TIGHT CONTACT STRUCTURES ON SMALL SEIFERT FIBERED SPACES

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

KÜRŞAT YILMAZ

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF MASTER OF SCIENCE
IN
MATHEMATICS

AUGUST 2018

Approval of the thesis:

TIGHT CONTACT STRUCTURES ON SMALL SEIFERT FIBERED SPACES

submitted by **KÜRŞAT YILMAZ** in partial fulfillment of the requirements for the degree of **Master of Science in Mathematics Department, Middle East Technical University** by,

Prof. Dr. Ahmet Oral
Dean, Graduate School of **Natural and Applied Sciences** ________________

Prof. Dr. Yıldıray Ozan
Head of Department, **Mathematics** ________________

Assoc. Prof. Dr. M. Fırat Arıkan
Supervisor, **Department of Mathematics, METU** ________________

Examining Committee Members:

Assoc. Prof. Dr. Sinem Onaran
Department of Mathematics, Hacettepe University ________________

Assoc. Prof. Dr. M. Fırat Arıkan
Department of Mathematics, METU ________________

Assoc. Prof. Dr. Mehmetcik Pamuk
Department of Mathematics, METU ________________

Date: ________________

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Name, Last Name: KÜRŞAT YILMAZ

Signature :

ABSTRACT

TIGHT CONTACT STRUCTURES ON SMALL SEIFERT FIBERED SPACES

Yılmaz, Kürşat
M.S., Department of Mathematics
Supervisor : Assoc. Prof. Dr. M. Fırat Arıkan

August 2018, 36 pages

Small Seifert fibered space is a Seifert fibered space with three exceptional fibers. There is an invariant of Seifert fibered spaces which is called Euler number (e_0). In this thesis, the classification of tight contact structures on some small Seifert fibered 3-manifolds will be studied. The classifications are based on understanding the interactions between different techniques and theories known as Dehn surgery, contact surgery, the bypass technique, and the convex surface theory. In particular, we will give the complete classification of the tight contact structures on small Seifert fibered spaces having e_0 less than or equal to -3, and greater than or equal to 1 by using the work of Wu. Moreover, we will give some partial results when e_0 is equal to -1 by using the work of Mark and Tosun.

Keywords: Tight contact structures, Seifert fibered 3-manifolds

ÖZ

KÜÇÜK SEİFERT LİF UZAYLARI ÜZERİNDEKİ SIKI KONTAKT YAPILAR

Yılmaz, Kürşat
Yüksek Lisans, Matematik Bölümü
Tez Yöneticisi : Doç. Dr. M. Fırat Arıkan

Ağustos 2018 , 36 sayfa

Sadece üç istisnai lifi bulunan Seifert lif uzayına küçük Seifert lif uzayı denir. Küçük Seifert lif uzayları üzerinde Euler sayısı (e_0) olarak adlandırılan bir değişmez vardır. Bu tezde 3-boyutlu küçük Seifert lif uzayları üzerinde sıkı kontakt yapıların sınıflandırılması çalışılacaktır. Sınıflandırma Dehn surgery, kontakt surgery, bypass teknikleri ve konveks yüzey teorisinin birbirleri ile etkileşimleri baz alınarak yapılacaktır. Özel olarak, Wu'nun çalışmaları kullanılarak Euler sayısı -3 den küçük eşit olan, ve 1 den büyük eşit olan küçük Seifert lif uzayları üzerindeki sıkı kontakt yapıların tam sınıflandırılması verilecektir. Ayrıca Mark ve Tosun'un çalışmaları kullanılarak Euler sayısı -1 olan küçük Seifert lif uzayları üzerindeki sıkı kontakt yapıların sınıflandırılması üzerine kısmi sonuçlar verilecektir.

Anahtar Kelimeler: Sıkı kontakt yapılar, Seifert lifli 3-boyutlu çokkatlılar

To my wife

ACKNOWLEDGMENTS

First, I would like to express my sincere gratitude to my supervisor, Assoc. Prof. Dr. Mehmet Fırat Arıkan, for his invaluable guidance, encouragement, and endless support throughout my thesis.

Also, I would like to thank Prof. Dr. Hurşit Önder and Assoc. Prof. Dr. Ali Ulaş Özgür Kişisel for many helpful conversations. I would also like to thank Prof. Dr. Mustafa Turgut Önder, without whose enthusiasm I would not be working on Geometric Topology.

Moreover, I would like to thank Asst. Prof. Dr. Bülent Tosun who first introduced the problem that I worked in this thesis.

My special thanks to my friends Dr. Merve Seçgin and Dr. Tülin Altınöz who have willingly helped me out with their knowledge.

Lastly, but not least, I would like to express my deepest thanks to my wife Yağmur Yılmaz for her patience and support throughout this thesis.

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CHAPTER 1

INTRODUCTION

One can trace the roots of the terminology related with contact geometry back to 1872, when Sophus Lie first introduced the notion of contact transformation. His work was a bit different than the modern understanding. He used this transformation as a geometric tool to understand the solution spaces of differential equations which will be later on called manifolds. Lie's transformations were extensively studied in the late 19^{th} century and at the first half of the 20^{th} century among the famous mathematicians H. Poincarè, F. Engel, E. Goursat, and E. Cartan. Nonetheless, contact geometry could not become as famous as its twin sister, symplectic geometry.

In contrast, contact topology is not old as the contact geometry. It has only started to been studied since 1970's. Afterwards, the contact geometry and topology have experienced some fruitful time, 3-dimensional contact manifolds became one of the main interest, and contact structures are defined. One of them is tight and the other one is overtwisted.

In 1989 [4], Eliashberg completed the classification of overtwisted contact structures on 3-manifolds. Then people concentrated on the classification of tight contact structures on 3-manifolds, which is much more subtle than the classification of overtwisted ones because of the various relations on the topology of underlying manifolds.

Gromov and Eliashberg [14],[5] showed that a fillable contact structure is tight. Since then, it became the main tool to determine that whether the given contact structure on a 3-manifold is tight or not. In addition, fillability is preserved by Legendrian surgery [22], [6] which in turn gives affluent source of tight contact structures. Gompf's work [13] on Legendrian surgery enables us to construct tight contact structures on Seifert fibered manifolds.

Instead of working with contact structure itself, people started to work on the singular foliation on the embedded surfaces which is induced by contact structure on the given manifold. In 1991 [12], Giroux came up with the idea of convex surface, which is an embedded surface whose characteristic foliation is transversely intersects with some curves. These curves are called dividing set which essentially determine the contact structure in a small neighborhood. Afterwards, it became one of the most important tool to study contact structures.

In 2000 [15], Honda developed a technique so-called bypass and he used this technique to split the manifold along convex surfaces into simpler pieces to analyze possible contact structures on them. By doing so, he classified all the contact structures on solid tori, and Lens spaces. Inspiring by this work, in [7] Etnyre and Honda proved the non-existence of tight contact structures on $M(-\frac{1}{2}, \frac{1}{3}, \frac{1}{5})$ which corresponds to the negatively oriented Poincarè homology sphere $-\Sigma(2, 3, 5)$. Moreover, Colin [1] showed that every oriented Seifert fibered spaces over a genus g surface (with $g \geq 1$) admits infinitely many non-isotopic tight contact structures.

As explained, it became really interesting to work on Seifert fibered manifolds over the 2-sphere S^2. Ghiggini and Schönenberger [10] showed that on the small Seifert fibered spaces $-\Sigma(2, 3, 11)$ and $\Sigma(2, 3, 11)$ there are exactly one and two contact structures respectively (one of them has Euler number $e_0 = -1$ and the other one has $e_0 = -2$ respectively). Furthermore, Wu [23] gave the complete classification for the small Seifert fibered spaces having $e_0 \leq -3$ and $e_0 \geq 1$. Also, in [20] Tosun showed that the Bireskorn homology sphere $-\Sigma(2, 3, 6n + 1)$ admits exactly $\frac{n(n-1)}{2}$ tight contact structure and in [19] Mark and Tosun showed that $\Sigma(2, 3, 6n+1)$ admits exactly two tight contact structures.

The outline of this thesis is as follows: In Chapter 2, some basic definitions and examples related with contact structures and contact 3-manifolds are given. In Chapter 3, some basics of convex surface theory and bypass technique are given. In Chapter 4, the definition of Seifert fibered manifolds and some classical invariants for Seifert fibered manifolds are given. In Chapter 5, the classifications of tight contact structures on some small Seifert fibered spaces is given.

CHAPTER 2

PRELIMINARIES

In this chapter, we will recall some basic definitions and facts about contact geometry in dimension three. In Subsection 2.1, we will define contact structures and give some basic examples related to the given definitions. After recalling front projection of Legendrian knots in Subsection 2.2, we will give some basics of surgery theory in Subsection 2.3.

2.1 Contact Structures

Definition 2.1.1. *Let M be a 3-dimensional manifold. A* ***contact structure*** *on M is a maximally non-integrable hyperplane field $\xi = ker(\alpha) \subset TM$ such that the defining differential 1-form α has to satisfy $\alpha \wedge d\alpha \neq 0$ in an open neighborhood of any point p in M. Such a 1-form α is called a* ***contact form****. The pair (M, ξ) is called a* ***contact manifold****.*

Example 2.1.1. *Consider the Euclidean Space $\mathbb{R}^3$ with the standard coordinates (x, y, z) and 1-form $\alpha_1 = dz + xdy$. $\xi_1 = ker(\alpha_1)$ is a contact structure on $\mathbb{R}^3$ (see Figure 2.1).*

Since $\alpha_1 \wedge d\alpha_1 = (dz + xdy) \wedge (dx \wedge dy) = dx \wedge dy \wedge dz \neq 0$, so $\xi_1 = ker(\alpha_1)$ is a contact structure on $\mathbb{R}^3$.

Example 2.1.2. *Similarly, consider $\mathbb{R}^3$ with the standard coordinates (x, y, z) and 1-form $\alpha_2 = dz + xdy - ydx$. Then $\xi_2 = ker(\alpha_2)$ is also a contact structure on $\mathbb{R}^3$.*

Example 2.1.3. *Let S^3 be the unit 3-sphere in $\mathbb{R}^4$ with standard coordinates*

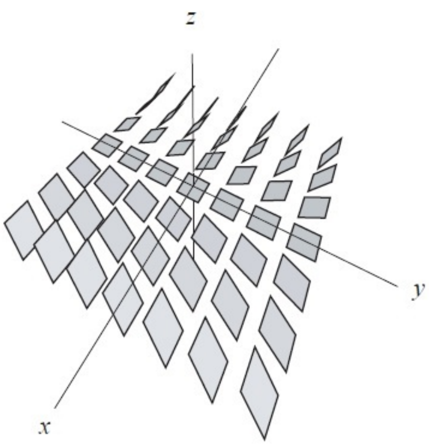

Figure 2.1: The contact structure $ker(dz + xdy)$.

(x_1, y_1, x_2, y_2). *The* 1*-form* $\alpha = x_1dy_1 - y_1dx_1 + x_2dy_2 - y_2dy_2$ *defines a contact structure* ξ_{st} *on* S^3*, which is called the standard contact structure on* S^3.

Definition 2.1.2. *Two contact manifolds* (M_1, ξ_1) *and* (M_2, ξ_2) *are called* ***contactomorphic*** *if there exists a diffeomorphism* $f : M_1 \to M_2$ *with* $Tf(\xi_1) = \xi_2$*, where* $Tf : TM_1 \to TM_2$ *denotes the derivative map of* f*. If* $\xi_i = ker(\alpha_i)$*,* $i = 1, 2$ *this is equivalent to the existence of nowhere zero function* $\lambda : M_1 \to \mathbb{R}$ *such that* $f^*\alpha_2 = \lambda\alpha_1$*. Such a map is called a* ***contactomorphism****.*

Example 2.1.4. *Two contact manifolds* $(\mathbb{R}^3, \xi_1)$ *and* $(\mathbb{R}^3, \xi_2)$ *given in the first two examples are contactomorphic via a contactomorphism given explicitly as* $f(x, y, z) = \Big(\frac{x+y}{2}, \frac{y-x}{2}, \frac{z+xy}{2}\Big)$*. One can easily see that* $f^*\alpha_2 = \alpha_1$*.*

Proof. Let us use the coordinates (u, v, w) in the range $\mathbb{R}^3$ to avoid confusion. Consider $\alpha_2 = dw + udv + vdu$. Then

$$\begin{aligned} f^*(\alpha_2) &= d\Big(z + \frac{xy}{2}\Big) + \Big(\frac{x+y}{2}\Big)d\Big(\frac{y-x}{2}\Big) - \Big(\frac{y-x}{2}\Big)d\Big(\frac{x+y}{2}\Big) \\ &= dz + \frac{ydx}{2} + \frac{xdy}{2} + \Big(\frac{x+y}{2}\Big)\Big(-\frac{dx}{2} + \frac{dy}{2}\Big) + \Big(\frac{x-y}{2}\Big)\Big(\frac{dx}{2} + \frac{dy}{2}\Big) \\ &= dz + xdy. \end{aligned}$$

□

Example 2.1.5. *S^3 is the one point compactification of $\mathbb{R}^3$. So, if we exclude one point from S^3 it will be diffeomorphic to $\mathbb{R}^3$. This diffeomorphism induces a contactomorphism between $(S^3 \setminus \{p\}, \xi_{st})$ and $(\mathbb{R}^3, \xi_2)$ and hence $(\mathbb{R}^3, \xi_1)$ of Example 2.1.1.*

An explicit contactomorphism is given in the proof of Proposition 2.1.8 in [8].

Definition 2.1.3. *An embedded disc D in a contact 3-manifold (M, ξ) is called an **overtwisted disk** if $T_pD = \xi|_p$ for every $p \in \partial D$. If such a disk exists in (M, ξ) then the contact structure on M is called **overtwisted**. If there is no such disk then the contact structure is called **tight**.*

Example 2.1.6. *Consider the $\mathbb{R}^3$ with the standard cylindrical coordinates (r, θ, z). The 1-form $\alpha_{ot} = cosrdz + rsinrd\theta$ gives a contact structure on $\mathbb{R}^3$ since*

$$\alpha_{ot} \wedge d\alpha_{ot} = \big(1 + \frac{sinr}{r} cosr\big) rdr \wedge \theta \wedge dz$$

is nowhere zero. The contact structure $\xi_{ot} = ker(\alpha_{ot})$ is called the standard overtwisted contact structure on $\mathbb{R}^3$.

Definition 2.1.4. *A 4-dimensional **Stein manifold** is a triple (X^4, J, ψ) where J is a complex structure on X, $\psi : X \to \mathbb{R}$ is a proper map and the closed 2-form $\omega_\psi = -d(d\psi \circ J)$ is non-degenerate. A contact 3-manifold (M, ξ) is said to be **Stein fillable** if there is a compact (necessarily with boundary) Stein manifold (X^4, J, ψ) such that $\partial X = M$ and the 1-form $\alpha := -(d\psi \circ J)|_{\partial X}$ defines ξ (i.e., $\xi = ker(\alpha)$).*

One way is to show that a given contact 3-manifold is tight, is to show it is holomorphically filled by a Stein 4-manifold, i.e., it is Stien fillable.

Remark 2.1.1. *We note that the contact structures given in Example 2.1.1, 2.1.2, 2.1.3 are all tight: In 1982, Douady [3] showed that standard contact structure on $\mathbb{R}^3$ has no overtwisted disk, hence it is tight. Also, the standard contact structure on S^3 is holomorphically filled by the unit 4-ball with standard complex structure, hence it is tight.*

Definition 2.1.5. *Two contact structures ξ_1 and ξ_2 on a given 3-manifold M is said to be **homotopic** if they are homotopic as tangent plane distributions, and they are called **isotopic** if they are homotopic through contact structures.*

Homotopy does not preserve the type of contact structure i.e., two contact structures having different type (one of them is tight and the other one is overtwisted) may be homotopic but they cannot be isotopic. For this reason, classification of contact structures are made up to isotopy.

We can induce by Gray's stability theorem that, on a closed contact 3-manifold M, two contact structures ξ_1 and ξ_2 are isotopic if and only if there exists an isotopy φ_t, $t \in [0,1]$, of M such that $\varphi_0 = id$ and $(\varphi_1)_*(\xi_1) = \xi_2$

Example 2.1.7. *Let us consider $\varphi_t : \mathbb{R}^3 \to \mathbb{R}^3$, given as*

$$\varphi_t(x,y,z) = \left(x, \frac{y}{(1+t)}, z + \frac{txy}{1+t}\right),$$

where $t \in [0,1]$. Clearly, φ_t is an isotopy of $\mathbb{R}^3$. Now consider the contact structures ξ_1 and ξ_2 on $\mathbb{R}^3$ in the previous examples. One can easily see that $\varphi_0 = id$ and $\varphi_1^*(\alpha_2) = \alpha_1$. Hence, the contact structures given as the kernels of α_1 and α_2 on $\mathbb{R}^3$ are isotopic.

In [4], Eliashberg showed that if two different overtwisted contact structures on a closed 3-manifold M have the same homotopy type, i.e., if they are homotopic, then they are isotopic as well. But it is not true in general.

Definition 2.1.6. *Given an orientation on M, if $\alpha \wedge d\alpha > 0$, then the orientation of ξ is said to be positive. If $\alpha \wedge d\alpha < 0$, then ξ is called negative. If there exists a global 1-form α, which defines ξ on M, then ξ is said to be co-orientable.*

In this thesis, since we are working on contact 3-manifolds, any contact structure will be positively co-oriented unless it is specified. So, it will be enough to fix the orientation of M.

2.2 Legendrian Knots

Definition 2.2.1. *A curve γ in a contact manifold (M, ξ) is called **Legendrian** if it is everywhere tangent to ξ.*

Definition 2.2.2. *A smoothly embedded S^1,i.e., a knot, in a contact manifold M is called a **Legendrian knot** if it is Legendrian.*

Definition 2.2.3. *Two Legendrian knots $L, L^{'}$ are called* ***Legendrian isotopic*** *if there is a smooth 1-parameter family L_t, $t \in [0, 1]$, such that $L_0 = L$ and $L_1 = L^{'}$.*

One of the basic invariants of Legendrian knots under Legendrian isotopies is the contact framing, which is inherited from the contact planes on the manifold. If a Legendrian knot L is given with a canonical framing $\mathcal{F}_r$, then its contact framing can be represented as a *(twisting) number*, $t(L, \mathcal{F}_r)$, which is the number of twists of the contact planes along L measured relative to $\mathcal{F}_r$.

Definition 2.2.4. *Let L be a Legendrian knot in a contact 3-manifold (M, ξ), and Σ be a Seifert surface of L. Since ξ is co-oriented, there necessarily exists a contact vector field X of M which is everywhere transverse to the contact structure ξ on M and hence transverse to the Legendrian knot L. Take the push off $L^{'}$ of L in the direction determined by X. Then the* ***Thurston-Bennequin number*** *$tb(L, \xi)$ is defined as the signed intersection of $L^{'}$ with Σ.*

Our aim is to construct tight contact structures on a 3-manifold. To do so, we start with S^3, and make some modifications which will be called surgery later on. For this reason, we are interested in knots and links in S^3 with its standard contact structure. However, any knot or link in S^3 misses at least one point. So, we will regard them as a knot or a link in $\mathbb{R}^3$ with the standard contact structure ξ_1 of it by using Example 2.1.5.

Definition 2.2.5. *The projection of a Legendrian knot in $(\mathbb{R}^3, \xi_1)$, where ξ_1 is the contact structure given in Example 2.1.1, on to the yz-plane is called* ***the front diagram of the Legendrian knot****.*

Let L be an oriented Legendrian knot in (S^3, ξ_{st}). Then using its front diagram Thurston-Bennequin number can be calculated as follows:

$$tb(L) = writhe(L) - \frac{1}{2}(\#cusps(L)).$$

Here, **writhe** is the sum of of all positive and negative intersections (see Figure 2.2), and **cusps** are the singular points in the front diagram of the Legendrian knot.

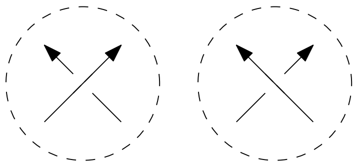

Figure 2.2: Positive and negative intersections.

There is another numerical invariant $r(L, \xi)$, namely rotation number, which is defined to be the obstruction to the extension of tangent vectors of L to a non-vanishing section of $\xi|_\Sigma$. Here, as before Σ is a Seifert surface of L.

Remark 2.2.1. *The Thurston-Bennequin number and rotation number of a null-homologous Legendrian knot L depend on the relative homology class of the chosen Seifert surface in $H_2(M; L)$. But, in the special situation where $H_1(M) = H_2(M) = 0$, all the Seifert surfaces are relatively homologous. And, hence, the Thurston-Bennequin number depends only on L and rotation number depends only on L and its orientation. In this case, we will denote them by $tb(L)$ and $r(L)$, respectively.*

2.3 Contact Dehn Surgery

Surgery is an essential tool to construct new manifolds. Instead of giving general theory, in this subsection we will focus on Dehn surgery since we are working in dimension three.

Let K be a knot in S^3 and denote the tubular neighborhood of K as νK. Then νK will be diffeomorphic to solid torus, $D^2 \times S^1$, since it is the only orientable D^2-bundle over S^1. Let C be the closure of $S^3 \setminus \nu K$ of νK in S^3. Consider $S^3 = \nu K \cup C$ and $\nu K \cap C = T^2$.

$$H_2(S^3) \longrightarrow H_1(T^2) \longrightarrow H_1(\nu K) \oplus H_1(C) \longrightarrow H_1(S^3)$$

$$0 \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z} \oplus H_1(C) \longrightarrow 0$$

Then we can use some part of the Mayer-Vietoris sequence to get $H_1(C) \cong \mathbb{Z}$. It is well known that $H_1(T^2) \cong \mathbb{Z}\oplus\mathbb{Z}$ and up to isotopy there are two distinct curves which generate $H_1(T^2)$. One of them is called the **meridian** μ which generates the kernel of the homomorphism $H_1(T^2) \rightarrow H_1(\nu K)$. The other one is called the **preferred longitude** λ which generates the kernel of the homomorphism $H_1(T^2) \rightarrow H_1(C)$.

We consider S^3 as the oriented boundary of D^4 which has the standard orientation. We give $T^2 = \partial(\nu K)$ the boundary orientation. Also, we assume that K is oriented as well. Then orientation of λ can be given in such a way that it is isotopic to K in νK as oriented curves and the orientation of μ is chosen such that it turns μ and λ into a positive basis for that homology group. (see Figure 2.3)

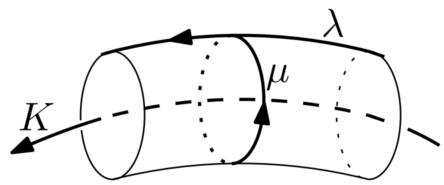

Figure 2.3: Meridian μ and longitude λ.

With a proper choice of generator for $H_1(C) = \mathbb{Z}$, the homomorphism $H_1(T^2) \rightarrow H_1(\nu K) \oplus H_1(C)$ can be identified by $\lambda \rightarrow (1, 0)$ and $\mu \rightarrow (0, 1)$.

Definition 2.3.1. *Let K be a knot in S^3. Remove the tubular neighborhood νK of K which is isomorphic to solid torus as mentioned before. Then, re-glue a solid torus $S^1 \times D^2$ by a diffeomorphism $\partial(S^1 \times D^2) \rightarrow \partial(\nu K)$. This removing and re-gluing operation is called* ***Dehn surgery****.*

Denote the meridian $* \times \partial D^2$ as μ_0 and the longitude $S^1 \times *$ as λ_0 of $S^1 \times D^2$. Here $*$ denotes a point in S^1 or in D^2. Then the gluing map can be described by

$$\mu_0 \rightarrow p\mu + q\lambda, \quad \lambda_0 \rightarrow m\mu + n\lambda$$

and $\begin{pmatrix} p & m \\ q & n \end{pmatrix} \in GL(2, \mathbb{Z})$ *is the matrix representation of this operation.*

A Dehn surgery along a knot K in S^3 is determined by the image of μ_0 since the curve on $\partial(\nu K)$ becomes homotopically trivial in the surgered manifold. As a matter of fact, it is completely determined by the surgery coefficient $p/q \in \mathbb{Q} \cup \infty$, since the diffeomorphism of $\partial(S^1 \times D^2)$ given by $(\lambda_0, \mu_0) \rightarrow (\lambda_0, -\mu_0)$ has the effect of changing the signs of both p and q, extends to a solid torus that we glue back. So there is no ambiguity to call this surgery as (p/q)-surgery. Here, p/q is called the **(topological) surgery coefficient**. Note that ∞-surgery ($p = \pm 1$, $q = 0$) has no effect on the manifold, so is to say it is topologically trivial.

Definition 2.3.2. *A finite collection of disjoint knots is called a **link**.*

Theorem 2.3.1 (**Likorish**[17], **Wallace** [21])**.** *Any closed, connected, orientable 3-manifold can be obtained by surgery along a link L in S^3.*

Definition 2.3.3. *Let K be a knot in S^3 with surgery coefficient k and let L be a Legendrian realization of K. The **contact surgery coefficient** of L is the number $l = k - tb(L)$, where $tb(L)$ denotes the Thurston-Bennequin number of L. Then (L, l) gives a new surgery description which is called **contact Dehn surgery**. A **Legendrian surgery** is a contact Dehn surgery with $l = -1$.*

By doing so, one can obtain another closed contact 3-manifold.

Example 2.3.1. 0*-surgery on an unknot in S^3 gives $S^1 \times S^2$. Similarly, $+1$ Legendrian surgery on Legendrian unknot with $tb = -1$ gives again $S^1 \times S^2$ but with a contact structure on it. A surgery description can be given as in Figure 2.4.*

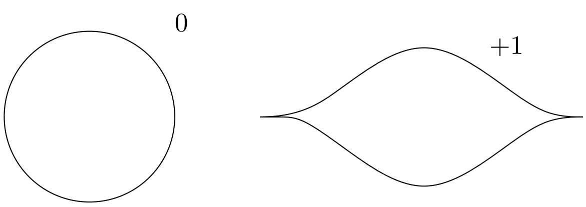

Figure 2.4: Topological 0- surgery and Legendrian +1 surgery on unknot.

CHAPTER 3

CONVEX SURFACE THEORY AND BYPASS

In this chapter, we will define some basic concepts of convex surface theory and one of the most useful technique which is introduced by Honda in [15], to understand tight contact structures on a given 3-manifold, namely bypass. Also, we will give some theorems which will be used in the last chapter.

Let us begin with defining contact vector fields:

Definition 3.0.1. *Let X be a vector field on a contact 3-manifold (M, ξ). Denote the local flow of X by ψ_t. Note that, if M is not closed, then the map ψ_t (for a fixed $t \neq 0$) is not defined globally on M in general. By any means, the vector field X is called a* ***contact vector field*** *if $(\psi_t)_*(\xi) = \xi$ for all $t \in \mathbb{R}$, i.e., ξ is preserved under the flow of X.*

Equivalently, a vector field X is a contact vector field if and only if $\mathcal{L}_X\alpha = \kappa\alpha$ for some function $\kappa : M \to \mathbb{R}$. Note that this condition is independent of the choice of the contact form α defining the given contact structure ξ.

Example 3.0.1. *The vector field $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2z\frac{\partial}{\partial z}$ is a contact vector field for the standard contact structure ξ_1 on $\mathbb{R}^3$ since $\mathcal{L}_X\alpha_1 = 2\alpha_1$.*

Definition 3.0.2. *An embedded surface Σ in a compact manifold (M, ξ) is said to be* ***convex*** *if there exists a contact vector field transverse to Σ.*

Let Σ be an embedded surface in a contact 3-manifold (M, ξ). Consider $\xi \cap T\Sigma$. This intersection gives a line field except at finitely many points where the tangent plane of those points coincide with the 2-plane distribution of ξ. If we integrate this line field, we will get a foliation of Σ with the singularities at those points of tangencies. This foliation is called **characteristic foliation** Σ_ξ of Σ in (M, ξ).

Here is an equivalent definition of a convex surface:

Definition 3.0.3. *An embedded surface Σ in a contact manifold (M, ξ) is said to be convex if there exists a collection of curves Γ_Σ for which the following conditions hold:*

(i) Γ_Σ divides Σ into two different types of subsurfaces which we denote as Σ^+ and Σ^-.

(ii) Γ_Σ and the characteristic foliation Σ_ξ of Σ intersect transversely.

(iii) There exists a vector field X in the direction of which the characteristic foliation flows and a volume form ω such that the vector field X expands ω on Σ^+ and shrinks ω on Σ^- and X point outward on Σ^+.

The collection of curves Γ_Σ is called the **dividing curves** which determines the contact structure in an small neighborhood of the convex surface Σ according to the following theorem, by which we can determine the contact structure in a tubular neighborhood of an embedded convex surface Σ.

Theorem 3.0.1 (Giroux Criterion). *In a contact manifold (M, ξ) a convex surface Σ has a tight neighborhood if and only if either one of the following holds:*

- *no component of dividing set bounds a disk*
- *Σ is sphere and the dividing set consists of just one connected component.*

Example 3.0.2. *Consider $\mathbb{R}^3$ with the contact structure ξ_1 given as the kernel of $\alpha_1 = dz + xdy - ydx$ and Let X be the contact vector field given in Example 3.0.1. This vector field is transverse to the unit sphere S^2 so under this construction S^2 is a convex surface in $\mathbb{R}^3$.*

In general, an embedded surface Σ may not be convex. However, Giroux [12] proved that any embedded closed surface can be made convex by some small C^∞-perturbation. The new surface will not exactly be the old one but it will be an isotopic copy.

Lemma 3.0.1 (Edge Rounding). *Let Σ_1 and Σ_2 be two convex surfaces in a contact 3-manifold (M, ξ) with collared Legendrian boundary and transversely intersecting along a Legendrian curve γ. If Σ_1 and Σ_2 are oriented, then smoothing the intersection gives another oriented convex surface Σ. Smoothing the corner can be done in such a way that it connects the dividing set of Σ_1 and Σ_2 such that positive regions of Σ_1 connect to positive regions of Σ_2 (negative regions connects with the negative regions respectively) which determines the dividing set of Σ.*

Now it is time to explain what bypass is.

Definition 3.0.4. *An embedded oriented overtwisted half disk D is called a **bypass disk** for the convex surface Σ if it intersects with Σ along a Legendrian arc γ and γ intersects with the set of dividing curves of Σ in 3 different points two of them are end points of γ and the other is in between. Moreover, this intersection points are exactly the elliptic points of γ. D may have other tangencies not along γ and they have to be positive tangencies and have to alternate between elliptic and hyperbolic.*

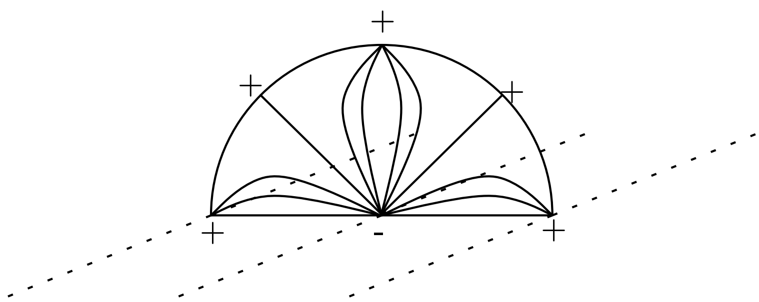

Figure 3.1: A bypass disk.

Proposition 1 (Imbalance Principle). *Consider $S^1 \times [0, 1]$ as a convex surface with Legendrian boundary in a tight contact 3-manifold. If $t_0(S^1 \times 0) < t_1(S^1 \times 1) \leq 0$ then there exists necessarily a bypass along the $S^1 \times 0$ side. Here t_i's are the reciprocals of the slopes in $S^1 \times 0$ and $S^1 \times 1$ respectively.*

The next lemma from [10] gives an idea about the slopes of the dividing curves.

Lemma 3.0.2 (Twisting Number). *Consider a Legendrian curve γ in a contact manifold (M, ξ) with twisting number n relative to a fixed framing and a standard neighborhood N of γ. If there exists a bypass attached to a Legendrian curve of ∂N with* *slope r and $\frac{1}{r} \geq n+1$, then there exists a Legendrian curve with twisting number $n+1$ isotopic to γ.*

CHAPTER 4

SEIFERT FIBERED MANIFOLDS

In this chapter, we will give the definition of Seifert fibered spaces and some properties that they have together with the surgery description of them.

Definition 4.0.1. *A* ***generalized Seifert fibration*** *is a triple* (M, Σ, ϕ) *where M is an oriented 3-manifold,* Σ *is an either oriented or non-oriented surface and* ϕ *is a map from* M *to* Σ *such that this triple is almost locally trivial* S^1*-bundle.*

For any element x of Σ there exists a local neighborhood of x, which can be seen as D^2, and $\phi^{-1}(D^2) \cong D^2 \times S^1$. Then the mapping $\phi : D^2 \times S^1 \to D^2$ is defined by, $(r\theta_1\theta_2) \to (r\theta_1^p\theta_2^q)$ where $\theta_i \in S^1 = \{\theta \in \mathbb{C} : |\theta| = 1\}$, $r \in [0, 1]$ and $p, q \in \mathbb{Z}$ and $gcd(p, q) = 1$. Here p, q are depending on the choice of x.

If $p \neq 0$ for every $x \in \Sigma$, then the triple is called a Seifert fibration and M is called a Seifert fibered space.

A fiber is called **regular** if $p = 1$, otherwise it is called a **singular fiber** or **exceptional fiber**. If $p \neq 0$, then locally an exceptional fiber can be considered as $D^2 \times I$ having the core $\{x\} \times I$ where the ends of solid cylinder is identified to form a solid torus $T^2 \times S^1$ with a $2\pi q/p$ twist.

In general Σ need not to be orientable or compact but in this thesis we fix our surface as S^2 which is compact without boundary to use the fact that if Σ is compact then the number of exceptional fibers is finite.

One can easily prove this fact: Since for every $x \in \Sigma$ there exists only one D^2 neighborhood of that point which has an exceptional lifting. By the assumption that Σ is compact, it can be covered by finitely many such neighborhoods.

By the reason above there is no ambiguity to consider the Seifert fibration (M, Σ, ϕ) with closed oriented and connected surface Σ. If the fibration is as stated, we can remove the solid torus neighborhood of the (finitely many) exceptional fibers of M and corresponding D^2 neighborhoods of Σ, to have a trivial S^1-bundle over a connected orientable surface with boundary. That is to say, M can be obtained from Dehn surgery on some fibers of trivial bundle $\Sigma \times S^1 \to \Sigma$. As mentioned in Definition 2.3.1, each Dehn surgery is determined with coprime integers (p, q), but in this case there are $\#$(exceptional fibers) many coprime integers (a_i, b_i).

Any such manifolds can be given as the following data: $(g; (a_1, b_1), ..., (a_n, b_n))$ where $g \geq 0$, $a_i, b_i \in \mathbb{Z}$. Here g stands for the genus of the surface Σ and the pairs (a_i, b_i) with $gcd(a_i, b_i) = 0$ stands for the surgery coefficients. Consider Σ_0 as an oriented surface of genus g with n punctures. In other words,

$$\Sigma_0 = \Sigma \setminus (D_1^2 \cup D_2^2 \cup ... \cup D_n^2).$$

Then M_0 can be defined as the trivial S^1-bundle over Σ_0 where $\partial M_0 = (S_1^1 \times S^1) \cup (S_2^1 \times S^1) \cup ... \cup (S_n^1 \times S^1)$. Let $R = \Sigma_0 \times \{1\}$, $Q_i = R \cap (S_i^1 \times S^1)$ and $H_i = \{1\} \times S^1 \subseteq S_i^1 \times S^1$. By using this trivial bundle, H_i and Q_i we can construct a Seifert fibered space (M, Σ, ϕ) by gluing a solid torus $T_i = D^2 \times S^1$ into the i'th boundary complement $S_i^1 \times S^1$ via a diffeomorphism $\begin{pmatrix} a_i & b_i \\ a_i' & b_i' \end{pmatrix} \in SL(2, \mathbb{Z})$ on the boundary torus. So, the meridian $\mu_i = S_i^1 \times \{1\} \subseteq \partial T_i$ satisfies the homology relation $\mu_i \cong a_i Q_i + b_i H_i$ in the homology of ∂T.

Here M_0 can be chosen in a way that $\mu_0 \cong Q_i + b_0 H_i$ where $b_0 \in \mathbb{Z}$. Then everything will change accordingly with a parameter b_0.

If we let $\lambda = \{1\} \times S^1 \subseteq \partial T$ and $\mu_i \cong a_i Q_i + b_i H_i$, then $\lambda_i \cong a_i' Q_i + b_i' H_i$. So, it is possible to solve H_i and Q_i in terms of λ_i and μ_i. Doing so we get $-a_i' \mu_i + a_i \lambda_i = H_i$ and $b_i' \mu_i - b_i \lambda_i = Q_i$. In T_i, μ_i is trivial, so we have $H_i \cong a_i \lambda_i$ and $Q_i \cong -b_i \lambda_i$ in the homology of T_i. So, the first row determines the surgery and a_i is the number of times H_i wraps around T_i and $-b_i$ is the number of times Q_i wraps around T_i.

The above construction gives a description of 3-dimensional Seifert fibered manifold $M(g; (a_1 b_1), (a_2, b_2), ..., (a_n, b_n))$. Here $\{g; (a_1 b_1), (a_2, b_2), ..., (a_n, b_n)\}$ are called the **Seifert invariants**.

Seifert invariants are not unique in general but unique up to the following operations [16]:

(i) adding or deleting any Seifert pair $(a, b) = (1, 0)$,

(ii) replacing any $(0, \pm 1)$ by $(0, \mp 1)$,

(iii) replacing each (a_i, b_i) by $(a_i, b_i + K_i)$ provided that $\sum K_i = 0$.

Definition 4.0.2. *The* ***Euler number*** *of a Seifert fibered space $M(g; (a_1 b_1), ..., (a_n, b_n))$ is the number $e_0 = -\big(b_0 + \sum \lfloor a_i/b_i \rfloor\big)$ where $b_0 \in \mathbb{Z}$ is a parameter stated as above and $\lfloor x \rfloor$ is the greatest integer function.*

Note that the Seifert manifold M is oriented and the corresponding Seifert invariants do not depend on the orientation of the base surface Σ. If the orientation of Σ is reversed, to preserve the orientation of M, the orientation of the fibers should be reversed as well. Since both Q_i and H_i are reversed, the homology relation $a_i Q_i + b_i H_i$, which determines (a_i, b_i) does not effected. In other words, there exists a fiber preserving self diffeomorphism of M preserving the orientation of M such that the induced map $\Sigma \to \Sigma$ reverses the orientation. Also, if the orientation of M is reversed then the sign of either M_i or Q_i is reversed. So, a_i/b_i is changed with $-a_i/b_i$.

Definition 4.0.3. *A Seifert fibered manifold M is called a* ***small Seifert fibered manifold*** *if it has exactly three exceptional fibers.*

Definition 4.0.4. *Let p, q, r be relatively prime positive integers. Then the link of singularity of $\{x^p + y^q + z^r = 0\} \cap S^5 \subset \mathbb{C}^3$ gives an oriented Seifert fibered 3-manifold having three singular fibers, which is called the* ***Brieskorn homology sphere*** *and denoted by $\Sigma(p, q, r)$.*

If one of p, q or r is equal to 1, then it is identically a homeomorphic copy of S^3. Moreover, the Seifert invariants of positively oriented Brieskorn homology spheres can be found by solving the following equation for the integers b_0, b_1, b_2, b_3:

$$b_0 pqr + b_1 qr + b_2 pr + b_3 pq = 1.$$

In this thesis, we work with small Seifert fibered manifolds over S^2. Since S^2 has no genus we abuse the notation: The manifold $M(e_0; (a_1, b_1), (a_2, b_2), (a_3, b_3))$ will represent the Seifert fibered space over S^2 having the Euler number e_0.

So, the Brieskorn homology sphere $\Sigma(p, q, r)$ can be seen as $M(e_0; -\frac{b_1}{p}, -\frac{b_2}{q}, -\frac{b_3}{r})$ and a surgery representation of a small Seifert fibered space can be given as in Figure 4.1.

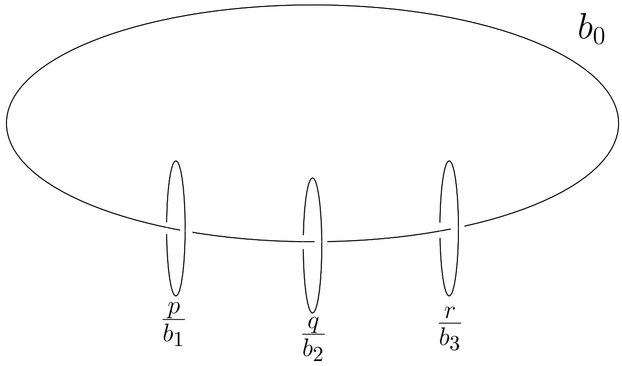

Figure 4.1: Standard surgery description for $\Sigma(p, q, r)$.

CHAPTER 5

CONSTRUCTING AND COUNTING THE TIGHT CONTACT STRUCTURES

In this chapter we construct the tight contact structures on some small Seifert fibered spaces. We start with topological surgery description of our manifolds. Then we Legendrian realize the topological surgery description to a contact surgery description having Legendrian surgeries only. This will give a lower bound for the number of tight contact structures. At the end, by using convex surface theory, we try to get an upper bound for the number of tight contact structures.

Case 1: $e_0 < -2$

Let Σ be a pair of pants and let $M(e_0; -\frac{q_1}{p_1}, -\frac{q_2}{p_2}, -\frac{q_3}{p_3})$ be a small Seifert fibered manifold having three singular fibers F_i, $i = 1, 2, 3$ satisfying the property that $p_i \geq 2$, $q_i \geq 1$ and $gcd(p_i, q_i) = 1$. Note that $\lfloor \frac{q_i}{p_i} \rfloor \leq -1$ and so $e_0(M) < -2$. Also, assume that for each $i = 1, 2, 3$ we have $\frac{q_i}{p_i} = [a_0^{(i)}, a_1^{(i)}, ..., a_{m_i}^{(i)}]$, where all $a_j^{(i)}$'s are integers, $a_0^{(i)} = -(\lfloor \frac{q_i}{p_i} \rfloor + 1) \leq -1$ and $a_j^{(i)} \leq -2$ for all $j \geq 1$. Denote the tubular neighborhood of each singular fiber by $V_i \cong D^2 \times S^1$ and identify ∂V_i with $\mathbb{R}^2 \setminus \mathbb{Z}^2$, having the basis $\{(1,0)^T, (0,1)^T\}$, such that the meridian direction of V_i is identified with $(1,0)^T$. Since $M \setminus (V_1 \cup V_2 \cup V_3) \cong \Sigma \times S^1$, we can choose an identification for $-\partial(M \setminus V_i) \cong \mathbb{R}^2 \setminus \mathbb{Z}^2$ such that $(0,1)^T$ represents the direction of S^1 fiber, and $(1,0)^T$ represents the direction of $-(\{\text{pt}\} \times \partial\Sigma)$. Let T_i be the standard solid torus and identify ∂T_i with $\mathbb{R}^2 \setminus \mathbb{Z}^2$ by identifying a meridian $\partial D^2 \times \{pt\}$ with $(1,0)^T$ and a longitude $\{pt\} \times S^1$ with $(0,1)^T$. Then by using an orientation preserving diffeomorphism A_i from the boundary ∂T_i of standard solid torus to $-\partial(M \setminus V_i)$, defined as

$$A_i = \begin{pmatrix} p_i & u_i \\ q_i & v_i \end{pmatrix} \in SL_2(\mathbb{Z}),$$

we can obtain M as $M \cong (\Sigma \times S^1) \cup_{A_1 \cup A_2 \cup A_3} (T_1 \cup T_2 \cup T_3)$.

Here A_i's are called the **attaching maps**. The first column of A_i is determined by the Seifert invariant corresponding to the exceptional fiber F_i and the second column is determined in a way that $p_i \geq u_i > 0$, $q_i \geq v_i > 0$ and $p_i v_i - q_i u_i = 1$.

It is well known that 0- surgery on an unknot in S^3 gives $S^2 \times S^1$ and it is S^1-bundle over S^2 which is a Seifert fibered space having no exceptional fiber. Take out three points from the base space S^2 to get Σ, pair of pants. The effect of this on the manifold $S^2 \times S^1$ is taking out three solid tori. If we glue back them via the diffeomorphisms stated as above we can get the following surgery diagram (see Figure 5.1) for the new Seifert fibered manifold M:

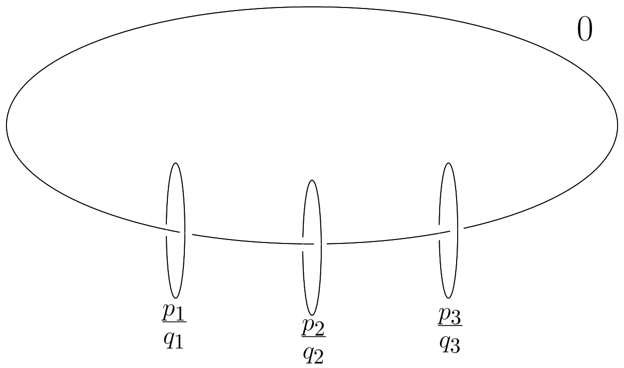

Figure 5.1: Standard surgery description for $M(e_0; -\frac{q_1}{p_1}, -\frac{q_2}{p_2}, -\frac{q_3}{p_3})$.

Consider the continued fraction expansion for each $\frac{p_i}{q_i} = [a_0^{(i)}, a_1^{(i)}, a_2^{(i)}, ..., a_{m_i}^{(i)}]$. After performing Rolfsen twist to each $\frac{p_i}{q_i}$ component on the diagram we get the surgery diagram as in Figure 5.2, since $a_0^{(1)} + a_0^{(2)} + a_0^{(3)} = e_0(M)$.

Since and $\frac{p_i}{q_i + a_0^{(i)} p_i} = [a_1^{(i)}, a_2^{(i)}, ..., a_{m_i}^{(i)}]$, after performing slum-dunks to the new coefficients, we get the surgery diagram as in Figure 5.3.

There are $|(e_0(M) + 1) \prod_{i=1}^{3} \prod_{j=1}^{m_i} (a_j^{(i)} + 1)|$ many ways to Legendrian realize the given topological surgery description in Figure 5.3, and all of them are non-isotopic and holomorphically fillable by Proposition 2.3 on [13] and Theorem 1.2 in [18].

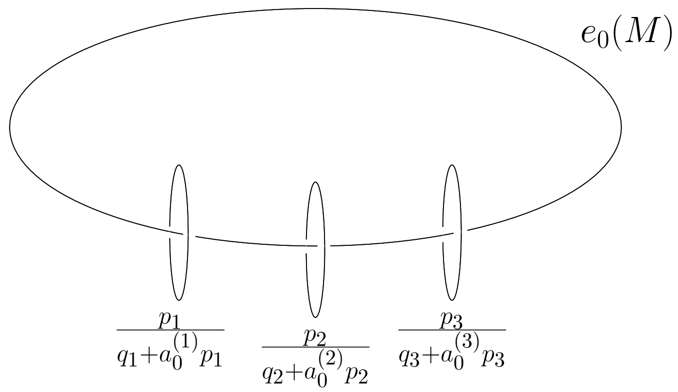

Figure 5.2: Surgery diagram after Rolfsen twist to each $\dfrac{p_i}{q_i}$.

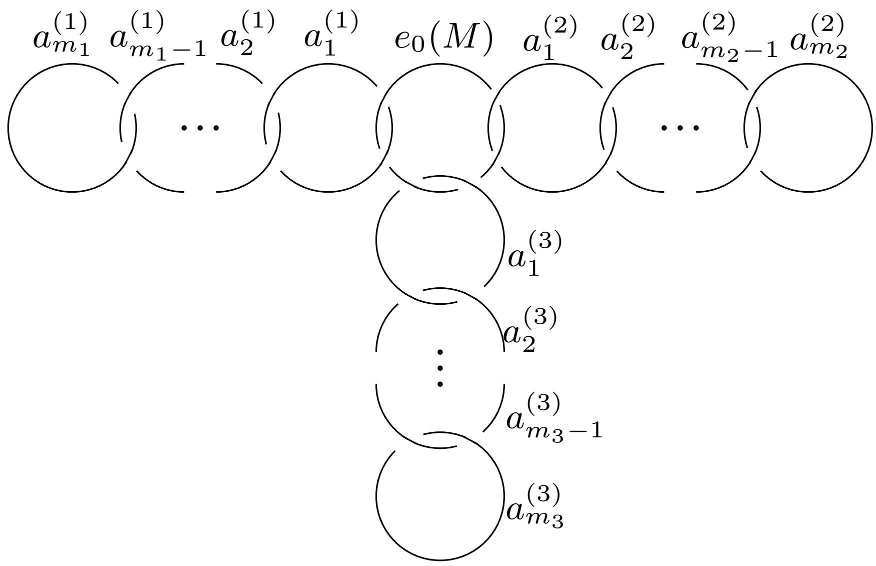

Figure 5.3: After slum-dunks and using continued fraction expansions.

So there are at least $|(e_0(M)+1)\prod_{i=1}^{3}\prod_{j=1}^{m_i}(a_j^{(i)}+1)|$ many tight contact structures on the manifold $M(e_0; -\frac{q_1}{p_1}, -\frac{q_2}{p_2}, -\frac{q_3}{p_3})$.

Now, by using Honda's bypass technique, we will find an upper bound for the number of the tight contact structures.

Consider $M \cong (\Sigma \times S^1) \cup_{A_1 \cup A_2 \cup A_3} (T_1 \cup T_2 \cup T_3)$ and let ξ be a tight contact structure on M. We begin with isotoping ξ to make each T_i a standard neighborhood of a Legendrian circle L_i which is isotopic to F_i for each $i = 1, 2, 3$ with twisting number $n_i \leq -2$, in other words by using Giroux flexibility theorem we make each ∂T_i convex with two dividing curves having the slope $\frac{1}{n_i}$ when measured with respect to the coordinates of ∂T_i as mentioned above. Let s_i be the slope of the dividing curves after applying the attaching maps, i.e, s_i's are the slope of the dividing curves on $-(M \setminus V_i)$. Applying the attaching maps we get s_i's as:

$$s_i = \frac{n_i q_i + v_i}{n_i p_i + u_i} = \frac{q_i}{p_i} + \frac{1}{p_i(n_i p_i + u_i)}.$$

By the reason that $n_i \leq -2$ we have $\lfloor \frac{q_i}{p_i} \rfloor < s_i < \frac{q_i}{p_i}$. Giroux flexibility theorem allows us to consider the slope of the Legendrain rulings as ∞ when measured in the coordinates of ∂T_i. For each i, pick a Legendrian ruling L_i on ∂T_i. Consider the vertical annulus $\mathcal{A}$ between T_1 and T_2 such that $\partial \mathcal{A} = L_1 \cup L_2$ and the interior of $\mathcal{A}$ is contained in the interior of $\Sigma \times S^1$. Theorem 1.4 of [24] says that ξ does not admit Legendrian vertical circles with twisting number 0. So $\mathcal{A}$ has dividing curves which connects two boundary components L_1 and L_2. So, we isotope ∂T_1 and ∂T_2 by attaching bypass disks corresponding to the boundary parallel dividing curves of $\mathcal{A}$. Since bypass attaching done in a small neighbourhood, T_1 and T_2 remains disjoint. Moreover, ∂T_i's remain minimal after each bypass. Doing so, we will end up with an isotopic copy of $\mathcal{A}$ having no boundary parallel dividing curve, i.e., all of the dividing curves of $\mathcal{A}$ connects two boundary components. So, after an isotopy, the slopes of the dividing curves of ∂T_1 and ∂T_2 become $s_1^{'} = \frac{k_1}{k}$ and $s_2^{'} = \frac{k_2}{k}$ where $k \geq 1$ and $gcd(k_i, k) = 1$ for $i = 1, 2$. Since $\lfloor \frac{q_i}{p_i} \rfloor < s_i$ we have $s_i^{'} \geq \lfloor \frac{q_i}{p_i} \rfloor \geq 0$ for each $i = 1, 2$ and hence $k_i \geq 0$. That is true because of the reason that by Lemma 3.15 of [15] if $s_i^{'} < \lfloor \frac{q_i}{p_i} \rfloor$, then $s_i^{'} = \infty$ which contradicts to Theorem 1.4 of [24]. Cut M open along $\mathcal{A} \cup \partial T_1 \cup \partial T_2$ and round the edges. So, we get a convex torus isotopic to the boundary of T_3 with two dividing curves and by using the Edge Rounding Lemma we can calculate the slope $s_3 = -\frac{k_1+k_2+1}{k}$ when measured in the coordinates of ∂T_3. By applying the inverse of the attaching map A_3 we get the slope of the boundary of the abstract solid torus as $n_3 = -\frac{kq_3+(k_1+k_2+1)p_3}{kv_3+(k_1+k_2+1)}$, but this quantity is less than $\frac{q_3}{p_3}$. So, by Theorem 4.16 of [15], we can isotope ∂T^3 so that it has two dividing curves and having the slope $-\frac{q_3}{p_3}$ But when measured in coordinates of ∂T_3 this slope is identically 0. Thus the maximal twisting number of a Legendrian vertical circle is -1.

So, there exists an isotopic copy of ξ which allows us to find a Legendrian circle L in the interior of $\Sigma \times S^1$ having twisting number -1, and consider each T_i as standard neighborhood of a Legendrian circle L_i isotopic to F_i and with twisting number less than or equal to 2. There is no ambiguity to assume that ∂T_i has Legendrian slope ∞ when measured as the standard coordinates of ∂T_i as mentioned before. Let L_i be the Legendrian ruling of ∂T_i. Then choose a convex vertical annulus $\mathcal{A}_i \in \Sigma \times S^1$ for each $i = 1, 2, 3$ having the property that $\partial \mathcal{A}_i = L \cup L_i$. So, the interior of each $\mathcal{A}_i$ is contained in the interior of $\Sigma \times S^1$, and $\mathcal{A}_i \cap \mathcal{A}_j = L$ for any $i \neq j$. Since L is maximally twisting, there is no boundary parallel arc on the L side in each $\mathcal{A}_i$. So, the dividing set of $\mathcal{A}$ contains only two curves connecting L to L_i and some boundary parallel curves on the L_i side. By adding bypasses along these boundary parallel curves on L_i sides, we isotope the ∂T_i to get the following convex decomposition:

$$M = M(e_0; -\frac{q_1}{p_1}, -\frac{q_2}{p_2}, -\frac{q_3}{p_3}) \cong (\Sigma \times S^1) \cup_{A_1 \cup A_2 \cup A_3} (T_1 \cup T_2 \cup T_3).$$

Here each ∂T_i is minimal convex with the slope $\lfloor \frac{q_i}{p_i} \rfloor$ when measured in the coordinates of T_i. when measured in the coordinates of $-\partial(M \setminus T_i)$ this slopes corresponds to $-\dfrac{q_i - \lfloor \frac{q_i}{p_i} \rfloor p_i}{v_i - \lfloor \frac{q_i}{p_i} \rfloor u_i} = -\dfrac{q_i + (^i_0 + 1)p_i}{v_i + (a^i_0 u_i)}$. By the 4-th part of Lemma 2.1 of [23], there are exactly $2 + \lfloor \frac{q_i}{p_i} \rfloor + \lfloor \frac{q_i}{p_i} \rfloor + \lfloor \frac{q_i}{p_i} \rfloor = |e_0(M) + 1|$ tight contact structures on $\Sigma \times S^1$ satisfying the boundary conditions and admitting no Legendrian vertical circle having twisting number 0. Also, by Theorem 1.6.4 of [15] and Lemma 2.1 of [23], there are exactly $| \prod_{j=1}^{m_i} (a_j^{(i)} + 1)$ tight contact structures on T_i satisfying the boundary conditions. Hence, up to isotopy, there are exactly $|(e_0(M) + 1) \prod_{i=1}^{3} \prod_{j=1}^{m_i} (a_j^{(i)} + 1)|$ tight contact structures on $M(e_0; -\frac{q_1}{p_1}, -\frac{q_2}{p_2}, -\frac{q_3}{p_3})$ that we have already been constructed. Moreover, all of them are holomorphically fillable, i.e., they can be seen as the boundary of some Stein 4-manifolds.

Case 2: $e_0 > 0$

Let $M(e_0; \frac{q_1}{p_1}, \frac{q_2}{p_2}, e_0 + \frac{q_3}{p_3})$ be a small Seifert fibered space, where $\frac{q_i}{p_i} \in (0, 1)$ are Seiefert invariants. Since $\lfloor \frac{q_i}{p_i} \rfloor = 0$ for all i, the Euler number of this manifold $e_0 = \lfloor \frac{q_1}{p_1} \rfloor + \lfloor \frac{q_2}{p_2} \rfloor + \lfloor \frac{q_3}{p_3} \rfloor + e_0$. Assume also that e_0 is positive. Then standard surgery diagram for this manifold is as in Figure 5.4.

After performing slam-dunk between the 0-framed component and $-\frac{1}{e_0 + \frac{q_3}{p_3}}$-framed component, we eliminate the $-\frac{1}{e_0 + \frac{q_3}{p_3}}$-framed component and the final framing of the 0- framed component becomes $e_0 + \frac{q_3}{p_3}$, see Figure 5.5.

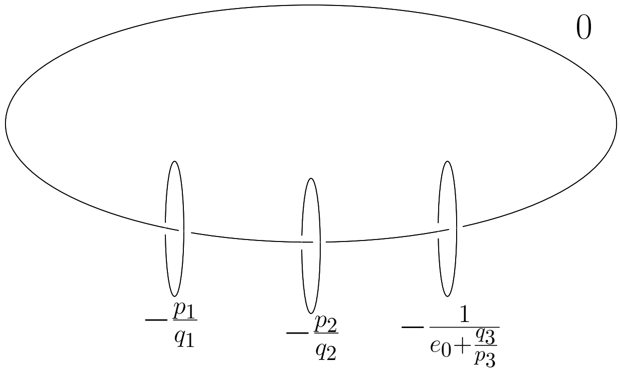

Figure 5.4: Standard surgery diagram for $M(e_0; \frac{q_1}{p_1}, \frac{q_2}{p_2}, e_0 + \frac{q_3}{p_3})$.

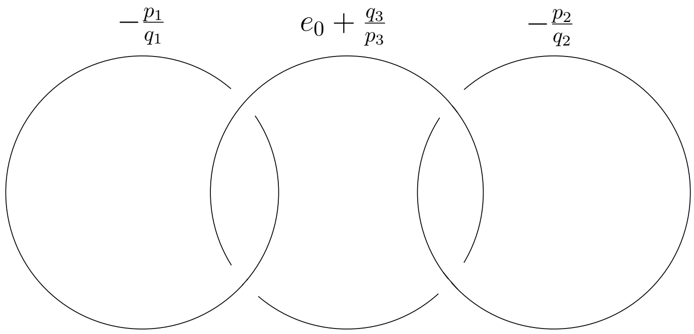

Figure 5.5: After slum-dunk between 0-component and $-\frac{1}{e_0+\frac{q_3}{p_3}}$-component.

Then perform -1-Rolfsen twist to the component having the framing $e_0 + \frac{q_3}{p_3}$. After doing so the new coefficients become $-\frac{p_1}{q_1} - 1$, $-\frac{p_2}{q_2}$ and $-\frac{q_3+e_0p_3}{q_3+(e_0-1)p_3}$ (see Figure 5.6).

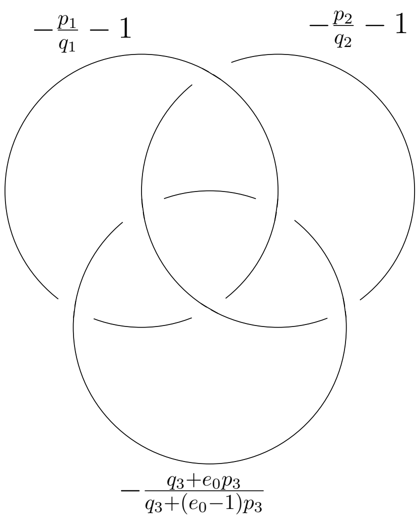

Figure 5.6: After -1-Rolfsen twist to $e_0 + \frac{q_3}{p_3}$-component.

On the other hand, we have the following continued fraction expansions:

$$-\frac{p_1}{q_1}-1=[a_0^{(1)}-1,a_1^{(1)},...,a_{l_1}^{(1)}],$$
$$-\frac{p_2}{q_2}-1=[a_0^{(2)}-1,a_1^{(2)},...,a_{l_2}^{(2)}],$$
$$-\frac{q_3+e_0p_3}{q_3+(e_0-1)p_3}=[-2,...,-2,a_0^{(3)}-1,a_1^{(3)},...,a_{l_3}^{(3)}].$$

Here, on the right hand side of the third equality we have e_0 many -2's just before the term $a_0^{(3)}-1$. After performing some blow-ups we get the diagram in Figure 5.7.

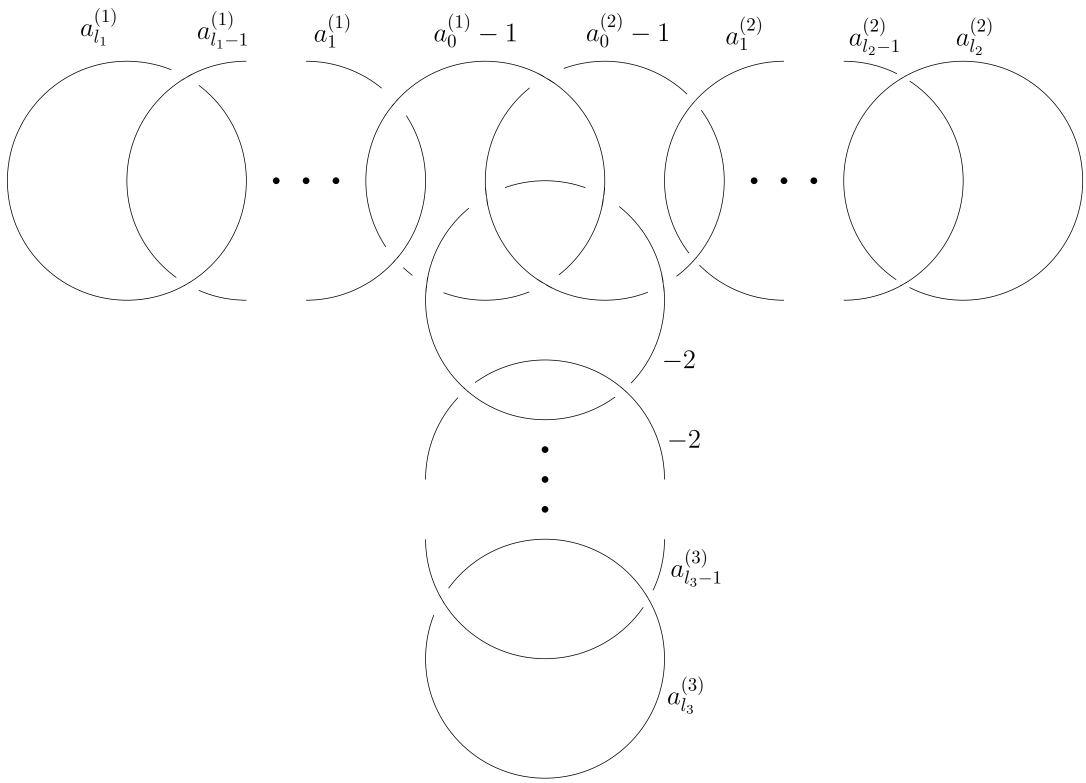

Figure 5.7: After reversed Rolfsen twists.

There are $|\prod_{i=1}^{3}a_0^{(i)}\prod_{j=1}^{l_i}(a_j^{(i)}+1)|$ many ways to Legendrian realize this diagram and all of which gives non-isotopic and holomorphically fillable tight contact structures. Hence on the manifold $M(e_0;\frac{q_1}{p_1},\frac{q_2}{p_2},e_0+\frac{q_3}{p_3})$ there are at least $|\prod_{i=1}^{3}a_0^{(i)}\prod_{j=1}^{l_i}(a_j^{(i)}+1)|$ non-isotopic tight contact structures.

Now it is time to show that the upper bound for the number of tight contact structures on $M(e_0;\frac{q_1}{p_1},\frac{q_2}{p_2},e_0+\frac{q_3}{p_3})$ is exactly $|\prod_{i=1}^{3}a_0^{(i)}\prod_{j=1}^{l_i}(a_j^{(i)}+1)|$. To do so, we use the idea in the first case. Let us begin with defining $\{p_j^{(i)}\}$ and $\{q_j^{(i)}\}$ as below to define the attaching maps.

$$\begin{cases} p_j^{(i)} = -a_j^{(i)} p_{j-1}^{(i)} - p_{j-2}^{(i)}, j = 0, 1, ..., l_i \\ p_{-2}^{(i)} = 0, p_{-1}^{(i)} = 1, \end{cases}$$

$$\begin{cases} q_j^{(i)} = -a_j^{(i)} q_{j-1}^{(i)} - q_{j-2}^{(i)}, j = 0, 1, ..., l_i \\ q_{-2}^{(i)} = -1, q_{-1}^{(i)} = 0. \end{cases}$$

From the equalities above we deduce that $p_i = p_{l_i-1}^{(i)}$ and $q_i = q_{l_i}^{(i)}$. If we choose $u_i = -p_{l_i}^{(i)}$ and $v_i = -q_{l_i-1}^{(i)}$ then $p_i v_i - q_i u_i = 1$. Now we can define orientation preserving diffeomorphisms $A_i : -\partial T_i \to \partial(M \setminus V_i)$ as:

$$A_i = \begin{cases} \begin{pmatrix} p_i & -u_i \\ -q_i & v_i \end{pmatrix}, i = 1, 2 \\ \begin{pmatrix} p_3 & -u_3 \\ -q_3 - e_0 p_3 & v_3 + e_0 u_3 \end{pmatrix}, i = 3. \end{cases}$$

Hence, our 3-manifold M can be seen as

$$M = M(e_0; \frac{q_1}{p_1}, \frac{q_2}{p_2}, e_0 + \frac{q_3}{p_3}) \cong (\Sigma \times S^1) \cup_{A_1 \cup A_2 \cup A_3} (T_1 \cup T_2 \cup T_3).$$

Now let ξ be a tight contact structure on M. We know from Theorem 1.3 in [24] that every tight contact structure ξ on $M(e_0; \frac{q_1}{p_1}, \frac{q_2}{p_2}, e_0 + \frac{q_3}{p_3})$ admits a Legendrian vertical circle L with 0 twisting. We start with isotoping ξ so that it contains a Legendrian circle L with 0 twisting in the interior of $\Sigma \times S^1$ and tubular neighborhoods V_i of each singular fiber F_i is isotopic to standard neighborhood T_i of Legendrian circles L_i with twisting number $n_i < 0$, i.e., each ∂T_i is convex with two dividing curves having the slope $-\frac{1}{n_i} < 0$ when measured in the coordinates of ∂T_i. As before let s_i be the slope of dividing curves of $-\partial(M \setminus V_i)$. Then by using the gluing maps A_i's we get s_i's as follows:

$$s_i = \begin{cases} \dfrac{-n_i q_i + v_i}{n_i p_i - u_i} = \dfrac{-q_i}{p_i} + \dfrac{1}{p_i(n_i p_i - u_i)}, i = 1, 2 \\ \dfrac{-n_3(q_3 + e_0 + p_3) + (v_3 + e_0 u_3)}{n_3 p_i - u_3} = -e_0 - \dfrac{q_i}{p_i} + \dfrac{1}{p_i(n_i p_i - u_i)}, i = 3. \end{cases}$$

Choose n_i's in a way that $n_i \ll -1$ which makes $\frac{1}{b_0^{(i)}+1} < s_i < -\frac{q_i}{p_i}$ for $i = 1, 2$ and

$e_0 - \frac{1}{b_0^{(3)}+1} < s_3 < -e_0 - \frac{q_3}{p_3}$. The Legendrian vertical circle L, which is mentioned as above, allows us to find a thickening $V_i^{'}$ of each V_i such that $V_i^{'}$'s are pairwise disjoint, and $-\partial(M \setminus V_i^{'})$ is minimally convex, i.e., it has only two vertical dividing curves when measured in the coordinates of $-\partial(M \setminus V_i^{'})$. It directly follows from the Bypass Attachment Lemma that we can find a minimal convex torus between V_i and $V_i^{'}$, of course which is contained in the interior of $V_i^{'} \setminus V_i$, which is isotopic to $-\partial(M \setminus V_i)$ that has dividing curve having the slope $\frac{1}{1_0^{(i)}+1}$ for $i = 1, 2$ and $-e_0 + \frac{1}{1_0^{(3)+1}}$ for $i = 3$. Let $V_i^{''}$ be the solid torus bounded by that minimally convex torus and $\Sigma^{''} \times S^1 = M \setminus (V_1^{''} \cup V_2^{''} \cup V_3^{''})$. We start with the $V_i^{''}$'s. In the coordinates of $-\partial(M \setminus V_i^{''})$ the dividing curves of the boundary torus has slope $\frac{(a_0^{(i)}+1)q_i+i}{(a_0^{(i)}+1)v_i+u_i}$. But by the definitions of u_i and v_i we have $\frac{(a_0^{(i)}+1)q_i+i}{(a_0^{(i)}+1)v_i+u_i} = [a_{l_i}^{(i)}, a_{l_i-1}^{(i)}, ..., a_2^{(i)}, a_1^{(i)} + 1]$. But from Honda's result in [15] we know that up to isotopy there are exactly $|\prod_{j=1}^{l_i}(a_j^{(i)} + 1)|$ tight contact structures on each $V_i^{''}$ satisfying the given boundary condition. To finish the proof we need Lemma 4.1 of [23] which says that if the three boundary component of $\Sigma \times S^1$ has dividing curves of slopes $-1, -1$ and $-n$, then $\Sigma \times S^1$ can be factorized as $L_1 \cup L_2 \cup L_3 \cup (\Sigma^{'} \times S^1)$ where L_i's are embedded thickened tori with minimal twisting and the orientation of this solid tori determine the contact structure uniquely.

Now it is time to count tight contact structures on $\Sigma^{''} \times S^1$ satisfying the boundary condition. Consider the thickened torus L_i bounded by the $\partial V_i^{'} - \partial V_i^{''}$ which has a continued fraction block which consists of $|a_0^{(i)}|$ basic slices. Let $L_i^{'}$ be the basic slice which is closest to ∂V_i and $\partial L_i^{'} = \partial V_i^{'} - \partial V_i^{'''}$, where $\partial V_i^{'''}$ is a minimal convex torus with dividing curves of slope -1 for $i = 1, 2$ and $-e_0 - 1$ for $i = 3$. Now let us consider $\Sigma^{'} \times S^1 = M \setminus (V_1^{'} \cup V_2^{'} \cup V_3^{'})$. By the above lemma, the tight contact structure on $(\Sigma^{'} \times S^1) \cup L_1^{'} \cup L_2^{'} \cup L_3^{'}$ is uniquely determined by the signs of the basic slices $L_i^{'}$. Since we are allowed to shuffle the signs of basic slices, let's do it in a way that all the basic slices closest to $\partial V_i^{'}$'s have positive signs. Then the sign of $L_i^{'}$ is uniquely determined by the number of positive basic slices in L_i, and so is the number of positive slices in $L_i \setminus L_i^{'}$. So, the tight contact structure on $(\Sigma^{'} \times S^1) \cup L_1^{'} \cup L_2^{'} \cup L_3^{'}$ and $L_i \setminus L_i^{'}$ is determined by this three numbers. However, there are $|a_0^{(1)} a_0^{(2)} a_0^{(3)}|$ many ways to choose these numbers. Hence, there are at most $|a_0^{(1)} a_0^{(2)} a_0^{(3)}|$ tight contact structures on $\Sigma^{''} \times S^1$ satisfying the given boundary condition. All together we have at most $|\prod_{i=1}^{3} a_0 \prod_{j=1}^{l_i}(a_j^{(i)} + 1)|$ tight contact structures on M. Since we have constructed $|\prod_{i=1}^{3} a_0 \prod_{j=1}^{l_i}(a_j^{(i)}+1)|$ non-isotopic tight contact structures M has exactly $|\prod_{i=1}^{3} a_0 \prod_{j=1}^{l_i}(a_j^{(i)}+1)|$ tight contact structures up to isotopy.

In [9] Ghiggini, Lisca and Stipsitcz extended the case to $e_0 \geq 0$ by using handle body decompositions which is not included in this thesis.

Case 3: $e_0 = -1$ **or** $e_0 = -2$

For the cases $e_0 = -1$ and $e_0 = -2$, there is no complete classification but some partial results. For instance, in [11] Ghiggini and Van Horn-Morris showed that on the Brieskorn homology sphere $-\Sigma(2, 3, 6n-1)$ for $n \geq 1$, which has Euler number $e_0 = -2$, there are $\frac{n(n-1)}{2}$ non-isotopic tight contact structures, by using the idea that the first author used in [10]. Also, in [20], Tosun showed that the family $M = (-2; \frac{1}{2}, \frac{2}{3}, \frac{5n+1}{6n+1})$ for $n \geq 1$ has exactly $\frac{n(n+1)}{2}$ strongly fillable non-isotopic tight contact structures at least n of which are Stein fillable and at least $\lfloor \frac{n}{2} \rfloor$ of them are not Stein fillable. In that paper, he also counted the number of tight contact structures on small Seifert fibered manifold with $e_0 = -2$, whose Seifert invariants satisfy some condition, and constructed them. For the case $e_0 = -1$, in [10] Ghiggini and Schönenberger showed that the exact number of tight contact structures on both Brieskorn homology spheres $\pm\Sigma(2, 3, 11)$. Here $\pm$ indicates the orientation of the homology sphere, and one of which corresponds to the small Seifert fibered manifold $M(-2; \frac{1}{2}, -\frac{1}{3}, -\frac{2}{11})$ which has $e_0 = -2$ and the other corresponds to the manifold $M(-1; -\frac{1}{2}, \frac{1}{3}, \frac{2}{11})$ which has $e_0 = -1$. By using a similar idea as Ghiggini and Schönenberger did in [10], Tosun and Mark [19] showed the following:

Theorem 5.0.1. *The Brieskorn homology sphere* $\Sigma(2, 3, 6n+1)$ *has exactly two tight contact structures for any* $n \geq 1$*, all of which are Stein fillable.*

Similar to the first two cases, we start with the basic surgery description of $\Sigma(2, 3, 6n+1)$. To find the Seifert invariants we begin with solving the equation $3(6n+1)b_1 + 2(6n+1)b_3 + 6b_3 = 1$ for the integers b_1, b_2 and b_3. To make it simple let us take $b_1 = 1, b_2 = -1$ and $b_3 = -n$. Then the basic surgery description for $\Sigma(2, 3, 6n+1)$ is as given in the left hand side of Figure 5.8.

After performing -1 Rolfsen twist to the component with framing 2 we get the new framing of this component as -2 and 0-framed unknot will now gets -1. After three

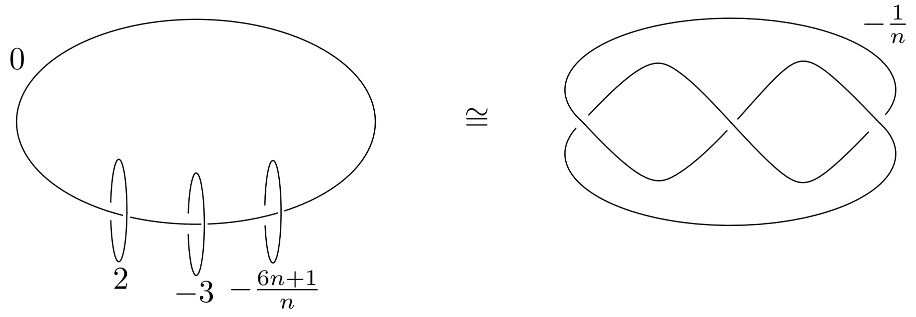

Figure 5.8: Surgery descriptions of $\Sigma(2, 3, 6n + 1)$.

handleslides we end up with a $-\frac{1}{n}$ surgery on the right trefoil, see the right hand side of Figure 5.8. To construct the contact structures, we Legendrian realize the right trefoil, but unlike the previous cases we end up with rational framing which is $-\frac{n+1}{n}$. We know from the work of Ding, Geiges and Stipsicz in [2] that it is possible to describe a rational surgery as a ± 1 contact surgery along a Legendrian link. There are two possible stabilizations of the given surgery description which are shown in the right hand side of Figure 5.9, and they give non-isotopic Stein fillable contact structures, since Legendrian links have different rotation numbers.

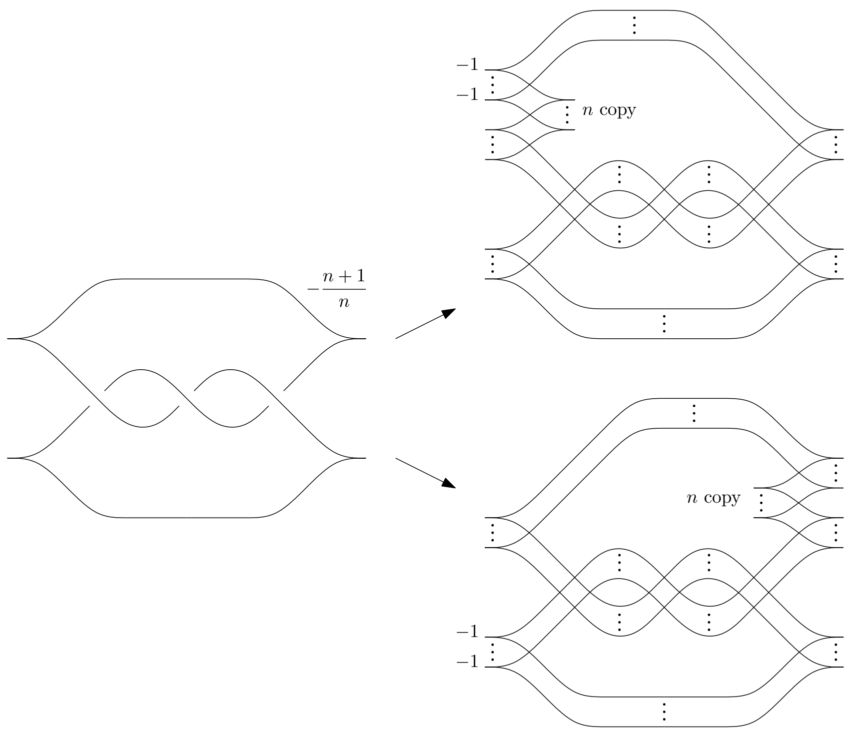

Figure 5.9: Non-isotopic tight contact structures on $\Sigma(2, 3, 6n + 1)$.

Now we need to show that the upper bound for the number of tight contact structures is exactly two to finish the proof. Let us denote the tubular neighborhoods of each singular fiber F_i as V_i for $i = 1, 2$ and 3. Then we make the same identification as in the previous cases to fix a convention, i.e., we identify $\partial V_i = \mathbb{R}^2 \setminus \mathbb{Z}^2$ where $(1, 0)^T$ is the direction of meridian. Since $M \setminus (V_1 \cup V_2 \cup V_3) \cong \Sigma \times S^1$, we fix the identification for $-\partial(M \setminus V_i) \cong \mathbb{R}^2 \setminus \mathbb{Z}^2$ such that $(0, 1)^T$ represents the S^1 fiber direction and $(1, 0)^T$ represents the $-(\{pt\} \times \partial\Sigma)$ direction. By doing so, we can see M as the following union $M = (\Sigma \times S^1) \cup_{A_1 \cup A_2 \cup A_3} (T_1 \cup T_2 \cup T_3)$ where T_i's are the standard solid torus. Where the attaching maps $A_i : \partial T_i \to -\partial(M \setminus V_i)$ defined by

$$A_1 = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix}, A_3 = \begin{pmatrix} 6n+1 & 6n-5 \\ -n & -n+1 \end{pmatrix}.$$

Let ξ be a tight contact structure on M. Using Giroux's flexibility theorem, we can assume each F_i Legendrian, and consider each T_i as a standard solid torus with boundary slopes $\frac{1}{n_i}$ where $n_i < 0$, and the number of dividing curves is exactly two. From the convention mentioned above this slope $\frac{1}{n_i}$ corresponds to the vector $(n_i, 1)^T$. We can assume that the slope of ruling curves of $-\partial(M \setminus V_i)$ to be infinity and we therefore call this curve vertical. Moreover, the slopes n_i correspond to the slopes s_i (measured in the coordinates of $-\partial(M \setminus V_i)$) as follows:

$$s_1 = \frac{n_1}{2n_1 - 1}, s_2 = -\frac{n_2}{3n_2 + 1}, s_3 = -\frac{nn_3 + n - 1}{(6n+1)n_3 + 6n - 5}.$$

Now, by thickening each V_i and by finding enough number of bypass disks, we show that the numbers n_i can be increase up to $n_1 = n_2 = -2$ and $n_3 = 0$. To do so, we consider the vertical annulus $\mathcal{A}$ between V_1 and V_2, which is assumed to be convex. Here, there are two cases to be analyzed based on the slope of tori that $\mathcal{A}$ connects and the configuration of the set of dividing curves of $\mathcal{A}$ (see Figure 5.10). Note that the number of end points of $\Gamma_{\mathcal{A}}$ on the boundary is exactly equal to the number of intersections with the set of dividing curves $\Gamma_{-\partial(M \setminus V_i)}$ of the relevant boundary component V_i. Since $\mathcal{A}$ is vertical and each of the boundary component has exactly two dividing curves, this number of intersection is equal to twice the denominator of the corresponding slope s_i.

Case 1: If $2n_1 - 1 \neq 3n_2 + 1$, then by imbalance principle there necessarily exists a bypass on one side. This bypass my occur either on V_1 side or V_2 side, which allows

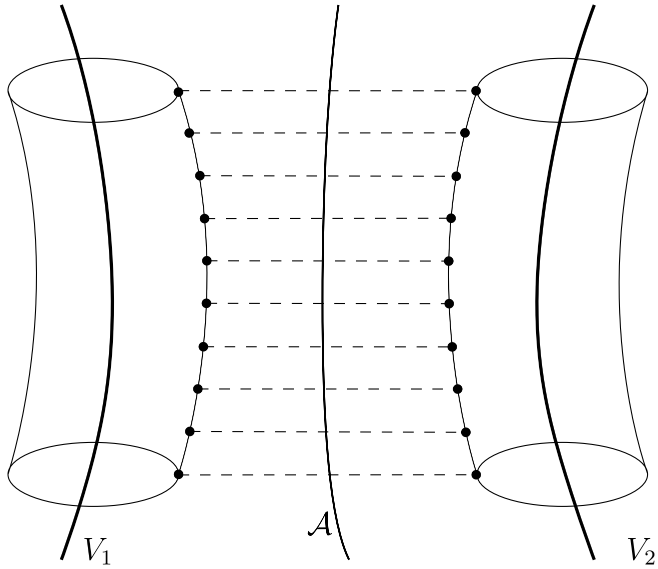

Figure 5.10: The dividing curves (dashed lines) configuration of the annulus $\mathcal{A}$.

us to increase either n_1 or n_2. On the other hand, since the ruling slopes on ∂V_1 and ∂V_2 are 2 and -3 respectively, Twisting Number Lemma allows us to increase n_1 and n_2 to 0 and -1 respectively (as long as we remain in the case 1).

Case 2: If $2n_1 - 1 = 3n_2 + 1$, and the set of dividing curves of $\mathcal{A}$ has no boundary parallel arcs, then the dividing curves of $\mathcal{A}$ run across from $-\partial(M \backslash V_1)$ to $-\partial(M \backslash V_2)$. We cut $\mathcal{A}$ and round the corners to get a smooth manifold $M \setminus (V_1 \cup V_2 \cup \mathcal{A})$ such that $\partial(M \setminus (V_1 \cup V_2 \cup \mathcal{A}))$ is smoothly isotopic to $\partial(M \setminus V_3)$ (see Figure 5.11). By Edge Rounding Lemma, we compute the slope of the dividing curves of it as

$$s(\Gamma_{\partial(M \backslash V_1 \cup V_2 \cup \mathcal{A})}) = \frac{n_1}{2n_1 - 1} - \frac{n_2}{3n_2 + 1} - \frac{1}{2n_1 - 1} = \frac{n_1 - 1}{6n_1 - 3}.$$

Then the corresponding slope on the ∂T_3 can be calculated by first reversing the sign and applying the inverse of the gluing map A_3.

We have $\partial(\Gamma_{\partial T_3}) = A_3^{-1}(6n_1 - 3, -n_1 + 1)^T = \dfrac{-n_1 + 3n + 1}{n_1 - 3n + 2}$. On the other hand, this quantity is less then -1 for all $n_1 \leq 0$ and $n \geq 1$. So, for any negative n_1 we can find a convex neighborhood $V_3' \subset V_3$ of the singular fiber F_3 such that $\Gamma_{\partial V_3'} = -1$. When measured in the coordinates of $-\partial(M \setminus V_3')$ this slope become $-\frac{1}{6}$ which corresponds to $n_3 = -1$. Now take a vertical annulus between V_1 and V_3' and compare the slopes of denominators. Note that $|2n_1 - 1| > 6$ as long as $n_1 < -2$. Therefore, by using imbalance principle we can find a bypass disk on V_1 side, and by Twisting Number

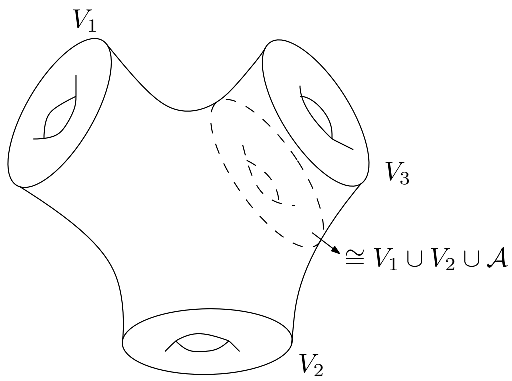

Figure 5.11: The isotopy between $\partial(M \setminus (V_1 \cup V_2 \cup \mathcal{A}))$ and $\partial(M \setminus V_3)$.

Lemma we can continue to do the same process till $n_1 = -2$. In a similar way, we can show that it is possible to increase n_2 to -2. So, the corresponding slopes, when measured in the coordinates of $-\partial(M \setminus V_i)$, becomes $s_1 = \frac{2}{5}$, $s_2 = -\frac{2}{5}$ and $s_3 = -\frac{1}{6}$.

Our claim is that the vertical annulus $\mathcal{A}$ between V_1 and V_2 has no boundary parallel arcs in its dividing set if $n_1 = n_2 = -2$ and if the contact structure under consideration is tight. To show this, assume that there exists a bypass disk on either V_1 side or the other. Since each boundary component of $\mathcal{A}$ has the same number of end points of $\Gamma_{\mathcal{A}}$, there must be a bypass on each side. After attaching this bypass disks to each V_i for $i = 1, 2$ resulting thickened convex neighborhoods, which we denote with the same symbols, have slopes $s_1 = s(\Gamma_{-\partial(M \setminus V_1)}) = \frac{1}{3}$ and $s_2 = s(\Gamma_{-\partial(M \setminus V_2)}) = -\frac{1}{2}$. Now, again by Imbalance Principle, there must be a bypass disk on V_1 side since the denominator of $s_1 > s_2$. Attach this bypass disk to find a further thickening of V_1 with the slope $s_1 = 0$. But this time the denominator of s_2 is larger, so there must exists a bypass on V_2 side. After attaching this we find a thickening of V_2 with slope $s_2 = -1$. At this point denominators agree again. In this case, there is no further bypass or there exists bypass on both sides. In the latter case we can increase s_1 up to ∞, and also s_2 to ∞. We end up with a vertical curve on $-\partial(M \setminus V_1)$ with zero twisting. But it contradicts to the fact in Lemma 4.11 of [10] that the maximal twisting number of any tight contact structure on Seifert manifold $M(-\frac{1}{2}, \frac{1}{3}, r)$ is negative for any $r \leq \frac{1}{5}$. On the other hand, if the vertical annulus $\mathcal{A}$ has no boundary parallel arc if $s_1 = 0$ and $s_2 = -1$. Then we can cut the manifold along the vertical annulus $\mathcal{A}$ and round edges to get a torus with slope 0. When measured in the coordinates of $\partial(V_3)$ this slope becomes $-\frac{n}{n-1}$. But this number is less than $-\frac{6n+1}{6n-5}$ for any $n \geq 1$.

Hence there exists a torus V_3' in V_3 whose boundary is convex and has slope $-\frac{6n+1}{6n-5}$ (when measured in the coordinates of $-\partial(M \setminus V_3')$ it corresponds to ∞) which yields the same contradiction that vertical Legendrian curve with twisting number 0.

So, the vertical annulus $\mathcal{A}$ cannot have boundary parallel arcs in its dividing set. Without loss of generality we can assume that $\Gamma_{\mathcal{A}}$ consists of horizontal arcs. Again, we cut our manifold M along $\mathcal{A}$ and round the edges. As mentioned before $\partial(M \setminus (V_1 \cup V_2 \cup \mathcal{A}))$ is isotopic to $\partial(M \setminus V_3)$ and by using Edge Rounding Lemma it has slope $\frac{2}{5} - \frac{2}{5} - \frac{1}{5} = -\frac{1}{5}$. Note that the slopes $s_1 = \frac{2}{5}$ and $s_2 = -\frac{2}{5}$ corresponds to slopes $\frac{1}{n_1} = -\frac{1}{2}$ and $\frac{1}{n_2} = -\frac{1}{2}$ respectively. In other words, V_1 and V_2 are the standard tubular neighborhoods of the singular fibers F_i's for $i = 1, 2$, and hence each carries unique tight contact structure. On the other hand, the slope $s_3 = -\frac{1}{5}$ corresponds (in the coordinates of ∂V_3) to $-\frac{n+1}{n}$ which has continued fraction $[-2, ..., -2]$ (the number of -2's are exactly n), and by Theorem 2.3 of [15] we know that the solid torus satisfying this boundary conditions admits exactly two tight contact structures, which implies that our manifold M carries at most two tight contact structures. Since we have constructed two non-isotopic tight contact structures on M, M has exactly two non-isotopic tight contact structures.

REFERENCES

[1] V. Colin. Sur la torsion des structures de contact tendues. In *Annales Scientifiques de l'École Normale Supérieure*, volume 34, pages 267–286. No longer published by Elsevier, 2001.

[2] F. Ding, H. Geiges, and A. I. Stipsicz. Surgery diagrams for contact 3-manifolds. *Turkish Journal of Mathematics*, 28(1):41–74, 2004.

[3] A. Douady. Noeuds et structures de contact en dimension 3. *Séminaire Bourbaki*, 25:129–148, 1982.

[4] Y. Eliashberg. Classification of overtwisted contact structures on 3-manifolds. *Inventiones Mathematicae*, 98(3):623–637, 1989.

[5] Y. Eliashberg. Filling by holomorphic discs and its applications. *Geometry of low-dimensional manifolds*, 2:45–67, 1990.

[6] Y. Eliashberg. Topological characterization of Stein manifolds of dimension > 2. *International Journal of Mathematics*, 1(01):29–46, 1990.

[7] J. B. Etnyre and K. Honda. On the nonexistence of tight contact structures. *Annals of Mathematics*, pages 749–766, 2001.

[8] H. Geiges. *An introduction to contact topology*, volume 109. Cambridge University Press, 2008.

[9] P. Ghiggini, P. Lisca, and A. I. Stipsicz. Classification of tight contact structures on small Seifert 3-manifolds with $e_0 \geq 0$. *Proceedings of the American Mathematical Society*, pages 909–916, 2006.

[10] P. Ghiggini and S. Schönenberger. On the classification of tight contact structures. In *Topology and geometry of manifolds (Athens, GA, 2001)*, volume 71 of *Proc. Sympos. Pure Math.*, pages 121–151. Amer. Math. Soc., Providence, RI, 2003.

[11] P. Ghiggini and J. Van Horn-Morris. Tight contact structures on the Brieskorn spheres $-\Sigma(2, 3, 6n-1)$ and contact invariants. *J. Reine Angew. Math.*, 718:1–24, 2016.

[12] E. Giroux. Convexité en topologie de contact. *Commentarii Mathematici Helvetici*, 66(1):637–677, 1991.

[13] R. E. Gompf. Handlebody construction of Stein surfaces. *Annals of Mathematics*, pages 619–693, 1998.

[14] M. Gromov. Pseudo holomorphic curves in symplectic manifolds. *Inventiones Mathematicae*, 82(2):307–347, 1985.

[15] K. Honda. On the classification of tight contact structures. I. *Geom. Topol*, 4:309–368, 2000.

[16] M. Jankins and W. D. Neumann. *Lectures on Seifert manifolds*, volume 2. Brandeis University, 1983.

[17] W. R. Lickorish. A representation of orientable combinatorial 3-manifolds. *Annals of Mathematics*, pages 531–540, 1962.

[18] P. Lisca and G. Matić. Tight contact structures and Seiberg-Witten invariants. *Inventiones Mathematicae*, 129(3):509–525, 1997.

[19] T. E. Mark and B. Tosun. Obstructing pseudoconvex embeddings and contractible Stein fillings for Brieskorn spheres. *arXiv preprint arXiv:1603.07710*, 2016.

[20] B. Tosun. Tight small Seifert fibered manifolds with $e_0 = -2$. *arXiv preprint arXiv:1510.06948*, 2015.

[21] A. H. Wallace. Modifications and cobounding manifolds. *Canad. J. Math*, 12:503–528, 1960.

[22] A. Weinstein et al. Contact surgery and symplectic handlebodies. *Hokkaido Mathematical Journal*, 20(2):241–251, 1991.

[23] H. Wu. Tight contact small Seifert spaces with $e_0 \neq 0, -1, -2$. *arXiv preprint math.GT/0402167*, 2004.

[24] H. Wu. Legendrian vertical circles in small Seifert spaces. *Communications in Contemporary Mathematics*, 8(02):219–246, 2006.