpha}. \tag{2.127}
$$

Now, let's compute the Christoffel symbol again,

$$
\begin{aligned}
\Gamma^\mu{}_{\nu\rho} &= \frac{1}{2}g^{\mu\alpha}\left(\partial_\nu g_{\alpha\rho} + \partial_\rho g_{\alpha\nu} - \partial_\alpha g_{\nu\rho}\right) \\
&= \frac{1}{2}\left(\bar{g}^{\mu\alpha} - h^{\mu\alpha} + h^\mu{}_\lambda h^{\lambda\alpha}\right)\left[\partial_\nu(\bar{g}_{\alpha\rho} + h_{\alpha\rho}) + \partial_\rho(\bar{g}_{\alpha\nu} + h_{\alpha\nu}) - \partial_\alpha(\bar{g}_{\nu\rho} + h_{\nu\rho})\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}\bar{g}^{\mu\alpha}\big(\partial_\nu \bar{g}_{\alpha\rho} + \partial_\rho \bar{g}_{\alpha\nu} - \partial_\alpha \bar{g}_{\nu\rho}\big) + \frac{1}{2}\bar{g}^{\mu\alpha}\big(\partial_\nu h_{\alpha\rho} + \partial_\rho h_{\alpha\nu} - \partial_\alpha h_{\nu\rho}\big) \\
&\quad - \frac{1}{2}h^{\mu\alpha}\big(\partial_\nu \bar{g}_{\alpha\rho} + \partial_\rho \bar{g}_{\alpha\nu} - \partial_\alpha \bar{g}_{\nu\rho}\big) - \frac{1}{2}h^{\mu\alpha}\big(\partial_\nu h_{\alpha\rho} + \partial_\rho h_{\alpha\nu} - \partial_\alpha h_{\nu\rho}\big) \\
&\quad + \frac{1}{2}h^{\mu}{}_{\lambda}h^{\lambda\alpha}\big(\partial_\nu \bar{g}_{\alpha\rho} + \partial_\rho \bar{g}_{\alpha\nu} - \partial_\alpha \bar{g}_{\nu\rho}\big).
\end{aligned}
$$

If we use the expression (2.124) instead of the first three terms, we will get

$$
\begin{aligned}
\Gamma^{\mu}{}_{\nu\rho} = \bar{\Gamma}^{\mu}{}_{\nu\rho} + \Gamma^{\mu(1)}_{\nu\rho} - \frac{1}{2}h^{\mu\alpha}\big(\partial_\nu h_{\alpha\rho} + \partial_\rho h_{\alpha\nu} - \partial_\alpha h_{\nu\rho}\big) \\
+ \frac{1}{2}h^{\mu}{}_{\lambda}h^{\lambda\alpha}\big(\partial_\nu \bar{g}_{\alpha\rho} + \partial_\rho \bar{g}_{\alpha\nu} - \partial_\alpha \bar{g}_{\nu\rho}\big),
\end{aligned}
$$

where

$$
\Gamma^{\mu(1)}_{\nu\rho} = \frac{1}{2}\bar{g}^{\mu\alpha}(\bar{D}_\nu h_{\alpha\rho} + \bar{D}_\rho h_{\alpha\nu} - \bar{D}_\alpha h_{\nu\rho}). \tag{2.128}
$$

Let's denote the quadratic order terms of the Christoffel symbol such as

$$
\begin{aligned}
\Gamma^{\mu(2)}_{\nu\rho} &= -\frac{1}{2}h^{\mu\alpha}\big(\partial_\nu h_{\alpha\rho} + \partial_\rho h_{\alpha\nu} - \partial_\alpha h_{\nu\rho}\big) + \frac{1}{2}h^{\mu}{}_{\lambda}h^{\lambda\alpha}\big(\partial_\nu \bar{g}_{\alpha\rho} + \partial_\rho \bar{g}_{\alpha\nu} - \partial_\alpha \bar{g}_{\nu\rho}\big) \\
&= -\frac{1}{2}h^{\mu\alpha}\big(\bar{D}_\nu h_{\alpha\rho} + \bar{D}_\rho h_{\alpha\nu} - \bar{D}_\alpha h_{\nu\rho}\big) - \frac{1}{2}h^{\mu\alpha}\big(\cancel{\bar{\Gamma}^{\lambda}{}_{\nu\alpha}h_{\lambda\rho}} + \bar{\Gamma}^{\lambda}{}_{\nu\rho}h_{\alpha\lambda} + \cancel{\bar{\Gamma}^{\lambda}{}_{\rho\alpha}h_{\lambda\nu}} \\
&\quad + \bar{\Gamma}^{\lambda}{}_{\rho\nu}h_{\alpha\lambda} - \cancel{\bar{\Gamma}^{\lambda}{}_{\alpha\nu}h_{\lambda\rho}} - \cancel{\bar{\Gamma}^{\lambda}{}_{\alpha\rho}h_{\nu\lambda}}\big) + \frac{1}{2}h^{\mu}{}_{\lambda}h^{\lambda}{}_{\gamma}\bar{g}^{\gamma\alpha}\big(\partial_\nu \bar{g}_{\alpha\rho} + \partial_\rho \bar{g}_{\alpha\nu} - \partial_\alpha \bar{g}_{\nu\rho}\big) \\
&= -\frac{1}{2}h^{\mu\alpha}\big(\bar{D}_\nu h_{\alpha\rho} + \bar{D}_\rho h_{\alpha\nu} - \bar{D}_\alpha h_{\nu\rho}\big) - \cancel{h^{\mu\alpha}h_{\alpha\lambda}\bar{\Gamma}^{\lambda}{}_{\nu\rho}} + \cancel{h^{\mu}{}_{\lambda}h^{\lambda}{}_{\gamma}\bar{\Gamma}^{\gamma}{}_{\nu\rho}} \\
&= -\frac{1}{2}h^{\mu}{}_{\beta}\bar{g}^{\beta\alpha}\big(\bar{D}_\nu h_{\alpha\rho} + \bar{D}_\rho h_{\alpha\nu} - \bar{D}_\alpha h_{\nu\rho}\big).
\end{aligned}
$$

Using the expression (2.128), we get

$$
\Gamma^{\mu(2)}_{\nu\rho} = -h^{\mu}{}_{\beta}\Gamma^{\beta(1)}_{\nu\rho}. \tag{2.129}
$$

Hence, the Christoffel symbol is

$$
\Gamma^{\mu}{}_{\nu\rho} = \bar{\Gamma}^{\mu}{}_{\nu\rho} + \Gamma^{\mu(1)}_{\nu\rho} - h^{\mu}{}_{\beta}\Gamma^{\beta(1)}_{\nu\rho} + O(h^3). \tag{2.130}
$$

By the definition of the Riemann tensor and using the a suitable coordinate system such that $\bar{\Gamma}^{\mu}{}_{\nu\rho}(x) = 0$,

$$
\begin{aligned}
R^{\mu}{}_{\nu\rho\sigma} &= \partial_\rho \Gamma^{\mu}{}_{\nu\sigma} - \partial_\sigma \Gamma^{\mu}{}_{\nu\rho} + \Gamma^{\mu}{}_{\rho\lambda}\Gamma^{\lambda}{}_{\sigma\nu} - \Gamma^{\mu}{}_{\sigma\lambda}\Gamma^{\lambda}{}_{\rho\nu} \\
&= \partial_\rho \bar{\Gamma}^{\mu}{}_{\nu\sigma} + \partial_\rho \Gamma^{\mu(1)}_{\nu\sigma} - \partial_\rho\big(h^{\mu}{}_{\beta}\Gamma^{\beta(1)}_{\nu\sigma}\big) - \partial_\sigma \bar{\Gamma}^{\mu}{}_{\nu\rho} - \partial_\sigma \Gamma^{\mu(1)}_{\nu\rho} + \partial_\sigma\big(h^{\mu}{}_{\beta}\Gamma^{\beta(1)}_{\nu\rho}\big) \\
&\quad + \Gamma^{\mu(1)}_{\rho\lambda}\Gamma^{\lambda(1)}_{\sigma\nu} - \Gamma^{\mu(1)}_{\sigma\lambda}\Gamma^{\lambda(1)}_{\rho\nu} + O(h^3) \\
&= \bar{R}^{\mu}{}_{\nu\rho\sigma} + R^{\mu(1)}_{\nu\rho\sigma} - \big(\partial_\rho h^{\mu}{}_{\beta}\big)\Gamma^{\beta(1)}_{\nu\sigma} + \big(\partial_\sigma h^{\mu}{}_{\beta}\big)\Gamma^{\beta(1)}_{\nu\rho} - h^{\mu}{}_{\beta}R^{\beta(1)}_{\nu\rho\sigma} + \\
&\quad + \Gamma^{\mu(1)}_{\rho\lambda}\Gamma^{\lambda(1)}_{\sigma\nu} - \Gamma^{\mu(1)}_{\sigma\lambda}\Gamma^{\lambda(1)}_{\rho\nu} + O(h^3).
\end{aligned}
$$

We can change the normal derivatives terms to covariant derivative thanks to the coordinate system which we chose. If we rewrite the last expression after multiplying with metric $g_{\alpha\mu}$, we have

$$
\begin{aligned}
g_{\alpha\mu}R^{\mu}{}_{\nu\rho\sigma} &= \left(\bar{g}_{\alpha\mu}+h_{\alpha\mu}\right)R^{\mu}{}_{\nu\rho\sigma}\\
R_{\alpha\nu\rho\sigma} &= \bar{g}_{\alpha\mu}\bar{R}^{\mu}{}_{\nu\rho\sigma}+\bar{g}_{\alpha\mu}R^{\mu(1)}_{\nu\rho\sigma}-\bar{g}_{\alpha\mu}\left(\bar{D}_{\rho}h^{\mu}{}_{\beta}\right)\Gamma^{\beta(1)}_{\nu\sigma}+\bar{g}_{\alpha\mu}\left(\bar{D}_{\sigma}h^{\mu}{}_{\beta}\right)\Gamma^{\beta(1)}_{\nu\rho}\\
&\quad-\bar{g}_{\alpha\mu}h^{\mu}{}_{\beta}R^{\beta(1)}_{\nu\rho\sigma}+\bar{g}_{\alpha\mu}\Gamma^{\mu(1)}_{\rho\lambda}\Gamma^{\lambda(1)}_{\sigma\nu}-\bar{g}_{\alpha\mu}\Gamma^{\mu(1)}_{\sigma\lambda}\Gamma^{\lambda(1)}_{\rho\nu}+h_{\alpha\mu}\bar{R}^{\mu}{}_{\nu\rho\sigma}\\
&\quad+h_{\alpha\mu}R^{\mu(1)}_{\nu\rho\sigma}+O(h^3).
\end{aligned}
$$

Now we can use the definition of the Ricci tensor,

$$
\begin{aligned}
g^{\rho\alpha}R_{\alpha\nu\rho\sigma} &= \left(\bar{g}^{\rho\alpha}-h_{\rho\alpha}+h^{\rho}{}_{\lambda}h^{\lambda\alpha}\right)R_{\alpha\nu\rho\sigma}\\
R_{\nu\sigma} &= \bar{g}^{\rho\alpha}\bar{g}_{\alpha\mu}\bar{R}^{\mu}{}_{\nu\rho\sigma}+\bar{g}^{\rho\alpha}\bar{g}_{\alpha\mu}R^{\mu(1)}_{\nu\rho\sigma}-\bar{g}^{\rho\alpha}\bar{g}_{\alpha\mu}\left(\bar{D}_{\rho}h^{\mu}{}_{\beta}\right)\Gamma^{\beta(1)}_{\nu\sigma}+\bar{g}^{\rho\alpha}\bar{g}_{\alpha\mu}\left(\bar{D}_{\sigma}h^{\mu}{}_{\beta}\right)\Gamma^{\beta(1)}_{\nu\rho}\\
&\quad-\bar{g}^{\rho\alpha}\bar{g}_{\alpha\mu}h^{\mu}{}_{\beta}R^{\beta(1)}_{\nu\rho\sigma}+\bar{g}^{\rho\alpha}\bar{g}_{\alpha\mu}\Gamma^{\mu(1)}_{\rho\lambda}\Gamma^{\lambda(1)}_{\sigma\nu}-\bar{g}^{\rho\alpha}\bar{g}_{\alpha\mu}\Gamma^{\mu(1)}_{\sigma\lambda}\Gamma^{\lambda(1)}_{\rho\nu}+\bar{g}^{\rho\alpha}h_{\alpha\mu}\bar{R}^{\mu}{}_{\nu\rho\sigma}\\
&\quad+\bar{g}^{\rho\alpha}h_{\alpha\mu}R^{\mu(1)}_{\nu\rho\sigma}-h^{\rho\alpha}\bar{g}_{\alpha\mu}\bar{R}^{\mu}{}_{\nu\rho\sigma}-h^{\rho\alpha}\bar{g}_{\alpha\mu}R^{\mu(1)}_{\nu\rho\sigma}+h^{\rho\alpha}h_{\alpha\mu}\bar{R}^{\mu}{}_{\nu\rho\sigma}\\
&\quad+h^{\rho}{}_{\lambda}h^{\lambda\alpha}\bar{g}_{\alpha\mu}\bar{R}^{\mu}{}_{\nu\rho\sigma}+O(h^3)\\
&= \bar{R}_{\nu\sigma}+R^{\rho(1)}_{\nu\rho\sigma}-\left(\bar{D}_{\rho}h^{\rho}{}_{\beta}\right)\Gamma^{\beta(1)}_{\nu\sigma}+\left(\bar{D}_{\sigma}h^{\rho}{}_{\beta}\right)\Gamma^{\beta(1)}_{\nu\rho}-h^{\rho}{}_{\beta}R^{\beta(1)}_{\nu\rho\sigma}+\Gamma^{\rho(1)}_{\rho\lambda}\Gamma^{\lambda(1)}_{\sigma\nu}\\
&\quad-\Gamma^{\rho(1)}_{\sigma\lambda}\Gamma^{\lambda(1)}_{\rho\nu}+\cancel{h^{\rho}{}_{\mu}\bar{R}^{\mu}{}_{\nu\rho\sigma}}+\cancel{h^{\rho}{}_{\mu}R^{\mu(1)}_{\nu\rho\sigma}}-\cancel{h^{\rho}{}_{\mu}\bar{R}^{\mu}{}_{\nu\rho\sigma}}-\cancel{h^{\rho}{}_{\mu}R^{\mu(1)}_{\nu\rho\sigma}}\\
&\quad+\cancel{h^{\rho\alpha}h_{\alpha\mu}\bar{R}^{\mu}{}_{\nu\rho\sigma}}+\cancel{h^{\rho}{}_{\lambda}h^{\lambda}{}_{\mu}\bar{R}^{\mu}{}_{\nu\rho\sigma}}+O(h^3)\\
&= \bar{R}_{\nu\sigma}+R^{\rho(1)}_{\nu\rho\sigma}-\left(\bar{D}_{\rho}h^{\rho}{}_{\beta}\right)\Gamma^{\beta(1)}_{\nu\sigma}+\left(\bar{D}_{\sigma}h^{\rho}{}_{\beta}\right)\Gamma^{\beta(1)}_{\nu\rho}-h^{\rho}{}_{\beta}R^{\beta(1)}_{\nu\rho\sigma}+\Gamma^{\rho(1)}_{\rho\lambda}\Gamma^{\lambda(1)}_{\sigma\nu}\\
&\quad-\Gamma^{\rho(1)}_{\sigma\lambda}\Gamma^{\lambda(1)}_{\rho\nu}+O(h^3).
\end{aligned}
$$

We can define the second order terms as $R^{(2)}_{\nu\sigma}$, so the Ricci tensor can be defined as

$$
R_{\nu\sigma}=\bar{R}_{\nu\sigma}+R^{\rho(1)}_{\nu\rho\sigma}+R^{(2)}_{\nu\sigma}+O(h^3). \tag{2.131}
$$

Now, let's find the $R^{(2)}_{\nu\sigma}$ explicitly

$$
\begin{aligned}
R^{(2)}_{\nu\sigma} &= -\left(\bar{D}_{\rho}h^{\rho}{}_{\beta}\right)\left[\frac{1}{2}\bar{g}^{\beta\alpha}\left(\bar{D}_{\nu}h_{\rho\alpha}+\bar{D}_{\rho}h_{\nu\alpha}-\bar{D}_{\alpha}h_{\nu\rho}\right)\right]\\
&\quad+\left(\bar{D}_{\sigma}h^{\rho}{}_{\beta}\right)\left[\frac{1}{2}\bar{g}^{\beta\alpha}\left(\bar{D}_{\nu}h_{\sigma\alpha}+\bar{D}_{\sigma}h_{\nu\alpha}-\bar{D}_{\alpha}h_{\nu\sigma}\right)\right]-h^{\rho}{}_{\beta}R^{\beta(1)}_{\nu\rho\sigma}\\
&\quad+\frac{1}{2}\bar{g}^{\rho\omega}\left(\bar{D}_{\rho}h_{\lambda\omega}+\bar{D}_{\lambda}h_{\rho\omega}-\bar{D}_{\omega}h_{\rho\lambda}\right)\frac{1}{2}\bar{g}^{\lambda\gamma}\left(\bar{D}_{\sigma}h_{\nu\gamma}+\bar{D}_{\nu}h_{\sigma\gamma}-\bar{D}_{\gamma}h_{\sigma\nu}\right)\\
&\quad-\frac{1}{2}\bar{g}^{\rho\omega}\left(\bar{D}_{\sigma}h_{\lambda\omega}+\bar{D}_{\lambda}h_{\sigma\omega}-\bar{D}_{\omega}h_{\sigma\lambda}\right)\frac{1}{2}\bar{g}^{\lambda\gamma}\left(\bar{D}_{\rho}h_{\nu\gamma}+\bar{D}_{\nu}h_{\rho\gamma}-\bar{D}_{\gamma}h_{\rho\nu}\right)\\
&= -\frac{1}{2}h^{\rho}{}_{\beta}\left(\bar{D}_{\rho}\bar{D}_{\sigma}h^{\beta}{}_{\nu}+\bar{D}_{\rho}\bar{D}_{\nu}h^{\beta}{}_{\sigma}-\bar{D}_{\rho}\bar{D}^{\beta}h_{\sigma\nu}-\bar{D}_{\sigma}\bar{D}_{\rho}h^{\beta}{}_{\nu}-\bar{D}_{\sigma}\bar{D}_{\nu}h^{\beta}{}_{\rho}+\bar{D}_{\sigma}\bar{D}^{\beta}h_{\rho\nu}\right)\\
&\quad+\frac{1}{2}\left(\bar{D}_{\sigma}h^{\rho\alpha}\bar{D}_{\nu}h_{\rho\alpha}+\bar{D}_{\sigma}h^{\rho\alpha}\bar{D}_{\rho}h_{\nu\alpha}-\bar{D}_{\sigma}h^{\rho\alpha}\bar{D}_{\alpha}h_{\nu\rho}-\bar{D}_{\rho}h^{\rho\alpha}\bar{D}_{\nu}h_{\sigma\alpha}-\bar{D}_{\rho}h^{\rho\alpha}\bar{D}_{\sigma}h_{\nu\alpha}\right.\\
&\quad\left.+\bar{D}_{\rho}h^{\rho\alpha}\bar{D}_{\alpha}h_{\nu\sigma}\right)+\frac{1}{4}\bar{g}^{\rho\omega}\bar{g}^{\lambda\gamma}\left(\bar{D}_{\rho}h_{\lambda\omega}\bar{D}_{\sigma}h_{\nu\gamma}+\bar{D}_{\rho}h_{\lambda\omega}\bar{D}_{\nu}h_{\sigma\gamma}-\bar{D}_{\rho}h_{\lambda\omega}\bar{D}_{\gamma}h_{\sigma\nu}\right.\\
&\quad+\bar{D}_{\lambda}h_{\rho\omega}\bar{D}_{\sigma}h_{\nu\gamma}+\bar{D}_{\lambda}h_{\rho\omega}\bar{D}_{\nu}h_{\sigma\gamma}-\bar{D}_{\lambda}h_{\rho\omega}\bar{D}_{\gamma}h_{\sigma\nu}-\bar{D}_{\omega}h_{\rho\lambda}\bar{D}_{\sigma}h_{\nu\gamma}-\bar{D}_{\omega}h_{\rho\lambda}\bar{D}_{\nu}h_{\sigma\gamma}
\end{aligned}
$$

$$+\bar{D}_\omega h_{\rho\lambda}\bar{D}_\gamma h_{\sigma\nu}-\bar{D}_\sigma h_{\lambda\omega}\bar{D}_\rho h_{\nu\gamma}-\bar{D}_\sigma h_{\lambda\omega}\bar{D}_\nu h_{\rho\gamma}+\bar{D}_\sigma h_{\lambda\omega}\bar{D}_\gamma h_{\rho\nu}-\bar{D}_\lambda h_{\sigma\omega}\bar{D}_\rho h_{\nu\gamma}$$
$$-\bar{D}_\lambda h_{\sigma\omega}\bar{D}_\nu h_{\rho\gamma}+\bar{D}_\lambda h_{\sigma\omega}\bar{D}_\gamma h_{\rho\nu}+\bar{D}_\omega h_{\sigma\lambda}\bar{D}_\rho h_{\nu\gamma}-\bar{D}_\omega h_{\sigma\lambda}\bar{D}_\nu h_{\rho\gamma}-\bar{D}_\omega h_{\sigma\lambda}\bar{D}_\gamma h_{\rho\nu}\big).$$

Let's rearrange the indices of the last equation,

$$\begin{aligned}
R^{(2)}_{\mu\nu}=&\frac{1}{4}\bar{g}^{\rho\sigma}\bar{g}^{\alpha\beta}\big(\bar{D}_\rho h_{\alpha\sigma}\bar{D}_\nu h_{\mu\beta}+\bar{D}_\rho h_{\alpha\sigma}\bar{D}_\mu h_{\nu\beta}-\bar{D}_\rho h_{\alpha\sigma}\bar{D}_\beta h_{\nu\mu}\\
&+\bar{D}_\alpha h_{\rho\sigma}\bar{D}_\nu h_{\mu\beta}+\bar{D}_\alpha h_{\rho\sigma}\bar{D}_\mu h_{\nu\beta}-\bar{D}_\alpha h_{\rho\sigma}\bar{D}_\beta h_{\nu\mu}-\bar{D}_\sigma h_{\rho\alpha}\bar{D}_\nu h_{\mu\beta}-\bar{D}_\sigma h_{\rho\alpha}\bar{D}_\mu h_{\nu\beta}\\
&+\bar{D}_\sigma h_{\rho\alpha}\bar{D}_\beta h_{\nu\mu}-\bar{D}_\nu h_{\alpha\sigma}\bar{D}_\rho h_{\mu\beta}-\bar{D}_\nu h_{\alpha\sigma}\bar{D}_\mu h_{\rho\beta}+\bar{D}_\nu h_{\alpha\sigma}\bar{D}_\beta h_{\rho\mu}-\bar{D}_\alpha h_{\nu\beta}\bar{D}_\rho h_{\mu\beta}\\
&-\bar{D}_\alpha h_{\nu\sigma}\bar{D}_\mu h_{\rho\beta}+\bar{D}_\alpha h_{\nu\sigma}\bar{D}_\beta h_{\rho\mu}+\bar{D}_\sigma h_{\nu\alpha}\bar{D}_\rho h_{\mu\beta}-\bar{D}_\sigma h_{\nu\alpha}\bar{D}_\mu h_{\rho\beta}-\bar{D}_\sigma h_{\nu\alpha}\bar{D}_\beta h_{\rho\nu}\big)\\
&+\frac{1}{2}\bar{g}^{\rho\sigma}\bar{g}^{\alpha\beta}\big(\bar{D}_\nu h_{\sigma\beta}\bar{D}_\mu h_{\rho\alpha}+\bar{D}_\nu h_{\sigma\beta}\bar{D}_\rho h_{\mu\alpha}-\bar{D}_\nu h_{\sigma\beta}\bar{D}_\alpha h_{\mu\rho}-\bar{D}_\rho h_{\sigma\beta}\bar{D}_\mu h_{\nu\alpha}\\
&-\bar{D}_\rho h_{\sigma\beta}\bar{D}_\nu h_{\mu\alpha}+\bar{D}_\rho h_{\sigma\beta}\bar{D}_\alpha h_{\mu\nu}\big)+\frac{1}{2}\bar{g}^{\rho\sigma}\bar{g}^{\alpha\beta}h_{\rho\alpha}\big(-\bar{D}_\sigma\bar{D}_\nu h_{\beta\mu}-\bar{D}_\sigma\bar{D}_\mu h_{\beta\nu}\\
&+\bar{D}_\sigma\bar{D}_\beta h_{\nu\mu}+\bar{D}_\nu\bar{D}_\sigma h_{\beta\mu}+\bar{D}_\nu\bar{D}_\mu h_{\beta\sigma}-\bar{D}_\nu\bar{D}_\beta h_{\sigma\mu}\big).
\end{aligned}$$

If we make the suitable cancellations in the last expression, we will get

$$\begin{aligned}
R^{(2)}_{\mu\nu}=\frac{1}{2}\bar{g}^{\rho\sigma}\bar{g}^{\alpha\beta}\Bigg[&\frac{1}{2}\bar{D}_\mu h_{\rho\alpha}\bar{D}_\nu h_{\sigma\beta}+\Big(\bar{D}_\rho h_{\nu\alpha}\Big)\Big(\bar{D}_\sigma h_{\mu\beta}-\bar{D}_\beta h_{\mu\sigma}\Big)\\
&+h_{\rho\alpha}\Big(\bar{D}_\nu\bar{D}_\mu h_{\sigma\beta}+\bar{D}_\beta\bar{D}_\sigma h_{\mu\nu}-\bar{D}_\beta\bar{D}_\nu h_{\mu\sigma}-\bar{D}_\beta\bar{D}_\mu h_{\nu\sigma}\Big)\\
&+\Big(\frac{1}{2}\bar{D}_\alpha h_{\rho\sigma}-\bar{D}_\rho h_{\alpha\sigma}\Big)\Big(\bar{D}_\nu h_{\mu\beta}+\bar{D}_\mu h_{\nu\beta}-\bar{D}_\beta h_{\mu\nu}\Big)\Bigg]. \qquad (2.132)
\end{aligned}$$

Now we can think about the question what the energy-momentum tensor of gravitational waves is. To understand the aim of this question, we firstly consider the situation where there is no external matter, $T_{\mu\nu}=0$ for the eqns.(2.121) and (2.122). Hence, we have

$$\bar{R}_{\mu\nu}=[R^{(2)}_{\mu\nu}]^{Low},$$

and we found that $R^{(2)}_{\mu\nu}=O\big((\partial h)^2\big)+O\big(h\partial^2 h\big)$ from eqn.(2.132). When there is no matter, we can write that

$$\bar{R}_{\mu\nu}\sim(\partial h)^2, \qquad (2.133)$$

so we can say that the $[R^{(2)}_{\mu\nu}]^{Low}$ has the order $(\partial h)^2$. Where the scale of variation of h is λ, and the scale of variation of the background metric is L_B, then we have

$$\partial\bar{g}_{\mu\nu}\sim\frac{1}{L_B}, \qquad (2.134)$$

and

$$\partial h\sim\frac{h}{\lambda}. \qquad (2.135)$$

The Ricci tensor of the background $\bar{R}_{\mu\nu}$ is constructed from the $\partial^2\bar{g}_{\mu\nu}$ then eqn.(2.134) implies that

$$\bar{R}_{\mu\nu} \sim \partial^2\bar{g}_{\mu\nu} \sim \frac{1}{L_B^2},$$

while from eqn.(2.135)

$$(\partial h)^2 \sim (\frac{h}{\lambda\!\!\!^-})^2.$$

Using the last two expressions in eqn.(2.133), we get

$$\frac{1}{L_B^2} \sim (\frac{h}{\lambda\!\!\!^-})^2, \tag{2.136}$$

which means that

$$h \sim \frac{\lambda\!\!\!^-}{L_B}. \tag{2.137}$$

On the other hand, the curvature is determined by the gravitational waves. Consider the situation $T_{\mu\nu} \neq 0$ which means there is matter, we can neglect the background curvature if we compare it to the contribution of matter sources. Then,

$$\frac{1}{L_B^2} \sim \frac{h^2}{\lambda\!\!\!^-{}^2} + (matter\ contribution) >> \frac{h^2}{\lambda\!\!\!^-{}^2}$$

implies that

$$h << \frac{\lambda\!\!\!^-}{L_B}. \tag{2.138}$$

The curvature is determined by the matter. As a result, we can understand why the linearized approximation of flat metric expansion cannot be used. In other words, to think $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$ makes $\frac{1}{L_B}$ zero. Also, from eqns.(2.137) and (2.138), it is obvious that $\frac{\lambda\!\!\!^-}{L_B}$ is at least order one.

Now, we consider eqn.(2.121). Suppose there is a clear-cut separation between $\lambda\!\!\!^-$ and L_B. To define a scale $\bar{l}$ such that $\lambda\!\!\!^- << \bar{l} << L_B$,and to average over a spatial volume with side $\bar{l}$, we can make the projection on the long-wavelength (or low-frequency) modes. If we find average the modes with wavelengths of order L_B, we will get a constant value. Because of this, we can say that there is no effect on them. On the other hand, modes with a reduced wavelength of order $\lambda\!\!\!^-$ oscillate very fast, and their average is zero. Next thing we will do is that introducing the time scale $\bar{t}$ which is $\bar{t} >> \frac{1}{f}$ (the period of the gravitational waves) and $\bar{t} << \frac{1}{f_B}$ (the typical time-scale of the background). We can therefore write eqn.(2.121) as

$$\bar{R}_{\mu\nu} = -\langle R^{(2)}_{\mu\nu}\rangle + \frac{8\pi G}{c^4}\langle T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\rangle, \tag{2.139}$$

where $\langle ... \rangle$ denote a spatial average.

Let's define an effective energy-momentum tensor of matter which is denoted by $\bar{T}_{\mu\nu}$,

$$\langle T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \rangle = \bar{T}_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} \bar{T} \tag{2.140}$$

where $\bar{T}$ denotes the trace of effective energy momentum tensor and defined as $\bar{T} = \bar{g}_{\mu\nu} \bar{T}^{\mu\nu}$. In the case, the fundamental energy momentum tensor $T^{\mu\nu}$ is constant when we average it. In this case,

$$\begin{aligned} \langle T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \rangle &\simeq T_{\mu\nu} - \frac{1}{2} \langle g_{\mu\nu} \rangle T \\ &= T_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} T, \end{aligned}$$

so $\bar{T}_{\mu\nu} \simeq T_{\mu\nu}$. Also, $\bar{T}_{\mu\nu}$ is a low-frequency quantity by the definition.

Let's define another quantity $t_{\mu\nu}$ as

$$t_{\mu\nu} = -\frac{c^4}{8\pi G} \langle R^{(2)}_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} R^{(2)} \rangle, \tag{2.141}$$

where

$$R^{(2)} = \bar{g}^{\mu\nu} R^{(2)}_{\mu\nu}, \tag{2.142}$$

and its trace is

$$\begin{aligned} t &= \bar{g}^{\mu\nu} t_{\mu\nu} \\ &= -\frac{c^4}{8\pi G} \langle \bar{g}^{\mu\nu} R^{(2)}_{\mu\nu} - \frac{1}{2} \bar{g}^{\mu\nu} \bar{g}_{\mu\nu} R^{(2)} \rangle \\ &= -\frac{c^4}{8\pi G} \langle R^{(2)} - \frac{1}{2} 4 R^{(2)} \rangle \\ &= \frac{c^4}{8\pi G} \langle R^{(2)} \rangle, \end{aligned} \tag{2.143}$$

where we used $\bar{g}^{\mu\nu} \bar{g}_{\mu\nu} = \delta^{\nu}{}_{\nu} = 4$. If we use eqn.(2.143) in eqn.(2.141), we will get

$$\begin{aligned} -\langle R^{(2)}_{\mu\nu} \rangle &= \frac{8\pi G}{c^4} (t_{\mu\nu}) - \frac{1}{2} \bar{g}_{\mu\nu} \langle R^{(2)} \rangle \\ &= \frac{8\pi G}{c^4} (t_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} t), \end{aligned} \tag{2.144}$$

where $\langle \bar{g}_{\mu\nu} R^{(2)} \rangle = \bar{g}_{\mu\nu} \langle R^{(2)} \rangle$ which means $\bar{g}_{\mu\nu}$ is constant under the averaging procedure. So eqn.(2.139) becomes

$$\bar{R}_{\mu\nu} = \frac{8\pi G}{c^4} (t_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} t) + \frac{8\pi G}{c^4} (\bar{T}_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} \bar{T}), \tag{2.145}$$

or, equivalently

$$\bar{R}_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} \bar{R} = \frac{8\pi G}{c^4} (\bar{T}_{\mu\nu} + t_{\mu\nu}). \tag{2.146}$$

2.4.3 The energy-momentum tensor of gravitational waves

In this subsection, we want to find the energy and momentum which are carried by the gravitational waves. Again, we will use the situation $T_{\mu\nu} = 0$ in which we consider the large distance from the sources. Suppose that the background space-time is flat, then $(\bar{D}_\mu \to \partial_\mu)$. In this case, we can write the second order Ricci tensor as

$$R^{(2)}_{\mu\nu} = \frac{1}{2}\eta^{\rho\sigma}\eta^{\alpha\beta}\Big[\frac{1}{2}\partial_\mu h_{\rho\alpha}\partial_\nu h_{\sigma\beta} + \partial_\rho h_{\nu\alpha}\big(\partial_\sigma h_{\mu\beta} - \partial_\beta h_{\mu\sigma}\big) + h_{\rho\alpha}\big(\partial_\nu\partial_\mu h_{\sigma\beta} + \partial_\beta\partial_\sigma h_{\mu\nu} - \partial_\beta\partial_\nu h_{\mu\sigma} - \partial_\beta\partial_\mu h_{\nu\sigma}\big) + \big(\frac{1}{2}\partial_\alpha h_{\rho\sigma} - \partial_\rho h_{\alpha\sigma}\big)\big(\partial_\nu h_{\mu\beta} + \partial_\mu h_{\nu\beta} - \partial_\beta h_{\mu\nu}\big)\Big].$$

If we make some suitable interchange between some dummy indices, we will easily get

$$R^{(2)}_{\mu\nu} = \frac{1}{2}\Big[\frac{1}{2}\partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta} + h^{\alpha\beta}\partial_\mu\partial_\nu h_{\alpha\beta} - h^{\alpha\beta}\partial_\nu\partial_\beta h_{\alpha\mu} - h^{\alpha\beta}\partial_\mu\partial_\beta h_{\alpha\nu} + h^{\alpha\beta}\partial_\alpha\partial_\beta h_{\mu\nu} + \partial^\beta h^\alpha{}_\nu\partial_\beta h_{\alpha\mu} - \partial^\beta h^\alpha{}_\nu\partial_\alpha h_{\beta\mu} - \partial_\beta h^{\alpha\beta}\partial_\nu h_{\mu\alpha} + \partial_\beta h^{\alpha\beta}\partial_\alpha h_{\mu\nu} - \partial_\beta h^{\alpha\beta}\partial_\mu h_{\alpha\nu} - \frac{1}{2}\partial^\alpha h\partial_\alpha h_{\mu\nu} + \frac{1}{2}\partial^\alpha h\partial_\nu h_{\alpha\mu} + \frac{1}{2}\partial^\alpha h\partial_\mu h_{\alpha\nu}\Big]. \quad (2.147)$$

In this section, one of our aim is to compute the $t_{\mu\nu}$ explicitly. It is the reason why we wrote $R^{(2)}_{\mu\nu}$ explicitly. We discussed before the fact that $h_{\mu\nu}$ has 10 degrees of freedom thanks to the symmetry property of it. In addition, in section (2.2), we saw that 2 of them are physical while the other eight of them are gauge modes. As a result, we can say that $t_{\mu\nu}$ has the contributions of both physical and gauge modes.

If we want to compute the contribution of the physical modes, we need to use the Lorenz gauge condition. Using it eliminates 4 degrees of freedom. Also, if we use the residual gauge conditions which are $\Box\xi_\mu = 0$ where $\xi_\mu = 0$ are the four gauge modes as discussed in section (2.2). Also, we had chosen the $\xi_\mu = 0$ such that $h = 0$. Then $\bar{h}_{\mu\nu} = h_{\mu\nu}$ implies that Lorenz condition becomes $\partial^\mu h_{\mu\nu} = 0$.

$$\langle R^{(2)}_{\mu\nu}\rangle = \int d^3x R^{(2)}_{\mu\nu},$$

where d^3x is the spatial volume element. Let's write the last equations explicitly

$$\langle R^{(2)}_{\mu\nu}\rangle = \frac{1}{2(volV)}\int_V dx^3\Big[\frac{1}{2}\partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta} + h^{\alpha\beta}\partial_\mu\partial_\nu h_{\alpha\beta} - h^{\alpha\beta}\partial_\nu\partial_\beta h_{\alpha\mu} - h^{\alpha\beta}\partial_\mu\partial_\beta h_{\alpha\nu} + h^{\alpha\beta}\partial_\alpha\partial_\beta h_{\mu\nu} + \partial^\beta h^\alpha{}_\nu\partial_\beta h_{\alpha\mu} - \partial^\beta h^\alpha{}_\nu\partial_\alpha h_{\beta\mu} - \partial_\beta h^{\alpha\beta}\partial_\nu h_{\mu\alpha} + \partial_\beta h^{\alpha\beta}\partial_\alpha h_{\mu\nu} - \partial_\beta h^{\alpha\beta}\partial_\mu h_{\alpha\nu} - \cancel{\frac{1}{2}\partial^\alpha h\partial_\alpha h_{\mu\nu}} + \cancel{\frac{1}{2}\partial^\alpha h\partial_\nu h_{\alpha\mu}} + \cancel{\frac{1}{2}\partial^\alpha h\partial_\mu h_{\alpha\nu}}\Big]$$

$$= \frac{1}{2(volV)}\int dx\int dy\int dz\Big[-\frac{1}{2}\partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta}+\partial_\mu\big(h^{\alpha\beta}\partial_\nu h_{\alpha\beta}\big)-\partial_\beta\big(h^{\alpha\beta}\partial_\nu h_{\alpha\mu}\big)$$
$$-\partial_\beta\big(h^{\alpha\beta}\partial_\mu h_{\alpha\nu}\big)+\partial_\beta\big(h^{\alpha\beta}\partial_\alpha h_{\mu\nu}\big)+\partial^\beta\big(h^\alpha{}_\nu\partial_\beta h_{\alpha\mu}\big)-h^\alpha{}_\nu\partial^\beta\partial_\beta h_{\alpha\mu}$$
$$-\partial^\beta\big(h^\alpha{}_\nu\partial_\alpha h_{\beta\mu}\big)+h^\alpha{}_\nu\partial^\beta\partial_\alpha h_{\beta\mu}\Big],$$

where $volV$ is the volume of the region V in which we take the integral over it. Using the fact that $h_{\mu\nu}=h_{\mu\nu}(t-\frac{z}{c})$, we can write

$$\langle R^{(2)}_{\mu\nu}\rangle = \frac{1}{2(volV)}\int dx\int dy\int dz\Big[-\frac{1}{2}\partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta}+\partial_0\big(h^{\alpha\beta}\partial_\nu h_{\alpha\beta}\big)+\partial_z\big(h^{\alpha\beta}\partial_\nu h_{\alpha\beta}\big)$$
$$-\partial_0\big(h^{\alpha 0}\partial_\nu h_{\alpha\mu}\big)-\partial_z\big(h^{\alpha z}\partial_\nu h_{\alpha\mu}\big)-\partial_0\big(h^{\alpha 0}\partial_\mu h_{\alpha\nu}\big)-\partial_z\big(h^{\alpha z}\partial_\mu h_{\alpha\nu}\big)$$
$$+\partial_0\big(h^{\alpha 0}\partial_\alpha h_{\mu\nu}\big)+\partial_z\big(h^{\alpha z}\partial_\alpha h_{\mu\nu}\big)+\partial^0\big(h^\alpha{}_\nu\partial_0 h_{\alpha\mu}\big)+\partial^z\big(h^\alpha{}_\nu\partial_z h_{\alpha\mu}\big)-\cancel{h^\alpha{}_\nu\Box h_{\alpha\mu}}$$
$$-\partial^0\big(h^\alpha{}_\nu\partial_\alpha h_{o\mu}\big)-\partial^z\big(h^\alpha{}_\nu\partial_\alpha h_{z\mu}\big)+\cancel{h^\alpha{}_\nu\partial_\alpha\partial^\beta h_{\beta\mu}}\Big],$$

where we used the plane wave equation $\Box h_{\alpha\mu}=0$ and the Lorenz gauge $\partial^\beta h_{\beta\mu}=0$. If we use the fact that the boundary terms vanish when the size of the box which we take integration over is infinitely larger than $\lambda\!\!\!^{-}$, we get

$$\langle R^{(2)}_{\mu\nu}\rangle = \frac{1}{2(volV)}\int dx\int dy\int dz\Big[-\frac{1}{2}\partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta}\Big].$$

To get the last expression, we also used that $\partial_0 h_{\mu\nu}=-\partial_z h_{\mu\nu}$ since $h_{\mu\nu}$ is the function of $(t-z)$. As a result,

$$\langle R^{(2)}_{\mu\nu}\rangle = -\frac{1}{4}\langle\partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta}\rangle. \tag{2.148}$$

Similarly, we can compute the value of the $\langle R^{(2)}\rangle$

$$\begin{aligned}\langle R^{(2)}\rangle &= \langle\bar{g}^{\mu\nu}R^{(2)}_{\mu\nu}\rangle\\ &= -\frac{1}{4}\bar{g}^{\mu\nu}\langle\partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta}\rangle\\ &= -\frac{1}{4}\langle\partial^\nu h_{\alpha\beta}\partial_\nu h^{\alpha\beta}\rangle\\ &= -\frac{1}{4}\langle\partial^\nu\big(h_{\alpha\beta}\partial_\nu h^{\alpha\beta}\big)-h_{\alpha\beta}\partial^\nu\partial_\nu h^{\alpha\beta}\rangle,\end{aligned}$$

by using same arguments which we used to find expression (2.48), so

$$\langle R^{(2)}\rangle = 0. \tag{2.149}$$

Using the results (2.148) and (2.149) in eqn.(2.141), we get

$$\begin{aligned}t_{\mu\nu} &= -\frac{c^4}{8\pi G}\big(-\frac{1}{4}\big)\langle\partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta}\rangle\\ &= \frac{c^4}{32\pi G}\langle\partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta}\rangle.\end{aligned} \tag{2.150}$$

Secondly, if we use the residual gauge (2.35) in the expression (2.150), we get

$$\begin{aligned}
\delta t_{\mu\nu} &= \frac{c^4}{32\pi G}\Big[\langle \partial_\mu h_{\alpha\beta}\partial_\nu(\delta h^{\alpha\beta}) + \partial_\nu h_{\alpha\beta}\partial_\mu(\delta h^{\alpha\beta})\rangle\Big] \\
&= \frac{c^4}{32\pi G}\Big[\langle \partial_\mu h_{\alpha\beta}\partial_\nu(\partial^\alpha\xi^\beta + \partial^\beta\xi^\alpha) + \partial_\nu h_{\alpha\beta}\partial_\mu(\partial^\alpha\xi^\beta + \partial^\beta\xi^\alpha)\rangle\Big] \\
&= \frac{c^4}{32\pi G}\Big[\langle \partial_\mu h_{\alpha\beta}\partial_\nu\partial^\alpha\xi^\beta\rangle + \langle \partial_\mu h_{\alpha\beta}\partial_\nu\partial^\beta\xi^\alpha\rangle + \langle \partial_\nu h_{\alpha\beta}\partial_\mu\partial^\alpha\xi^\beta\rangle + \langle \partial_\nu h_{\alpha\beta}\partial_\mu\partial^\beta\xi^\alpha\rangle\Big] \\
&= \frac{c^4}{16\pi G}\Big[\langle \partial_\mu h_{\alpha\beta}\partial_\nu\partial^\alpha\xi^\beta\rangle + \langle \partial_\nu h_{\alpha\beta}\partial_\mu\partial^\alpha\xi^\beta\rangle\Big] \\
&== \frac{c^4}{16\pi G}\Big[\cancel{\langle \partial^\alpha(\partial_\mu h_{\alpha\beta}\partial_\nu\xi^\beta)\rangle} + \cancel{\langle \partial^\alpha(\partial_\nu h_{\alpha\beta}\partial_\mu\xi^\beta)\rangle}\Big] \\
&= 0,
\end{aligned}$$

where we again used the Lorenz gauge. Consequently, we eliminated four degrees of freedom using the Lorenz gauge and eliminated the other four of them using infinitesimal gauge choice that means there are only two physical modes h_{ij}^{TT}. Then expression (2.150) becomes

$$t_{\mu\nu} = \frac{c^4}{32\pi G}\langle \partial_\mu h_{ij}^{TT}\partial_\nu h_{ij}^{TT}\rangle. \tag{2.151}$$

Now, we can compute the gauge invariant energy density t^{00},

$$\begin{aligned}
t^{00} &= \frac{c^4}{32\pi G}\langle \partial_0 h_{ij}^{TT}\partial_0 h_{ij}^{TT}\rangle \\
&= \frac{c^2}{32\pi G}\langle \dot{h}_{ij}^{TT}\dot{h}_{ij}^{TT}\rangle,
\end{aligned} \tag{2.152}$$

where $\dot{h}_{ij}^{TT} = \frac{1}{c}\partial_0 h_{ij}^{TT} = \partial_t h_{ij}^{TT}$, and using eqn.(2.148)

$$\begin{aligned}
t^{00} &= \frac{c^2}{32\pi G}\langle \dot{h}_{11}^{TT}\dot{h}_{11}^{TT} + \dot{h}_{12}^{TT}\dot{h}_{12}^{TT} + \dot{h}_{21}^{TT}\dot{h}_{21}^{TT} + \dot{h}_{22}^{TT}\dot{h}_{22}^{TT}\rangle \\
&= \frac{c^2}{32\pi G}\langle \dot{h}_+^2 + \dot{h}_x^2 + \dot{h}_x^2 + (-\dot{h}_+)^2\rangle \\
&= \frac{c^2}{16\pi G}\langle \dot{h}_+^2 + \dot{h}_x^2\rangle.
\end{aligned} \tag{2.153}$$

Since h_{ij}^{TT} is a function of the $(t - \frac{z}{c})$ for a plane wave traveling along the z-direction, it is obvious $t^{01} = t^{02} = 0$. Also, since $\partial_z h_{ij}^{TT} = -\partial_0 h_{ij}^{TT} = \partial^0 h_{ij}^{TT}$, we have $t^{00} = t^{03}$. Using the Bianchi identity, we can get

$$\bar{D}^\mu\big(\bar{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{R}\big) = 0,$$

so from eqn.(2.146), we have

$$\bar{D}^\mu\big(\bar{T}_{\mu\nu} + \bar{t}_{\mu\nu}\big) = 0. \tag{2.154}$$

If we look at the last expression at large distances from the source with flat background, we will get

$$\partial^{\mu} t_{\mu\nu} = 0 \tag{2.155}$$

since $\bar{T}_{\mu\nu} = 0$ and $\bar{D}^{\mu} \to \partial^{\mu}$.

The energy flux

Let's start with the conservation of the energy-momentum tensor which is given by eqn.(2.155),

$$\partial^0 t_{0\nu} + \partial^i t_{i\nu} = 0.$$

If we look at it in the situation $\nu = 0$, then

$$\partial_0 t^{00} + \partial_i t_{i0} = 0$$

implies that

$$\int_V d^3x \, (\partial_0 t^{00} + \partial_i t_{i0}) = 0 \tag{2.156}$$

where V is a spatial volume in the far region, which has the boundary $\partial V = S$. The gravitational energy inside the volume V is given by

$$E_V = \int_V d^3x \, t^{00}. \tag{2.157}$$

Taking the time derivative of last expression, we get

$$\partial_0 E_V = \int_V d^3x \, \partial_0 t^{00}. \tag{2.158}$$

Using eqn.(2.157) and $\partial_0 = \frac{1}{c}\frac{d}{dt}$, we get

$$\begin{aligned} \frac{1}{c}\frac{dE_V}{dt} &= -\int_V d^3x \, \partial_i t^{0i} \\ &= -\int_S dA \, n_i t^{0i} \end{aligned} \tag{2.159}$$

where n^i is the outer normal to the surface and dA is the surface area element. Say S be a spherical surface at a large distance r from the source, then $dA = r^2 \sin\theta d\theta d\phi = r^2 d\Omega$. Then eqn.(2.159) becomes

$$\frac{dE_V}{dt} = -c \int_S dA \, t^{0r} \tag{2.160}$$

where

$$t^{0r} = \frac{c^4}{32\pi G} \langle \partial^0 h_{ij}^{TT} \frac{\partial}{\partial_r} h_{ij}^{TT} \rangle. \tag{2.161}$$

A gravitational wave which propagates radially outward has the form

$$h_{ij}^{TT}(t,r) = \frac{1}{r} f_{ij}(t - \frac{r}{c}). \tag{2.162}$$

To compute eqn.(2.161), we firstly compute some arguments which we need

$$\frac{\partial}{\partial_r} h_{ij}^{TT}(t,r) = -\frac{1}{r^2} f_{ij}(t-\frac{r}{c}) + \frac{1}{r}\frac{\partial}{\partial_r} f_{ij}(t-\frac{r}{c}).$$

Also,

$$\begin{cases} \frac{\partial}{\partial_r} f_{ij}(t-\frac{r}{c}) = \frac{\partial f_{ij}}{\partial_u}\frac{\partial u}{\partial_r} = -\frac{1}{c}\frac{\partial f_{ij}}{\partial_u} \\ \frac{\partial}{\partial_t} f_{ij}(t-\frac{r}{c}) = \frac{\partial f_{ij}}{\partial_u}\frac{\partial u}{\partial_t} = \frac{\partial f_{ij}}{\partial_u} \end{cases}$$

imply that

$$\frac{\partial}{\partial_r} f_{ij}(t-\frac{r}{c}) = -\frac{1}{c}\frac{\partial}{\partial_t} f_{ij}(t-\frac{r}{c}), \tag{2.163}$$

where $u(t,r) = t - \frac{r}{c}$. Therefore,

$$\begin{aligned} \frac{\partial}{\partial_r} h_{ij}^{TT}(t,r) &= -\frac{1}{r^2} f_{ij}(t-\frac{r}{c}) - \frac{1}{r}\frac{1}{c}\frac{\partial}{\partial_t} f_{ij}(t-\frac{r}{c}) \\ &= -\partial_0\big[\frac{1}{r} f_{ij}(t-\frac{r}{c})\big] + O(\frac{1}{r^2}) \\ &= -\partial_0 h_{ij}^{TT}(t,r) + O(\frac{1}{r^2}) \\ &= \partial^0 h_{ij}^{TT}(t,r) + O(\frac{1}{r^2}). \end{aligned} \tag{2.164}$$

Using this expression in eqn.(2.161), we get

$$\begin{aligned} t^{0r} &= \frac{c^4}{32\pi G}\langle \partial^0 h_{ij}^{TT} \partial^0 h_{ij}^{TT}\rangle \\ &= t^{00}. \end{aligned}$$

Now, we can go back to eqn.(2.160) which becomes

$$\frac{dE_V}{dt} = -c\int_S dA\, t^{00}. \tag{2.165}$$

It is obvious that $\frac{dE_V}{dt} < 0$ from the last equation, so there is an important result which is E_V decreases in time. Since E_V decreases, we can say that the outward-propagating gravitational waves carry away an energy flux

$$\begin{aligned} \frac{dE}{dAdt} &= ct^{00} \\ &= \frac{c^3}{32\pi G}\langle \dot{h}_{ij}^{TT} \dot{h}_{ij}^{TT}\rangle, \end{aligned} \tag{2.166}$$

or,

$$\frac{dE}{dt} = \frac{c^3 r^2}{32\pi G}\int d\Omega \langle \dot{h}_{ij}^{TT} \dot{h}_{ij}^{TT}\rangle. \tag{2.167}$$

Using eqn.(2.153) in the last expression, we get

$$\frac{dE}{dAdt} = \frac{c^3}{16\pi G}\langle \dot{h}_+^2 + \dot{h}_x^2\rangle. \tag{2.168}$$

As a result, the total energy which flows through dA is

$$\frac{dE}{dA} = \frac{c^3}{16\pi G}\int_{-\infty}^{\infty} dt\langle \dot{h}_+^2 + \dot{h}_x^2\rangle. \tag{2.169}$$

The reason why we take the integration over from $t = -\infty$ to $t = \infty$ is that we want to resolve all possible frequencies. However, we will integrate the signal only over a certain interval $\triangle t$. We saw that the average in eqn.(2.169) is a temporal average over a few periods in the previous section. Therefore, we can omit the average in eqn.(2.169),

$$\frac{dE}{dA} = \frac{c^3}{16\pi G}\int_{-\infty}^{\infty} dt\big(\dot{h}_+^2 + \dot{h}_x^2\big). \tag{2.170}$$

From (2.86) and (2.90),

$$h_+ = \int_{-\infty}^{\infty} df\tilde{h}_+(f)e^{-2i\pi ft},$$

$$\dot{h}_+ = \frac{d}{dt}\Big(\int_{-\infty}^{\infty} df\tilde{h}_+(f)e^{-2i\pi ft}\Big) = \int_{-\infty}^{\infty} df(-2i\pi f)\tilde{h}_+(f)e^{-2i\pi ft}.$$

Similarly,

$$\dot{h}_x = \int_{-\infty}^{\infty} df(-2i\pi f)\tilde{h}_x(f)e^{-2i\pi ft}.$$

To use the last two result in eqn.(2.170), we will firstly compute

$$\begin{aligned}
\int_{-\infty}^{\infty} dt\dot{h}_+^2 &= \int_{-\infty}^{\infty} dt\left[\Big(\int_{-\infty}^{\infty} df(-2i\pi f)\tilde{h}_+(f)e^{-2i\pi ft}\Big)^2\right]\\
&= \int_{-\infty}^{\infty} dt\Big(\int_{-\infty}^{\infty} df(-2i\pi f)\tilde{h}_+(f)e^{-2i\pi ft}\Big)\Big(\int_{-\infty}^{\infty} df'(2i\pi f')\tilde{h}_+^*(f')e^{2i\pi f't}\Big)\\
&= \int_{-\infty}^{\infty} df(-2i\pi f)\tilde{h}_+(f)\int_{-\infty}^{\infty} df'(2i\pi f')\tilde{h}_+^*(f')\int_{-\infty}^{\infty} dte^{2i\pi(f'-f)t}\\
&= \int_{-\infty}^{\infty} df(-2i\pi f)\tilde{h}_+(f)\int_{-\infty}^{\infty} df'(2i\pi f')\tilde{h}_+^*(f')\delta(f'-f)\\
&= \int_{-\infty}^{\infty} df(2\pi f)^2|\tilde{h}_+(f)|^2,
\end{aligned}$$

where we used the definition of the Dirac-delta function and superscript (*) denotes the complex conjugation. Similarly, we can find

$$\int_{-\infty}^{\infty} dt\dot{h}_x^2 = \int_{-\infty}^{\infty} df(2\pi f)^2|\tilde{h}_x(f)|^2,$$

by using the same technique. Hence, eqn.(2.170) becomes

$$\begin{aligned}
\frac{dE}{dA} &= \frac{c^3}{16\pi G}\int_{-\infty}^{\infty} df(2\pi f)^2\big(|\tilde{h}_+(f)|^2 + |\tilde{h}_x(f)|^2\big)\\
&= \frac{\pi c^3}{4G}\int_{-\infty}^{\infty} dff^2\big(|\tilde{h}_+(f)|^2 + |\tilde{h}_x(f)|^2\big). \qquad (2.171)
\end{aligned}$$

If we look at only the physical situations which is $f > 0$, then it becomes

$$\frac{dE}{dA} = \frac{\pi c^3}{2G}\int_0^{\infty} dff^2\big(|\tilde{h}_+(f)|^2 + |\tilde{h}_x(f)|^2\big). \tag{2.172}$$

Hence,

$$\frac{dE}{dAdf} = \frac{\pi c^3}{2G} f^2 \big(|\tilde{h}_+(f)|^2 + |\tilde{h}_x(f)|^2\big). \tag{2.173}$$

The integration of $\frac{dE}{df}$ which is the energy spectrum over the positive frequencies gives us the total energy. From eqn.(2.173),

$$\frac{dE}{df} = \frac{\pi c^3}{2G} f^2 r^2 \int d\Omega \big(|\tilde{h}_+(f)|^2 + |\tilde{h}_x(f)|^2\big). \tag{2.174}$$

Now, we can compute the flux of momentum using the same way which we used to find the energy flux. By definition, the momentum of the gravitational waves inside a spherical shell which has the volume V at large distance from the source is

$$P_V^k = \frac{1}{c} \int_V d^3x \, t^{0k}, \tag{2.175}$$

where k is the spatial index ($k = 1, 2, 3$).

2.5 Propagation in Curved Space-time

In the last section, we gave our attention to low-modes eqn.(2.121). Now, we will focus on eqn.(2.121) which is the high-modes equation. First of all, we again look at the case where there is no matter, $T_{\mu\nu} = 0$, so

$$R^{(1)}_{\mu\nu} = -\big[R^{(2)}_{\mu\nu}\big]^{High}. \tag{2.176}$$

There are two small parameters $h \equiv O(h_{\mu\nu})$ and $\frac{\lambda}{L_B}$ (or $\frac{f_B}{f}$) in the short-wave (high-frequency) expansion. From (2.137) the Einstein equation shows us these two parameters have the same order of the magnitude, $h \sim \frac{\lambda}{L_B}$. Let's define a single parameter which we denote by ε such that

$$\varepsilon = O(h) = O(\frac{\lambda}{L_B}). \tag{2.177}$$

For simplicity, we use units $L_B = 1$ when we compare the orders. Hence, we have $\varepsilon \sim h \sim \lambda$. If we look at eqn.(2.126), we will see that

$$R^{(1)}_{\mu\nu} \sim \partial^2 h \sim \frac{h}{\lambda^2} \sim \frac{1}{\varepsilon} \tag{2.178}$$

where we used the fact that the scale of variation of h is λ. Also, since

$$R^{(2)}_{\mu\nu} \sim \partial^2 h^2 \sim \frac{h^2}{\lambda^2} \sim 1, \tag{2.179}$$

$\big[R^{(2)}_{\mu\nu}\big]^{High}$ is at most $O(1)$. It can be omitted in eqn.(2.165) when we compare it with the leading term of $R^{(1)}_{\mu\nu}$ which is $O(\frac{1}{\varepsilon})$. Also, if we look at eqn.(2.122) with this leading term, then it becomes

$$\left[R^{(1)}_{\mu\nu}\right]_{\frac{1}{\varepsilon}} = 0 \tag{2.180}$$

where subscript $\frac{1}{\varepsilon}$ means that we must extract the $O(\frac{1}{\varepsilon})$ part. Let's write eqn.(2.180) explicitly

$$\eta^{\rho\sigma}\left(\partial_\rho\partial_\nu h_{\mu\sigma} + \partial_\rho\partial_\mu h_{\nu\sigma} - \partial_\nu\partial_\mu h_{\rho\sigma} - \partial_\rho\partial_\sigma h_{\mu\nu}\right) \simeq 0, \tag{2.181}$$

where we changed the covariant derivative with regular derivative, and changed the metric $\bar{g}_{\mu\nu}$ with Minkowski flat space metric $\eta_{\mu\nu}$.

In section (1.1), we constructed the propagation equation for the field $h_{\mu\nu}$ in a flat background by using the linearized theory, it is obvious that eqn. (2.181) is the same equation with it. Defining $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$ and using the Lorenz gauge condition, we can easily get

$$\Box\bar{h}_{\mu\nu} \simeq 0, \tag{2.182}$$

where $\Box = \partial_\mu\partial^\mu$ is the flat space d'Alembertian. As a result, since this equation is the same equation as eqn.(2.60) where $T_{\mu\nu} = 0$, we can say that the high-frequency eqn.(2.122) is a wave equation for the perturbation $h_{\mu\nu}$.

Now we can look at the situation there is matter which means $T_{\mu\nu} \neq 0$. This matter will dominate the curvature, so eqn.(2.121) which is known as low-frequency equation becomes

$$\bar{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{R} \simeq \frac{8\pi G}{c^4}\bar{T}_{\mu\nu}. \tag{2.183}$$

For the frequency case, we know that $h << \frac{\lambda}{L_B} << 1$ from eqn.(2.138) that means the expansion in h and in $\frac{\lambda}{L_B}$ are not same. If we only use the liner terms with respect to h, and we make an expansion with respect to $\frac{\lambda}{L_B}$, eqn.(2.122) becomes

$$R^{(1)}_{\mu\nu} = 0, \tag{2.184}$$

where we limit only to the leading and next-to-leading order in $\frac{\lambda}{L_B}$. Since $\left[R^{(2)}_{\mu\nu}\right]^{High}$ has the square power of h, we can omit it. Also, if we use the fact that $g_{\mu\nu}T = (\bar{g}_{\mu\nu} + h_{\mu\nu})T$, we can say that it has high-frequency part $O(h)$, and there is another high-frequency part with order h which based on multiplying $h_{\mu\nu}$ with low-frequency part of T. Hence,

$$\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)^{High} = O(\frac{h}{L_B^2}). \tag{2.185}$$

Since $R^{(1)}_{\mu\nu} \sim \partial^2 h \sim \frac{h}{\lambda^2}$, the order of eqn.(2.85) is smaller than the order of $R^{(1)}_{\mu\nu}$. When there is matter, the background metric $\bar{g}_{\mu\nu}$ will be different from the flat metric $\eta^{\mu\nu}$. Hence, $R^{(1)}_{\mu\nu}$ is a covariant quantity with respect to background metric. From eqn.(2.126), we can write $R^{(1)}_{\mu\nu}$ explicitly,

$$\bar{g}^{\rho\sigma}\big(\bar{D}_\rho \bar{D}_\nu h_{\mu\sigma} + \bar{D}_\rho \bar{D}_\mu h_{\nu\sigma} - \bar{D}_\nu \bar{D}_\mu h_{\rho\sigma} - \bar{D}_\rho \bar{D}_\sigma h_{\mu\nu}\big) = 0. \tag{2.186}$$

We already made discussions of the flat space metric version of this last equation, in section (1.1). In here, $\eta_{\mu\nu}$ will be replaced by $\bar{g}_{\mu\nu}$. Defining $h = \bar{g}^{\mu\nu} h_{\mu\nu}$ and

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu} h, \tag{2.187}$$

eqn.(1.186) will be simpler. Also, we can use the condition

$$\bar{D}^\nu \bar{h}_{\mu\nu} = 0 \tag{2.188}$$

which is still called Lorenz gauge. Let's use this gauge condition, definition (2.187) and in eqn.(2.186),

$$\begin{aligned}
0 &= \bar{D}^\rho \bar{D}_\nu h_{\mu\rho} + \bar{D}^\rho \bar{D}_\mu h_{\nu\rho} - \bar{D}_\nu \bar{D}_\mu h - \bar{D}^\rho \bar{D}_\rho h_{\mu\nu} \\
&= \bar{D}^\rho \bar{D}_\rho \big(h_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu} h\big) + \frac{1}{2}\bar{g}_{\mu\nu}\bar{D}^\rho \bar{D}_\rho h + \bar{D}_\nu \bar{D}_\mu h - \bar{D}^\rho \bar{D}_\nu \big(h_{\mu\rho} - \frac{1}{2}\bar{g}_{\mu\rho} h\big) \\
&\quad - \frac{1}{2}\bar{g}_{\mu\rho}\bar{D}^\rho \bar{D}_\nu h - \bar{D}^\rho \bar{D}_\mu \big(h_{\nu\rho} - \frac{1}{2}\bar{g}_{\nu\rho} h\big) - \frac{1}{2}\bar{g}_{\nu\rho}\bar{D}^\rho \bar{D}_\mu h \\
&= \bar{D}^\rho \bar{D}_\rho \bar{h}_{\mu\nu} + \frac{1}{2}\bar{g}_{\mu\nu}\bar{D}^\rho \bar{D}_\rho h + \frac{1}{2}\cancel{\big[\bar{D}_\nu, \bar{D}_\mu\big]} h - \big[\bar{D}^\rho, \bar{D}_\nu\big]\bar{h}_{\mu\rho} - \big[\bar{D}^\rho, \bar{D}_\mu\big]\bar{h}_{\nu\rho}
\end{aligned}$$

to cancel the third term of the last expression, we used the fact that covariant derivative commutes for the scalar. Also, if we use the expression

$$\big[\bar{D}_\nu, \bar{D}_\rho\big]\bar{h}_\mu{}^\rho = h^{\rho\alpha}\bar{R}_{\rho\mu\alpha\nu} - \bar{h}_\mu{}^\tau \bar{R}_{\tau\nu},$$

then we get

$$\bar{D}^\rho \bar{D}_\rho \bar{h}_{\mu\nu} + 2\bar{R}_{\mu\rho\nu\sigma}\bar{h}^{\rho\sigma} - \bar{R}_{\mu\rho}\bar{h}^\rho{}_\nu - \bar{R}_{\nu\rho}\bar{h}^\rho{}_\mu = 0, \tag{2.189}$$

where we used the residual gauge to make $h_{\mu\nu}$ traceless.

If we look at the situation outside the matter which means $\bar{T}_{\mu\nu} = 0$, then the Einstein equation (2.183) for the background implies that $\bar{R}_{\mu\nu} = 0$. In other words, $\bar{R}_{\mu\nu}$ has the terms which are contributions only from $\big[R^{(2)}_{\mu\nu}\big]^{Low}$ if we look at eqn.(2.121), so $\bar{R}_{\mu\nu} = O(\frac{h^2}{\lambda^2})$. Since we are only interested the linear order in h, $\bar{R}_{\mu\rho}\bar{h}^\rho{}_\nu$ and $\bar{R}_{\nu\rho}\bar{h}^\rho{}_\mu$ can be canceled in eqn.(2.189). In addition, $\bar{R}_{\mu\rho\nu\sigma} = O(\frac{1}{L_B^2})$ implies that $\bar{R}_{\mu\rho\nu\sigma}\bar{h}^{\rho\sigma} = O(\frac{h}{L_B^2})$, but on the other hand $\bar{D}^\rho \bar{D}_\rho \bar{h}_{\mu\nu} = O(\frac{h}{\lambda^2})$. We already have a restriction which we study order in $\frac{\lambda}{L_B}$, so we have

$$\bar{D}^{\rho}\bar{D}_{\rho}\bar{h}_{\mu\nu} = 0. \tag{2.190}$$

Equations (2.188) and (2.190) give us the propagation of gravitational waves in the curved background. Consequently, if we separate the Einstein equations into two parts where the first part is a low-frequency and second one is a high-frequency part, then we can say that the low-frequency part gives us the information about the effect of gravitational waves and effect of matter on the background space-time; on the other hand, the high-frequency part gives us a wave equation in curved space. We can solve this curved-space equation using the eikonal approximation of geometric optics. In the next section, we will do this.

2.5.1 Geometric Optics in Curved Space

Electromagnetic waves

The action of the electromagnetic field in the curved space is

$$S = -\frac{1}{4}\int d^4x\sqrt{-\bar{g}}\bar{g}_{\mu\alpha}\bar{g}_{\nu\beta}F^{\mu\nu}F^{\alpha\beta}, \tag{2.191}$$

and it is known that its variation gives the equation of motion

$$\bar{D}_{\mu}\left(\bar{D}^{\mu}A^{\nu} - \bar{D}^{\nu}A^{\mu}\right) = 0, \tag{2.192}$$

where we use $\bar{g}_{\mu\nu}$ to raise and lower the indices [5]. The result (2.192) is the generalization of Maxwell equations $\partial_{\mu}F^{\mu\nu}$. The curved space generalization of the Lorenz gauge on the four vector potential A^{μ} is

$$\bar{D}_{\mu}A^{\mu} = 0. \tag{2.193}$$

Lets now compute the expression $\left[\bar{D}_{\mu}, \bar{D}^{\nu}\right]A^{\mu}$. To do this, let's firstly compute

$$\begin{aligned}
\bar{D}_{\mu}\bar{D}^{\nu}A^{\mu} &= \bar{g}^{\nu\alpha}\bar{D}_{\mu}\bar{D}_{\alpha}A^{\mu} \\
&= \bar{g}^{\nu\alpha}\left[\partial_{\mu}\left(\bar{D}_{\alpha}A^{\mu}\right) + \bar{\Gamma}^{\mu}{}_{\mu\lambda}\left(\bar{D}_{\alpha}A^{\lambda}\right) - \bar{\Gamma}^{\lambda}{}_{\mu\alpha}\left(\bar{D}_{\lambda}A^{\mu}\right)\right] \\
&= \bar{g}^{\nu\alpha}\left[\partial_{\mu}\left(\partial_{\alpha}A^{\mu} + \bar{\Gamma}^{\mu}{}_{\alpha\beta}A^{\beta}\right) + \bar{\Gamma}^{\mu}{}_{\mu\lambda}\left(\partial_{\alpha}A^{\lambda} + \bar{\Gamma}^{\lambda}{}_{\alpha\beta}A^{\beta}\right)\right. \\
&\quad \left. - \bar{\Gamma}^{\lambda}{}_{\mu\alpha}\left(\partial_{\lambda}A^{\mu} + \bar{\Gamma}^{\mu}{}_{\lambda\beta}A^{\beta}\right)\right] \\
&= \bar{g}^{\nu\alpha}\left[\partial_{\mu}\partial_{\alpha}A^{\mu} + \bar{\Gamma}^{\mu}{}_{\alpha\beta}\partial_{\mu}A^{\beta} + \bar{\Gamma}^{\mu}{}_{\mu\lambda}\partial_{\alpha}A^{\lambda} - \bar{\Gamma}^{\lambda}{}_{\mu\alpha}\partial_{\lambda}A^{\mu}\right. \\
&\quad \left. + \left(\partial_{\mu}\bar{\Gamma}^{\mu}{}_{\alpha\beta} + \bar{\Gamma}^{\mu}{}_{\mu\lambda}\bar{\Gamma}^{\lambda}{}_{\alpha\beta} - \bar{\Gamma}^{\lambda}{}_{\mu\alpha}\bar{\Gamma}^{\mu}{}_{\lambda\beta}\right)A^{\beta}\right].
\end{aligned}$$

Secondly, if we use same technique, we will get

$$\begin{aligned}\bar{D}^{\nu}\bar{D}_{\mu}A^{\mu} = \bar{g}^{\nu\alpha}\Big[&\partial_{\alpha}\partial_{\mu}A^{\mu} + \bar{\Gamma}^{\mu}{}_{\mu\beta}\partial_{\alpha}A^{\beta} + \bar{\Gamma}^{\mu}{}_{\alpha\lambda}\partial_{\mu}A^{\lambda} - \bar{\Gamma}^{\lambda}{}_{\alpha\mu}\partial_{\lambda}A^{\mu} \\ &+ \left(\partial_{\alpha}\bar{\Gamma}^{\mu}{}_{\mu\beta} + \bar{\Gamma}^{\mu}{}_{\alpha\lambda}\bar{\Gamma}^{\lambda}{}_{\mu\beta} - \bar{\Gamma}^{\lambda}{}_{\alpha\mu}\bar{\Gamma}^{\mu}{}_{\lambda\beta}\right)A^{\beta}\Big].\end{aligned}$$

If we subtract the second term from first one, we can get

$$\begin{aligned}\left[\bar{D}_{\mu},\bar{D}^{\nu}\right]A^{\mu} &= \bar{g}^{\nu\alpha}\left(\partial_{\mu}\bar{\Gamma}^{\mu}{}_{\alpha\beta} - \partial_{\alpha}\bar{\Gamma}^{\mu}{}_{\mu\beta} + \bar{\Gamma}^{\mu}{}_{\mu\lambda}\bar{\Gamma}^{\lambda}{}_{\alpha\beta} - \bar{\Gamma}^{\mu}{}_{\alpha\lambda}\bar{\Gamma}^{\lambda}{}_{\mu\beta}\right)A^{\beta} \\ &= \bar{g}^{\nu\alpha}\bar{R}^{\mu}{}_{\beta\mu\alpha}A^{\beta} \\ &= \bar{R}^{\nu}{}_{\mu}A^{\mu}\end{aligned}$$

where $\bar{R}^{\nu}{}_{\mu}$ is the Ricci tensor of the background metric $\bar{g}_{\mu\nu}$. From the last expression,

$$\bar{D}_{\mu}\bar{D}^{\nu}A^{\mu} = \cancel{\bar{D}^{\nu}\bar{D}_{\mu}A^{\mu}} + \bar{R}^{\nu}{}_{\mu}A^{\mu} = \bar{R}^{\nu}{}_{\mu}A^{\mu}$$

where we used the gauge condition (2.193). Hence, eqn.(2.181) becomes

$$\begin{aligned}&\bar{D}_{\rho}\bar{D}^{\rho}A^{\mu} - \bar{D}_{\rho}\bar{D}^{\mu}A^{\rho} = 0, \\ &\Rightarrow \bar{D}_{\rho}\bar{D}^{\rho}A^{\mu} - \bar{R}^{\mu}{}_{\rho}A^{\rho} = 0.\end{aligned} \tag{2.194}$$

If $\lambda\!\!\!^{-}$ is much smaller than the other scalers in the problem, geometric optics will be valid. Hence, we must have $\lambda\!\!\!^{-} << L_B$, where L_B is the scale of variation of the background metric. In addition, we must have $\lambda\!\!\!^{-} << L_c$ where L_c is the characteristic length-scale over which the amplitude, polarization or wavelength of the electromagnetic field change. In particular, the curvature radius of the wavefront must be much bigger than $\lambda\!\!\!^{-}$. Say

$$A^{\mu}(x) = \left[a^{\mu}(x) + \mathscr{E}b^{\mu}(x) + \mathscr{E}^2c^{\mu}(x) + ...\right]e^{\frac{i\theta(x)}{\mathscr{E}}}, \tag{2.195}$$

where $\mathscr{E}$ is a fictitious parameter, which reminds us that the term which has $\mathscr{E}^n$ is of order $\left(\frac{\lambda\!\!\!^{-}}{L}\right)^n$ where L is the $min(L_B, L_c)$.

Since $R^{\mu}{}_{\rho}A^{\rho} = O(A/L_B^2)$ where A is the amplitude of the A^{μ}, and $\bar{D}^{\rho}\bar{D}_{\rho}A^{\mu} = O(A/\lambda\!\!\!^{-2})$, to leading and next-to-leading order in $\lambda\!\!\!^{-}/L_B$ we can omit $R^{\mu}{}_{\rho}A^{\rho}$. Hence, the equations of motion (2.194) can be rewritten as

$$\bar{D}_{\rho}\bar{D}^{\rho}A^{\mu} = 0. \tag{2.196}$$

Let's define the wave-vector $k_{\mu} \equiv \partial_{\mu}\theta$, and use this and eqn.(2.195) in eqn.(2.193) in order to find the lowest order term:

$$\bar{D}_{\mu}A^{\mu} = \bar{D}_{\mu}\left[\left(a^{\mu}(x) + \mathscr{E}b^{\mu}(x) + \mathscr{E}^2c^{\mu}(x) + ...\right)e^{\frac{i\theta(x)}{\mathscr{E}}}\right]$$

$$
\begin{aligned}
0 &= \left(\bar{D}_\mu a^\mu(x) + \mathscr{E}\bar{D}_\mu b^\mu(x) + \mathscr{E}^2\bar{D}_\mu c^\mu(x) + \ldots\right) e^{\frac{i\theta(x)}{\mathscr{E}}} \\
&\quad + \left(a^\mu(x) + \mathscr{E} b^\mu(x) + \mathscr{E}^2 c^\mu(x) + \ldots\right) \frac{i\partial_\mu \theta}{\mathscr{E}} e^{\frac{i\theta(x)}{\mathscr{E}}} \\
0 &= \left[\mathscr{E}^{-1}\left(ik_\mu a^\mu\right) + \mathscr{E}^0\left(\bar{D}_\mu a^\mu + ik_\mu b^\mu\right) + \mathscr{E}\left(\bar{D}_\mu b^\mu + ik_\mu c^\mu\right) + O(\mathscr{E}^2)\right] e^{\frac{i\theta(x)}{\mathscr{E}}}.
\end{aligned}
\tag{2.197}
$$

The lowest order term implies that

$$
k_\mu a^\mu = 0 \Rightarrow \bar{g}_{\mu\nu} k^\mu a^\nu = 0, \tag{2.198}
$$

where we use $\bar{D}_\mu \theta = \partial_\mu \theta$ since θ is a scalar. Similarly, let's find the lowest order term from eqn.(2.196)

$$
\begin{aligned}
\bar{D}^\rho \bar{D}_\rho A^\mu &= \bar{D}^\rho \bar{D}_\rho \left[\left(a^\mu + \mathscr{E} b^\mu + \ldots\right) e^{\frac{i\theta(x)}{\mathscr{E}}}\right] \\
&= \bar{D}^\rho \left\{ \left[\left(\bar{D}_\rho a^\mu + \mathscr{E}\bar{D}_\rho b^\mu + \ldots\right) + \left(a^\mu + \mathscr{E} b^\mu + \ldots\right)\frac{ik_\rho}{\mathscr{E}}\right] e^{\frac{i\theta(x)}{\mathscr{E}}} \right\} \\
&= \left(\bar{D}^\rho \bar{D}_\rho a^\mu + \mathscr{E}\bar{D}^\rho \bar{D}_\rho b^\mu + \ldots\right) e^{\frac{i\theta(x)}{\mathscr{E}}} + \frac{ik^\rho}{\mathscr{E}}\left(\bar{D}_\rho a^\mu + \mathscr{E}\bar{D}_\rho b^\mu + \ldots\right) e^{\frac{i\theta(x)}{\mathscr{E}}} \\
&\quad + \frac{i\bar{D}^\rho k_\rho}{\mathscr{E}}\left(a^\mu + \mathscr{E} b^\mu + \ldots\right) e^{\frac{i\theta(x)}{\mathscr{E}}} + \frac{ik_\rho}{\mathscr{E}}\left(\bar{D}^\rho a^\mu + \mathscr{E}\bar{D}^\rho b^\mu + \ldots\right) e^{\frac{i\theta(x)}{\mathscr{E}}} \\
&\quad + \frac{ik_\rho}{\mathscr{E}}\frac{ik^\rho}{\mathscr{E}}\left(a^\mu + \mathscr{E} b^\mu + \ldots\right) e^{\frac{i\theta(x)}{\mathscr{E}}},
\end{aligned}
\tag{2.199}
$$

so it is

$$
k_\rho k^\rho = 0 \Rightarrow \bar{g}_{\mu\nu} k^\mu k^\nu = 0. \tag{2.200}
$$

which is called the *eikonal equation*. Also, we have

$$
\begin{aligned}
\bar{D}_\nu\left(k_\mu k^\mu\right) &= \bar{D}_\nu\left(\bar{g}_{\mu\nu} k_\mu k^\mu\right) \\
&= \bar{g}_{\mu\nu}\left[\left(\bar{D}_\nu k^\mu\right) k^\alpha + k^\mu\left(\bar{D}_\nu k^\alpha\right)\right] \\
&= k^\mu\left(\bar{D}_\nu k_\mu\right) + k^\mu\left(\bar{D}_\nu k_\mu\right) \\
&= 2k^\mu \bar{D}_\nu k_\mu,
\end{aligned}
\tag{2.201}
$$

which is zero from eqn.(2.200). There is another important expression which is

$$
\bar{D}_\nu \partial_\mu \theta = \bar{D}_\nu \bar{D}_\mu \theta = \bar{D}_\mu \bar{D}_\nu \theta = \bar{D}_\mu \partial_\nu \theta,
$$

since θ is a scalar and it is known that covariant derivatives always commute on the scalars. If we use the definition of the wave vector on the last expression, we get

$$
\bar{D}_\nu k_\mu = \bar{D}_\mu k_\nu. \tag{2.202}
$$

In addition, using eqns.(2.200) and (2.202) in eqn.(2.201), we can get

$$
k^\mu \bar{D}_\mu k_\nu = 0. \tag{2.203}
$$

The expression (2.203) is the geodesic equation in the space-time of the background metric $\bar{g}_{\mu\nu}$. It can be seen easily to write $k^\mu = \frac{dx^\mu}{d\lambda}$ where λ is the affine parameter along the geodesic,

$$\begin{aligned}\bar{g}_{\alpha\nu}\frac{dx^\mu}{d\lambda}\bar{D}_\mu\frac{dx^\alpha}{d\lambda} &= \bar{g}_{\alpha\nu}\frac{dx^\mu}{d\lambda}\Big(\partial_\mu\frac{dx^\alpha}{d\lambda} + \Gamma^\alpha{}_{\mu\beta}\frac{dx^\beta}{d\lambda}\Big)\\ &= \bar{g}_{\alpha\nu}\Big(\frac{d^2x^\alpha}{d\lambda^2} + \Gamma^\alpha{}_{\mu\beta}\frac{dx^\mu}{d\lambda}\frac{dx^\beta}{d\lambda}\Big) = 0,\end{aligned}$$

implies that

$$\frac{d^2x^\alpha}{d\lambda^2} + \Gamma^\alpha{}_{\mu\beta}\frac{dx^\mu}{d\lambda}\frac{dx^\beta}{d\lambda} = 0. \tag{2.204}$$

As a result, the curves which are orthogonal to the surfaces move along the null geodesic of background metric.

Now, we can find the next-to-leading order from eqn.(2.196). If we look at the expression (2.199), the terms which have the order of $\mathscr{E}^{-1}$ is

$$2k_\rho\bar{D}^\rho a^\mu + \big(\bar{D}^\rho k_\rho\big)a^\mu = 0. \tag{2.205}$$

Let's define the real scalar amplitude $a = (a^\mu a^*_\mu)^{1/2}$ and the polarization vector e^μ such that $a^\mu = ae^\mu$, so we have $e^\mu e^*_\mu = 1$. It is trivial that $k^\mu\partial_\mu(a^2) = 2ak^\mu\partial_\mu a$. Using the fact that normal derivative is equivalent to covariant derivative when we apply it to a scalar, we get

$$\begin{aligned}k^\mu\partial_\mu(a^2) &= k^\mu\bar{D}_\mu(a^\rho a^*_\rho)\\ &= k^\mu\big(\bar{D}_\mu a^\rho\big)a^*_\rho + k^\mu a^\rho\big(\bar{D}_\mu a^*_\rho\big)\\ &= -\frac{1}{2}\big(\bar{D}^\mu k_\mu\big)a^\rho a^*_\rho - \frac{1}{2}\big(\bar{D}^\mu k_\mu\big)a^*_\rho a^\rho\\ &= -\big(\bar{D}^\mu k_\mu\big)a^2,\end{aligned} \tag{2.206}$$

where we used eqn.(2.205) to write $k^\mu\big(\bar{D}_\mu a^\rho\big) = -\frac{1}{2}\big(\bar{D}^\mu k_\mu\big)a^\rho$. From the result (2.206), we get

$$\begin{aligned}k^\mu\partial_\mu(a^2) &= -\big(\bar{D}^\mu k_\mu\big)a^2\\ 2ak^\mu\partial_\mu a &= -\big(\bar{D}^\mu k_\mu\big)a^2\\ \Rightarrow k^\mu\partial_\mu a &= -\frac{1}{2}\big(\bar{D}^\mu k_\mu\big)a.\end{aligned} \tag{2.207}$$

Also, we can find an equation for e^μ by writing $a^\mu = ae^\mu$ in eqn.(2.205) and using eqn.(2.207),

$$2k^\rho\bar{D}_\rho\big(ae^\mu\big) + \big(\bar{D}_\rho k^\rho\big)ae^\mu = 0$$

$$2k^{\rho}\left(\bar{D}_{\rho}a\right)e^{\mu}+2ak^{\rho}\bar{D}_{\rho}e^{\mu}+\left(\bar{D}_{\rho}k^{\rho}\right)ae^{\mu}=0$$
$$\left(2k^{\rho}\partial_{\rho}a\right)e^{\mu}+\left(\bar{D}_{\rho}k^{\rho}\right)ae^{\mu}+2ak^{\rho}\bar{D}_{\rho}e^{\mu}=0$$
$$-\cancel{\left(\bar{D}_{\rho}k^{\rho}\right)ae^{\mu}}+\cancel{\left(\bar{D}_{\rho}k^{\rho}\right)ae^{\mu}}+2ak^{\rho}\bar{D}_{\rho}e^{\mu}=0$$

implies that

$$k^{\rho}\bar{D}_{\rho}e^{\mu}=0. \tag{2.208}$$

The main results of the geometric optics of electromagnetic waves in curved spaces are the equations (2.198), (2.200), (2.203), (2.207) and (2.208). Eqn.(2.198) means $k_{\mu}e^{\mu}=0$, and eqn.(2.208) means that it is parallel-transported along the null geodesic. Lastly, eqn.(2.207) gives us information about the conservation of the number of photons (in the quantum language) in the limit of geometric optics. Let's look at $\bar{D}^{\mu}\left(a^2k^{\mu}\right)$ explicitly

$$\begin{aligned}
\bar{D}^{\mu}\left(a^2k^{\mu}\right) &= \bar{D}^{\mu}\left(ae^{\rho}ae^{*}_{\rho}k^{\mu}\right)\\
&= (\bar{D}^{\mu}a)e^{\rho}ae^{*}_{\rho}k^{\mu}+\cancel{a(\bar{D}^{\mu}e^{\rho})ae^{*}_{\rho}k^{\mu}}+ae^{\rho}(\bar{D}^{\mu}a)e^{*}_{\rho}k^{\mu}+\cancel{a^2e^{\rho}(\bar{D}^{\mu}e^{*}_{\rho})k^{\mu}}\\
&\quad +ae^{\rho}ae^{*}_{\rho}(\bar{D}^{\mu}k^{\mu})\\
&= 2ak^{\mu}(\partial^{\mu}a)+a^2(\bar{D}^{\mu}k^{\mu})\\
&= -a^2\bar{D}^{\mu}k^{\mu}+a^2\bar{D}^{\mu}k^{\mu}\\
&= 0
\end{aligned} \tag{2.209}$$

As a result, if we define a current such that $j=a^2k^{\mu}$, then we can say that the current is covariantly conserved. Noether's theorem says us that its relevant conserved charge is the spatial surface integral of a^2k^0 at constant time. Also energy density is proportional to $(k^0a)^2$. Because the each photon have the energy k^0 and we have eqn.(2.209), the number of photons is conserved in the limit of the geometric optics.

Gravitational waves

The question is what the eikonal approximation is for gravitational waves. Say

$$\bar{h}_{\mu\nu}(x)=\Big[A_{\mu\nu}(x)+\mathscr{E}B_{\mu\nu}(x)+...\Big]e^{\frac{i\theta(x)}{\mathscr{E}}}. \tag{2.210}$$

Similar to the situation in electromagnetic waves, there is the definition $k_{\mu}=\partial_{\mu}\theta$, and we also write $A_{\mu\nu}=Ae_{\mu\nu}$ where $e_{\mu\nu}$ is the polarization tensor which is normalized as $e^{\mu\nu}e^{*}_{\mu\nu}=1$, and A is the scalar amplitude. If we substitute eqn.(2.210) into eqn.(2.188),

$$
\begin{aligned}
\bar{D}^{\nu}\bar{h}_{\mu\nu} &= \bar{D}^{\nu}\Big\{\big[A_{\mu\nu}(x)+\mathscr{E}B_{\mu\nu}(x)+...\big]e^{\frac{i\theta(x)}{\mathscr{E}}}\Big\} \\
&= \big(\bar{D}^{\nu}A_{\mu\nu}(x)+\mathscr{E}\bar{D}^{\nu}B_{\mu\nu}(x)+...\big)e^{\frac{i\theta(x)}{\mathscr{E}}}+\big(A_{\mu\nu}(x)+\mathscr{E}B_{\mu\nu}(x)+...\big)\frac{ik^{\nu}}{\mathscr{E}}e^{\frac{i\theta(x)}{\mathscr{E}}} \\
&= 0,
\end{aligned}
$$

then the lowest order term gives us

$$k^{\nu}A_{\mu\nu}=0. \tag{2.211}$$

Let's focus on eqn.(2.190) this time,

$$
\begin{aligned}
\bar{D}^{\rho}\bar{D}_{\rho}\bar{h}_{\mu\nu} = \bar{D}^{\rho}\Big\{&\big(\bar{D}_{\rho}A_{\mu\nu}(x)+\mathscr{E}\bar{D}_{\rho}B_{\mu\nu}(x)+...\big)e^{\frac{i\theta(x)}{\mathscr{E}}} \\
&+\big(A_{\mu\nu}(x)+\mathscr{E}B_{\mu\nu}(x)+...\big)\frac{ik_{\rho}}{\mathscr{E}}e^{\frac{i\theta(x)}{\mathscr{E}}}\Big\} \\
= \big(&\bar{D}^{\rho}\bar{D}_{\rho}A_{\mu\nu}(x)+\mathscr{E}\bar{D}^{\rho}\bar{D}_{\rho}B_{\mu\nu}(x)+...\big)e^{\frac{i\theta(x)}{\mathscr{E}}} \\
&+\big(\bar{D}_{\rho}A_{\mu\nu}(x)+\mathscr{E}\bar{D}_{\rho}B_{\mu\nu}(x)+...\big)\frac{ik^{\rho}}{\mathscr{E}}e^{\frac{i\theta(x)}{\mathscr{E}}} \\
&+\big(\bar{D}^{\rho}A_{\mu\nu}(x)+\mathscr{E}\bar{D}^{\rho}B_{\mu\nu}(x)+...\big)\frac{ik_{\rho}}{\mathscr{E}}e^{\frac{i\theta(x)}{\mathscr{E}}} \\
&+\big(A_{\mu\nu}(x)+\mathscr{E}B_{\mu\nu}(x)+...\big)\frac{i\bar{D}^{\rho}k_{\rho}}{\mathscr{E}}e^{\frac{i\theta(x)}{\mathscr{E}}} \\
&-\big(A_{\mu\nu}(x)+\mathscr{E}B_{\mu\nu}(x)+...\big)\frac{k_{\rho}k^{\rho}}{\mathscr{E}^2}e^{\frac{i\theta(x)}{\mathscr{E}}}.
\end{aligned} \tag{2.212}
$$

Hence, from the lowest order term of eqn.(2.212), we have

$$k_{\rho}k^{\rho}=0 \Rightarrow \bar{g}_{\mu\nu}k^{\mu}k^{\nu}=0. \tag{2.213}$$

The eqns.(2.211) and (2.212) are the same as for the electromagnetic waves version of the geometric optics which we found before. As a result, we can say that gravitons use the null geodesic of background metric hence they travel like the photons. In addition, $A_{\mu\nu}$ satisfies, if we look at the next-to-leading term from eqn.(2.121)

$$
\begin{aligned}
k^{\rho}\bar{D}_{\rho}A_{\mu\nu}+k_{\rho}\bar{D}^{\rho}A_{\mu\nu}+(\bar{D}^{\rho}k_{\rho})A_{\mu\nu}&=0 \\
2k_{\rho}\bar{D}^{\rho}A_{\mu\nu}+(\bar{D}^{\rho}k_{\rho})A_{\mu\nu}&=0
\end{aligned}
$$

which implies that

$$k_{\rho}\bar{D}^{\rho}A_{\mu\nu}=-\frac{1}{2}(\bar{D}^{\rho}k_{\rho})A_{\mu\nu}. \tag{2.214}$$

Also, let's look at the expression

$$
\begin{aligned}
k^{\mu}\partial_{\mu}(A^2) &= k^{\mu}\bar{D}_{\mu}\big(A^{\alpha\beta}A^{*}_{\alpha\beta}\big) \\
&= k^{\mu}\big(\bar{D}_{\mu}A^{\alpha\beta}\big)A^{*}_{\alpha\beta}+k^{\mu}A^{\alpha\beta}\big(\bar{D}_{\mu}A^{*}_{\alpha\beta}\big)
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{1}{2}\big(\bar{D}^{\mu}k_{\mu}\big)A^{\alpha\beta}A^{*}_{\alpha\beta} - -\frac{1}{2}\big(\bar{D}^{\mu}k_{\mu}\big)A^{\alpha\beta}A^{*}_{\alpha\beta} \\
&= -\big(\bar{D}^{\mu}k_{\mu}\big)A^{2} \qquad (2.215)
\end{aligned}
$$

From the last expression, we can get

$$
\begin{aligned}
2Ak^{\mu}\partial_{\mu}A &= -\big(\bar{D}^{\mu}k_{\mu}\big)A^{2} \\
\Rightarrow k^{\mu}\partial_{\mu}A &= -\frac{1}{2}\big(\bar{D}^{\mu}k_{\mu}\big)A. \qquad (2.216)
\end{aligned}
$$

Now, we can look at the expression

$$
\begin{aligned}
\bar{D}^{\mu}\big(A^{2}k^{\mu}\big) &= \partial^{\mu}\big(A^{2}\big)k^{\mu} + A^{2}\big(\bar{D}^{\mu}k^{\mu}\big) \\
&= 0 \qquad (2.217)
\end{aligned}
$$

where we used eqn.(2.215). As a result, we can say that the number of the gravitons are conserved in this approximation of geometric optics.

CHAPTER 3

INTRODUCTION TO GRAVITATIONAL WAVES' MEMORY

To observe the gravitational waves, we can use a detector which consists of freely falling test particles. The gravitational wave passing through a detector causes a relative motion of these test particles. The Laser Interferometer Gravitational-Wave Observatory (LIGO) has this physical mechanism. For example, assume that there are four test particles which make a circle shape before the plane wave arrives. When the plane wave is passing, the shape of it changes as

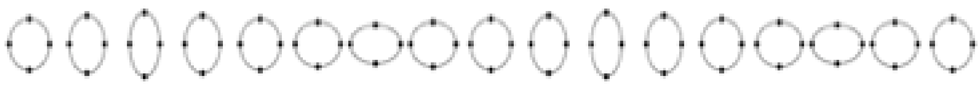

Figure 3.1: A simplified gravitational wave detector [2].

Astrophysical realistic radiation has the form of finite pulses instead of endlessly repeating plane waves [9]. Such a pulse which is created by interactions of sources such as stars or black holes in a galactic nucleus, can cause a finite, permanent change in the separations of the particles. This is known as the *gravitational wave memory effect*. Recently there are many publications about this phenomenon [3, 4, 12–23] . If we look at this phenomenon with same analogy which is shown in Fig.(3.1). It starts with the perfect circle shape, then it will oscillate for a finite amount of time, and it finally will stop. However, this time it cannot be a circle again.

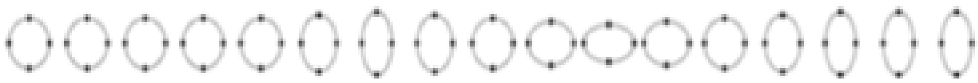

Figure 3.2: A representation of the memory effect [2].

3.1 What Are the Mathematical Properties of the Memory Effect?

There is a reason for use of null infinity. The reason is that it gives us a permission to isolate the radiation from other forms of gravitation due to the peeling theorem, since we can identify different portions of curvature using the peeling theorem [10]. We can accept that the source is very far away from the detector, then the space-time is asymptotically flat. Let the source be at a point p in extended space-time; however, the detector is near a point q on null infinity. Then we can expand the gravitational fields in powers of $\frac{1}{r}$, and we will only focus on the leading term.

Let d^a be the spatial separation between two detector particles, and let's define $\triangle d^a$ as the change in the separation, then we can find that the memory is

$$\triangle d^a = \frac{1}{r}\triangle^a{}_b d^b \tag{3.1}$$

where $\triangle_{ab}$ is the memory tensor which has some interesting information about the memory-angular dependence, energy and mass scales, etc. If $\hat{r}$ points to the location of the detector on the sphere at null infinity, the memory tensor is

$$\triangle_{ab} = 2\sum_{(i),out}\left[\frac{m_{(i)}}{\sqrt{1-v_{(i)}^2}}\frac{(v_{(i)})_a(v_{(i)})_b}{1-\hat{r}.\vec{v}}\right]^{TT} - 2\sum_{(j),in}\left[\frac{m_{(j)}}{\sqrt{1-v_{(j)}^2}}\frac{(v_{(j)})_a(v_{(j)})_b}{1-\hat{r}.\vec{v}}\right]^{TT} \tag{3.2}$$

where

$$[X_{ab}]^{TT} = q_a{}^c q_b{}^d X_{cd} - \frac{1}{2}q^{cd}X_{cd}q_{ab} \tag{3.3}$$

where q_{ab} known as the transverse-traceless projection operator which is given explicit form of it in section (1.2).

3.2 Notation and Convention

In the memory part of the thesis, we use the geometrized units $(G = c = 1)$, and the abstract index notation for the tensors. The indices a, b take the values 1,2 which we have already used them in Chapter 1. A Latin index in the parentheses (i) tells us which particle we are interest in. Also, $t^a = -\partial^a t$ is future-pointing, and $r^a = \partial^a r$ is outward-pointing vector. $U = t - r$ is the retarded time, and

$$K^a = -\partial^a U = t^a + r^a.$$

we have the definition such that the n^{th} derivative of the Dirac delta function is denoted by $\delta^{(n)}$, and m-dimensional coordinate Delta function is denoted δ_m, and $\int d^m x \delta_m(x) = 1$. We use the symbol Θ for the Heaviside step function.

Lastly, q_{ab} denote the projection of the 4-dimensional Minkowski metric onto S^2.

CHAPTER 4

THE GRAVITATIONAL WAVES MEMORY OF SCATTERING PARTICLES IN MINKOWSKI SPACE-TIME

In this Chapter, we use [2], [3] and [4] as a guide to construct the idea of gravitational memory.

4.1 Scalar Fields

The scalar wave equation is

$$\partial^a \partial_a \varphi = -4\pi S, \tag{4.1}$$

where φ is a scalar field and S is a scalar charge distribution. Our aim is to find the retarded solution of the wave eqn.(4.1). Here, S represents a system of charged point-particles. They follow inertial trajectories except at the point P which is called "interaction vertex". At the point P, the particles may interact, and they can be created, or destroyed. For example, we can imagine that six incoming point particles go on to a interaction vertex P, and then three point particles are born, which is shown in the Fig.(4.1). Their worldliness can be time-like or light-like (null).

Let $(t, \vec{x})$ be a global inertial coordinate system (GICS) such that the point P can be chosen as the origin $(t = 0, \vec{x} = 0)$ of the GISC. In this coordinate system, we can write the charge distribution S such as

$$S(x) = \sum_{(i) \quad in} q_{(i)} \frac{d\tau_{(i)}}{dt} \delta_3 \left(\vec{x} - \vec{y}_{(i)}(t) \right) \Theta(-t) + \sum_{(j) \quad out} q_{(j)} \frac{d\tau_{(j)}}{dt} \delta_3 \left(\vec{x} - \vec{y}_{(j)}(t) \right) \Theta(t), \tag{4.2}$$

where $q_{(i)}$ are the scalar charges of the particles which are measured in their rest frame, and $(t, \vec{y}_{(i)}(t))$ with $(\vec{y}_{(i)}(0) = 0)$ are the particles worldliness which are parametrized with the GICS time coordinate. In here, the particles which arrive the point P are de-

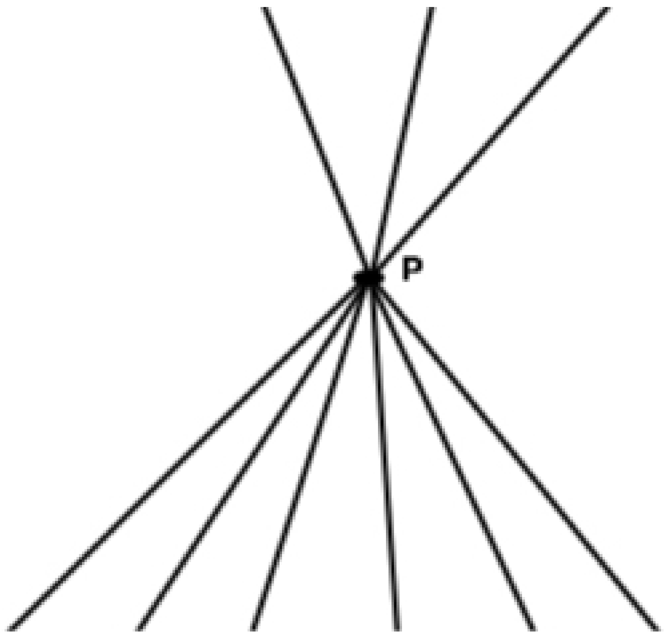

Figure 4.1: A space-time diagram of the sort of radiation source we will consider. [2]

stroyed by the $\Theta(-t)$ factor, and the particles which leave the point P are created by the $\Theta(t)$ factor.

We can find the retarded solution of eqn. (4.1) by using the charge distribution S given by eqn. (4.2) with retarted Green's function G of eqn. (4.1), such that

$$\varphi(x) = 4\pi \int d^4x' G(x,x')S(x'), \tag{4.3}$$

where the retarded Green's function $G(x,x')$ is

$$G(x,x') = \frac{1}{2\pi}\delta\Big(\sigma^2(x,x')\Big)\Theta(t-t'), \tag{4.4}$$

in which $\sigma^2(x,x') = -(t-t')^2 + |\vec{x}-\vec{x}'|^2$ is the squared geodesic distance between field point x and the charge point x'.

Let's think a source $S_{in,\vec{v}=0}(x)$ which is a single massive particle at rest, and is destroyed at point P, so it is

$$S_{in,\vec{v}=0}(x) = q\delta_3(\vec{x})\Theta(-t). \tag{4.5}$$

If we write the Green's function (4.4) and the charge density (4.5) on the retarded field which is given at eqn.(4.3), we will get

$$\begin{aligned}\varphi_{in,\vec{v}=0}(x) &= 4\pi \int d^4x' \Big[\frac{1}{2\pi}\delta\Big(\sigma^2(x,x')\Big)\Theta(t-t')\Big]\Big[q\delta_3(\vec{x'})\Theta(-t')\Big] \\ &= 2q\int d^4x'\delta\Big(-(t-t')^2+|\vec{x}-\vec{x}'|^2\Big)\Theta(t-t')\delta_3(\vec{x'})\Theta(-t').\end{aligned}$$

First of all, we can take the spatial part of the integral, so

$$\varphi_{in,\vec{v}=0}(x) = 2q\int dt'\delta\Big(-(t-t')^2+r^2\Big)\Theta(t-t')\Theta(-t'). \tag{4.6}$$

Then, to obtain the value of the integral (4.6), we will give an interlude about the delta function. The definition of the delta function is

$$\delta(f) = \sum_i \frac{\delta(x-x_i)}{\left|\left(\frac{df}{dx}\right)_{x=x_i}\right|}. \tag{4.7}$$

where the x_i is the zeros of the function f. In our problem, $f = r^2-(t-t')^2$,and its zeros are $t' = t-r$ and $t' = t+r$. Since we have the factor $\Theta(t-t')$, $t' = t-r$ is the only zero point of f which give us a value different from the zero when we take the integral (4.6). Thus,

$$\begin{aligned}\delta(-(t-t')^2+r^2) &= \frac{\delta(t'-(t-r))}{\left|\Big(-2(t-t')\Big)_{t'=t-r}\right|}\\ &= \frac{\delta(t'-(t-r))}{2r}.\end{aligned} \tag{4.8}$$

Using the result (4.8) in the integral eqn. (4.6), we get

$$\begin{aligned}\varphi_{in,\vec{v}=0}(x) &= 2q\int dt'\frac{\delta(t'-(t-r))}{2r}\Theta(t-t')\Theta(-t')\\ &= \frac{q}{r}\Theta\big(-(t-r)\big)+O(\frac{1}{r^2})\\ &= \frac{q}{r}\Theta\big(-U\big)+O(\frac{1}{r^2}),\end{aligned} \tag{4.9}$$

where

$$U = t-r. \tag{4.10}$$

Hence, the leading order of $\varphi_{in,\vec{v}=0}$ is $\frac{1}{r}$. The terms which have the higher order of $\frac{1}{r}$ can be neglected since ve have the detectors in the radiation zone. We can find the field of a particle which is created with velocity $\vec{v}$ by boosting eqn.(4.9). Let's $(t,\vec{y}(t))$ be the geodesic of a particle with coordinate velocity $\vec{v} = \frac{d\vec{y}}{dt}$, then the leading term of field is

$$\varphi_{in,\vec{v}}(x) = \frac{q}{r}\frac{d\tau}{dt}\frac{1}{1-\hat{r}.\vec{v}}\Theta\big(-U\big). \tag{4.11}$$

Now, let's consider about the retarded solution of $S_{out,\vec{v}=0}$ which is a single massive particle at rest, and is created at P. Thus, if we make same calculation, then we will get

$$\varphi_{out,\vec{v}=0}(x) = \frac{q}{r}\Theta\big(U\big)+O(\frac{1}{r^2}). \tag{4.12}$$

The only difference of last equation from eqn.(4.9) is that it does not have the minus sign in the factor of Θ, since there will be term $\Theta(t)$ to create a particle instead of $\Theta(-t)$ in the source density $S_{out,\vec{v}=0}$. Again, to find the leading term of a retarded solution for a particle which is created with velocity $\vec{v}$, we can boost eqn.(4.12) with the same approach we did before for the particle which is destroyed with velocity $\vec{v}$. Thus,

$$\varphi_{out,\vec{v}}(x) = \frac{q}{r}\frac{d\tau}{dt}\frac{1}{1-\hat{r}.\vec{v}}\Theta\big(U\big). \tag{4.13}$$

If we consider a general source like (4.2), we can write the field as a linear superposition of the created and destroyed particles. Thus, the general retarded field solution is the superposition of the solution of the every particle in the system,

$$\varphi = \frac{1}{r}\Big(\Theta\big(U\big)\alpha(\hat{r}) + \Theta\big(-U\big)\beta(\hat{r})\Big), \tag{4.14}$$

with leading order $\frac{1}{r}$, where

$$\alpha(\hat{r}) = \sum_{(i),out}\frac{d\tau_{(i)}}{dt}\frac{q_{(i)}}{1-\hat{r}.\vec{v}_{(i)}}, \quad \beta(\hat{r}) = \sum_{(j),in}\frac{d\tau_{(j)}}{dt}\frac{q_{(j)}}{1-\hat{r}.\vec{v}_{(j)}}. \tag{4.15}$$

There are two cases for an observer;

Case 1 ($U < 0$, or equivalently, $t < r$): He/she will observe a collection of charges which have several constant velocities. Thus, he/she measure a superposition of boosted Coulomb-like fields.

Case 2 ($U > 0$, or equivalently, $t > r$): He/she will observe a collection of charges which are different from the particles described in case 1 with different velocities.

As a result, we can find a "scalar wave" propagating with a Heaviside step wavefront on the future light cone of the interaction point P between these two regions.

4.1.1 Scalar Memory

Now, we have the tools to find the effect of the scalar field (4.14) on a "scalar wave detector" which made of a massive test charge at rest in the GICS near future null infinity. The scalar force on a test particle which has mass M_0 and charge Q is

$$f^a = Q\partial^a\varphi. \tag{4.16}$$

Then, the leading order term of the force at large distance from the source is

$$f^a(U,x) = -\frac{Q}{r}(\alpha-\beta)\delta(U)K^a, \tag{4.17}$$

where

$$K^a = -\partial^a U. \tag{4.18}$$

We neglect the terms in eqn.(4.17) which come from derivative of $1/r$, α and β, since they are order in $1/r^2$.

Assume that the test particle is initially at rest, then the change of the momentum is

$$\begin{aligned}\Delta P^a(U) &= \int_{-\infty}^{U} dU' f^a(U',\vec{x}) \\ &= \int_{-\infty}^{U} dU' \frac{Q}{r}(\alpha-\beta)\delta(U)\partial^a U \\ &= -\frac{Q}{r}(\alpha-\beta)\Theta(U)K^a \end{aligned} \tag{4.19}$$

Because of the scalar radiation which is emitted by interactions of particles, a test particle will have a momentum kick. There will be a change in mass because of mass which is

$$M_1^2 = -\eta_{ab}\left(P_0^a + \Delta P^a\right)\left(P_0^b + \Delta P^b\right) = M_0^2 - 2P_0^a \Delta P_a = M_0^2 - 2Q(\alpha-\beta)\frac{M_0}{r}, \tag{4.20}$$

where we only write the leading order term [6, 7].

4.2 Electromagnetic Fields

We can write the Maxwell's equation for the four-potential A^a with the form of a wave equation which is like eqn.(4.1),

$$\partial^b \partial_b A^a = -4\pi J^a, \tag{4.21}$$

where we have the Lorentz gauge

$$\partial_a A^a = 0, \tag{4.22}$$

and J^a is the electromagnetic current density. The current density has the property $\partial_a J^a = 0$ which is known as the charge conservation law. If we use the retarded integral (4.3) on each GICS component of (4.21), we can compute the retarded electromagnetic field for a given current density. Let's again assume that there is a event P where ingoing particles destroyed, and outgoing particles created. Say the event P is the origin of the our GICS, then we can write the charge-current density for ingoing particles with the form

$$J^a_{(i)} = q_{(i)} \frac{d\tau_{(i)}}{dt} u^a_{(i)} \delta_3\left(\vec{x} - \vec{y}_{(i)}(t)\right)\Theta(-t), \tag{4.23}$$

where $\tau_{(i)}$ is the proper time for the i th particle which has the world-line $(t, \vec{y}_{(i)}), u^a_{(i)}$ is the tangent vector which is normalized, and $q_{(i)}$ is the charge of it. Also, the massless version of the charge-current density of ingoing charges is

$$J^a_{(j)} = q_{(i)} w^a_{(j)} \delta_3\big(\vec{x} - \vec{y}_{(j)}(t)\big)\Theta(-t) \tag{4.24}$$

where $\omega^a_{(j)}$ is the tangent vector for the particle's null world-line(geodesic). We have the normalization such that an observer who has four-velocity t^a measures $\omega^a_{(j)} t_a = -1$. Changing the factor $\Theta(-t)$ with $\Theta(t)$, we can get the charge-current density for the outgoing particles. In other words, the factor $\Theta(t)$ creates the outgoing particles at interaction points. As a result, we can write the general charge-current density as

$$J^a = \sum_{(i)in,massive} J^a_{(i)} + \sum_{(j)in,null} J^a_{(j)} + \sum_{(k)out,massive} J^a_{(k)} + \sum_{(l)out,null} J^a_{(l)}. \tag{4.25}$$

Also,if we consider the conservation of J^a, then we have another important expression about conservation of charge which is

$$\sum_{(i)out} q_{(i)} = \sum_{(j)in} q_{(j)}. \tag{4.26}$$

We will only consider the massive charges for the simplicity. It is easy to generalize our solution which include the massless charged particles. Let's find the retarded solution of eqn.(4.21) for the only ingoing particles charge-current density which belongs to a single massive particle at rest which is destroyed at point P

$$J^a_{in,\vec{v}^a=0}(x) = q\frac{d\tau}{dt} u^a \delta_3(\vec{x})\Theta(-t). \tag{4.27}$$

Since the every component of the charge-current density has scalar nature, we can use same integral which we used in the scalar part. Then the leading order term of a^{th} component of the vector potential is

$$\begin{aligned}
A^a_{in,\vec{v}=0} &= 4\pi \int d^4x' G(x,x') J^a(x') \\
&= 4\pi \int d^4x' \frac{1}{2\pi}\big[-(t-t')^2 + |\vec{x} - \vec{x}'|^2\big]\Theta(t-t') q u^a \delta_3(\vec{x})\Theta(-t') \\
&= 2qu^a \int d^4t' \delta\big[-(t-t')^2 + r^2\big]\Theta(t-t')\Theta(-t') \\
&= 2qu^a \int d^4t' \frac{\delta\left[t' - (t-r)\right]}{2|t-t'|}\Theta(t-t')\Theta(-t') \\
&= \frac{q}{r} u^a \Theta(-(t-r)) \\
&= \frac{q}{r} u^a \Theta(-U).
\end{aligned} \tag{4.28}$$

Then,the a^{th} component of the vector potential of the particle which is created with velocity $\vec{v}$ is

$$A^a_{in,\vec{v}} = \frac{q}{r}\frac{d\tau}{dt}\frac{u^a}{1-\hat{r}.\vec{v}}\Theta(-U). \tag{4.29}$$

Using the same technique, we can write solution for the outgoing particles

$$A^a_{out,\vec{v}} = \frac{q}{r}\frac{d\tau}{dt}\frac{u^a}{1-\hat{r}.\vec{v}}\Theta(U). \tag{4.30}$$

As a result, we can write the retarded solution of eqn.(4.21) with leading order

$$A^a = \frac{1}{r}\big(\Theta(U)\alpha^a + \Theta(-U)\beta^a\big), \tag{4.31}$$

where

$$\alpha^a(\hat{r}) = \sum_{(i),out}\frac{d\tau_{(i)}}{dt}\frac{q_{(i)}u^a_{(i)}}{1-\hat{r}.\vec{v}_{(i)}}, \tag{4.32}$$

$$\beta^a(\hat{r}) = \sum_{(j),in}\frac{d\tau_{(j)}}{dt}\frac{q_{(j)}u^a_{(j)}}{1-\hat{r}.\vec{v}_{(j)}}. \tag{4.33}$$

4.2.1 Electromagnetic Memory

By the definition, the field tensor $F_{ab} = \partial_a A_b - \partial_b A_a$ is

$$\begin{aligned}
F^{ab} &= \partial^a\Big(\frac{1}{r}\big(\Theta(U)\alpha^b + \Theta(-U)\beta^b\big)\Big) - \partial^b\Big(\frac{1}{r}\big(\Theta(U)\alpha^a + \Theta(-U)\beta^a\big)\Big)\\
&= \frac{1}{r}(\alpha^b - \beta^b)\delta(U)\partial^a U - \frac{1}{r}(\alpha^a - \beta^a)\delta(U)\partial^b U\\
&= -\frac{1}{r}\big(K^a(\alpha^b - \beta^b) - K^b(\alpha^a - \beta^a)\big)\delta(U)\\
&= -\frac{2}{r}K^{[a}(\alpha^{b]} - \beta^{b]})\delta(U).
\end{aligned} \tag{4.34}$$

Using the conservation of the charge (4.26), we get

$$K^{[a}(\alpha^{b]} - \beta^{b]}) = \sum_{(i),out,in}\frac{d\tau_{(i)}}{dt}\frac{\eta_{(i)}q_{(i)}}{1-\hat{r}.\vec{v}_{(j)}}K^{[a}q^{b]c}u_{(i)c}. \tag{4.35}$$

The force acting on the test particle with charge Q and four-velocity V^a is

$$\begin{aligned}
f^a &= QF^{ab}V_b\\
&= Q\frac{2}{r}K^{[a}(\alpha^{b]} - \beta^{b]})\delta(U)V_b\\
&= Q\frac{2}{r}\sum_{(i),out,in}\frac{d\tau_{(i)}}{dt}\frac{\eta_{(i)}q_{(i)}}{1-\hat{r}.\vec{v}_{(j)}}K^{[a}q^{b]c}u_{(i)c}V_b\delta(U).
\end{aligned} \tag{4.36}$$

If the test particle initially at rest in our GICS, $V^a = t^a$, then

$$f^a = \frac{Q}{r}\left[\sum_{(i),out,in}\frac{d\tau_{(i)}}{dt}\frac{\eta_{(i)}q_{(i)}}{1-\hat{r}.\vec{v}_{(j)}}q^{ab}u_{(i)b}\right]\delta(U), \tag{4.37}$$

where we used

$$q^{bc}t_b = 0,$$

and

$$K^b t_b = -1.$$

Its change in momentum is

$$\triangle P^a = \int_{-\infty}^{U} dU' f^a(U', \vec{x})$$
$$= \frac{Q}{r}\left[\sum_{(i),out,in} \frac{d\tau_{(i)}}{dt} \frac{\eta_{(i)}q_{(i)}}{1-\hat{r}.\vec{v}_{(j)}} q^{ab}u_{(i)b}\right]\Theta(U). \quad (4.38)$$

Since $f_a V^a = 0$, there is no any change in mass due to the electromagnetic force [8].

4.3 Gravitational Fields

Let's remember the Einstein-Hilbert action which is

$$S = \frac{1}{\kappa}\int d^4x\sqrt{-g}R, \quad (4.39)$$

which yields the field equation

$$G_{ab} = \kappa T_{ab}, \quad (4.40)$$

as we have shown in chapter 1. If we consider the linearization around 4-dimensional flat background $g_{ab} = \eta_{ab} + h_{ab}$, then the Einstein tensor can be given

$$G^{(1)}_{ab} = -\frac{1}{2}(\partial^c\partial_c h_{ab} - \frac{1}{2}\eta_{ab}\partial^c\partial_c h) \quad (4.41)$$

where we used the harmonic gauge $\partial^a h_{ab} = \frac{1}{2}\partial_b h$. Then, from eqn.(4.36), we have

$$\partial^c\partial_c h_{ab} - \frac{1}{2}\eta_{ab}\partial^c\partial_c h = -2\kappa T_{ab}. \quad (4.42)$$

Taking the trace of the last equation, we get

$$\partial^c\partial_c h - 2\partial^c\partial_c h = -2\kappa T$$
$$\Rightarrow \partial^c\partial_c h = 2\kappa T. \quad (4.43)$$

Then using this expression in eqn.(4.38), we obtain

$$\partial^c\partial_c h_{ab} = -2\kappa T_{ab} + \frac{1}{2}\eta_{ab}(2\kappa T)$$
$$= -2\kappa(T_{ab} - \frac{1}{2}\eta_{ab}T), \quad (4.44)$$

which is known as the linearized field equation. Taking $G = 1$ natural unit as we did chapter 1, $\kappa = 8\pi$, then

$$\partial^c\partial_c h_{ab} = -16\pi\tilde{T}_{ab}, \quad (4.45)$$

where we define $\tilde{T}_{ab} = T_{ab} - \frac{1}{2}\eta_{ab}T$. The general solution of this last expression is

$$h_{ab} = 16\pi \int d^4x' G_{ab}{}^{cd}(x,x')\tilde{T}_{cd}(x'), \tag{4.46}$$

where the retarded Green's function

$$G_{ab}{}^{cd}(x,x') = \eta_a{}^c \eta_b{}^d G(x,x'). \tag{4.47}$$

To find the retarded solution, let us consider the following energy-momentum tensor

$$\begin{aligned} T_{ab} = &\sum_{(i),in} m^{(i)} u_a^{(i)} u_b^{(i)} \frac{d\tau^{(i)}}{dt} \delta(\vec{x} - \vec{y}^{(i)}(t))\Theta(-t) \\ &+ \sum_{(j),out} m^{(j)} u_a^{(j)} u_b^{(j)} \frac{d\tau^{(j)}}{dt} \delta(\vec{x} - \vec{y}^{(j)}(t))\Theta(t), \end{aligned} \tag{4.48}$$

where $m^{(i)}$ is the rest mass of the i^{th} particle which follows the geodesic $(t, \vec{y}^{(j)}(t))$ with four velocity $u^{(i)}$. Now, let's find the retarded solution of eqn.(4.42) for only the outgoing particles at rest which is created at P

$$\begin{aligned} h_{ab} &= 16\pi \int d^4x' \eta_a{}^c \eta_b{}^d G(x,x')\tilde{T}_{cd}(x') \\ &= 16\pi \int d^4x' G(x,x')\tilde{T}_{ab}(x') \\ &= 16\pi \int d^4x' G(x,x') \Big[\sum_{(i),out} m^{(i)} u_a^{(i)} u_b^{(i)} \frac{d\tau^{(i)}}{dt} \delta_3(\vec{x'})\Theta(t') \\ &\quad - \frac{1}{2}\eta_{ab} \sum_{(i),out} m^{(i)} \eta^{cd} u_c^{(i)} u_d^{(j)} \frac{d\tau^{(i)}}{dt} \delta_3(\vec{x'})\Theta(t') \Big]. \end{aligned}$$

If we use the fact that $\eta^{ab} u_a^{(i)} u_b^{(j)} = -1$, we get

$$\begin{aligned} h_{ab} &= 16\pi \int d^4x' G(x,x') \sum_{(i),out} m^{(i)} \frac{d\tau^{(i)}}{dt} \big(u_a^{(i)} u_b^{(i)} + \frac{1}{2}\eta_{ab}\big) \delta_3(\vec{x'})\Theta(t') \\ &= 16\pi \sum_{(i),out} m^{(i)} \frac{d\tau^{(i)}}{dt} \big(u_a^{(i)} u_b^{(i)} + \frac{1}{2}\eta_{ab}\big) \int d^4x' G(x,x') \delta_3(\vec{x'})\Theta(t'). \end{aligned}$$

We have already computed the integral in the last term in the scalar part, so we can use the result of it.

$$\begin{aligned} h_{ab} &= 16\pi \sum_{(i),out} m^{(i)} \frac{d\tau^{(i)}}{dt} \big(u_a^{(i)} u_b^{(i)} + \frac{1}{2}\eta_{ab}\big) \frac{1}{4\pi r}\Theta(U) \\ &= \frac{4}{r} \sum_{(i),out} m^{(i)} \frac{d\tau^{(i)}}{dt} \big(u_a^{(i)} u_b^{(i)} + \frac{1}{2}\eta_{ab}\big)\Theta(U). \end{aligned}$$

Now, we can generalize the last result for the particles which is created with velocity $\vec{v}$ such that

$$h_{ab} = \frac{4}{r} \sum_{(i),out} \frac{m^{(i)}}{1 - \hat{r}.\vec{v}} \frac{d\tau^{(i)}}{dt} \big(u_a^{(i)} u_b^{(i)} + \frac{1}{2}\eta_{ab}\big)\Theta(U)$$

$$= \frac{4}{r}\alpha_{ab}\Theta(U), \tag{4.49}$$

where

$$\alpha_{ab} := \sum_{(i),out} \frac{m^{(i)}}{1-\hat{r}.\vec{v}} \frac{d\tau^{(i)}}{dt}\big(u_a^{(i)}u_b^{(i)} + \frac{1}{2}\eta_{ab}\big). \tag{4.50}$$

Making same computation, we can find the following retarded solution for the ingoing particles

$$h_{ab} = \frac{4}{r}\beta_{ab}\Theta(-U), \tag{4.51}$$

where

$$\beta_{ab} := \sum_{(j),out} \frac{m^{(j)}}{1-\hat{r}.\vec{v}} \frac{d\tau^{(j)}}{dt}\big(u_a^{(j)}u_b^{(j)} + \frac{1}{2}\eta_{ab}\big). \tag{4.52}$$

As a result, the retarded metric perturbation can be obtained as

$$h_{ab} = \frac{4}{r}\big(\alpha_{ab}\Theta(U) + \beta_{ab}\Theta(-U)\big). \tag{4.53}$$

4.3.1 The Gravitational Memory

We computed the Riemann tensor for the linearized theory in chapter 1 which was

$$\begin{aligned} R_{abcd} &= \frac{1}{2}\big(\partial_c\partial_b h_{ad} - \partial_c\partial_a h_{bd} - \partial_d\partial_b h_{ac} + \partial_d\partial_a h_{ac}\big) \\ &= \partial_c\partial_{[b}h_{a]d} - \partial_d\partial_{[b}h_{a]c}. \end{aligned} \tag{4.54}$$

Now, let us start with computing the term,

$$\partial_c\partial_b h_{ad} = \partial_c\partial_b\big[\frac{4}{r}\big(\alpha_{ab}\Theta(U) + \beta_{ab}\Theta(-U)\big)\big]. \tag{4.55}$$

In here, we can neglect the derivative of $\frac{1}{r}$ and $\frac{1}{1-\hat{r}.\vec{v}}$ which is $O(\frac{1}{r^2})$,since we study with $O(\frac{1}{r})$. We have already defined $K^a = -\partial^a U = t^a + r^a$ in scalar part, so we have

$$\begin{aligned} \partial_c\partial_b h_{ad} &= \partial_c\big[\frac{4}{r}\big(\alpha_{ad}\partial_b\Theta(U) + \beta_{ad}\partial_b\Theta(-U)\big)\big] \\ &= \partial_c\big[\frac{4}{r}\big(\alpha_{ad}\delta(U)\partial_b U - \beta_{ad}\delta(U)\partial_b U\big)\big] \\ &= \partial_c\Big[-\frac{4}{r}K_b\big(\alpha_{ad} - \beta_{ad}\big)\delta(U)\Big] \\ &= \frac{4}{r}K_cK_b\big(\alpha_{ad} - \beta_{ad}\big)\delta'(U) \end{aligned} \tag{4.56}$$

where we used the fact that $\partial_c K_b = 0$. Then

$$\begin{aligned} R_{abcd} = \frac{2}{r}\Big[&K_cK_b\big(\alpha_{ad} - \beta_{ad}\big) - K_cK_a\big(\alpha_{bd} - \beta_{bd}\big)\delta'(U) - K_dK_b\big(\alpha_{ac} - \beta_{ac}\big) \\ &+ K_dK_a\big(\alpha_{ac} - \beta_{ac}\big)\Big]\delta'(U). \end{aligned}$$

$$ \tag{4.57} $$

Now, let us define

$$\triangle_{ab} = 2(\alpha_{ab} - \beta_{ab})^{TT}, \tag{4.58}$$

where we use the TT gauge condition since our solution (4.53) is not transverse-traceless. We have already discussed this technique in Chapter 1. To find the expression (4.54), we can look at only the first term of it, since second term will give us a similar result. Then

$$\begin{aligned}
\alpha^{TT}_{ab} &= \left(\sum_{(i),out} \frac{m^{(i)}}{1-\hat{r}.\vec{v}} \frac{d\tau^{(i)}}{dt} (u^{(i)}_a u^{(i)}_b + \frac{1}{2}\eta_{ab}) \right)^{TT} \\
&= \left(q_a{}^c q_b{}^d - \frac{1}{2} q^{cd} q_{ab} \right) \left(\sum_{(i),out} \frac{m^{(i)}}{1-\hat{r}.\vec{v}} \frac{d\tau^{(i)}}{dt} (u^{(i)}_c u^{(i)}_d + \frac{1}{2}\eta_{cd}) \right) \\
&= \sum_{(i),out} \frac{m^{(i)}}{1-\hat{r}.\vec{v}} \frac{d\tau^{(i)}}{dt} \left(q_a{}^c q_b{}^d - \frac{1}{2} q^{cd} q_{ab} \right) \left(u^{(i)}_c u^{(i)}_d + \frac{1}{2}\eta_{cd} \right) \\
&= \sum_{(i),out} \frac{m^{(i)}}{1-\hat{r}.\vec{v}} \frac{d\tau^{(i)}}{dt} \left(q_a{}^c q_b{}^d u^{(i)}_c u^{(i)}_d + \frac{1}{2}\eta_{cd} q_a{}^c q_b{}^d - \frac{1}{2} q^{cd} q_{ab} u^{(i)}_c u^{(i)}_d - \frac{1}{4}\eta_{cd} q^{cd} q_{ab} \right) \\
&= \sum_{(i),out} \frac{m^{(i)}}{1-\hat{r}.\vec{v}} \frac{d\tau^{(i)}}{dt} \left(q_a{}^c q_b{}^d u^{(i)}_c u^{(i)}_d + \cancel{\frac{1}{2} q_{ab}} - \frac{1}{2} q^{cd} q_{ab} u^{(i)}_c u^{(i)}_d - \cancel{\frac{1}{4} 2 q_{ab}} \right) \\
&= \sum_{(i),out} \frac{m^{(i)}}{1-\hat{r}.\vec{v}} \frac{d\tau^{(i)}}{dt} \left(q_a{}^c q_b{}^d u^{(i)}_c u^{(i)}_d - \frac{1}{2} q^{cd} q_{ab} u^{(i)}_c u^{(i)}_d \right).
\end{aligned} \tag{4.59}$$

Similarly, the second term of the (4.54) is

$$\beta^{TT}_{ab} = \sum_{(i),in} \frac{m^{(i)}}{1-\hat{r}.\vec{v}} \frac{d\tau^{(i)}}{dt} \left(q_a{}^c q_b{}^d u^{(i)}_c u^{(i)}_d - \frac{1}{2} q^{cd} q_{ab} u^{(i)}_c u^{(i)}_d \right).$$

Hence, the expression (4.54) becomes

$$\triangle_{ab} = 2 \sum_{(i),out,in} \frac{\eta_{(i)} m^{(i)}}{1-\hat{r}.\vec{v}} \frac{d\tau^{(i)}}{dt} \left(q_a{}^c u^{(i)}_c q_b{}^d u^{(i)}_d - \frac{1}{2} q^{cd} u^{(i)}_c u^{(i)}_d q_{ab} u^{(i)}_c u^{(i)}_d \right) \tag{4.60}$$

where $\eta_{(i)}$ is +1 for out-going and -1 for incoming particles.

We can apply transverse-traceless gauge to the linearized Riemann tensor (4.53), since it is gauge invariant in the General Relativity case. To do this, we can apply the TT-gauge to terms α and β as it can be seen eqn.(4.55). Then, the linearized Riemann tensor is

$$\begin{aligned}
R_{abcd} &= \frac{1}{r} \left[K_c \triangle_{ad} K_b - K_c \triangle_{bd} K_a - K_d \triangle_{ac} K_b + K_d \triangle_{bc} K_a \right] \delta'(U) \\
&= \frac{4}{r} \left[K_{[a} \triangle_{b][c} K_{d]} \right] \delta'(U).
\end{aligned} \tag{4.61}$$

The memory effect relates to the curvature via the geodesic deviation equation which has been reached in Chapter (1.3.1), since the relative motion of the test particles is described by the geodesic deviation equation. Let us consider a detector which consists of two freely falling test particles which are initially at rest with respect to each other, so their 4-velocities are both t^a, and say the spatial separation between them be d^a, then the coordinate version of the geodesic deviation equation is

$$\frac{d^2 d^i}{dt^2} = -R^i{}_{0j0} d^j. \tag{4.62}$$

If we use eqn.(4.57) in eqn.(4.58), and integrate it twice, then we have

$$\triangle d^i(U) = \int_{-\infty}^{U} dU' \int_{-\infty}^{U'} dU'' \frac{d^2 d^i}{dU''^2} \tag{4.63}$$

$$= \frac{1}{r} \triangle_k{}^i d^k. \tag{4.64}$$

Thus, there is nontrivial memory effect in 4-dimensional space-time.

4.4 Conclusion

Using the linearized theory and some gauges such that Lorenz and TT gauges we have transformed the Einstein equations to a simple wave equations. Then we have found the solution of this wave equations which describes the gravitational waves. By using the geodesic deviation we have found that there is non-trivial memory effect in 4-dimensional flat Minkowski space. The gravitational memory is already in data obtained by advanced LIGO, but it cannot be distinguished from low frequency background noise. In the future, one could expect that it can be measurable.

Also we have found that there are memory effects for the scalar fields and the electromagnetic fields.

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