ON EQUIVELAR TRIANGULATIONS OF SURFACES

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submitted by **EBRU ADIGÜZEL** in partial fulfillment of the requirements for the degree of **Master of Science in Mathematics Department, Middle East Technical University** by,

Prof. Dr. Halil Kalıpçılar
Dean, Graduate School of **Natural and Applied Sciences** _______________

Prof. Dr. Yıldıray Ozan
Head of Department, **Mathematics** _______________

Assoc. Prof. Dr. Semra Pamuk
Supervisor, **Department of Mathematics, METU** _______________

Examining Committee Members:

Assoc. Prof. Dr. İbrahim Ünal
Department of Mathematics, METU _______________

Assoc. Prof. Dr. Semra Pamuk
Department of Mathematics, METU _______________

Assist. Prof. Dr. Çisem Güneş Aktaş
Engineering Faculty, Abdullah Gül University _______________

Date: _______________

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Name, Last Name: EBRU ADIGÜZEL

Signature :

ABSTRACT

ON EQUIVELAR TRIANGULATIONS OF SURFACES

Adıgüzel, Ebru
M.S., Department of Mathematics
Supervisor : Assoc. Prof. Dr. Semra Pamuk

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Triangulation on a given surface S is an embedded graph G such that the intersection of two triangle can be either one edge, one vertex or empty set. A surface S is classified with respect to Euler characteristic and orientability. If the degree of each vertex of 2-simplex is q, then it is called as q-equivelar triangulation. If X_1 is an simplicial complex, $V(X_1)$ is the vertex set of X_1 and $Aut(X_1)$ acts on the set $V(X_1)$, then this triangulation is called as vertex-transitive triangulation. The Ringel's cyclic $7mod(12)$ series, Altshuler series and the class of cyclic equivelar tori are the only infinite series of vertex transitive triangulations of the surfaces. In this thesis, we survey classification of vertex-transitive and q-equivelar triangulations with respect to the number q, Euler characteristic and the number of vertices that will be used to triangulate the given surface S.

Keywords: equivelar triangulation, closed surfaces, vertex-transitive triangulation

ÖZ

YÜZEYLERİN EŞDEĞERLİ ÜÇGENLEMELERİ ÜZERİNE

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Verilen bir S yüzeyi üzerindeki üçgenleme iki üçgenin kesişimlerinin bir kenar, bir köşe veya boş küme olduğu bir G grafının S yüzeyine gömülmesidir. Bir S yüzeyi yüzeyin Euler karakteristiğine ve yönlendirilebilirliğine göre sınıflandırılır. Eğer S yüzeyi üzerindeki her 2-tekyönlü'nün derecesi q ise, bu üçgenleme q-eşdeğerli üçgenleme olarak adlandırılır. Eğer X_1 bir basit komplex, $Aut(X_1)$ köşe kümesi olan $V(X_1)$ kümesi üzerinde etki ediyorsa, bu üçgenleme köşe-değişmeli üçgenleme olarak adlandırılır. Ringel'in $7mod(12)$ serisi, Altshuler serisi ve halkalı eşdeğerli torus sınıfları bir yüzeyin sonsuz serili köşe-değişmeli üçgenlemelerinin tek örnekleridir. Bu tezde, biz köşe-değişmeli ve q-eşdeğerli üçgenlemelerin Euler karakteristiğine, S yüzeyinin üçgenlemesinde kullanılacak olan q-derecesine ve köşe sayılarına göre sınıflandırılmasını araştıracağız.

Anahtar Kelimeler: eşdeğerli üçgenleme, kapalı yüzeyler, köşe-değişmeli üçgenleme

To my family...

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CHAPTER 1

INTRODUCTION

A convex d-polytope is defined to be convex hull of any $d+1$ points in $\mathbb{R}^d$ and if the $d+1$ points are chosen as affinely independent points, it is called as d-simplex. The theory of the polytopes is one of the oldest topics that has been studied in mathematics and there exists a lot of crucial properties of the d-polytopes such as Euler's theorem which we state in section 2.1. If a simplicial complex X is homeomorphic to a topological space S, then one can say that the topological space S can be triangulated. A triangulation of a topological space enables us to compute some algebraic topological invariants such as homology and cohomology groups, and also numeric invariants such as the Euler characteristic. In this thesis, we will be interested in triangulations of surfaces only.

The main problem of triangulations of surfaces is the classification up to combinatorial equivalence of these triangulations on a fixed surface. Some specific triangulations of surfaces are of special interest, such as vertex-transitive, q-equivelar or d-covered triangulations which are defined in section 3. Historically, the first restriction on the number of vertices for triangulations is given by Heawood in 1890, [1]. He proved that a closed surface M not including Klein bottle, the orientable surface of genus 2, the non-orientable surface of genus 3 has a triangulation on n vertices if and only if $n \geq \lceil \frac{1}{2}(7+\sqrt{49-24\chi(M)}) \rceil$, where $\chi(M)$ is its Euler characteristic.

In addition to the bound for n, some restrictions such as the degree q of any vertex v is considered for getting q-equivelar triangulations on a given surface. For a q-equivelar triangulation with n-vertices on a given surface M, the bound for q is given as follows: $q \leq \lfloor \frac{1}{2}(5+\sqrt{49-24\chi(M)}) \rfloor$, where $q = \frac{6-6\chi(M)}{n}$ [1]. If the degree of at least one vertex of each edge is d, then it is called as d-covered triangulation which is studied first by Negami and Nakamoto [5]; if $d \geq 13$ and M is a closed surface then M has only finitely many d-covered triangulations. If the $Aut(X)$ acts transitively on the vertex set then this triangulation is called vertex-transitive. There are well-known infinite families of vertex-transitive triangulations of surfaces, such as Ringel series and cyclic torus triangulations.

This thesis is organized as follows: In chapter 2, we give some basic definitions on convex polytopes and we state the classification of surfaces. In chapter 3, we give a survey on q-equivelar, d-covered and vertex transitive triangulation of surfaces. In the last chapter, we give some classification results about q-equivelar and vertex transitive triangulations of surfaces.

CHAPTER 2

GENERALITIES

2.1 Convex Polytopes

Let us start this section by some basic notions based on [2] and [3]. A family of n-points $\{x_1, x_2, ...x_n\}$ in $\mathbb{R}^d$ is called *affinely independent* if the linear combination $\lambda_1 x_1 + \cdots + \lambda_n x_n = 0$ with $\lambda_1 + \lambda_2 + \cdots + \lambda_n = 0$ can be only satisfied when $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$. In particular, when $n = 0$ the empty set is affinely independent. When a point x is affine combination of $\{x_1, ...x_n\}$ such that $x = \lambda_1 x_1 + \lambda_2 x_2 \cdots + \lambda_n x_n$ with $\lambda_1 + \lambda_2 \cdots + \lambda_n = 1$ and $\lambda_1, \lambda_2, \cdots \lambda_n$ can be chosen in a unique way if and only if $\{x_1, x_2, \cdots x_n\}$ is affinely independent. Otherwise, it is called as affinely dependent. An affinely independent set of n-points $\{x_1, ...x_n\}$ is equivalent to linear independence of all of $(n-1)$-families of vectors $\{x_1 - x_i, \cdots, x_{i-1} - x_i, \cdots, x_n - x_i\}$, $i = 1, 2, \cdots n$. The unique smallest affine subspace of $\mathbb{R}^d$ containing the set A which consists of affinely independent n-points is called the *affine hull* of A, denoted as $aff(A)$. If we have affinely independent n points $\{x_1, ...x_n\}$ in a set A and any affine subspace of $\mathbb{R}^d$ is spanned by A, then $dim(aff(A))$ is equal to $dim(L) + 1$, where L is the linear subspace of $\mathbb{R}^d$ spanned by $\{x_1 - x_i, \cdots, x_{i-1} - x_i, \cdots, x_n - x_i\}$, since $A = L + x$, $x \in \mathbb{R}^d$. It is clear that;

- Each point in $aff(A)$ has unique representation as an affine combination of $\{x_1, ...x_n\}$.
- The 0-dimensional affine spaces are the 1-point sets.
- The 1-dimensional affine spaces are the lines. When the two points x_1 and x_2 are affinely independent, i.e the set $\{x_1, x_2\}$ is affinely independent and the

dimension of $aff(\{x_1, x_2\})$ is equal to 1, which is a line. Conversely, the line through two points x_1 and x_2 is in fact 1-dimensional affine space.

A *hyperplane* in an n-dimensional affine space A for $n \geq 1$ is an $(n-1)$-dimensional affine subspace of A. If A is an affine subspace of $\mathbb{R}^d$, a hyperplane in A is a set $H \cap A$, where H is a hyperplane in $\mathbb{R}^d$ such that $H \cap A \neq \emptyset$.
Recall that a set $S \subset \mathbb{R}^d$ is called *convex* if it has the property that for any pair of points $x, y \in S$, the line segment $\{\lambda x + (1-\lambda)y : 0 \leq \lambda \leq 1\}$ with end points x and y is in S. A *convex hull* is the smallest convex set containing S. A *convex d-polytope* P is defined to be convex hull of any $d+1$ points in $\mathbb{R}^d$. For example an edge is a 1-polytope, an n-gon in $\mathbb{R}^2$ is a 2-polytope and a tetrahedron is a 3-polytope. The convex hull of $d+1$ affinely independent points is a d-polytope known as d-*simplex*. If all the proper faces of d-polytope are simplicies, it is called as a *simplicial polytope.*

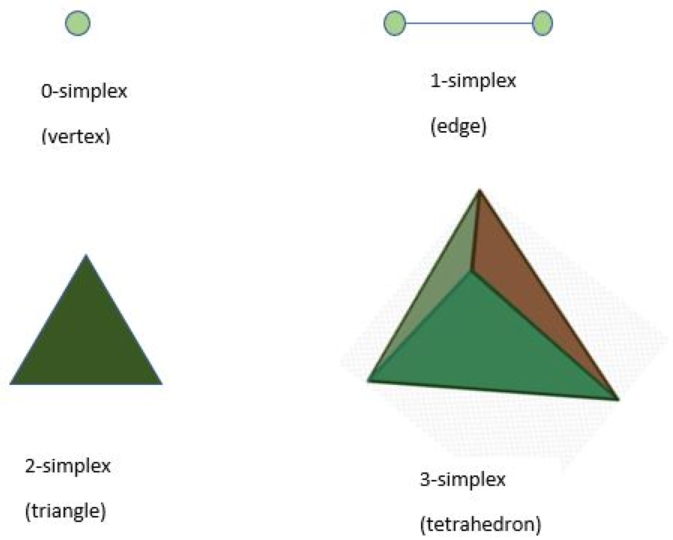

Figure 2.1: Affinely independent points in $\mathbb{R}^3$

2.1.1 Combinatorial Theory Of Polytopes:

A hyperplane H *supports* a closed bounded convex set S if $H \cap S \neq \emptyset$, and S lies in one of two closed half-spaces bounded by H. If H supports S then $H \cap S$ is called as a *face* of a polytope. For a d-polytope P, the following properties hold:

1. The faces of P are polytopes.
2. The dimensions of faces of P can be $0, 1, ...n - 1$. A 0-face is called a *vertex*. A 1-face is called an *edge*, a $(d-1)$-face is called a *facet* of P. The empty set $\emptyset$ is called as an *improper face* and $dim(\emptyset) = -1$.
3. Every face of a face is also a face of P.
4. For every two faces F_1 and F_2 of P, $F_1 \cap F_2$ is also a face of P. It will be denoted as $F_1 \wedge F_2$.
5. For every two faces F_1 and F_2 of P, there exists uniquely defined smallest face which is denoted by $F_1 \vee F_2$.

The combinatorial theory of polytopes is concerned with their facial structures. Let $\mathcal{F}(P)$ denote the set of all faces of the polyhedron P. Then, $\mathcal{F}(P)$ becomes a lattice with the operations $\vee, \wedge$ defined in the above items 4 and 5 and it is called as the *face lattice* of P. Two polytopes P_1 and P_2 are called *combinatorially equivalent* and denoted by $P_1 \approx P_2$, if their face lattices are isomorphic, i.e. $\mathcal{F}(P_1) \approx \mathcal{F}(P_2)$. That is, there exists a one-to-one inclusion-preserving mapping from $\mathcal{F}(P_1)$ onto $\mathcal{F}(P_2)$. If σ is a subset of $\mathcal{F}(P_1)$, $\sigma \subset \mathcal{F}(P_1)$, and ϕ is an affine map from $\mathcal{F}(P_1)$ onto $\mathcal{F}(P_2)$, then $\phi(\sigma) \subset \mathcal{F}(P_2)$. Roughly speaking, if two polytopes P_1 and P_2 are combinatorially equivalent, they have the same number of faces for each dimension, but possibly of different shapes.

Definition 2.1.1. *For each d-polytope, there exists a polytope P^*, called dual of P, if $\mathcal{F}(P)$ and $\mathcal{F}(P^*)$ are anti-isomorphic, i.e there exists an inclusion-reserving mapping from $\mathcal{F}(P)$ onto $\mathcal{F}(P^*)$.*

Note that the meaning of the inclusion-reserving mapping is as follows: if $\sigma \subset \mathcal{F}(P_1)$ and ϕ is an affine map from $\mathcal{F}(P_1)$ onto $\mathcal{F}(P_2)$, then $\phi(\sigma) \supset \mathcal{F}(P_2)$.) The central problem of combinatorial theory is about finding necessary and sufficient conditions for a given lattice L to be isomorphic to its face lattice $\mathcal{F}(P)$ of some polytope P. The f-vector of any d-polytope P is defined as $f(P) = (f_0(P), f_1(P), ...f_{d-1}(P))$,where

$f_j(P)$ represent the number of j-faces of P for all values of $j = 0, 1 \cdots d - 1$. Since the problem of characterising face lattices of polytopes is so harder than characterising f-vectors of polytopes, the problem about f-vectors of a polytope is more suitable problem to find isomorphic face lattices of some polytope P. Recall that the Euler characteristic $\chi(L)$ of a subcomplex L (or any subset of L) is defined by

$$\chi(L) = \sum_{i=0}^{i=n} (-1)^i f_i(L). \tag{2.1}$$

The main result that is relevant with Euler's characteristic is the following:

Theorem 2.1.1. *(Euler's Theorem, [27]) Let P be any d-polytope, then we have*

$$f_0(P) - f_1(P) + \cdots (-1)^{d-1} f_{d-1}(P) = 1 + (-1)^{d-1}$$

The generalization for a surface with boundary is the following:

Theorem 2.1.2. *(Dehn-Sommerville Relations, [2]) Let M be a triangulated surface with boundary ∂M. For $k = 0, 1, ..., m$;*

$$f_k(M) - f_k(\partial M) = \sum_{i=k}^{i=m} (-1)^{i+m} \binom{i+1}{k+1} f_i(M)$$

Recall that a moment curve in $\mathbb{R}^d$ is defined as follows:

$$X : \mathbb{R} \to \mathbb{R}^d$$
$$t \to [t, t^2, ..., t^d]$$

The convex hull of n distinct points on the moment curve $\{m(t) = (t, t^2, ..., t^d)^T : t \in \mathbb{R}\}$ in $\mathbb{R}^d$ is called as a cyclic polytope. To investigate the number of k-dimensional faces in a polytope P, we will search the number of faces of cyclic polytope, since the number of faces of k-dimensional cyclic polytope is the upper bound for the number of k-dimensional faces of any polytope P. McMullen's Upper Bound Theorem [4] shows that the maximum number of k-dimesional faces of a polytope P, denoted $f_k(P)$, with n vertices is attained by the cyclic polytopes for all values of $k = 1, 2, ..., d - 1$.

Theorem 2.1.3. *(Upper Bound Theorem, [4]) For any d-polytope with n vertices;*

$$f_k(P) \leq f_k(c(d,n)), \forall k = 1, ...d-1$$

The number of k-faces of a cyclic polytope $c(d,n)$ can be evaluated and thus one can find the upper bound for the k-faces of a cyclic polytope. Another crucial point about finding the number of facets of cyclic polytope is Gale's evenness condition that is given below.

Theorem 2.1.4. *(Gale's evenness condition, [6])*

Let $T = \{t_1, t_2, ...t_n\}$ with $t_1 < t_2... < t_n$. $T_d \subset T$ forms a facet of $c(n,d)$ if and only if any two elements in $T \backslash T_d$ are seperated by an even number of elements from T_d.

Example 2.1.1. *If $T = \{1,2,3,4,5,6\}$ and $T_d = \{1,2,3\}$ are given sets then $T \backslash T_d = \{4,5,6\}$.*

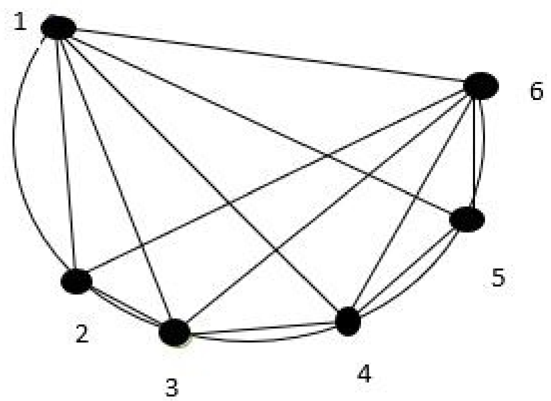

Figure 2.2: Cyclic polytope $c(6,3)$

1. *There is no elements between any element of $T \backslash T_d$ which are in T_d. So, (123) forms a facet.*

2. *Let $T_d = \{3,4,6\}$. Then, $T \backslash T_d = \{1,2,5\}$. There is no element between 1 and 2 which is element of T_d. Also, there exists two elements between 2 and 5 which are elements of T_d. So, (346) will be a facet.*

3. *If $T_d = \{1,2,5\}$, then $T \backslash T_d = \{3,4,6\}$. There exists one element between 4 and 6 which is element of T_d. So, it contradicts with the Gale's evenness condition and it will not be a facet of the cyclic polytope $c(6,3)$.*

4. *If $T_d = \{1,3,4\}$, then $T \backslash T_d = \{2,5,6\}$. There exists two element between 2 and 5. Also, there is no element between 5 and 6. So, (134) forms a facet.*

5. *If $T_d = \{1,4,5\}$, then $T \backslash T_d = \{2,3,6\}$. There is no element between 2 and 3 and there are two elements which are 4 and 5 between 3 and 6. So, (145) forms a facet.*

6. *If $T_d = \{2,3,6\}$, then $T \backslash T_d = \{1,4,5\}$. There exists two element between 1 and 4. Also, there is no element between 4 and 5. So, (236) forms a facet.*

7. *If $T_d = \{4,5,6\}$, then $T \backslash T_d = \{1,2,3\}$. There is no element between any element of the set $T \backslash T_d$. So, (456) forms a facet.*

2.2 Clasification Of Surfaces:

In this section we will state the very well known classification theorem of connected, closed surfaces. All the statements and proofs can be found in a standard geometric topology book such as [7]. Recall that, a Hausdorff, second countable topological space M in which every point has an open neighbourhood homeomorphic to an open disc in $\mathbb{R}^n$ is called an *n-manifold*. In particular, a 2-manifold is called a *surface*.

Theorem 2.2.1. *[7] Every connected, compact surface is homeomorphic to a sphere, or a connected sum of tori, or connected sum of projective planes.*

The proof of this theorem is based on a well known lemma from geometric topology, which says that every connected, compact surface has a planar diagram; a polygon with $2k$-edges, where a pair of edges are identified with either same or opposite orientation and some certain cutting and pasting operations on these diagrams which do not change the topology of the surface. One can find the details in [7].

Now, one needs to show that the surfaces; sphere S^2, connected sum of tori mT^2, and connected sum of projective planes $k\mathbb{R}P^2$, are not homeomorphic.

Definition 2.2.1. *Let K be a finite cell complex of dimension n, the Euler characteristic of K, is*

$$\chi(K) = \sum_{i=0}^{n} (-1)^n |i - cells|$$

where $|i - cells|$ denotes the number of i-cells in K.

Theorem 2.2.2. *For finite complexes K and L, if their geometric realizations are homeomorphic $|K| \cong |L|$, then they have the same Euler characteristic , i.e. $\chi(K) = \chi(L)$.*

The Euler characteristic of a connected, compact surface S is defined as the Euler characteristic of any cell complex K such that $|K| \cong S$. So, by the above theorem Euler characteristic is a well-defined topological invariant. It can also be shown that the Euler characteristic of connected sum of two surfaces S_1 and S_2 is $\chi(S_1 \sharp S_2) = \chi(S_1) + \chi(S_2) - 2$. Now, we can compute the Euler characteristics of the above surfaces:

$$\chi(S^2) = 2, \ \ \chi(mT^2) = 2 - 2m, \ \ \chi(k\mathbb{R}P^2) = 2 - k$$

In the case of a torus T^2 and a Klein bottle $K = \mathbb{R}P^2 \sharp \mathbb{R}P^2$ their Euler characteristics are same 0, but we know that they are not homeomorphic since one of them orientable and the other is not. Recall that a surface is called *non-orientable* if it contains a Möbius band, otherwise it is called *orientable*. For example, $\mathbb{R}P^2$ can be obtained by attaching a disk to the boundary of the Möbius band, so it is non-orientable. So, we can state the following theorem:

Theorem 2.2.3. *Up to homeomorphisms, every connected, compact surface is uniquely determined by its Euler characteristic and orientability.*

The number m of tori in the connected sum of mT^2 is called genus of the surface. More generally, for orientable surfaces we can think the genus as the number of closed non-intersecting curves, so that when we cut the surface along these curves the surface does not become disconnected. For example, genus of a sphere is 0. So for orientable surfaces the relation between genus and Euler characteristic is:

$$\chi(S) = 2 - 2m \ \ then \ \ m = \frac{2 - 2\chi(S)}{2}$$

For non-orientable surfaces, the number k of $\mathbb{R}P^2$'s in the connected sum of $k\mathbb{R}P^2$ is called cross caps or sometimes non-orientable genus of the surface. The relation between the Euler characteristic is:

$$\chi(S) = 2 - k \ \ then \ \ k = 2 - \chi(S)$$

Therefore, the genus and/or the number of cross caps characterize the surfaces.

CHAPTER 3

TRIANGULATIONS OF SURFACES

Definition 3.0.1. *An abstract simplicial complex A is a finite collection of non-empty finite sets such that*
(i) if $\tau \in A$ and $\sigma \subseteq \tau$, then $\sigma \in A$,
(ii) if $\sigma, \tau \in A$, then either $\sigma \cap \tau = \emptyset$ or $\sigma \cap \tau \in A$.

A *geometric realization or carrier* of an abstract simplicial complex A is defined by a function $f : A \to \mathbb{R}^d$ that maps each element in A to an affinely independent set in $\mathbb{R}^d$. The geometric realization of an abstract simplicial complex A is denoted by $|A|$. If any topological space P is homeomorphic to the geometric carrier $|A|$, then one can say that the topological space P can be *triangulated*. It is easy to see that any finite abstract simplicial complex has a geometric realization into a space of dimension of $|A|$. Menger showed that any n-dimensional abstract simplicial complex can be geometrically realized in $\mathbb{R}^{2n+1}$, [8]. As an example, let's take the set A as follows: $A = \Big\{\{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}, \{1,2,3\}, \{1,3,4\}, \{1,2,4\}, \{2,3,4\}\Big\}$. It can be seen from the Figure 3.1, the geometric realization of the abstract simplicial complex A gives us a tetrahedron. We have another definition of the triangulation of a surface as follows:

Definition 3.0.2. *A triangulation of a closed surface S is an embedded graph G such that*
(i) all regions (faces) are topological triangles (elements of geometric carier)
(ii) each edge in $E(G)$ belongs to exactly two triangles;
(iii) each vertex in $V(G)$ belongs to at least 3 triangles, arranged in a cyclic order around it.

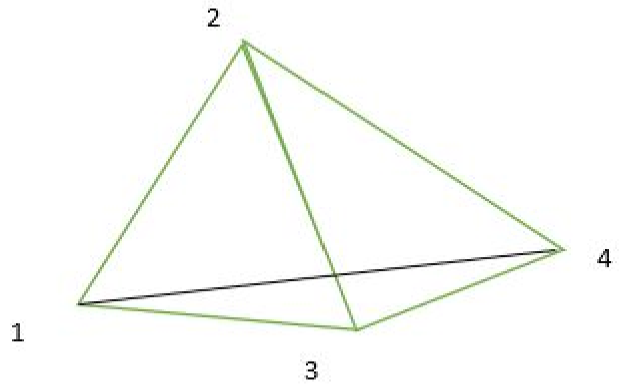

Figure 3.1: The geometric realization of the set A in $\mathbb{R}^3$

Let K be a triangulation of a closed surface S. The f-vector of K is the vector $f(K) = (f_0(K), f_1(K), f_2(K))$, where $f_0(K)$ is the number of vertices, $f_1(K)$ is the number of edges and $f_2(K)$ is the number of triangles.

Definition 3.0.3. *(q-equivelar triangulation, [1]) If all vertices of the triangulation have the same vertex-degree q, a triangulation K of a surface S is called a q-equivelar or type $\{3, q\}$.*

Let $f_0(K) = n$ and clearly we have $n \geq q$. Since each edge belongs to two triangle, then $f_1(K) = \frac{nq}{2}$ and since each triangle has three vertices with vertex-degree q, then $f_2(K) = \frac{nq}{3}$. So, we have $nq = 2f_1(K) = 3f_2(K)$. Moreover, by Euler equation we have

$$\chi(M) = n - f_1 + f_2 = n - \frac{nq}{2} + \frac{nq}{3} = \frac{n(6-q)}{6}$$

Since q is a positive integer, n has to be a divisor of $6\chi(M)$, if $\chi(M) \neq 0$. If $\chi(M) \neq 0$, there exists only finitely many equivelar triangulations.

Definition 3.0.4. *(Neighbourly triangulation, [1]) If there exists an edge between any two vertices of simplicial complex K, then K is called a neighbourly triangulation.*

That is, $f_1 = \binom{n}{2}$ and K has at most $f_1 \leq \binom{n}{2}$ edges. When we have neighbourly triangulations with $q = n - 1$ and therefore

$$\chi(M) = \frac{n(7-n)}{6} \tag{3.1}$$

$n \equiv 0, 1, 3, 4\,(mod 6)$, where $n \geq 4$.

$$q = 6 - \frac{6\chi(M)}{n} \tag{3.2}$$

For instance; if $\chi(M) = 2$ then $q = 6 - \frac{12}{n}$ and $nq = 6n - 12$, i.e., $12 = n(6 - q)$. So, the possibilities of (n, q) are $(4, 3)$, $(6, 2)$ or $(12, 5)$ and the boundary of tetrahedron, octahedron or icosahedron are the only equivelar triangulations of the 2-sphere.

There is only one equivelar triangulation of the projective plane $\mathbb{R}P^2$, since when $\chi(M) = 1$, $6 = n(6 - q)$ so we get $(n, q) = (6, 5)$.

In the case of $\chi(M) = 0$, $n(6 - q) = 0$ if and only if $q = 6$ with $n \geq 7$. So, there are infinitely many equivaler triangulation of the torus and Klein bottle. The complete classification of equivelar triangulations of the torus and Klein bottle can be found in [11].

If $\chi(M) = -1$ then $n(6 - q) = -6$. So, there is no q-equivelar triangulation of the surface M with $\chi(M) = -1$.

Definition 3.0.5. *(d-covered triangulation, [5]) A triangulation K of a surface M is called d-covered, if at least one vertex of each edge of K has degree d.*

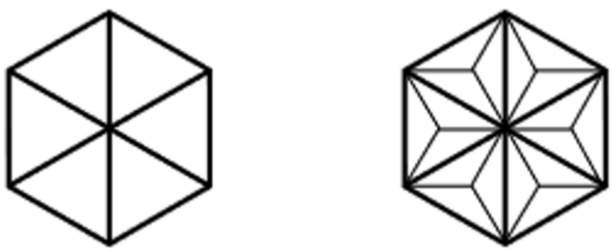

Figure 3.2: A vertex-star of degree six and its subdivision.

The study of d-covered triangulations was initiated by Negami and Nakamato in [5]. We can get a $2q$-covered triangulation from the q-equivelar triangulation; for each triangle we will add a new vertex inside the triangle and connect the new vertex inside with the three vertices of the triangle by an edge. Clearly, any q-equivelar triangulation is q-covered triangulation.

Theorem 3.0.1. *(Jungerman and Ringel, [9],[10]) Let M be a surface different from the orientable surface of genus 2, the Klein bottle, and the non-orientable surface of genus 3 then there is a triangulation on M with n vertices if and only if*

$$n \geq \lceil \frac{1}{2}(7 + \sqrt{49 - 24\chi(M)}) \rceil$$

with equality if and only if the triangulation is neighbourly. For the three exceptional cases, one extra vertex has to be added to obtain the lower bound.

Proof. *Let n be the number of vertices of triangulation of M. If the triangulation is neighbourly, $f_1 = \binom{n}{2}$. Otherwise; $f_1 \leq \binom{n}{2}$. Also, a face consists of 3 edges and each edge counts twice, i.e $f_2 = \frac{2f_1}{3}$. So, by the Euler equation we have*

$$\chi(M) = f_0 - f_1 + f_2 \leq n - \binom{n}{2} + \frac{2}{3}\binom{n}{2} \leq n - \frac{1}{3}\binom{n}{2} \leq n - \frac{n(n-1)}{6}.$$

From this inequality, we have $\chi(M) \leq \frac{7n-n^2}{6}$ and $n^2 - 7n + 6\chi(M) \leq 0$. Since

$$\Delta = 49 - 24\chi(M) \text{ then } n_{1,2} = \frac{7 \pm \sqrt{49 - 24\chi(M)}}{2},$$

also n must be smaller than or equal to the number which is the least integer greater than or equal to x, we get

$$n \geq \lceil \frac{1}{2}(7 + \sqrt{49 - 24\chi(M)}) \rceil.$$

Note that, the bound for the Klein bottle triangulations, we need to have at least 8 vertices [24], in particular, for the q-equivelar triangulation of the Klein bottle, n must be a composite number and $n \geq 9$. The proof of this fact can be found in [11]. Also, the bound of the orientable surface of genus 2 was proved by Huneke in [12] and the bound of the non-orientable surface of genus 3 was proved by Ringel in [10].

Recall that the 1-skeleton graph of a simplicial 2-complex K consists of only the vertices and edges of K. Let $G = Sk_1(K)$ be the 1-skeleton of K, $deg_G(v)$ be the degree of a vertex v in the graph G, and $\delta(G) = min_{v \in G} deg_G(v)$. Since the degree of any vertex in a triangulation K can be at least 3 then $3 \leq \delta(G) \leq n - 1$ and $n\delta(G) \leq \sum_{v \in G} deg_G(v) = 2f_1$ so $\delta(G) \leq \dfrac{2f_1}{n}$ with $f_1 = 3(n - \chi(K))$ and this gives the upper bound for the minimum degree $\delta(G)$ as $6 - \dfrac{6\chi(K)}{n}$. If $\chi(K) > 0$, then $\delta(G) \leq 5$. If $\chi(K) \leq 0$, then

$$\delta(G) \leq 6 - \frac{6\chi(K)}{n} \text{ so } \delta(G) \leq 6 - \frac{6\chi(K)}{\delta(G) + 1}.$$

So,

$$\delta(G) \leq \frac{6(\delta(G)+1) - 6\chi(K)}{\delta(G)+1}$$

implies

$$\delta^2(G) + \delta(G) \leq 6\delta(G) + 6 - 6\chi(K)$$

and it gives us the following

$$\delta^2(G) - 5\delta(G) - 6 + 6\chi(K) \leq 0.$$

We know that $\Delta = 49 - 24\chi(K)$ and $\delta_{1,2}(G) = \dfrac{5 \pm \sqrt{49 - 24\chi(K)}}{2}$. Therefore, the upper bound for the minimum degree $\delta(G)$ is as follows:

$$\delta(G) \leq \lfloor \frac{1}{2}(5 + \sqrt{49 - 24\chi(K)}) \rfloor.$$

In particular, we have $\delta(G) = q$ for any q-equivelar triangulation K of a surface M, and thus,

$$q \leq \lfloor \frac{1}{2}(5 + \sqrt{49 - 24\chi(M)}) \rfloor \tag{3.3}$$

Let K be a d-covered triangulation of a surface M, one can also find an upper bound for the number d. Let H be the subgraph of $Sk_1(K)$, which is induced by the vertices of degree d. Since $Sk_1(K)$ is d-covered, i.e for each edge at least one vertex has degree d, the link of any vertex v is contained in H. So, $deg_H(v) \geq \dfrac{d}{2}$. Let w be a vertex of $Sk_1(K)$ with $deg_{Sk_1K}(w) \neq d$. Then all vertices of the link of w have degree d and thus so are the vertices of H. We get a cellular decomposition C of M with $H = Sk_1(C)$ by deleting all vertices w of degree $deg_{Sk_1(K)}(w) \neq d$ and the edges. Let $f^c = (f_0^c, f_1^c, f_2^c)$ be the face vector of the cellular decomposition C and $2f_1^c = \sum_{p \in \{2-faces\ of\ C\}} p(P) \geq 3f_2^c$, where $p(P)$ is the number of the polygon P whose the number of edges of P will be at least 3. So, we can evaluate the Euler characteristic $\chi(M)$ of the cellular decomposition C of M by using the Euler theorem and the relation that was known about f_0^c and f_1^c. In this manner, we have

$$\chi(M) = f_0^c - f_1^c + f_2^c \leq f_0^c - f_1^c + \frac{2f_1^c}{3} = f_0^c - \frac{f_1^c}{3}$$

or equivalently $f_1^c \leq 3(f_0^c - \chi(M))$. Since

$$\frac{d}{2} \leq \delta(H) = \frac{f_0^c \delta(H)}{f_0^c} \leq \frac{\sum_{v \in H} deg_H(v)}{f_0^c} = \frac{2f_1^c}{f_0^c} \leq 6 - \frac{6\chi(M))}{f_0^c}$$

and the minimum degree of any vertex of H denoted by $\delta(H)$ can be at most $f_0^c - 1$ so we have $\dfrac{d}{2} \leq \delta(H) \leq 6 - \dfrac{6\chi(M))}{f_0^c} \leq 6 - \dfrac{6\chi(M)}{\delta(H)+1}$ and from this inequality, the following can be obtained:

$\delta^2(H) + \delta(H) \leq 6\delta(H) + 6 - 6\chi(M)$ and then $\delta^2(H) - 5\delta(H) + 6\chi(M) - 6 \leq 0$.

So, if we find the roots of the $\delta(H)$, we get

$$\Delta = 49 - 24\chi(M) \text{ and } (\delta(H))^c_{1,2} \leq \lfloor \frac{1}{2}(5 + \sqrt{49 - 24\chi(M)}) \rfloor.$$

Since

$$\frac{d}{2} \leq \delta(H) \leq \lfloor \frac{1}{2}(5 + \sqrt{49 - 24\chi(M)}) \rfloor, \text{ then } d \leq 2\lfloor \frac{1}{2}(5 + \sqrt{49 - 24\chi(M)}) \rfloor$$

and the interior of the $\sqrt{49 - 24\chi(M)}$ is always positive when $\chi(M) < 0$, so we get

$$d \leq 2\lfloor \frac{1}{2}(5 + \sqrt{49 - 24\chi(M)}) \rfloor.$$

If $\chi(M) > 0$, $\sqrt{49 - 24\chi(M)}$ can be at most 5 and so $d \leq 10$. Since the equation 3.2 is satisfied and when $\chi(M) = 0$, we have infinitely many q-equivelar triangulations of torus and Klein bottle with $q = 6$. Since, any q-equivelar triangulation is also q-covered, there exists infinitely many q-covered triangulations when $\chi(M) = 0$. So, in the case $d = 12$, there exists infinitely many d-covered triangulations. If $\chi(M) \leq 0$ and so

$$d \geq 13 \text{ and } 7 \leq \lceil \frac{d}{2} \rceil \leq \delta(H) \leq 6 - \frac{6\chi(M)}{f_0^c}$$

or equivalently $f_0^c \leq -6\chi(M)$. Therefore, there are only finitely many cellular decomposition for $d \geq 13$. From this inequalities, one can easily obtain the following theorems:

Theorem 3.0.2. *(Negami and Nakamato, [5]) Let K be a d-covered triangulation of a surface M.*
(i) If $\chi(M) > 0$, then $d \leq 10$.
(ii) If $\chi(M) \leq 0$, then $d \leq 2\lfloor \frac{1}{2}(5 + \sqrt{49 - 24\chi(M)}) \rfloor$.

Theorem 3.0.3. *(Negami and Nakamato, [5]) Let $d \geq 13$ and M be a closed surface. Then M has only finitely many d-covered triangulations.*

Since we have $\frac{d}{2} \leq \delta(H) \leq 6 - \frac{6\chi(M)}{f_0^c}$, the following theorems are easily obtained.

Theorem 3.0.4. *(Frank Lutz, [1]) Let C be a regular cell decomposition of a surface M and $H = Sk_1(C)$.*
(i)If $\chi(M) > 0$, then $\delta(H) \leq 5$.
(ii)If $\chi(M) \leq 0$, then $\delta(H) \leq \lfloor \frac{1}{2}(5 + \sqrt{49 - 24\chi(M)}) \rfloor$

Theorem 3.0.5. *(Frank Lutz, [1]) Let M be a closed surface.Then M has only finitely many regular cell decompositions C with $\delta(Sk_1(C)) \geq 7$.*

3.1 The action of a group on the set of vertices $V(X)$

Definition 3.1.1. *An action of a group G on a set X is defined as follows: $\forall g \in G$, $\Pi_g : X \to X$ satisfies the following two conditions:*
(i)Π_e is the identity: $\Pi_e(x) = x$ for each $x \in X$
(ii)For every $g_1, g_2 \in G$, $\Pi_{g_1} \circ \Pi_{g_2} = \Pi_{g_1 g_2}$

Example 3.1.1. *The symetric group S_n acts on $X = \{1, 2, ...n\}$ in the usual way. Since $\Pi_g : X \to X$ with $\Pi_e(x) = x$ for every $x \in X$ and for every $g_1, g_2 \in S_n$, the second property $\Pi_{g_1} \circ \Pi_{g_2} = \Pi_{g_1 g_2}$ is also satisfied, S_n acts on the set X.*

Example 3.1.2. *D_3 acts on $X = \{1, 2, 3\}$: Let $D_3 = \{e, (123), (132), (12), (13), (32)\}$.*

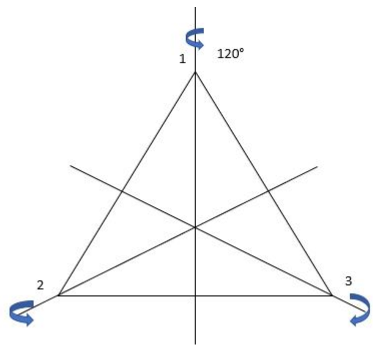

Figure 3.3: The action of D_3 on the set X

The group D_3 has 6 elements that consists of 3 rotations and 3 reflections. The first three elements are the rotations of the triangle, and the last three elements are the reflections of the triangle. So, the action of D_3 on X is all the rigid motions of the triangle.

Recall that, if the group G acts on $X, \forall x \in X$, its orbit is $Orb_x = \{g.x|g \in G\} \subset X$. Let G acts on $X, \forall x \in X$, its stabilizer is $Stab_x = \{g \in G|gx = x\} \subset G$.

Example 3.1.3. *Let* $G = \{(1), (132)(465)(78), (132)(465), (123)(456), (123)(456)(78), (78)\}$. $Stab_G(1) = \{(1), (78)\}$, $Stab_G(2) = \{(1), (78)\}$, $Stab_G(3) = \{(1), (78)\}$. $Orb_G(1) = \{1, 2, 3\}$, $Orb_G(2) = \{1, 2, 3\}$, $Orb_G(3) = \{1, 2, 3\}$.

Example 3.1.4. *Consider the action of* $D_4 = \{p_0, p_1, p_2, p_3, \mu_1, \mu_2, \delta_1, \delta_2\}$ *on the square:*
where p_0*:*$0°$ *rotation,* p_1*:*$90°$ *rotation,* p_2*:*$180°$ *rotation,* p_3*:*$270°$ *rotation,* μ_1*:reflection*

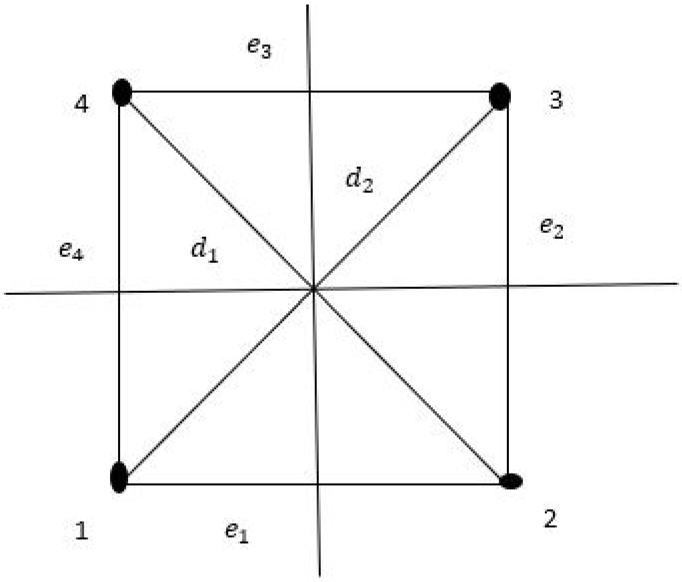

Figure 3.4: The action of D_4 on the set X

with respect to vertical line, μ_2*:reflection with respect to horizontal line,* δ_1*:reflection with respect to* d_1*,* δ_2*:reflection with respect to* d_2*.*
$Orb_G(1) = \{1, 2, 3, 4\}$, $Orb_G(e_3) = \{e_1, e_2, e_3, e_4\}$
$Orb_G(d_1) = \{d_1, d_2\}$, $Orb_G(d_2) = \{d_1, d_2\}$.
$Stab_G(2) = \{g \in D_4|g \vee 2 = 2\} = \{p_0, \delta_1\}$,
$Stab_G(e_2) = \{g \in D_4|g \vee e_2 = e_2\} = \{e_2, \mu_2\}$
$Stab_G(d_1) = \{g \in D_4|g \vee d_1 = d_1\} = \{p_0, p_2, \delta_1, \delta_2\}$
$Stab_G(d_2) = \{p_0, p_2, \delta_1, \delta_2\}$.

3.2 Vertex-Transitive Triangulations

Recall that the set $Aut(X)$ consists of the functions $f : X \to X$ which is an isomorphism with the induced bijective inclusion-preserving function ϕ on the vertex set $V(X)$ of the simplicial complex X.

Definition 3.2.1. *[13] If X is a simplicial complex and $\phi : V(X) \to V(X)$ is an isomorphism such that $Aut(X)$ acts on the set $V(X)$ transitively, then X has vertex-transitive triangulation.*

That is, every vertex which is an element of X must go to another vertex which is again an element of X under the action of automorphism group operation.

Example 3.2.1. *Let $n = 7$ and K be a triangulation on the torus, denoted $T^2(7)$. Then, let's consider the action of Z_7 on the elements $\{0, 1, ...6\}$ generated by the permutation $(0, 1, ...6)$. The orbits of the generating triangles $[0, 1, 3]$ and $[0, 2, 3]$ are as follows:*

$$Orb_{Z_7}([0,1,3]) = \{[0,1,3],[1,2,4],[2,3,5],[3,4,6],[4,5,0],[5,6,1],[6,0,2]\}$$

$$Orb_{Z_7}([0,2,3]) = \{[0,2,3],[1,3,4],[2,4,5],[3,5,6],[4,6,0],[5,0,1],[6,1,2]\}$$

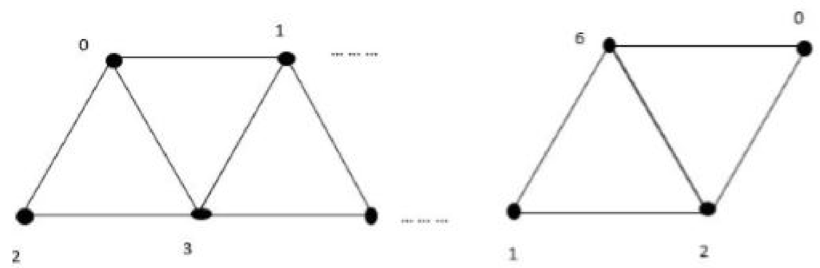

Figure 3.5: The action of the cyclic group Z_7 on the vertex set $\{0, 1, ...6\}$.

It can be generalized for the action of any cyclic group Z_n on the elements $0, 1, ...n-1$ generated by the permutation $(0, 1, ...n-1)$ with $n \geq 7$. This triangulation gives us a vertex-transitive triangulation of the torus and this series is called as cyclic torus triangulation [1].

There are more examples of vertex-transitive triangulations on the orientable surfaces as known *Ringel's cyclic* $7(mod12)$. Now, we will give a background to explain why the Ringel's cyclic $7(mod12)$ gives us a vertex-transitive triangulation when $n \equiv 7(mod12)$.

3.2.1 Ringel's cyclic $7mod12$ series

The existence of Ringel's cyclic $7mod12$ series is concerning with the four colour problem given in graph theory. The motivation began with the following question. Can we color the countries on the map with four colors such that any adjacent country will not have same color? We will answer this question by using some constructions that were defined in the graph theory.

Definition 3.2.2. *If S is a surface and each map on S are colorable with n colours, while not every map on S is colorable with $n-1$ colours, then the chromatic number of S will be $\gamma(S) = n$*

Heawood [15] proved the following inequality about the chromatic number $\gamma(S)$ on the orientable surface of genus $g \geq 1$.

$$\gamma(S) \leq \lfloor \frac{7+\sqrt{1+48g}}{2} \rfloor$$

The proof of the above inequality is related with "Thread Problem". Now,we will explain the relation between "Thread Problem" and the "Heawood's bound".
Thread Problem: Can we choose n points on the surface and connect each of these n points to each other by a simple curves(thread) such that these curves do not intersect? Note that, finding the chromatic number and finding non-intersecting curves(thread) are completely same. So, we will use the same notations for these two problems. In the first shape, there exists 4 points on a plane or a surface that can be connected by non-intersecting simple curves. The second shape illustrates a torus with 7 points on it that can be connected by non-intersecting simple curves. The above question was solved in 1968 by proving the following formula;

$$\gamma(n) = \lceil \frac{7+\sqrt{1+48g}}{2} \rceil, \ for \ \ n \geq 3$$

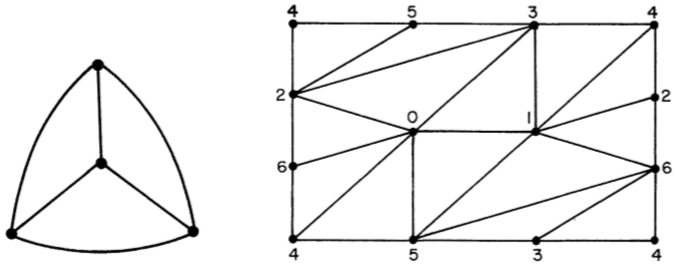

Figure 3.6: Minimal triangulation of sphere and torus

In 1891, L.Hefter [16] proved the formula for all $n \leq 12$ and for the every specific numbers in the given sequence: $n = 19, 31, 55, 67, 139, 175, 199....$ These are the numbers n of the form $n = 12s + 7$, where $g = 4s + 3$ is a prime number.

3.2.1.1 Rotation Of Graphs

Definition 3.2.3. *Let G be a graph and a rotation of a vertex A is an oriented cyclic order of all arcs incident with A.*

Example 3.2.2. *If the vertex 0 is adjacent to three vertices 1,2,3, the rotation of the vertex 0 is given as follows:*

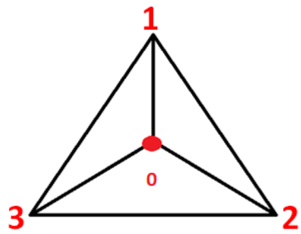

Figure 3.7: Rotations on the triangle

$$(123) = (231) = (312)$$

or

$$(132) = (213) = (321)$$

In the plane, a clockwise(or counter-clockwise) reading of the arcs incident with each vertex gives the rotation of a graph G. If the reading is clockwise(counter-clockwise), then the vertex is represented by a small filled-in(empty) circle such as $\bullet(\circ)$. The number of possible rotations of a vertex of valance n is $(n-1)! = 1.2...(n-1)$. A rotation σ of a graph G is a rotation for each vertex of G. The notation (G, σ) means a graph G with a certain rotation σ. Let's write a rotation for the above graph. Denote the

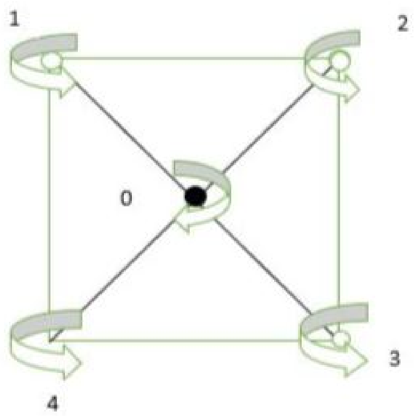

Figure 3.8: A square graph with rotation

vertices by the numbers $0, 1, 2, 3, 4$. Then write down the cyclic permutation of the neighbours of each vertex i, where $i = 0, 1, 2, 3, 4$. The cyclic permutation of the arcs incident with i gives us a rotation of the vertex i.

$$
\begin{array}{l}
0.1 \;\; 2 \;\; 3 \;\; 4 \\
1.4 \;\; 0 \;\; 2 \\
2.0 \;\; 3 \;\; 1 \\
3.2 \;\; 0 \;\; 4 \\
4.3 \;\; 0 \;\; 1
\end{array}
$$

For instance, the above scheme represents the graph given in Figure 3.8 with the rotation. Recall that, a *walk* of length k from $v_0 \in V$ to $v_k \in V$ is a sequence of vertices $v_0 v_1 ... v_{k-1} v_k$ such that the adjacent pairs $v_0 v_1$,$v_1 v_2$...$v_{k-1} v_k$ are all edges. A *circuit* is a closed walk with all edges distinct. The circuit $1, 2, 0, 3, 4, 0$ is induced by the graph given with the rotation of the square graph. Our first aim is to construct a circuit that can be written by the rotation of graph G. Let A_0 be a vertex incident with an arc c_0 in a graph G with a rotation (G, σ). Then one can construct a closed walk as follows:

$$A_0c_0A_1c_1...A_tc_tA_0c_0$$

We denote the number of circuits induced by a rotation σ of a graph G by $\alpha_2(G, \sigma)$. Let's denote the numbers as follows:

$$\mu(G) = max\alpha_2(G, \sigma)$$

$$\nu(G) = min\alpha_2(G, \sigma)$$

where the maximum number(or minimum number) is taken over all possible rotations σ of a graph G. A *dead-end-arc* is incident with a vertex of valance 1. Suppose (G, σ) be a graph with rotation σ and c be an arc of G, we define $(G - c, \sigma)$ as follows: If A is a vertex not incident with the arc c, then rotation of A in (G, σ) is equal to rotation of A in $(G - c, \sigma)$. If A is a vertex incident with the arc c, then the rotation of A in $(G - c, \sigma)$ is obtained as follows: the arc c incident with A will be omitted from the rotation of A in (G, σ).

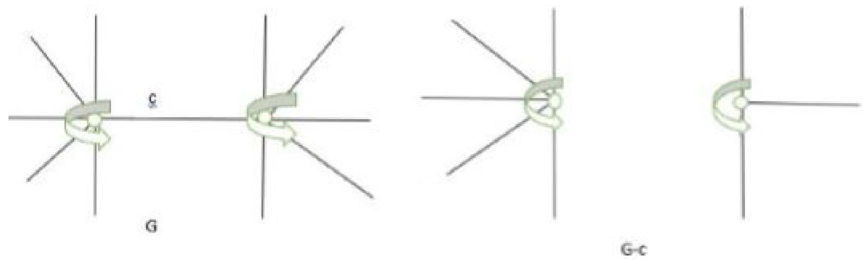

Figure 3.9: The graph G with dead-end-arc

Theorem 3.2.1. *[15] Let G be a graph with rotation (G, σ) and c an arc of G which is not a dead-end-arc. Then $\alpha_2(G, \sigma)$ differs from $\alpha_2(G - c, \sigma)$ by exactly 1.*

Proof. *Suppose A and B are two vertices incident with the arc c. Let α and β be two circuits which are generated by the vertices A and B, respectively. If we remove the arc c, we get either $\alpha = \beta$ or $\alpha \neq \beta$. So, we have two cases.*
Case 1: *If $\alpha \neq \beta$ then two circuits α and β will be as follows:*
As one can see from the Figure 3.10; there exists two circuit in the first graph but one remove the arc c from the second graph, there is only one circuit.

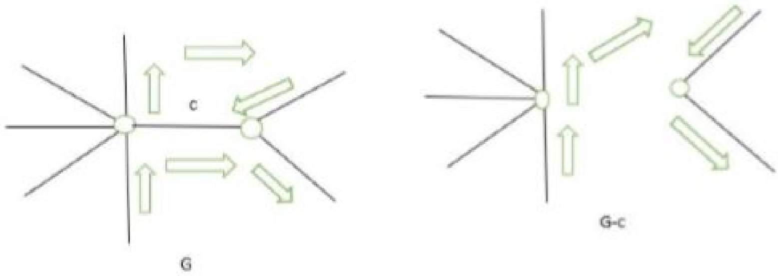

Figure 3.10: Two different circuit in the graph G with the arc c.

Case 2: *If $\alpha = \beta$, then one circuit will be divided into two parts as one can see from the below graph. So, we have two different circuits in $G - c$, while there exists one circuit in G.*

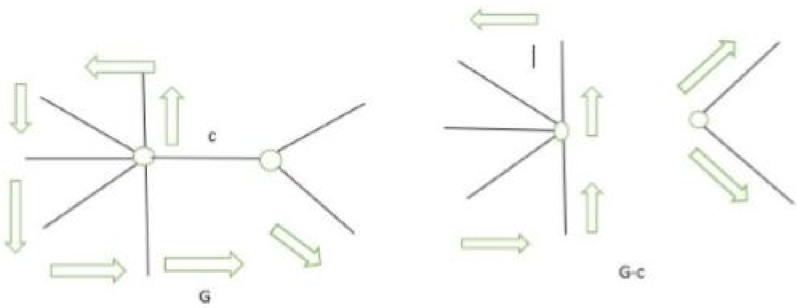

Figure 3.11: One circuit in the graph G with the arc c.

□

Theorem 3.2.2. *[15] If G is a graph with the vertices α_0, whose degrees are not equal to 0, and the arcs α_1, then*

$$\alpha_0 - \alpha_1 + \alpha_2(G, \sigma) \equiv 0\,(mod 2)$$

holds for every rotation σ of G.

Proof. *Without loss of generality, G is connected. We will prove this theorem by using induction on the number of arcs. If G has one or two arcs, then $2 - 1 + 1 \equiv 0\,(mod 2)$(or $2 - 2 + 2 \equiv 0\,(mod 2)$). So, it is automatically satisfied. If $n \geq 2$, then since this statement is true for $n - 1$, when we remove an arc from G we have two cases:*

Case 1: *If G contains a vertex P of valance one, we obtain $G - P$ by removing the vertex P and the dead-end-arc incident with P. The rotation in G can be carried over in a natural way to $G - P$. In this case, $\alpha_2(G - P, \sigma) = \alpha_2(G, \sigma)$. So, we have $(\alpha_0 - 1) - (\alpha_1 - 1) + \alpha_2(G - P, \sigma) \equiv 0\,(mod 2)$.*

Case 2: *Assume G contains an arc c which is not a dead-end-arc. By removing the arc c, we get $G - c$ and the graph $G - c$ has less arcs and by induction hypothesis, the above theorem is true for $G - c$. By theorem 3.2.2, the circuit number $\alpha_2(G, \sigma)$ differs from $\alpha_2(G - c, \sigma)$ by one. So, we have $(\alpha_0) - (\alpha_1 - 1) + \alpha_2(G - c, \sigma) \equiv 0\,(mod 2)$. This shows that theorem 3.2.3 holds.*

□

If each circuit induced by a rotation σ for a graph has length three, then σ is called as *triangular rotation*. The following theorem guarantees that there exists a triangular rotation whenever $n \equiv 0, 3, 4$ *or* $7 (mod 12)$.

Theorem 3.2.3. *[15] If there exists a triangular rotation of the complete graph K_n, then $n \equiv 0, 3, 4$ or $7 (mod 12)$.*

Proof. *Assume that σ is a triangular rotation of the graph K_n. One triangle uses three arcs and each arc belongs to two of these triangles. Also, each triangle will be produced by the rotation σ. Therefore, $3\alpha_2(K_n, \sigma) = 2\alpha_1$. After multiplying the formula of theorem 3.2.3 by 6 and substituting $\alpha_0 = n$ and $\alpha_1 = \frac{n(n-1)}{2}$, we have*

$$6\alpha_0 - 6\alpha_1 + 6\alpha_2(K_n, \sigma) \equiv 0\,(mod 12)$$

Since the equation $3\alpha_2(K_n, \sigma) = 2\alpha_1$ is satisfied, we get the following:

$$6\alpha_0 - 6\alpha_1 + 6\frac{2}{3}\alpha_1 \equiv 0\,(mod 12)$$

$$6\alpha_0 - 2\alpha_1 \equiv 0\,(mod 12)$$

$$6n - n(n-1) \equiv 0\,(mod 12)$$

$$(7 - n)n \equiv 0\,(mod 12) \tag{3.4}$$

Therefore, we have four solutions of (3.4) which are $n \equiv 0, 3, 4$ or $7\,(mod 12)$.

$\square$

Actually, the converse of this theorem is more important. That is, there exists a triangular rotation for K_n for all $n \equiv 0, 3, 4,\ or\ 7(mod12)$. As an example, consider the complete graph K_7 with the vertices $0, 1, 2, 3, 4, 5, 6$. The following scheme shows a rotation for K_7.

$$
\begin{array}{cccccc}
0.1 & 3 & 2 & 6 & 4 & 5 \\
1.2 & 4 & 3 & 0 & 5 & 6 \\
2.3 & 5 & 4 & 1 & 6 & 0 \\
3.4 & 6 & 5 & 2 & 0 & 1 \\
4.5 & 0 & 6 & 3 & 1 & 2 \\
5.6 & 1 & 0 & 4 & 2 & 3 \\
6.0 & 2 & 1 & 5 & 3 & 4
\end{array}
$$

Of course, the lines read cyclically, i.e in the line 6 the number 0 follows 4 and in the line 5 the number 2 follows 3. This graph K_7 with respect to above rotation is the embedding into torus. That is, it gives us a triangulation of the torus with 7 vertices. One can easily check that the scheme for rotation of K_7 satisfies the following rule, the rule of triangles:

Rule Δ^*:If in line i one has $i.\ \cdots jk \cdots$ then in line k one must have $k.\ \cdots ij \cdots$ Rule Δ^* guarantees that the given rotation of K_n is a triangular one. We intend to show that there exists a triangular rotation of K_n if and only if $n \equiv 0, 3, 4,\ or\ 7\ (mod12)$.

Theorem 3.2.4. *[15] For each positive integer n such that $n \equiv 0, 3, 4,\ or\ 7\ (mod12)$, there exists a triangular rotation of the complete graph K_n.*

For every $n \equiv 0, 3, 4\, or\ 7(mod12)$, the aim is to check whether the given rotation for any vertex i satisfy the Rule Δ^* or not. One can easily check that the rotation for K_7 which was given above satisfy the rule Δ^*. Now, we will construct a row 0 so that the rule Δ^* will be satisfied for all n of the form $n = 12s + 7$. The vertices of K_n will be identified by the elements of $\mathbb{Z}_n$. Now, we will construct a suitable row for the row 0 that will satisfy the Rule Δ^*.

Let's consider the case $n = 19$. The vertices of the graph K_n will be identified by the $0, 1, 2, ...18$ or $\mp 0, \mp 1, ... \mp 9$ and also row 0 will be the permutations of these vertices.

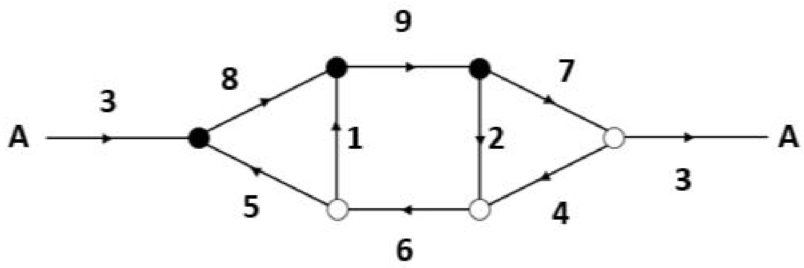

Figure 3.12: The directed graph $G_{1,19}$

The above graph has an orientation and a value called its current. Now, we will consider a graph with an orientation and currents for $n = 31$, and then we will construct a general rule for $n = 12s + 7$.

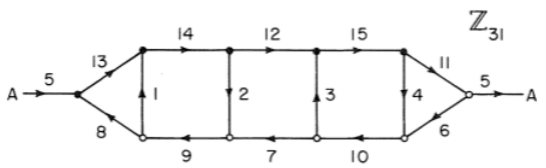

Figure 3.13: Directed graphs $G_{2,31}$

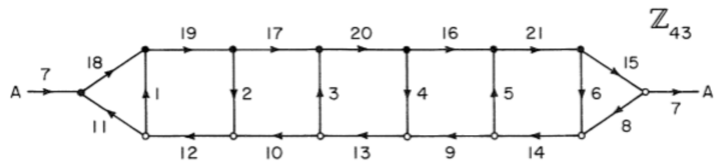

Figure 3.14: Directed graphs $G_{3,43}$

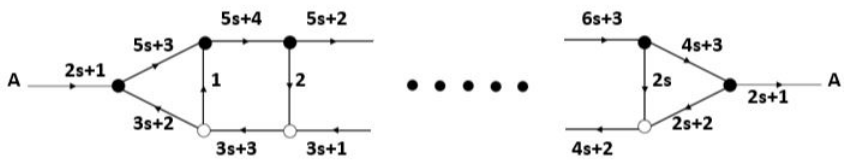

Figure 3.15: Directed graphs $G_{k,7+12k}$

All graphs that are given in the Figure 3.13, 3.14 and 3.15 satisfy the following conditions:

(i)Each vertex has valance 1 or 3.

(ii)Each rotation that was given on graphs can be induced one single circuit(circular rotation). Recall that, the vertex is represented by a small filled-in(empty) circle $\circ$ if the rotation is to be clockwise, otherwise; it is to be denoted with $\bullet$.

(iii) Each element $1, 2, ...6s + 3$ of $\mathbb{Z}_{12s+7}$ will appear exactly once as a current of an arc.

(iv) Kirchhoff's Current Law must be satisfied: The inward flowing currents must equal to the outward flowing currents at each vertex whose degree is equal to 3.

Now, let's consider the first shame of $\mathbb{Z}_{19}$. Let's start the arc carrying current 9 in the direction indicated by the arrow. It does not matter where we begin. The next vertex is rotated clockwise so the next arc carries 7 and we will continue in the counter-clockwise direction that gives the arc carrying 4. Now, we will progress in the counter-clockwise direction that gives the arc carrying 2. But, in this case, we will continue in the opposite direction of the arrow. So it will be -2. If we continue like this, we get the following sequence:

$$9\,7\,4\,\bar{2}\,\bar{9}\,\bar{1}\,5\,\bar{3}\,\bar{7}\,2\,6\,1\,\bar{8}\,\bar{5}\,\bar{6}\,\bar{4}\,3\,8$$

We used $\bar{a}$ instead of $-a$, also, the element -2 is same as with the element 17 in $\mathbb{Z}_{19}$. So the following sequence is same with the above sequence.

$$9\,7\,4\,17\,10\,18\,5\,16\,12\,2\,6\,11\,1\,14\,13\,15\,3\,8$$

We can take this sequence as the row 0 and we can get the row i by adding i to the elements of row 0. Then, our construction must satisfy the rule Δ^*. Before showing the given sequence satisfy the rule Δ^*, we have another property called as additive rule.

Additive Rule: To get the i-th row, one can add i to the elements of row 0. Our binary operation is additive in $mod(n)$. Let us assume that we have the following:

$$i.\ \cdots jk \cdots$$

The additive rule says that;

$$0.\ \cdots (j-i)(k-i) \cdots$$

appears in row 0. Let's consider the local graph of current. Let h be outward flowing current and $j-i$, $i-k$ are the inwards flowing currents. So the diagram will be as follows:

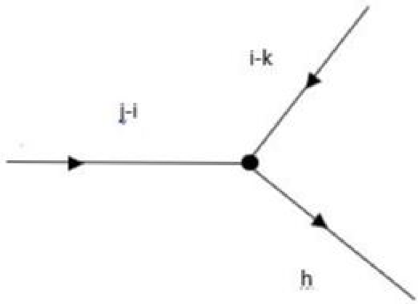

Figure 3.16: Local graph of the current

It follows that;

$$0 \cdots (j-i)(k-i)...(i-k)h \cdots$$

To get row k, we add k to the elements of row 0:

$$k \cdots ih+k \cdots$$

Since the Kirchhoff's Current Law must be satisfied, $h = (j-i)+(i-k) = j-k$ and then we have:

$$k \cdots ij \cdots$$

and the rule Δ^* holds. So, we have triangular rotation of K_{19}. We can easily generalize this construction to have a triangular rotation for $n = 12s+7$. The general sequence for the scheme whose the number of vertex is $12s+7$ is as follows:

$$3s+2, 3s+3, ..., 4s, 2s+4, 4s+1, 2s+3, 4s+2, 2s+2.$$

In this sequence, since we have $2s$ arcs in the middle of two horizontal line, to get the current of upper horizontal line we will add $2s+1$ to the lower horizontal line. One can easily check that the four conditions of having triangular rotation are satisfied.

Example 3.2.3. *As we can see the figure below, the construction that mentioned above gives us a triangulation with* $n \equiv 0, 3, 4\, or\, 7 (mod 12)$.

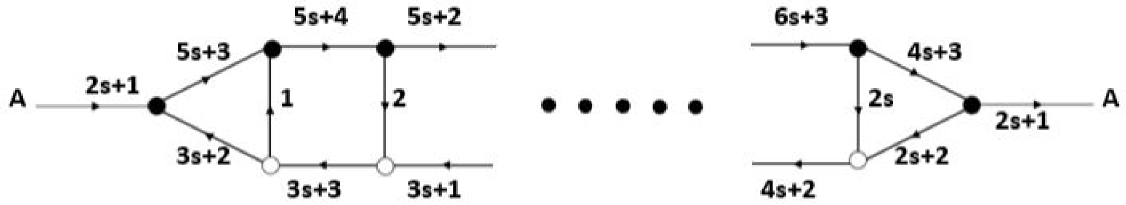

Figure 3.17: The directed graph $G_{7,7+12k}$

Note that it has been described explicitly in the literature that the Altshuler series [14], the Ringel series, and the class of cyclic equivelar tori [17] are the only infinite series of vertex-transitive triangulations of surfaces.

We can obtain the rotation of K_7 by using the following figure:

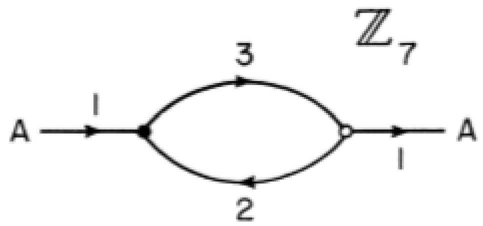

Figure 3.18: The directed graph $G_{0,7}$

If we start with 1 and continue with the direction of the arrow, we will walk along the arc carrying 3. Then, we will continue in the clockwise direction, so our new arc will carry 2. We go along in the counter-clockwise direction which is the arc carrying 1. But, the direction of the arrow is inverse of walking direction so, it will be -1. Then, we will follow -3 and -2. The sequence of walking will be as follows: $1\,3\,2\,(-1)\,(-3)\,(-2)$. Since $-a$ is the inverse of a in $mod(7)$, one can write $6, 4, 5$ instead of the numbers $(-1), (-3), (-2)$, respectively. So, we have the sequence $1\,3\,2\,6\,4\,5$. This sequence is same with the row 0 in the rotation of K_7. One can easily get the row i by using the additive rule. In addition to the rule Δ^*, we have the following rule:

Rule R^***:** If in the row i there exists $i.\ \cdots jkl \cdots$ then row k will be as follows: $k.\ \cdots lij \cdots$.

If the scheme for the graph G satisfy the rule R^*, then it will satisfy the rule Δ^*. The

converse of this statement is also true: Suppose the rule Δ^* is satisfied:
We have $i.\ \cdots jkl \cdots$ and then we can obtain $k.\ \cdots ij \cdots$ and $l.\ \cdots ik \cdots$ by using the rule Δ^*. If one applies the rule Δ^* to the row l, one gets, $k.\ \cdots li \cdots$. Since the graph G has no multiple edge the row k will appear as follows: $k.\ \cdots lij \cdots$. So, the rule R^* is satisfied.

3.3 Search of Vertex-Transitive Triangulation

In the previous section, we gave examples of vertex-transitive triangulations on surfaces. Now, our aim is to find different vertex-transitive triangulations on the given surface. First of all, we will consider the actions of the transitive permutation groups on the vertex set and we will use the program which is called as GAP [18]. In the GAP library, there exists a complete list of transitive permutation groups of small degree n with $n \leq 31$. By (n, i) we denote the i-th permutation group of degree n in the GAP library. Then, we will find the action of the i-th permutation group on the given triangulation of a surface and we will compute the size of the orbit of each generating triangle by using GAP program. In 1985, Kühnel and Lassmann [19] found the enumeration algorithm to find the vertex transitive triangulations of the 3-manifolds by investigating *vertex transitive* cyclic and dihedral group actions on the given vertex set of the triangulation. The enumeration algorithm can be used to find vertex-transitive triangulations on the higher dimensional manifolds d with $d \leq 5$. But, there is no enumeration algorithm to find vertex-transitive triangulation for $d \geq 6$. The steps of the enumeration algorithm are as follows:

3.3.0.1 Enumeration Algorithm

Recall that, if the $Aut(X)$ acts on the vertices of the simplicial complex X, then this triangulation is called as vertex-transitive triangulation. Before starting to enumeration algorithm, we need to determine all vertex-transitive group actions on the given number n of vertices. As we mentioned before, all vertex-transitive groups are available with $n \leq 31$ in the GAP library. Also, the number of distinct actions of all vertex-transitive groups for the number of vertices n is given in [13]. If we consider the action of symmetric group S_n and alternating group A_n on the unordered $(d+1)$-subsets of $\{1, ...n\}$ when $n = d + 2$,we get $(n-1)$-simplex whose boundary is the $(n-2)$- sphere, S^{n-2}. The groups with the largest order which acts transitively on n vertices are S_n and A_n. Let us consider when $n \geq d + 2$

Step-I: We will fix the number of vertices n with $n \geq 4$ and the dimension $2 \leq d \leq n - 2$.

Step-II: We will find the all d-dimensional manifolds on n vertices which have the following property:

Pseudo-manifold Property: Every $(d-1)$-dimensional face of the manifold M will be contained twice in d-dimensional facet.

As an example, let us consider the pair $(16, 189)$ and the induced action of the given 189-th transitive group on 3 subsets of $\{1, 2, ...16\}$ which was defined in the program GAP. We compute the orbits of the generating triangle $[1, 3, 5]$ and $[1, 3, 7]$ by using the following command in GAP.

$g := TransitiveGroup(16, 189);$

$t16n189$

$brk > h := Orbit(g, [1, 3, 5], OnSets);$

$[[1, 3, 5], [10, 11, 14], [3, 6, 16], [9, 12, 13], [7, 12, 14], [1, 6, 15], [4, 5, 15],$

$[2, 4, 6], [7, 10, 13], [8, 10, 12], [8, 11, 13], [2, 5, 16], [1, 4, 16], [8, 9, 14],$

$[7, 9, 11], [2, 3, 15]]$

$Size(h);$

16

$k := Orbit(g, [1, 3, 7], OnSets);$

$Size(k);$

$[[1, 3, 7], [2, 10, 14], [6, 9, 16], [4, 12, 13], [7, 12, 15], [6, 11, 15], [4, 5, 13], [2, 4, 8],$

$[7, 10, 16], [3, 8, 12], [5, 11, 13], [2, 5, 14], [1, 9, 13], [2, 7, 16], [1, 4, 14], [1, 8, 10],$

$[3, 5, 11], [5, 9, 14], [6, 9, 11], [5, 10, 15], [3, 11, 14], [2, 7, 9], [4, 7, 15], [6, 10, 13],$

$[2, 3, 13], [1, 10, 16], [1, 5, 9], [3, 12, 15], [8, 11, 16], [5, 12, 16], [3, 6, 14], [2, 9, 15],$

$[4, 7, 11], [2, 8, 13], [6, 12, 14], [10, 11, 15], [4, 8, 14], [8, 9, 15], [1, 6, 13], [4, 11, 16],$

$[4, 6, 12], [3, 8, 16], [9, 12, 16], [1, 8, 15], [5, 10, 12], [3, 7, 13], [2, 6, 10], [1, 7, 14]]$

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As another example of different induced group actions on the vertex set $\{1, 2, ..., n\}$, let us consider the induced group actions of S_n and A_n on the $(d+1)$-subsets of the set $\{1, 2, ..., n\}$. It gives us a $(n-2)$-sphere which is the boundary of the $(n-1)$-simplex. For example, let us consider when $n = 4$ and $d = 2$. Now, we will find the action of S_4 on the 3-subsets of $\{1, 2, 3, 4\}$ by using GAP.

$H := TransitiveGroup(4, 5);$

$S4$

$brk_3 > Orbit(H, [1, 2, 3], OnSets);$

$[[1, 2, 3], [2, 3, 4], [1, 3, 4], [1, 2, 4]]$

$gap > h := AlternatingGroup(4);$

$Alt([1..4])$

$gap > Orbit(h, [1, 2, 3], OnSets);$

$[[1, 2, 3], [1, 3, 4], [1, 2, 4], [2, 3, 4]]$

These induced actions on the set $\{1, 2, 3\}$ gives us a sphere S^2 which is the boundary of a tetrahedron. Also, the induced action of the dihedral group on the $(d+1)$ subsets of the set $\{1, 2, ..., n\}$ can be considered. For instance; let $d = 2$,$n = 7$ and D_7 be the i-th permutation group. Now, our aim is to check whether 1-dimensional faces are contained precisely two in 2-dimensional faces or not. That is, we will search whether an edge is contained twice in a triangle. The following commands for the orbits of the triangles and edges were found by using GAP.

$h := TransitiveGroup(7,2);$

$D(7) = 7:2$

$brk_3 > Orbit(h,[1,2,3],OnSets);$

$[[1,2,3],[2,3,4],[4,5,6],[3,4,5],[5,6,7],[1,6,7],[1,2,7]]$

$brk_3 > Orbit(h,[1,2,4],OnSets);$

$[[1,2,4],[2,3,5],[3,5,6],[3,4,6],[2,4,5],[4,6,7],[4,5,7],$

$[1,3,4],[1,5,7],[1,3,7],[1,5,6],[2,3,7],$

$[1,2,6],[2,6,7]]$

$brk_3 > Orbit(h,[1,2,5],OnSets);$

$[[1,2,5],[2,3,6],[2,5,6],[3,4,7],[1,4,5],[3,6,7],[1,4,7]]$

$brk_3 > Orbit(h,[1,3,5],OnSets);$

$[[1,3,5],[2,4,6],[3,5,7],[1,4,6],[2,4,7],[2,5,7],[1,3,6]]$

$brk_3 > Orbit(h,[1,2],OnSets);$

$[[1,2],[2,3],[5,6],[3,4],[4,5],[6,7],[1,7]]$

$brk_3 > Orbit(h,[1,3],OnSets);$

$[[1,3],[2,4],[4,6],[3,5],[5,7],[1,6],[2,7]]$

$brk_3 > Orbit(h,[2,5],OnSets);$

$[[2,5],[3,6],[4,7],[1,4],[1,5],[3,7],[2,6]]$

$brk_3 > Orbit(h,[1,7],OnSets);$

$[[1,7],[1,2],[6,7],[2,3],[5,6],[3,4],[4,5]]$

$brk_3 > Orbit(h,[3,7],OnSets);$

$[[3,7],[1,4],[4,7],[2,5],[3,6],[1,5],[2,6]]$

$brk_3 > Orbit(h,[1,4],OnSets);$

$[[1,4],[2,5],[3,6],[4,7],[1,5],[3,7],[2,6]]$

The next step is to check whether the pseudomanifold property is satisfied or not. For this, we will compose a matrix A which is defined as follows:
If all edges contained in one of the generating edges are contained twice in all triangles contained in one of the generating triangles , then the entries of the matrix A will be 2. If it is contained once, then it will be 1. Otherwise; it will be 0.

$a : 123, 234, 456, 345, 567, 167, 127$

$b : 124, 235, 356, 346, 245, 467, 457$

$134, 157, 137, 156, 237, 126, 267$

$c : 125, 236, 256, 347, 145, 367, 147$

$d : 135, 246, 357, 146, 247, 257, 136$

$e : 12, 23, 56, 34, 45, 67, 17$

$f : 13, 24, 46, 35, 57, 16, 27$

$g : 25, 36, 47, 14, 15, 37, 26$

$h : 17, 12, 67, 23, 56, 34, 45$

$k : 37, 14, 47, 25, 36, 15, 26$

$$A = \begin{bmatrix} 2 & 2 & 1 & 0 \\ 1 & 2 & 0 & 2 \\ 0 & 2 & 2 & 1 \\ 2 & 2 & 1 & 0 \\ 0 & 2 & 2 & 1 \end{bmatrix}$$

According to the matrix A, the 1-simplexes consisting of the edges $12,\ 23,\ 56,\ 34,\ 45,$ $67,\ 17$ satisfy the pseudomanifold property in the triangles of the row a. Also the 1-simplexes consisting of the edges $17,\ 12,\ 67,\ 23,\ 56,\ 34,\ 45$ satisfy pseudomanifold property in the triangles of the row b. However, these edges can be combinatorially equivalent. To solve this problem, one will use the following steps:

Step-III:(Combinatorial Test) If the complexes that disagree with the pseudomanifold property, then we will remove these complexes.

Step-IV:(Combinatorial Equivalence)

Delete the complexes that has already found before. To determine the combinatorially equivalent complexes, we have two invariants which are *f-vectors* and Altshuler-Steinberg determinant which is given by $det(AA^T)$, where A is the vertex-facet incidence matrix. For instance; let us consider a tetrahedron with the facets $123,\ 234,$ $134,\ 124$ and the edges $12,\ 23,\ 13,\ 24,\ 34,\ 14$. Then, the vertex-facet incident matrix will be as follows:

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}, \ A^T = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}, and \ AA^T = \begin{bmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \end{bmatrix}$$

So, the determinant of the matrix AA^T is equal to 9 for the given tetrahedron given above. If one can say that two complexes are different from each other, the determinant of the matrix AA^T should be different.

Step-V:(Homology Computation:)

In this step, we will remove all complexes which are not *Poincare duality complex* with respect to Z_2 coefficients. One can also remove the complexes with vertex links whose homology is different from homology of S^{d-1}. There are programs doing these which are given in [25], [26].

Step-VI:(Recognition of the Vertex-Links)

Recall that when v is a vertex of the simplicial complex X, the link of the vertex v, $lk_X(v)$, is given as follows: $lk_X(v) : \{\tau \in X \mid \{v\} \notin \tau, \{v\} \cup \tau \in X\}$. Also, the star of the vertex v is defined as follows: $St_X(v) : \{\{v\}, \tau, \tau \cup \{v\} \mid \tau \in lk_X(v)\}$. The vertex-links should be a sphere for getting a vertex-transitive manifold. In this step, BISTELLAR [21] will be used.

Step-VII:(Topological Type)

As the last step of the enumeration algorithm, the topological types of the complexes should be found according to the *topological classification theorem* [7] in the program BISTELLAR EQUIVALENT [21].

The enumeration algorithm gives that; the 2-manifolds which have vertex-transitive automorphism groups can be S^2, T^2, the orientable surfaces of genus 2, 3, 4, 5, 6, $\mathbb{R}P^2$, or the non-orientable surfaces of genus 2, 4, 5, 7, 8, 15.

The results of the enumeration algorithm for the higher dimensional manifolds when $d = 3, 4, \ or \ 5$ can be found in [14].

CHAPTER 4

CLASSIFICATION

4.1 Classification of Vertex-Transitive Triangulation:

As we mentioned in the section called enumeration algorithm, one can find the vertex-transitive manifold on n vertices with $n \leq 15$. The vertex-transitive triangulations with up to 15 vertices can be classified by using the program GAP. There is no vertex-transitive triangulations of non-orientable surfaces when $\chi = -4$ and $n = 12$. Also, there is no vertex transitive triangulations with $\chi = -7$, $n = 14$ and $\chi = -27$, $n = 18$, [[13], Corollary 3].

Example	Group: (n, i)	Orbit generating triangles
E2	(16,189)	$[1,3,5]_{16}$, $[1,3,7]_{48}$
E3	(16,28)	$[1,3,6]_{32}$, $[1,3,9]_{32}$
E5	(18,3)	$[1,2,5]_{18}$, $[1,3,8]_{18}$, $[1,4,9]_{18}$, $[1,4,17]_{6}$, $[1,5,12]_{18}$, $[1,6,11]_{18}$
E6	(18,3)	$[1,2,3]_{18}$, $[1,4,6]_{18}$, $[1,5,11]_{18}$, $[1,5,15]_{6}$, $[1,6,16]_{18}$, $[1,7,13]_{18}$
E7	(20,35)	$[1,3,5]_{120}$
E8	(20,2)	$[1,3,5]_{20}$, $[1,3,7]_{20}$, $[1,6,13]_{20}$, $[1,6,15]_{20}$, $[1,7,13]_{20}$, $[1,10,19]_{20}$
E9	(21,7)	$[1,4,10]_{21}$, $[1,4,11]_{63}$, $[1,5,12]_{21}$, $[1,6,16]_{21}$
E10	(21,7)	$[1,4,11]_{63}$, $[1,4,13]_{63}$
E11	(24,84)	$[1,4,7]_{24}$, $[1,4,14]_{72}$, $[1,5,10]_{72}$
E14	(24,21)	$[1,3,8]_{48}$, $[1,3,23]_{16}$, $[1,4,13]_{48}$, $[1,4,15]_{48}$, $[1,5,16]_{16}$
E15	(26,5)	$[1,3,5]_{78}$, $[1,3,13]_{78}$, $[1,4,10]_{26}$, $[1,6,18]_{26}$
E17	(27,23)	$[1,4,10]_{81}$, $[1,4,11]_{81}$, $[1,8,18]_{27}$, $[1,9,17]_{27}$
E18	(30,9)	$[1,2,21]_{30}$, $[1,3,8]_{60}$, $[1,4,9]_{60}$, $[1,5,14]_{60}$, $[1,9,28]_{60}$
E20	(30,9)	$[1,3,8]_{60}$, $[1,4,9]_{60}$, $[1,5,14]_{60}$, $[1,9,30]_{30}$, $[1,10,30]_{10}$, $[1,15,17]_{60}$

Figure 4.1: Examples of vertex-transitive and nonvertex-transitive triangulations.

In the above table [1], $E2 - E3$, $E5 - E11$, $E14 - E15$, $E17 - E18$ and $E20$ are the examples of vertex-transitive triangulations. The non-transitive examples are $E1$, $E4$, $E12$, $E13$, $E16$ and $E19$. As an example, let us consider the 28-th permutation group in the GAP library of order 16 which is given by $E3$ in the above table. The subscripts of the generating triangles show the size of orbits.

$brk_3 > h := TransitiveGroup(16, 28);$
$t16n28$
$brk_3 > Orbit(h, [1, 3, 6], OnSets);$
$[[1, 3, 6], [7, 10, 12], [9, 13, 15], [2, 6, 15], [6, 7, 11], [4, 13, 16], [2, 4, 5],$
$[8, 12, 13], [4, 9, 14], [1, 6, 12], [8, 9, 11], [2, 3, 16], [10, 14, 16], [2, 7, 12],$
$[3, 9, 16], [1, 5, 16], [7, 9, 13], [5, 8, 12], [8, 10, 14], [3, 14, 15], [2, 6, 8],$
$[7, 11, 14], [3, 5, 15], [3, 10, 13], [4, 6, 16], [2, 5, 11], [1, 4, 15], [9, 12, 14],$
$[1, 8, 11], [10, 11, 13], [4, 10, 15], [1, 5, 7]]$
$brk_3 > Orbit(h, [1, 3, 9], OnSets);$
$[[1, 3, 9], [7, 10, 16], [5, 13, 15], [6, 14, 15], [3, 6, 11], [4, 11, 13], [2, 4, 10],$
$[3, 12, 13], [1, 4, 9], [1, 10, 12], [8, 9, 15], [2, 8, 16], [6, 14, 16], [2, 7, 15],$
$[7, 9, 16], [5, 13, 16], [6, 7, 13], [4, 5, 12], [5, 8, 14], [3, 12, 14], [6, 8, 13],$
$[4, 11, 14], [3, 5, 11], [2, 3, 10], [4, 6, 12], [2, 9, 11], [1, 7, 15], [2, 9, 12],$
$[1, 8, 16], [1, 10, 11], [8, 10, 15], [5, 7, 14]]$
$brk_3 > IsTransitive(h, [1, 3, 6]);$
$true$
$brk_3 > IsTransitive(h, [1, 3, 9]);$
$true$

The orbits of the generating triangles in the vertex-transitive triangulation $E11$ are as follows:
$k := TransitiveGroup(24, 84);$
$t24n84$
$brk_3 > Orbit(k, [1, 4, 7], OnSets);$
$[[1, 4, 7], [3, 5, 24], [11, 17, 21], [2, 9, 22], [12, 13, 20], [10, 16, 20], [2, 4, 23],$
$[14, 16, 19], [13, 18, 21], [3, 6, 9], [12, 18, 19], [1, 6, 22], [1, 8, 24], [11, 15, 19],$
$[12, 14, 17], [5, 7, 22], [11, 13, 16], [4, 9, 24], [3, 7, 23], [10, 14, 21], [15, 17, 20],$
$[2, 5, 8], [10, 15, 18], [6, 8, 23]]$
$brk_3 > Orbit(k, [1, 4, 14], OnSets);$

$[[1, 4, 14], [3, 14, 24], [7, 11, 21], [2, 9, 14], [7, 13, 20], [5, 16, 20], [2, 4, 17],$
$[7, 16, 19], [5, 13, 18], [6, 9, 17], [12, 18, 22], [6, 12, 22], [1, 10, 24], [11, 19, 23],$
$[5, 12, 17], [5, 17, 22], [11, 13, 22], [9, 12, 24], [4, 10, 23], [10, 16, 23], [7, 21, 23],$
$[3, 10, 14], [3, 9, 19], [1, 13, 21], [8, 17, 20], [2, 5, 15], [10, 21, 22], [3, 12, 23],$
$[7, 10, 22], [12, 14, 23], [4, 9, 21], [3, 13, 16], [8, 19, 24], [1, 11, 15], [8, 12, 19],$
$[1, 15, 22], [5, 8, 16], [4, 15, 17], [2, 14, 19], [1, 7, 20], [6, 10, 18], [6, 18, 23],$
$[2, 8, 21], [3, 15, 20], [5, 7, 19], [1, 14, 17], [8, 14, 21], [3, 7, 15], [4, 16, 24],$
$[4, 11, 16], [2, 12, 13], [9, 20, 22], [6, 17, 21], [3, 5, 18], [10, 15, 24], [8, 11, 23],$
$[1, 6, 16], [4, 18, 19], [2, 15, 18], [6, 8, 20], [6, 14, 16], [4, 7, 18], [12, 20, 24],$
$[2, 11, 22], [9, 15, 19], [1, 8, 13], [11, 17, 24], [5, 11, 24], [9, 18, 21], [3, 6, 13],$
$[9, 10, 20], [2, 13, 23]]$
$brk_3 > Orbit(k, [1, 5, 10], OnSets);$
$[[1, 5, 10], [3, 19, 22], [1, 12, 21], [2, 7, 17], [5, 14, 20], [3, 16, 21], [3, 4, 21],$
$[17, 19, 23], [14, 18, 22], [6, 7, 12], [2, 12, 16], [4, 20, 22], [2, 16, 24], [4, 11, 20],$
$[8, 10, 17], [5, 15, 23], [7, 11, 14], [7, 14, 24], [5, 21, 23], [3, 10, 17], [7, 18, 24],$
$[6, 11, 14], [1, 9, 12], [13, 19, 22], [18, 20, 24], [2, 6, 11], [6, 10, 19], [1, 18, 23],$
$[8, 15, 22], [8, 12, 15], [5, 9, 14], [7, 13, 17], [8, 16, 22], [4, 12, 15], [2, 10, 19],$
$[1, 20, 23], [5, 9, 11], [13, 17, 24], [14, 20, 23], [2, 7, 10], [9, 11, 18], [6, 13, 24],$
$[3, 8, 11], [15, 21, 24], [6, 7, 15], [8, 14, 18], [6, 15, 21], [3, 8, 18], [4, 14, 22],$
$[7, 12, 16], [5, 10, 13], [9, 17, 23], [1, 17, 19], [1, 5, 19], [9, 10, 13], [9, 13, 23],$
$[2, 6, 20], [2, 18, 20], [9, 15, 16], [4, 8, 13], [15, 16, 23], [4, 8, 10], [5, 12, 21],$
$[3, 17, 22], [4, 13, 19], [1, 9, 16], [1, 11, 18], [6, 19, 24], [16, 21, 22], [3, 4, 12],$
$[3, 11, 20], [2, 21, 24]]$ $brk_3 > IsTransitive(k, [1, 4, 7]);$
$true$
$brk_3 > IsTransitive(k, [1, 4, 14]);$
$true$
$brk_3 > IsTransitive(k, [1, 5, 10]);$
$true$

The non-transitive examples $E1$, $E4$, $E12$, $E13$, $E16$, and $E19$ are listed in [1]. Also, the classification of non-neighbourly vertex-transitive triangulations with the equality (3.3) was done in [1]. In this case, we have four undetermined cases given below:

1. For a orientable surface with $\chi = -64$, $n = 24$ and $q = 22$,

2. For a orientable surface with $\chi = -110$, $n = 30$ and $q = 28$,

3. For a non-orientable surface with $\chi = -60$, $n = 24$ and $q = 21$,

4. For a non-orientable surface with $\chi = -78$, $n = 26$ and $q = 24$.

However, there exists q-equivelar triangulations of the given surface with the numbers n, q and χ that were given above.

4.2 Classification of Equivelar Triangulations:

The classification of equivelar triangulations of closed surfaces on at most 11 vertices were done in [28]. There exists 27 such equivelar triangulations and there exists 240914 equivelar triangulations with $n = 12$, the result was obtained by using enumeration algorithm in [13]. When a given surface is non-orientable of genus 3 with $\chi = -1$, since $q = 6 - \frac{6\chi}{n}$ and $6 = n(q - 6)$ there is no equivelar triangulation. For $16 \leq n \leq 30$, the vertex transitive non-neighbourly triangulations of the 2-manifold with $-230 \leq \chi < 0$ and $q = \lfloor \frac{1}{2}(5 + \sqrt{49 - 24\chi(M)}) \rfloor$ were classified in [1]. For the Klein bottle and torus of type $\{3, 6\}$ with 15 vertices, the classification of equivelar triangulations of these surfaces can be found in [11]. Also, the enumeration algorithm up to 100 vertices of the q-equivelar triangulation of Klein bottle was done in [22]. For the double torus with 12 vertices, we have six combinatorially different equivelar triangulations [23]. The enumeration algorithms were done for the following surfaces with the numbers n, q, and χ in [13] as given below:

- When $n = 12$, $q = 7$, and M is a non-orientable surface with $\chi(M) = -2$, there exists 28 different 7-equivelar triangulations on the surfaces M.

- When $n = 9$, $q = 8$, and M is a non-orientable surface with $\chi(M) = -3$, there exists 2 examples and if we change the numbers n and q as $n = 18$, $q = 7$, there exits 1041 different equivelar triangulations of the surface M.

- When $n = 24$, $q = 7$ and M is a non-orientable surface with $\chi(M) = -4$, there exists 600946 different examples and if we change the numbers n, q as $n = 12$, $q = 8$, there exists 6500 different equivelar triangulations of the surface M.

- When $n = 12$, $q = 8$ and M is an orientable surface with $\chi = -4$, and if we change the numbers n, q as $n = 24$, $q = 7$, there exists 11277 different equivelar triangulations of the surface M.

For the torus; we have the following important theorem:

Theorem 4.2.1. *[11] Any q-equivelar triangulation of the torus is a vertex-transitive triangulation.*

To show this, we need the following facts and lemmas which are given in [11]:

Fact 1: Let E be a triangulation of the plane $\mathbb{R}^2$ such that the vertex set $V(E)$ consists of the following elements: $u_{m,2n} = (m, n\sqrt{3})$ and $u_{m,2n-1} = (m + \frac{1}{2}, \frac{(2n-1)\sqrt{3}}{2})$, where m, $n \in \mathbb{Z}$. The group $Aut(E)$ consists of translations, reflections and glide reflections (combination of translation and reflection), and let us define H be a subgroup of $Aut(E)$ as the groups of translations generated by $\alpha_1 : u \to u + u_{1,0}$ and $\alpha_2 : u \to u + u_{0,1}$. If one fixes any vertex on E and translates the given fix vertex under the maps α_1 and α_2, then we have a hexagon around the fix vertex. That is, we have all symmetries of the fix vertex which is given by the group D_6 of order 12 and so the stabilizer of the fixed vertex will be isomorphic to the dihedral group D_6. Let S be the group of the stabilizer of the vertex $u_{0,0}$. Since the group $Aut(E)$ acts transitively on the vertex set $V(X)$ and $|Aut(E)| = |Orb_{Aut(E)}(x)||Stab_{Aut(E)}(x)|$, for any $x \in V(X)$, then $Aut(E) =< H, S >$. If there is no element fixing any vertex, edge, or face, then any element $\sigma \in Aut(E)$ will be either in $H - \{Id\}$ or a glide reflection.

Lemma 4.2.1. *[11] There is no triangulation of the closed* 2*-disc that satisfy the following conditions:*
(i)the degree of each vertex(except one) on the boundary is 4,
(ii)the degree of each interior vertex is 6.

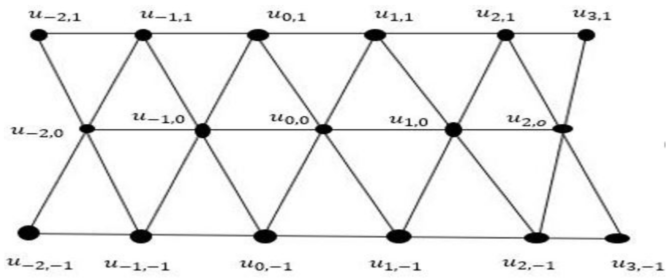

Figure 4.2: A triangulations of the plane $\mathbb{R}^2$.

Sketch of the proof: Suppose the contrary that there exists such a triangulation K of the closed 2-disc on $k + m + 1$ vertices. Let k be the number of interior vertices with degree 6 and m be the number of vertices on the boundary with degree 4. Also, the degree of one vertex on the boundary is $n \geq 2$. Then $f_0(K) = k + m + 1$ and $f_1(K) = \frac{6k+4m+n}{2}$. To find the number of triangles, we need to find the number of edges. Since the degree of any vertex on the boundary is 4 and the edge contained in $St_X(v)$ counted twice, the only edges contained in $lk_X(v)$ will be counted and there will be $3m$ edges. Also, we have one vertex whose degree is n, so we have $n - 1$ edges by same reason. That is, we have $6k + 3m + n - 1$ edges and then $f_2(K) = \frac{6k+3m+n-1}{3}$. Since the Euler's characteristic of any closed disc is equal to 1, $1 = \chi(K) = k + m + 1 - \frac{6k+4m+n}{2} + \frac{6k+3m+n-1}{3}$ then $1 = 1 - \frac{n}{2} + \frac{n-1}{3}$, so $n = -2$ which is indeed a contradiction. $\square$

Lemma 4.2.2. *[11] Let E be a triangulation in the above figure 4.2 and M be a triangulation of the plane $\mathbb{R}^2$. If the degree of each vertex of M is 6 then M is isomorphic to E.*

Sketch of the proof:

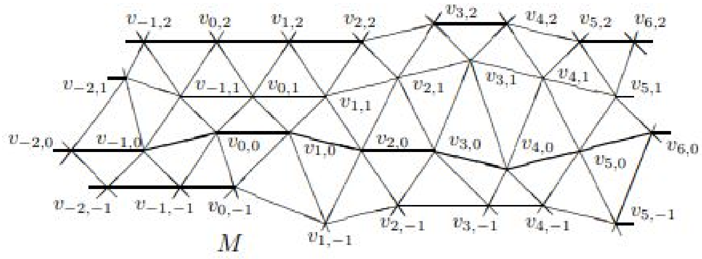

Figure 4.3: Triangulation of $\mathbb{R}^2$ with the degree of vertices are 6.

Let us choose an edge, say $v_{0,0}v_{1,0}$ then the vertex $v_{2,0}$ is contained in $lk(v_{1,0})$ such that each side of the segment $v_{0,0}v_{1,0}v_{2,0}$ contains three faces from $st(v_{1,0})$ and the vertex $v_{2,0}$ that satisfy this condition can be chosen, uniquely. Now, for the given vertices $v_{1,0}$ and $v_{2,0}$, the vertex $v_{3,0}$ will be our new vertex that is contained in $lk(v_{2,0})$ such that each side of the segment $v_{1,0}v_{2,0}v_{3,0}$ contains three faces from $st(v_{2,0})$. Continuing this way, we get a line segment $v_{i-1,0}v_{i,0}v_{i+1,0}$ such that there exists three faces on the each side of the segment contained in the $st(v_{i,0})$ for all $i \in \mathbb{Z}$. These points $v_{i,0}$ for $i \in \mathbb{Z}$ must be different from each other. Since, if any two of the vertices $v_{i,0}$ and $v_{j,0}$ is the same vertex, then we obtain a closed subset of $\mathbb{R}^2$ which is isomorphic to 2-disc with the induced triangulation as follows: The degrees of the interior vertices are 6 and the degrees of vertices on the boundary(except one) are 4, since it is a common vertex of three faces in the interior, but the vertex represented by $v_{i,0} = v_{j,0}$ has degree 3, because it is a common vertex of two interior faces. But this contradicts with the previous lemma, so these points must be distinct. If we choose the edge $v_{0,1}v_{1,1}$, there will be unique vertex $v_{2,1}$ that satisfy the property given above. That is, each side of the line segment $v_{i-1,1}v_{i,1}v_{i+1,1}$ contains three faces contained in $st(v_{i,1})$ for all $i \in \mathbb{Z}$. Also, it is satisfied when we choose the edge as $v_{i-1,-1}v_{i,-1}$. Continuing this way we get a general rule; each side of the line segment $v_{i-1,j}v_{i,j}v_{i+1,j}$ contains three faces contained in $st(v_{i,j})$ for all $i, j \in \mathbb{Z}$. Since M is connected, the vertex set of M is $\{v_{i,j}, i, j \in \mathbb{Z}\}$ and there exists an isomorphism $\varphi : V(M) \to V(E)$ given by $\varphi(v_{i,j}) = u_{i,j}$. □

Fact 2: Let $Q : \mathbb{R}^2 \to \mathbb{R}^2/\sim$ be a quotient map with the equivalence relation $\sim$: $(x, y) \sim (x + m, y + n)$, where m, n are in $\mathbb{Z}$. Since the identification of the opposite sides of the square gives us a torus which is image of the map Q, the map Q will be covering map and $\mathbb{R}^2$ is the universal cover of torus.

Proof (Theorem 17)**.** *Let S be a q-equivelar triangulation of the torus and M be a triangulation of the plane $\mathbb{R}^2$. Since we have a q-equivelar triangulation of torus, q should be 6 and by the previous fact given above, $M = E$ which is given in Figure 4.2. Now, consider the covering map $Q : M \to S$, since $\mathbb{R}^2$ is the universal covering of torus. By Spanier, this covering map is a simplicial covering. Let π be the group of all covering transformations, then $|S| = |E|/\pi$. Being a covering transformation, any element $\sigma \in \pi$ commutes with the covering projection; $Q \circ \sigma = Q$. So σ takes a* *simplicial complex to any other simplicial complex. Therefore, σ is an automorphism of E and π can be identified with a subgroup of $Aut(E)$. Then, the triangulation S will be quotient of the triangulation E by the subgroup π of $Aut(E)$. π does not fix any edges, vertices or faces. As we noted before, if the quotient map has no fixed element, it will be either $H - \{Id\}$ or a glide reflection, where H is the group of all translations which is a subset of $Aut(E)$. Since a torus is orientable ($S = M/\pi$) and any glide reflection reverses the orientation, but torus is orientable so, the group π will not include any glide reflection. Thus, the group π must be a subgroup of the group H. Since $h_1 \circ h_2 = h_1 \circ h_2$, for any element h_1, h_2 in H, i.e, H is commutative group, π will be a normal subgroup of H. Also, we know that the group H acts on the vertex set $V(X)$ transitively, so H/π will act transitively on the vertices of E/π. Since Q is a covering map and $S = E/\pi$ then S will be a vertex-transitive triangulation.*

□

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