CAUSALITY IN CURVED SPACE

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ABSTRACT

CAUSALITY IN CURVED SPACE

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The notion of Causality is a key concept in formulating theories that describe real physical systems. So when a proposed model violates this concept this usually signals the existence of an error in the conceptual formulation or in the physical understanding of the results, which might indicate to us which models should we pursue, modify or drop all together. In this work the possibility of causality violations introduced by modifying Einstein gravity is studied using the idea of the time-delay introduced to a particle while crossing the shock-wave generated by another particle, known as Shapiro-like time delay. We will see that some of the apparent causality violations are just a mathematical artifact of our formulations. While in other cases these violations can be lifted off if the theory is embedded in another theory with a better behaved UV completion. Is it also noticed that when we study local causality in (2+1) General Relativity the notions of causality and unitarity are compatible with each other.

Keywords: General Relativity, Local Causality, Shapiro time delay, shock-wave solutions.

ÖZ

BÜKÜLÜM UZAYDA NEDENSELİK

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Nedensellik ilkesi, reel fiziksel sistemleri açıklayan teorilerin formülleştirilmesini sağlayan anahtar bir kavramdır. Bu sebep ile, önerilen bir model bu kavramı ihlal ettiğinde, bu durum genellikle kavramsal formüllemede veya sonuçların fizik açısından yorumlanmasında bir hata olduğuna işaret eder ki böylece bize hangi modeller ile yola devam etmemiz, hangi modellerde değişiklik yapmamız ya da hangi modelleri tamamen bırakmamız gerektiğini gösterir. Bu çalışmada, Einstein'ın gravitasyon teorisinin değiştirilmesi ile ileri sürülen nedensellik ihlalleri ihtimali, -Shapiro zaman gecikmesi olarak da bilinen- bir partikülerin, başka bir partikül tarafından meydana getirilen şok dalgası ile çakıştığında ortaya çıkan zaman gecikmesi fikri kullanılarak incelenmiştir. Gördüğümüz üzere, aşikar nedensellik ihlallerinin bazıları sadece formüllerimizin matematiksel eserleridir. Diğer durumlarda ise, bu ihlaller, söz konusu teorinin daha iyi Ultraviyole tamamlama olan diğer bir teoriye entegre edilmesi ile ortadan kaldırılabilmektedir. Aynı zamanda, (2+1) Genel görelilik teorisindeki yerel nedensellik incelendiğinde, nedensellik ve bütünsellik ilkelerinin birbirleriyle uyumlu olduğu da fark edilmiştir.

Anahter Kelimeler: genel görelilik Teorisi, Yerel Nedensellik, Shapiro Zaman Gecikmesi, Şok Dalga Çözümler

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I would never be able to go down the path of scientific research without the full support of my family behind me and the scientific curiosity they implanted and encouraged in me since my early childhood. And for that I will be forever in their debt.

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CONVENTIONS AND ABBREVIATIONS

- in this work we adopt the flat metric $\eta_{\mu\nu}$ mostly positive signature (- + + +), or (- + +) in 3-dimensions.

- The Rieman curvature tensor: $R^{\sigma}{}_{\rho\mu\nu} = \left(\partial_{\mu}\Gamma^{\sigma}_{\nu\rho} + \Gamma^{\sigma}_{\mu k}\Gamma^{k}_{\nu\rho}\right) - \left(\partial_{\nu}\Gamma^{\sigma}_{\mu\rho} + \Gamma^{\sigma}_{\nu k}\Gamma^{k}_{\mu\rho}\right)$

- The Ricci tensor: $R_{\mu\nu} = R^{\sigma}_{\mu\sigma\nu} = \left(\partial_{\sigma}\Gamma^{\sigma}_{\mu\nu} + \Gamma^{\sigma}_{k\sigma}\Gamma^{k}_{\mu\nu}\right) - \left(\partial_{\nu}\Gamma^{\sigma}_{\mu\sigma} + \Gamma^{\sigma}_{k\nu}\Gamma^{k}_{\mu\sigma}\right)$

- The Ricci scalar $R = g^{\mu\nu}R_{\mu\nu}$

- The Levi-Civita Tensor: $\epsilon^{\mu\alpha\beta} = \frac{\tilde{\epsilon}^{\,\mu\alpha\beta}}{\sqrt{-g}}$ where the numbers in the numerator are the Levi-Civita symbols, with the normalization $\tilde{\epsilon}^{\,012} = +1$.

- The Cotton Tensor: $C_{\mu\nu} = \epsilon_{\mu}{}^{\alpha\beta}\nabla_{\alpha}\left(R_{\nu\beta} - \frac{1}{4}g_{\nu\beta}R\right)$

GR: General Relativity

QM: Quantum mechanics

QFT: Quantum Field Theory

TMG: Topological Massive Gravity

QG: Quantum Gravity

SEP: Strong Equivalence principle

UV: Ultraviolet (energy scale)

IR: Infrared (energy scale)

CHAPTER 1

INTRODUCTION

1.1. General Relativity and modifying gravity:

Ever since it was fully proposed in 1915, General Relativity (GR) [1] succeeded in describing and explaining the gravitational phenomena, fitting and explaining the discrepancy in observations with the previous (Newtonian) theory of gravity as in the case of Mercury perihelion, or predicting new phenomena which later was confirmed by the observations as like the effect of gravity in bending the light passing near massive objects which was first confirmed by Eddington during the solar eclipse in 1919, Or like the effect of a gravitational field on a photon climbing up that field, known as the Gravitational Red Shift effect. Other tests were proposed later that was confirmed by observations, which we will come back to later.

Though this theory might seem to offer answers to many questions about our universe, this doesn't mean that there are no troubles. Being one of the pillars in our understanding of the universe, one would hope that it would be compatible with the other pillar in the foundation of theoretical physics: Quantum Mechanics. But the two seem to be in conflict. Trying to fit them together in one mainframe has many motivations: for instance it is necessary to understand the first moments of our universe where the dimensional scale is so small and the mass density is too high for us to ignore either the quantum effects or the gravitational interactions. We run into the same situation when we try to understand the physics beyond the event horizon of a black hole. But here the troubles come in many flavors.

On one hand Einstein gravity in it's pure form with the so called Einstein-Hilbert action:

$$S = \frac{1}{\kappa}\int d^4x\sqrt{-g}\,R \tag{1.1}$$

is not renormalizable [2].

On the other hand, when we try to study the observations of an infalling observer into a black hole we see that GR assumptions (mainly the equivalence principle) and QM assumptions (mainly conservation of information) are in conflict [3] [1].

For many researchers these concerns present a strong motivation to seek a modification of Einstein's gravity presented in (1.1). More motivations come from different observations. The data from the study of the luminosity of supernova explosions [5][6] led in the late 90's to conclude that the universe is not only expanding, but it is doing so in an accelerating rate. If we assume that Einstein's gravity is correct then this can be accommodated by assuming the existence of some dark energy density $\rho \sim 10^{-29}$g/cm^3 which can be mathematically obtained by adding a constant term Λ in (1.1). But the problem is that when we evaluate this constant from the quantum field theory perspective its value is much larger. To find a way out from this discrepancy it might be tempting to assume that the form (1.1) of GR needs to be modified to allow this late stage of accelerated expansion.

But as it turned out to be, doing this modification is a tricky business that could easily introduce new problems if one was not careful. For example modifying the theory by adding higher curvature terms can solve the renormalizability problem but we would lose unitarity [7,8].

In this work we will focus on the effect of some modifications on the notion of causality: causality is one of the most cherished ones in physical theories. It can be described as the idea that the cause must precede the effect. Otherwise one can run into all kinds of inconsistencies and illogical phenomena. In flat space time, in theories where we don't take gravity into account, such as the quantum field theories, one can impose causality in a very strict sense, with the notion called microscopic causality as maintained by commutation relations or anti-commutation relations of bosonic or fermionic fields, respectively. But when gravity is switched on, we don't have the necessary and sufficient conditions that guarantee the causality. In fact, on the contrary, one can arrange the matter distribution $T_{\mu\nu}$in the universe in such a way

[1] A recent work [4] claims to settle down the issue put forward by [3]. But till the moment of writing this work there is no consensus.

that one can easily get a violation of causality. For example in July 1949, Kurt Gödel suggested a new cosmological solution to the Einstein equations [9,10] which was different than all the known solutions at that time. This solution allowed for the universe to have closed time-like curves, which potentially lead to causality violations. In this work we will not be interested in such a global causality issue which stems from the overall behavior of the solution to the Einstein equations which in itself is a deep unsolved problem. Instead we will be interested in the notion of local causality and whether our modification of the pure Einstein gravity leads to a time advance where in a theoretical scenario we could, for example, receive a signal before it is sent. Is the notion of causality in agreement or in conflict with the notion of unitarity in some of the modifications that we will discuss?. And how some of the causality violations might be solved. Until recently, it was not clear if causality would bring additional conditions beyond unitarity. Since unitarity is related to the non-existence of ghosts and tachyons which in flat spacetime would guarantee causality. But the recent work of Camanho *et.al* [11] showed that higher order corrections to gravity, albeit being unitary corrections, can cause causality violations. In fact, the problem seems to be so acute that causality can be guaranteed only in string theory with all sort of particles taking part in gravity. That and the ensuing works have been in $D > 3$ dimensions. It turns out that $D = 3$ dimensions is rather remarkable that causality and unitarity are not violated and one does not need to go to the string theory picture.

This thesis is mostly devoted to an understanding of this point which was studied in [12]. To study the causality of some of the modified theories of gravity we will rely on a thought experiment that will exploit what is known as the Shapiro-like time delay.

1.2. Shapiro time delay:

Other than the three tests of general relativity that were mentioned, a fourth test was suggested by Irwin I. Shapiro [13] in 1964. The idea simply was that if the positions of the Sun, the Earth and Venus were similar to the figure below (an exaggerated figure though) then a radar signal emitted from the Earth and bouncing back from Venus passing close to the sun will suffer a time delay (about 200 microsecond) as opposed to a signal travelling the same distance in the absence of a gravitational field. Even though this delay is small but it was measurable when the idea was proposed. The observations later confirmed this prediction which was calculated based on the Schwarzschild solution of Einstein equations. The emitted photon through its trip was under the effect of the Sun's gravitational field.

In our analysis we will employ a similar situation. Where the gravitational field is generated by a massless particle moving at the speed of light. This field is referred to as the Shock-wave.

The geometry of the shock-wave was first obtained [14] by writing the Shwarzschild solution for a particle m, then boosting this solution, then taking the limit where $m \rightarrow 0$. This solution was also studied further in [15]. And it was proven to be also a solution in the case of higher order gravity theories [16].

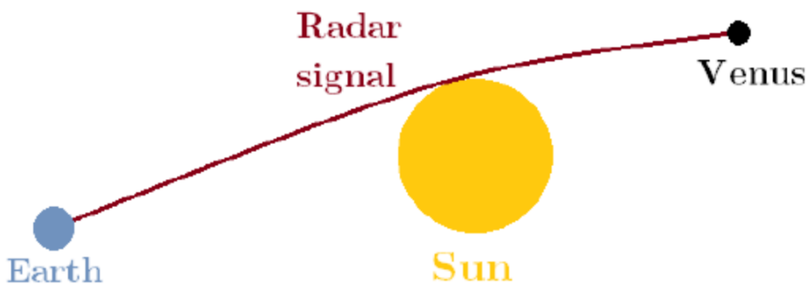

Figure 1: a figurative illustration of the experiment that Shapiro proposed as a fourth test to General relativity where a radar signal is sent to Venus and received back passing close to the sun. [17]

The layout of this work is as follows: we will study the shock-wave geometry and find the metric and use it to calculate the time delay for the case of scattering of a massless particle then a spin-1 particle (photon) with the shock-wave. Then we will use the same evaluating which cases present a possibility to a physics causality violation and whether these inconsistencies can be solved.

CHAPTER 2

THE SHOCK-WAVE METRIC

First we will find the metric corresponding to the gravitational field resulting from a point particle moving along the x-axis with momentum p. As we mentioned before this can be done by writing the Shwarzschild solution corresponding to a point particle, then boosting this solution in the x-direction, which is the approach adopted in [14]. In this chapter we will start from the flat spacetime metric then assume the form of the shock-wave metric and use the Einstein field equations to find this full form.

2.1 The D-dimensional case:

The flat spacetime metric in D dimensions is can be written as:

$$ds^2 = -dt^2 + dx^2 + \sum_{i=1}^{D-2} \left(dx^i\right)^2. \tag{2.1}$$

We can rewrite this metric in terms of the light-cone coordinates defined as:

$$u = t - x \quad , \quad v = t + x\,. \tag{2.2}$$

Then (2.1) becomes:

$$ds^2 = -du\,dv + \sum_{i=1}^{D-2} \left(dx^i\right)^2. \tag{2.3}$$

to find the shock-wave metric in these coordinates we start from the ansatz:

$$ds^2 = -du\,dv + H\left(u, x^i\right)du^2 + \sum_{i=1}^{D-2} \left(dx^i\right)^2. \tag{2.4}$$

Which becomes again the flat spacetime metric if we set the profile function $H\left(u, x^i\right)$ to be zero. We will see in a moment that the assumption about H being a function of only u and the transverse coordinates x^i is sufficient and there is no need to assume that it also v-dependent.

The components of the metric $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$ can be arranged as:

$$g_{\mu\nu}=\begin{pmatrix} H(u,x) & -\frac{1}{2} & 0 & \cdots & 0 \\ -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \qquad g^{\mu\nu}=\begin{pmatrix} 0 & -2 & 0 & \cdots & 0 \\ -2 & -4H(u,x) & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \quad (2.5)$$

The Christoffel symbols are:

$$\Gamma^{\rho}_{\mu\nu}=\frac{1}{2}g^{\rho\lambda}\left(\partial_\mu g_{\nu\lambda}+\partial_\nu g_{\mu\lambda}-\partial_\lambda g_{\mu\nu}\right) \quad (2.6)$$

The only non-zero Christoffel symbols can be computed as:

$\Gamma^{v}_{uu}=-\partial_u H \quad , \quad \Gamma^{i}_{uu}=-\frac{1}{2}\partial_i H \quad , \quad \Gamma^{v}_{iu}=\Gamma^{v}_{ui}=-\partial_i H.$

The Ricci Tensor is:

$$R_{\mu\nu}=R^{\sigma}_{\mu\sigma\nu}=\left(\partial_\sigma\Gamma^{\sigma}_{\mu\nu}+\Gamma^{\sigma}_{k\sigma}\Gamma^{k}_{\mu\nu}\right)-\left(\partial_\nu\Gamma^{\sigma}_{\mu\sigma}+\Gamma^{\sigma}_{k\nu}\Gamma^{k}_{\mu\sigma}\right) \quad (2.7)$$

The only non-zero component is R_{uu} which means that the Ricci scalar vanishes. The Einstein Tensor is:

$$G_{uu}=R_{uu}=-\frac{1}{2}\sum_{i=1}^{D-2}\partial_i^2 H(u,x). \quad (2.8)$$

now we will substitute this tensor in the Einstein field equations sourced with the energy-momentum tensor corresponding to a massless particle moving on the geodesic $u=0$. The non-vanishing component of the energy-momentum tensor can be written as:

$$T_{uu}=|p|\,\delta(u)\prod_{i=1}^{D-2}\delta(x^i)\ . \quad (2.9)$$

Then we get the equation:

$$\sum_{i=1}^{D-2}\partial_i^2 H(u,x)=-16\pi G\,|p|\,\delta(u)\prod_{i=1}^{D-2}\delta(x^i)\ .$$

Assuming: $H(u,x)=\delta(u)h(x^i)$ we get

$$\nabla_\perp^2 h\left(x^i\right)=\sum_{i=1}^{D-2}\partial_i^2 h\left(x^i\right)=-16\pi G\,|p|\,\prod_{i=1}^{D-2}\delta(x^i)\,. \quad (2.10)$$

Where $\nabla_{\perp}^{2}$ is the Laplacian in the transverse direction. Now since both sides are functions of x^i we can see that the ansatz (2.4) was enough and there was no need to assume a dv^2 term.

The problem is spherically symmetric in x^i so $h(x^i)$ depends only on r

$$r = \sqrt{\sum (x^i)^2} .$$

Integrating both sides and using the generalized form of Stokes theorem we get:

$$\int_{Sr} \nabla_{\perp}^{2} h(x^i) = \nabla_{\perp} h(x^i) \hat{r} ds = \frac{2\pi^{\frac{D-2}{2}}}{\Gamma(\frac{D-2}{2})} r^{D-3} \frac{dh}{dr} = -16\pi G|p| .$$

$$\Rightarrow \frac{dh}{dr} = -8\pi \frac{\Gamma(\frac{D-2}{2})}{\pi^{\frac{D-4}{2}}} G \frac{|p|}{r^{D-3}} . \qquad (2.11)$$

This is the general case for any number of dimensions. For our case here $D = 4$. By integrating (2.8) again, the profile function becomes the logarithmic function:

$$h(r) = -4\pi G|p| ln\left(\frac{r^2}{l^2}\right), \qquad (2.12)$$

where l here is a constant with the dimensions of length. It has been discussed that this l could be a UV cut-off limit of the theory [18] or even the Planck length l_p[15] but we will not go through this assumption for a reason that will be clear in a moment.

Substituting this back into (2.4) we get the metric:

$$ds^2 = -du \left[dv + 4\pi G|p| ln\left(\frac{r^2}{l^2}\right) \delta(u) du \right] + dy^2 + dz^2 . \qquad (2.13)$$

Which is the shock-wave metric resulting from a null source. That is, the massless point particle moving along $u = 0$ with momentum p.

2.2 The null geodesic time delay

We now examine the case of a massless particle moving with momentum q along the geodesic $v = v_0$ and crossing the shock-wave. The impact parameter of this crossing is $r = b$.

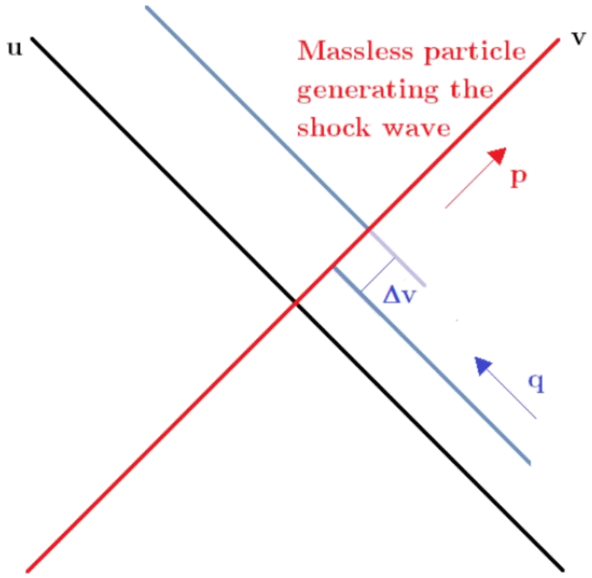

Figure.2 In this diagram we see the massless particle (blue line) crossing the shock-wave of another particle moving with momentum p at $u = 0$. The diagram represents a projection onto the (u, v) plane.

We can see that:

$$u = \lambda \quad , \quad v = v_0 - 4\pi G|p|ln\left(\frac{b^2}{l^2}\right)\theta(\lambda),$$

Where θ is the step function, satisfies the null geodesic equation $(\frac{ds}{d\lambda})^2 = 0$ for some parameter λ. We can see that crossing the shock-wave introduces a shift in the v coordinate corresponding (physically) to a shift in time:

$$\Delta v = -4\pi G|p|ln\left(\frac{b^2}{l^2}\right). \tag{2.14}$$

Now l introduces a constant shift in v that can be eliminated by a suitable transformation of coordinates. And while for a moment this seems like a time advance we should notice that whether b is larger or smaller than 1 is subject to our choice of units. So it does not represent a physical causality violation. The discussion of this result in [14] follows a different path but it leads to the same conclusion.

2.3 Scalar particle's time delay

A massless scalar particle is described by the Klein-Gordon equation:

$$\Box\phi = \frac{1}{\sqrt{-g}}\partial_\mu\left(g^{\mu\nu}\sqrt{-g}\partial_\nu\right)\phi = 0. \tag{2.15}$$

Using the form (2.4) of the metric then (2.15) reduces to

$$\partial_u\partial_v\phi + H\partial_v^2\phi - \frac{1}{4}\partial_i^2\phi = 0.$$

Considering here the case where the momentum of the scalar particle in question is in the u direction. Meaning that we can drop the terms where the derivatives are not along the null directions. So we have

$$\partial_u\partial_v\phi + H\partial_v^2\phi = 0. \tag{2.16}$$

Integrating once over v and assuming that the constant term on the left side is zero to have zero fields at infinity we get

$$\partial_u\phi + H\partial_v\ \phi = 0 \tag{2.17}$$

By using the separation of variables technique and assuming

$$\phi(u, v, \mathbf{x}) = U(u)V(v)\mathbf{X}(\mathbf{x}).$$

Substituting in (2.17) gives:

$$\frac{1}{H(u,x)}\frac{U^{'}(u)}{U(u)} + \frac{V^{'}(v)}{V(v)} = 0\ .$$

We can choose any term in this equation to equal any constant we desire. Now due to the link between the angular momentum and the derivation via: $p_v = -i\partial_v$ it would be wise to choose:

$$\frac{V^{'}(v)}{V(v)} = -ip_v \quad \Rightarrow \quad V(v) = V(0)e^{-ivp_v}\ ,$$

$$\frac{1}{H(u,x)}\frac{U^{'}(u)}{U(u)} = ip_v \quad \Rightarrow \quad U(u) = U(u_0)e^{ip_v\int_{u_0}^{u} duH(u,x)}\ .$$

And we finally have:

$$\phi(u, v, \mathbf{x}) = \mathbf{X}(\mathbf{x})V(0)U(u_0)e^{ip_v\left[v-\int_{u_0}^{u} duH(u,x)\right]}.$$

Then we use this result to see the effect of the shock-wave on the value of the scalar field around $u = 0$ when the impact parameter is b. It is easy to see that we get a phase shift of the form:

$$\phi(u = 0^+, v, \mathbf{b}) = e^{ip_v \int_{0^-}^{0^+} duH(u,x)} \phi(u = 0^-, v, \mathbf{b}) . \qquad (2.18)$$

Using the form of H(u, **x** = b) that we find:

$$\int_{0^-}^{0^+} duH(u, b) = -4G|p| ln\left(\frac{b^2}{l^2}\right) = \Delta v , \qquad (2.19)$$

$$\phi(u = 0^+, v, \mathbf{b}) = e^{ip_v \Delta v} \phi(u = 0^-, v, \mathbf{b}) .$$

Since p_v is the generator of translation in the v direction then the previous relation means that the scalar particle suffers a discontinuity Δv when it crosses the shock-wave. Since we did not seek here a geometric solution to find a geodesic, this treatment is more general because we did not have to assume that the massless particle will follow a geodesic and the previous computation in general serves as a starting point to more general fields.

Now we move to studying the case of a photon crossing the same shock-wave.

2.4 Photon in a shock-wave

Let's consider the case of the following action

$$S = -\frac{1}{4} \int d^4x \sqrt{-g} (F_{\mu\nu} F^{\mu\nu} + \gamma R^{\mu\nu}{}_{\rho\sigma} F_{\mu\nu} F^{\rho\sigma}). \qquad (2.20)$$

where γ is a parameter with the dimensions of $(\text{length})^2$. This action describes a photon nonminimally coupled to gravity. Which can be obtained after integrating the massive fermions[12]

Here $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$, and $A_\mu$is the vector potential. The equations of motion would be:

$$\nabla_\lambda F^{\lambda\alpha} + \gamma \nabla_\lambda \left(R^{\mu\nu\lambda\alpha}\right) F_{\mu\nu} + \gamma R^{\mu\nu\lambda\alpha} \nabla_\lambda F_{\mu\nu} = 0 .$$

Considering the high energy limit, where we assume the photon is moving along the u-direction. So only the light-cone coordinate derivatives matters and we neglect the others.

$$\partial^\lambda F_{\lambda\alpha} - \gamma R_\alpha{}^{\lambda\mu\nu} \partial_\lambda F_{\mu\nu} = 0 . \qquad (2.21)$$

Using the form of metric (2.4) and the resulting components of the Riemann tensor it can be shown that (2.21) will become:

$$\partial_u F_{vi} + \sum_{J=y,z} \left[H\delta_{ij} + \gamma \partial_i \partial_j H\right] \partial_v F_{vj} = 0 \qquad i = y, z.$$

Considering a photon polarized in the transverse plane with ε_i being the polarization vector. A_i can be written as: $A_i = \varepsilon_i f(u,v)$ and the field strength would be: $F_{vi} = \varepsilon_i \partial_v f(u,v)$

$$\varepsilon_i \partial_u \partial_v f(u,v) + \sum_{J=y,z} \left[H\delta_{ij} + \gamma \partial_i \partial_j H\right] \varepsilon_i \partial_v^2 f(u,v) = 0\,.$$

Multiplying by ϵ_i and summing over i

$$(\vec{\epsilon}.\vec{\epsilon}) \partial_u \partial_v f(u,v) + \left[(\vec{\epsilon}.\vec{\epsilon}) H + \gamma \sum_{j=y,z} \varepsilon_i \varepsilon_j \partial_i \partial_j H\right] \partial_v^2 f(u,v) = 0\,.$$

And finally we have:

$$\partial_u \partial_v f(u,v) + \left[H + \gamma \sum_{j=y,z} \frac{\varepsilon_i \varepsilon_j}{\vec{\epsilon}.\vec{\epsilon}} \partial_i \partial_j H\right] \partial_v^2 f(u,v) = 0. \qquad (2.17)$$

We can notice the similarity between this equation and (2.16). So in this case also when the photon cross the shock-wave it suffers a shift Δv given by:

$$\Delta v = \int_{0^-}^{0^+} du \left[H + \gamma \sum_{j=y,z} \frac{\varepsilon_i \varepsilon_j}{\vec{\epsilon}.\vec{\epsilon}} \partial_i \partial_j H\right]_{x=b}\,. \qquad (2.18)$$

The sum can be rewritten as:

$$\sum_{j=y,z} \frac{\varepsilon_i \varepsilon_j}{\vec{\epsilon}.\vec{\epsilon}} \partial_i \partial_j H = 4\pi G|p| \left[\frac{4}{\vec{\epsilon}.\vec{\epsilon}} \frac{x^i \epsilon_i x^j \epsilon_j}{r^4} - \frac{2}{r^2}\right].$$

After doing the integration at $r = b$ and introducing the impact parameter vector **b** and the magnitude $b^2 = r^2 = \sum_i (b_i)^2$ and the unit vector $\hat{\boldsymbol{n}} = \frac{\vec{\boldsymbol{b}}}{b}$

The Shapiro-like time shift is given by:

$$\Delta v = -4\pi G|p| ln\left(\frac{b^2}{l^2}\right) - \frac{8\pi G|p|\gamma}{b^2}[1 - 2(\vec{\epsilon}.\vec{n})^2]. \qquad (2.18)$$

Now let's discuss this result to better understand its physical implications.

Discussion: now while we already showed that the first term can be transformed away to become a time-delay term by a suitable choice of coordinates and length units, the second term present a real possibility of a time advance in the right circumstances. For example if the polarization vector $\vec{\epsilon}$ is perpendicular to the direction $\vec{n}$ of the impact parameter then the second term is negative (for $\gamma > 0$). This could happen as well if the impact parameter is small to the point where it is of the same magnitude as γ then it is also possible that the second term overwhelms the first one and we get a time advance.

At a first glance, we might think that this time advance, which can be thought of as related to a superluminal (larger than c) propagation of the electromagnetic wave, leads to a violation of causality. But the situation is more subtle as was shown in [19]. The fact that the electromagnetic field here is nonminimally coupled to the curvature means that a superluminal propagation does not mean a causality violation because this coupling means that the laws of physics are now curvature-dependent which breaks down the strong equivalence principle (SEP) which, in turn, takes away the possibility of having a local frame where this backward in time signal can be sent back to its source so it can arrive before it was even sent. So this break of SEP saves the day for causality. It was also shown in [18] that when we take the UV complete theory then the high frequency limit this time shift goes to zero and there is no time advance to begin with.

CHAPTER 3

(2+1) GRAVITY THEORIES

3.1 Introduction

The problem of quantizing gravity goes beyond the fact that GR is non-renormalizable. GR is a geometrical theory of spacetime so quantizing it would involve concepts describing the dynamics of spacetime in a quantum mechanical framework. But even in the basic and conceptual sense this still has a vague meaning. In [20] Steven Carlip mentions some of the issues:

- Quantum field theory (QFT) is a local theory while the fundamental physical observables in quantum gravity must be nonlocal.
- Causality in ordinary QFT is postulated. But in quantum gravity the causal structure and light cones are effected by quantum fluctuations.
- In quantum mechanics, time evolution is determined by the Hamiltonian operator. But usually in quantum gravity it is assumed that for a closed universe this Hamiltonian is identically zero. A theory that allows arbitrary parameterization of the time coordinate have a vanishing Hamiltonian.
- In a fixed time the probabilities in quantum mechanics add up to unity. But in GR there is no preferred time-slicing on which we can chose a fixed temporal point and check if the probabilities sum up to unity.
- To be able to do perturbative QFT we assume that the spacetime background is smooth and approximately flat. But we don't have any reason to assume that at the short distance limit of quantum gravity the space time will look like a smooth manifold.

So it is natural and instructional to approach those problems in a more simple setting where the calculations are easier and the conceptual complications are either the same or simpler. GR in (2+1) dimensions – one temporal and 2 spatial- is such a case.

Now since the general formulation of GR doesn't take in account the number of dimension we can see that (2+1) dimensional gravity has the same foundations in concept as (3+1) gravity. But it is vastly simpler both mathematically and physically. We will see now that the (2+1) dimensional case differs physically in some aspects from (3+1) gravity but when it comes to the analysis of conceptual problems like the nature of time, the role of topology and the effect of topology changes, and the relationships between different approaches to quantizing gravity it has proven to be useful in getting some insights.

The study of (2+1) dimensional gravity dates back at least to the 1960's but in the mid 80's a number of investigations by Deser, Jackiw, 't Hooft and Witten [21,22,23,24,25,26] gave the field a big boost into the insights that GR in (2+1) dimensions can offer.

3.2 The physical simplicity of (2+1) GR:

A fundamental physics difference in (2+1) GR is that every solution to the Einstein equations in the vacuum is locally either flat (when $\Lambda = 0$) , de Sitter (when $\Lambda > 0$) or Anti-de Sitter (when $\Lambda < 0$).

To see this we start from the fact that in any spacetime, the Riemannian tensor can be decomposed as the following:

$$R_{\rho\sigma\mu\nu} = \frac{1}{D-2}\left(g_{\rho\mu}R_{\nu\sigma} - g_{\rho\nu}R_{\mu\sigma} - g_{\sigma\mu}R_{\nu\rho} + g_{\sigma\nu}R_{\mu\rho}\right) - \frac{1}{(D-1)(D-2)}\left(g_{\rho\mu}g_{\nu\sigma} - g_{\rho\nu}g_{\mu\sigma}\right)R + C_{\rho\sigma\mu\nu}\,, \qquad (3.1)$$

where $C_{\rho\sigma\mu\nu}$ is the Weyl tensor that capture all the information about the trace-free parts of the Riemannian tensor.

In $D = 3$ this tensor is zero and we can write Riemann tensor as:

$$R_{\rho\sigma\mu\nu} = g_{\rho\mu}R_{\nu\sigma} - g_{\rho\nu}R_{\mu\sigma} - g_{\sigma\mu}R_{\nu\rho} + g_{\sigma\nu}R_{\mu\rho} - \frac{1}{2}\left(g_{\rho\mu}g_{\nu\sigma} - g_{\rho\nu}g_{\mu\sigma}\right)R\,. \qquad (3.2)$$

This means that the solutions of the field equations with a cosmological constant Λ

$$R_{\mu\nu} = 2g_{\mu\nu}\Lambda. \tag{3.3}$$

has a constant curvature [20].

A simple counting argument can show that the theory does not have local degrees of freedom and gravitational waves in (2+1) GR. Simply put, in D-dimensions the full number of entries in the metric are D^2 but due to the symmetries they are actually $\frac{1}{2}D(D-1)$ entry which means we have the same number of field equations. Now after linearizing the equation we can eliminate another D equation from our gauge choice and another D equations from the freedom in coordinates transformations [27] so we end up with $\frac{1}{2}D(D-3)$ degrees of freedom. When $D = 4$ we have 2 degrees of freedom which are the two possible polarization of the graviton. However when D = 3 we have no local degrees of freedom.

What we have just shown is the local triviality of General Relativity in (2+1) dimensions. For example according to this picture if there are two point masses m_1 and m_2, the Newtonian potential derived from the weak field limit of GR is trivial, namely the masses does not attract each other in (2+1) dimensions as long as they are stationary with respect to each other. But if the masses start moving, there will be an interaction between them. Furthermore, the local triviality of gravity in (2+1) GR does not means that gravity is globally trivial. One can still have non-local and global problems in GR. Moreover, the triviality is immediately lost if one adds higher curvature terms to the theory, modifying GR. In fact a great deal of research has gone into building locally nontrivial (2+1) dimensional gravity theories. We shall discuss some of these theories from the vantage point of causality below.

3.3 Einstein Gravity

Going back to the equation: $\nabla_{\perp}^2 h(x^i) = \sum_{i=1}^{D-2} \partial_i^2 h(x^i) = -16\pi G\ |p|\ \prod_{i=1}^{D-2} \delta(x^i)$. In the 3-dimnesional case it becomes:

$$\partial_y^2 h(y) = -16\pi G|p|\delta(y). \tag{3.4}$$

The most general solution to this differential equation is given by:

$$h(y) = -16\pi G|p|\theta(y)y + c_1(u)y + c_2(u). \tag{3.5}$$

Depending on our choice of c_1 and c_2 we can get either a non trivial profile function for y > 0 and zero function for y < 0 or vice versa. We can even chose them in a way where the profile is symmetric under the transformation y → -y [30]. But since we are interested in an asymptotic observer measuring the time delay we pick c_1 and c_2 so that we can have asymptotically flat spacetime and Cartesian coordinates for y > 0 [12]. We must also remember that Einstein's gravity is trivial in (2+1) dimensions so our choice should lead to a trivial profile beyond the shock-wave. For this purpose we chose: $c_1(u) = 16\pi G|p|\delta(u)$ and $c_2 = 0$.

Then $H(u, y)$ can be given as:

$$H(u, y) = 16\pi G|p|\theta(-y)y\delta(u). \tag{3.6}$$

Our interest now is to consider theories beyond General relativity that does not share its triviality in (2+1) dimensions. Like Topologically Massive Gravity and New Massive Gravity.

3.4 Topologically Massive Gravity in 3-dimensions (TMG)

There are many generalizations of (2+1) gravity that restore back the local degrees of freedom and give us a more interesting dynamics than the pure Einstein gravity in (2+1) dimensions. One of those generalizations is the topological massive gravity where a Chern-Simons term is added to the action. It has been shown that this theory is perturbativly renormalizable [28,29]. The Lagrangian density is given by:

$$\mathcal{L} = \sqrt{-g}\left(\sigma R + \frac{1}{2\mu}\varepsilon^{\mu\nu\alpha}\Gamma^{\beta}_{\mu\sigma}\left(\partial_\nu \Gamma^{\sigma}_{\alpha\beta} + \frac{2}{3}\Gamma^{\sigma}_{\nu\lambda}\Gamma^{\lambda}_{\alpha\beta}\right)\right). \tag{3.7}$$

where σ is the sign of Newton constant G, and μ has the dimension of mass. Taking the variation of this action and coupling it with matter we get the equations of motion:

$$\sigma G_{\mu\nu} + \frac{1}{\mu}C_{\mu\nu} = T_{\mu\nu}\,. \tag{3.8}$$

Where $C_{\mu\nu}$is the Cotton Tensor given by:

$$C_{\mu\nu} = \epsilon_{\mu}{}^{\alpha\beta}\nabla_\alpha\left(R_{\nu\beta} - \frac{1}{4}g_{\nu\beta}R\right) \;\; ; \;\; \epsilon^{\mu\alpha\beta} = \frac{\tilde{\epsilon}^{\;\mu\alpha\beta}}{\sqrt{-g}} = 1\,. \tag{3.9}$$

The Cotton tensor $C_{\mu\nu}$ plays the role of the Weyl tensor in $D = 3$ dimensions. The Weyl tensor, for $D \geq 4$ vanishes if and only if the metric is conformally flat, i.e $g_{\mu\nu} = \Omega\eta_{\mu\nu}$. In 3 dimensions the Weyl tensor automatically vanishes and so it does

not yield any information in the geometry. Instead of the Weyl tensor , one can study the Cotton tensor and establish a theory that says: if they Cotton tensor vanishes for a metric then it is conformally flat and vice versa.
In the normal coordinates (or locally inertial frame) since $\Gamma^{\mu}_{\alpha\beta} = 0$ at each point, the Chern-Simons part of the action is locally trivial, but globally it is not so. Hence it is a topological term. Moreover the Lagrangian density (3.7) does change when the coordinates change. But the change is in the form of a boundary term and does not affect the field equations of the theory.
Calculating the components of this tensor we find that the only non-zero one is:

$$C_{uu} = g_{uv}\epsilon^{vyu}\nabla_y R_{uu} = \frac{1}{2}\partial_y^3 H$$

And the equations of motion becomes:

$$\frac{-\sigma}{2}\partial_y^2 H + \frac{1}{2\mu}\partial_y^3 H = |p|\delta(y)\delta(u) \qquad (3.10)$$

The most general solution to this differential equation can be written as [12]:

$$H(u,y) = -\frac{2\sigma|p|}{m_g}\theta(y)\delta(u)\left[e^{-m_g y} + m_g y - 1\right] + C_1\frac{e^{-m_g y}}{{m_g}^2} + C_2 y + C_3$$

Where $m_g = -\sigma|\mu|$, and the C_i functions here depend arbitrarily on u. Now we can fix them to have an asymptotically flat metric, but it was indicated in [30] and following the discussion in [12] we can't expect to bring this metric into Cartesian form for both $y \to +\infty$ and $y \to -\infty$ at the same time. So by demanding that for the first case the metric is given in Cartesian form and for $\mu > 0$ we get:

$$H(u,y) = -\frac{2\sigma|p|}{m_g}\theta(y)\delta(u)e^{-m_g y} + \frac{2\sigma|p|}{m_g}\theta(-y)\delta(u)\left(m_g y - 1\right) \qquad (3.11)$$

And again using the same technique we used in analyzing the case of a massless scalar particle passing the shock-wave we see that the shift in the v coordinate is:

$$\Delta v = \int_{0^-}^{0^+} du H(u, y = b) = -\frac{2\sigma|p|}{m_g}e^{-m_g b}. \qquad (3.12)$$

So to get a time delay here we must choose σ to be negative. Which is the same requirement for the theory to be unitary. Now it is not in the scoop of this work to discuss the unitarity of the theory, but the crux of the argument is the following: the Einstein-Hilbert term R in the action does not yield a kinetic energy on its own because there are no degrees of freedom. But once the Chern-Simon term is added the Chern-Simon part together with the Einstein-Hilbert part yield a massive graviton as a single massive degree of freedom. But the kinetic energy of this photon is positive only for $\sigma = -1$.

Photon in TMG shock-wave

Taking again the case of a photon nonminimally coupled this time in TMG, in our previous discussion, our treatment will not be different for the 3-dimensional case. The only difference is in the profile function $H(u, y)$. So when the photon crosses the shock-wave the v coordinate will be shifted by:

$$\Delta v = \int_{0^-}^{0^+} du(H + \gamma \partial_y^2 H) = -\frac{2\sigma|p|}{m_g}\left(1 + \gamma m_g^2\right) e^{-m_g b}. \tag{3.13}$$

Setting σ to be negative will not guarantee that we will have a time delay, because γm_g^2 can overwhelm the first term. But this is similar to the 4-dimensional case that we already discussed.

3.5 Quadratic Gravity

The general form of Lagrangian density for quadratic gravity is given by:

$$\mathcal{L} = \sqrt{-g}\left(\sigma R + \alpha R^2 + \beta R_{\mu\nu}^2 + \mathcal{L}_{matter}\right) \tag{3.14}$$

the field equations, obtained from the variation of the action ,are:

$$\sigma\left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right) + 2\alpha R\left(R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R\right) + (2\alpha + \beta)\left(g_{\mu\nu}\Box - \nabla_\mu \nabla_\nu\right) R$$
$$+ \beta\Box\left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right) + 2\beta\left(R_{\mu\sigma\nu\rho} - \frac{1}{2} g_{\mu\nu} R_{\sigma\rho}\right) R^{\sigma\rho} = T_{\mu\nu}. \tag{3.15}$$

This theory has a rich particle content: it has two different gravitons. One of them is

a massive spin-2 particle, and the other is a massless spin-0 particle. Their masses are given in [31] respectively as:

$$m_g^2 = -\frac{\sigma}{\beta} \quad , \quad m_s^2 = -\frac{\sigma}{8\alpha + 3\beta}\,. \tag{3.16}$$

For us to have a unitary theory canonical analysis shows that one of these massive modes has to decouple. With the spin-2 graviton we do so by setting $8\alpha + 3\beta = 0$ and choose $\sigma < 0$ and $\beta > 0$. We call this theory New Massive Gravity (NMG). For our shock-wave ansatz in 3-dimnesions we have:

$$R = 0 \quad , R_{uu} = -\frac{1}{2}\partial_y^2 H(u,y) \quad , R^{vv} = g^{vu} g^{vu} R_{uu} = 2\partial_y^2 H(u,y) \; .$$

So the field equations become:

$$\sigma R_{\mu\nu} + \beta \Box R_{\mu\nu} = T_{\mu\nu} \; .$$

or:

$$-\sigma\partial_y^2 H(u,y) - \beta\partial_y^4 H(u,y) = 2|p|\delta(y)\delta(u). \qquad (3.17)$$

which has the solution:

$$H(u,y) = -\frac{\sigma|p|\delta(u)}{m_g}\left(e^{-m_g|y|} + m_g|y|\right) + c_1 y + c_2 \; . \qquad (3.18)$$

The theory is parity invariant so we can have a vanishing profile on both sides of the source. We can choose C_1 and C_2 in such a way and the solution reads:

$$H(u,y) = -\frac{\sigma|p|\delta(u)}{m_g} e^{-m_g|y|} + 2\sigma|p|\delta(u)\theta(-y)y \qquad (3.19)$$

Using the same method we used in the scalar massless particle case we find that the time shift in this case is:

$$\Delta v = -\frac{\sigma|p|}{m_g} e^{-m_g|b|}, \qquad (3.20)$$

which is positive (time delay) for $\sigma < 0$ so the theory can be made causal and unitary.

Photon in NMG

In this case as well we follow the same method for the case of a nonminimally coupled photon where we found the time shift to be:

$$\Delta v = \int_{0^-}^{0^+} du(H + \gamma\partial_y^2 H) = -\frac{\sigma|p|}{m_g}\left(1 + \gamma m_g^2\right)e^{-m_g|b|}. \qquad (3.21)$$

Since we set $\sigma < 0$ we will get a time delay provided: $\gamma > \frac{-1}{m_g^2}$

3.6 Born-Infeld Gravity

Considering the Born-Infeld extension of New Massive Gravity described by the action[32]:

$$I=-4m^2\int dx^3\left[\sqrt{-\det\left(g+\frac{\sigma}{m^2}G\right)}-\left(1-\frac{\lambda_0}{2}\right)\sqrt{-g}\right]. \qquad (3.22)$$

G is a matrix whose components are those of Einstein tensor $G_{\mu\nu}=R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R$. This theory describes a massive spin-2 graviton as its only excitation, just like the new massive gravity. But it has the added property that its vacuum is unique, compared to the new massive gravity which has two vacua. The field equations are given by [33]

$$\begin{aligned}-\frac{\kappa^2}{8m^2}T_{\mu\nu}=&-\frac{1}{2}Fg_{\mu\nu}+\left(g_{\mu\nu}\Box-\nabla_\mu\nabla_\nu\right)F_R+F_RR_{\mu\nu}\\&-\frac{\sigma}{m^2}\{2\nabla_\alpha\nabla_\mu(F_RR^\alpha_{\ \nu})-g_{\mu\nu}\nabla_\beta\nabla_\alpha(F_RR^{\alpha\beta})-\Box(F_RR_{\mu\nu})\\&-2F_RR_\nu{}^\alpha R_{\mu\alpha}+g_{\mu\nu}\Box(F_RR)-\nabla_\mu\nabla_\nu(F_RR)+F_RRR_{\mu\nu}\}\\&-\frac{1}{2m^4}\Big\{4F_RR^\rho{}_\mu R_{\rho\alpha}R^\alpha_{\ \nu}+2g_{\mu\nu}\nabla_\alpha\nabla_\beta(F_RR^{\beta\rho}R^\alpha_{\ \rho}+2\Box F_RR_\nu^{\ \rho}R_{\mu\rho}\\&-4\nabla_\alpha\nabla_\mu\left(F_RR_\nu{}^\rho R^\alpha_{\ \rho}\right)+2\nabla_\alpha\nabla_\mu(F_RRR^\alpha_{\ \nu})-g_{\mu\nu}\nabla_\alpha\nabla_\beta\left(F_RRR^{\alpha\beta}\right)\\&-\Box\left(F_RRR_{\mu\nu}\right)-2F_RRR_\nu^{\ \rho}R_{\mu\rho}-g_{\mu\nu}\Box\left(F_RR^2_{\alpha\beta}\right)+\nabla_\nu\nabla_\mu\left(F_RR^2_{\alpha\beta}\right)\\&-F_RR^2_{\alpha\beta}R_{\mu\nu}+\frac{1}{2}\left(g_{\mu\nu}\Box-\nabla_\mu\nabla_\nu\right)(F_RR^2)+\frac{1}{2}F_RR^2R_{\mu\nu}\Big\}, \qquad (3.24)\end{aligned}$$

where F, K, and $F_R$are defined as:

$$F(R,K,S)\equiv\sqrt{1-\frac{\sigma}{2m^2}\left(R+\frac{\sigma}{m^2}K-\frac{1}{12m^4}S\right)}-\left(1-\frac{\lambda_0}{2}\right), \qquad (3.25)$$

$$K\equiv R^2_{\mu\nu}-\frac{1}{2}R^2\ ,\quad S\equiv 8R^{\mu\nu}R_{\mu\alpha}R^\alpha_{\ \nu}-6RR^2_{\mu\nu}+R^3, \qquad (3.26)$$

$$F_R=\frac{\partial F}{\partial R}=-\frac{\sigma}{4m^2\left[F+\left(1-\frac{\lambda_0}{2}\right)\right]}. \qquad (3.27)$$

In the case of our ansatz: $R^2_{\mu\nu}=R^{\mu\nu}R_{\mu\nu}=0$, $R^{\mu\nu}=R^{vv}=4\partial^2H$, $R=0$.

And for the case of $\lambda_0=0$ we have: $S=0\ \ and\ \ K=0\ \ \Rightarrow\ \ F=0\ ,\ F_R=\frac{-\sigma}{4m^2}$

Substituting these in the equation of motion it reduces to

$$-\frac{k^2}{8m^2}T_{uu} = -\frac{\sigma}{4m^2}R_{uu} - \frac{\sigma}{m^2}\left(\frac{-\sigma}{4m^2}\right)\Box R_{uu},$$

$$\sigma R_{uu} + \frac{1}{m^2}\Box R_{uu} = \frac{1}{2}T_{uu}, \tag{3.28}$$

which can read as :

$$-\frac{\sigma}{2}\partial_y^2 H - \frac{1}{2m^2}\partial_y^4 H = \frac{1}{2}T_{uu}. \tag{3.29}$$

This is similar to the equations we obtained in the case of NMG. So again in this case here causality and unitarity are in agreement.

CHAPTER 4

GRAVITY THEORIES IN ANTI-DE SITTER SPACE

4.1 Shock-wave geometry in AdS$_3$:

Even though our universe is not an anti-de Sitter space (since the cosmological constant is not negative), nor it is 3-dimensional for that matter, a space with these two qualities provides an interesting test ground for many modified gravity theories to ponder on their implications and their asymptotic behavior.

Here we will continue to check the causality properties of some of those theories in the 3-dimnesional Anti-de Sitter space AdS$_3$. Where the metric written in the Poincaré coordinates is :

$$ds^2_{\mathrm{Ads3}} = \frac{l^2}{y^2}(-2dudv + dy^2). \tag{4.1}$$

To find the metric generated by a particle moving on the $u = 0$ axis with $y = y_0$we start from the ansatz:

$$ds^2 = \frac{l^2}{y^2}(-2dudv - F(u,y)du^2 + dy^2). \tag{4.2}$$

The components of the metric $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$can be arranged as:

$$g_{\mu\nu} = \frac{l^2}{y^2}\begin{pmatrix} -F & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad g^{\mu\nu} = \frac{y^2}{l^2}\begin{pmatrix} 0 & -1 & 0 \\ -1 & F & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.3}$$

The only non-vanishing Christoffel symbols are:

$$\Gamma^v_{uu} = \frac{1}{2}\partial_u F \ , \ \Gamma^u_{uy} = \Gamma^u_{yu} = -\frac{1}{y} \ , \ \Gamma^v_{vy} = \Gamma^v_{yv} = -\frac{1}{y} \ , \ \Gamma^v_{uy} = \Gamma^v_{yu} = \frac{1}{2}\partial_y F \ ,$$

$$\Gamma^y_{uu} = \frac{1}{2}\partial_y F - \frac{F}{y} \ , \ \Gamma^y_{yy} = -\frac{1}{y} \ , \ \Gamma^y_{vu} = \Gamma^y_{uv} = -\frac{1}{y} \, .$$

The resulting non-vanishing components of the curvature tensor are:

$$R^v{}_{uuv} = -\frac{F}{y^2} \ , \ R^v{}_{yyv} = -\frac{1}{y^2} \, , R^v{}_{yyu} = \frac{1}{2}\left(\partial_y^2 F - \frac{\partial_y F}{y}\right) ,$$

$R^{y}{}_{uuy} = -\frac{1}{2}\partial_y^2 F + \frac{1}{2}\frac{\partial_y F}{y} - \frac{F}{y^2}$, $R^{y}{}_{uvy} = -\frac{1}{y^2}$, $R^{y}{}_{vuy} = -\frac{1}{y^2}$, $R^{u}{}_{yuy} = -\frac{1}{y^2}$,

$R^{u}{}_{uuv} = \frac{1}{y^2}$, $R^{v}{}_{vuv} = \frac{-1}{y^2}$.

The Ricci tensor components:

$R_{uu} = \frac{1}{2}\partial_y^2 F - \frac{1}{2}\frac{\partial_y F}{y} + \frac{2F}{y^2}$, $R_{yy} = -\frac{2}{y^2}$, $R_{vv} = 0$, $R_{uv} = R_{vu} = \frac{2}{y^2}$.

And the Ricci scalar is:

$$R = g^{\mu\nu} R_{\mu\nu} = \frac{-6}{l^2}.$$

And

we know that in maximally symmetric spaces the cosmological constant is given by:

$$\Lambda = \frac{D-2}{2D} R = \frac{-1}{l^2}.$$

4.2 Einstein gravity in AdS$_3$:

We start with the simplest case which is the Einstein's gravity without any modification. The equations of motion are:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{4.4}$$

which reduces to the following equation for our case:

$$\frac{1}{2}\partial_y^2 F - \frac{1}{2}\frac{\partial_y F}{y} = 8\pi G T_{\mu\nu} \ . \tag{4.5}$$

The homogenous solution for this equation is:

$$F = y^2 \ .$$

For the particular solution we assume:

$$F_p = (\alpha y^2 + \beta y + \gamma)\theta(y - y_0) \ .$$

Substituting in the equation we can find that:

$\beta = 0$, $\alpha = \frac{1}{4y_0^2}$, $\gamma = \frac{-1}{4}$ $\Rightarrow F_p = \left(\frac{1}{4y_0^2} y^2 - \frac{1}{4}\right)\theta(y - y_0)$.

And the general solution:

$$F = \kappa\left[y^2 + \frac{1}{4}\left(\left(\frac{y}{y_0}\right)^2 - 1\right)\theta(y - y_0)\right] \quad , \quad \kappa = |p|\frac{l}{y_0}\delta(u) 8\pi G \tag{4.6}$$

As we have seen before for the case of a particle obeying the Klein-Gordon equation, the time-shift is:

$$\Delta v = \int_{0^-}^{0^+} du\, F(u, y = b) = 8\pi G|p| \frac{l}{y_0}\left[b^2 + \frac{1}{4}\left(\left(\frac{b}{y_0}\right)^2 - 1\right)\right] . \qquad (4.7)$$

The matter of whether what is inside the parenthesis is positive or negative is subject to our choice of units and length scale so it doesn't constitute a physical causality violation.

But the important factor is the sign of the Newton's constant G. to keep the theory causal this will require us to take G to be negative.

4.3 Topologically Massive Gravity (TMG):

Starting from the Lagrangian [34]:

$$\mathcal{L} = \sqrt{-g}\left(\sigma R - \frac{2}{l^2} + \frac{1}{2\mu}\varepsilon^{\mu\nu\alpha}\Gamma^{\beta}{}_{\mu\sigma}\left(\partial_\nu \Gamma^{\sigma}{}_{\alpha\beta} + \frac{2}{3}\Gamma^{\sigma}{}_{\nu\lambda}\Gamma^{\lambda}_{\alpha\beta}\right)\right). \qquad (4.8)$$

In the absence of sources the equations of motion are

$$\sigma G_{\mu\nu} + \frac{1}{l^2} g_{\mu\nu} + \frac{1}{\mu} C_{\mu\nu} = 0, \qquad (4.9)$$

Where $C_{\mu\nu}$is the Cotton tensor, and for the AdS_3 case here:

$$C_{\mu\nu} = \epsilon_{\mu}{}^{\alpha\beta}\nabla_\alpha\left(R_{\nu\beta} - \frac{1}{4}g_{\nu\beta}R\right) \;\; ; \;\; \epsilon^{\mu\alpha\beta} = \frac{\tilde{\epsilon}^{\mu\alpha\beta}}{\sqrt{-g}} = \frac{y^3}{l^3},$$

Which can be calculated in this case as:

$$C_{uu} = g_{u\lambda}\eta^{\lambda\alpha\beta}\nabla_\alpha\left(R_{u\beta} - \frac{1}{4}g_{u\beta}R\right),$$

$$C_{uu} = g_{uu}\eta^{uvy}\nabla_v\left(R_{uy} - \frac{1}{4}g_{uy}R\right) + g_{uu}\eta^{uyv}\nabla_y\left(R_{uv} - \frac{1}{4}g_{uv}R\right)$$

$$+ g_{uv}\eta^{vuy}\nabla_u\left(R_{uy} - \frac{1}{4}g_{uy}R\right) + g_{uv}\eta^{vyu}\nabla_y\left(R_{uu} - \frac{1}{4}g_{uu}R\right),$$

$$= \frac{y^3}{l^3}\{\left(-F\frac{l^2}{y^2}\right)[\nabla_v R_{uy}] — F\frac{l^2}{y^2}[\nabla_y R_{uv}] — \frac{l^2}{y^2}\left[\frac{1}{2}\frac{\partial_y^2 F}{y} - \frac{1}{2}\frac{\partial_y F}{y^2}\right] + 2\left(-\frac{l^2}{y^2}\right)[\frac{1}{2}\partial_y^3 F - \frac{1}{2}\frac{\partial_y^2 F}{y} - \frac{1}{2}\frac{\partial_y F}{y^2}],$$

$$C_{uu} = -\frac{y}{2l}\partial_y^3 F.$$

After setting $\sigma = -1$ (we will go back to this point in a moment) the equation of motion becomes:

$$-\frac{1}{2}\partial_y^2 F+\frac{1}{2}\frac{\partial_y F}{y}-\frac{1}{\mu}\frac{y}{2l}\partial_y^3 F=|p|\frac{l}{y_0}\delta(u)8\pi G\,\delta\big(y-y_0\big). \qquad (4.10)$$

We can write the general solution as [12]:

$$F(y)=l^2\mu\frac{\delta(u)|p|}{1-(l\mu)^2}\left[2\left(\frac{y}{y_0}\right)^{1-l\mu}-(1-l\mu)\left(\frac{y}{y_0}\right)^2-(1+l\mu)\right]\theta(y-y_0)$$

$$+l^2\mu\frac{\delta(u)|p|}{1-(l\mu)^2}[2c_1\left(\frac{y}{y_0}\right)^{1-l\mu}+(1-l\mu)c_2\left(\frac{y}{y_0}\right)^2$$

$$+(1+l\mu)c_3\ . \qquad (4.11)$$

Assuming certain asymptotic behavior of this function the values of C_1, C_2 and C_3 can be fixed to $C_1=0$, $C_2=C_3=1$ [12]

Now to check if TMG can manifest a causality violation, we study as before the case of a massless particle crossing the shock-wave. For the same studied case of a scalar particle we can write:

$$\Delta v=\int_{0^-}^{0^+}duF(u,y)=\frac{2\mu|p|}{\mu^2-\frac{1}{l^2}}(\frac{y}{y_0})^{-l(\mu-\frac{1}{l})} \qquad (4.12)$$

This is a positive time-shift meaning that we have a time delay and the theory is causal. We obtained this after setting σ to be negative. A requirement necessary also to have a ghost free theory. So the requirement for unitarity (ghost free and tachyon free) and causality are compatible.

4.4 The New Massive Gravity (NMG)

Let us take the field equations of the new massive gravity given in the form:

$$-G_{\mu\nu}+|\Lambda|g_{\mu\nu}+\frac{1}{2m^2}K_{\mu\nu}=0 \qquad (4.13)$$

where

$$K_{\mu\nu}=2\Box R_{\mu\nu}-\frac{1}{2}\nabla_\mu\nabla_\nu R-\frac{1}{2}g_{\mu\nu}\Box R+4R_{\mu\alpha\nu\beta}R^{\alpha\nu}-\frac{3}{2}RR_{\mu\nu}$$

$$-g_{\mu\nu}\left(R_{\mu\nu}R^{\mu\nu}-\frac{3}{8}R^2\right) \qquad (4.14)$$

We will calculate each term in this expression and then combine them at the end.

$$\Box R_{uu}=g^{yy}\nabla_y\nabla_y R_{uu}+g^{vv}\nabla_v\nabla_v R_{uu}+g^{uv}\nabla_u\nabla_v R_{uu}+g^{vu}\nabla_v\nabla_u R_{uu}\,,$$

- $\nabla_y R_{uu} = \partial_y R_{uu} - 2\Gamma^k_{yu} R_{ku} = \frac{1}{2}\partial^3 F + \frac{1}{2}\frac{\partial^2 F}{y} - \frac{1}{2}\frac{\partial F}{y^2}$,
- $\nabla_y \nabla_y R_{uu} = \partial_y\left(\nabla_y R_{uu}\right) - \Gamma^y_y \nabla_y R_{uu} - \Gamma^\mu_{yu}\nabla_y R_{\mu u} - \Gamma^\mu_{yu}\nabla_y R_{u\mu}$,

$$= \frac{1}{2}\partial^4 F + 2\frac{\partial^3 F}{y} + \frac{1}{2}\frac{\partial^2 F}{y^2} - \frac{1}{2}\frac{\partial F}{y^3},$$

- $\nabla_v R_{uu} = 0 - \Gamma^k_{vu} R_{ku} - \Gamma^k_{vu} R_{uk} = 0$,
- $\nabla_v \nabla_v R_{uu} = 0 - \Gamma^\lambda_{vv}\nabla_\lambda R_{uu} - \Gamma^\lambda_{vu}\nabla_v R_{\lambda u} - \Gamma^\lambda_{vu}\nabla_v R_{u\lambda}$

$$= -2\Gamma^y_{vu}\nabla_v R_{yu} = \frac{2}{y}\nabla_v R_{uy} = 0,$$

- $\nabla_u R_{uu} = 0 - \Gamma^\lambda_{vu} R_{\lambda u} - \Gamma^\lambda_{vu} R_{u\lambda} = -\frac{2\partial_u F}{y^2}$,
- $\nabla_v \nabla_u R_{uu} = 0 - \Gamma^\lambda_{vu}(\nabla_\lambda R_{uu}) - \Gamma^\lambda_{vu}\nabla_u R_{\lambda u} - \Gamma^\lambda_{vu}\nabla_u R_{u\lambda}$

$$= \frac{1}{2}\frac{\partial^3 F}{y} + \frac{3}{2}\frac{\partial^2 F}{y^2} - \frac{3}{2}\frac{\partial F}{y^3},$$

- $\nabla_v R_{uu} = -2\Gamma^k_{vu} R_{ku} = 0$,
- $\nabla_u \nabla_v R_{uu} = 0 - \Gamma^\lambda_{uv}(\nabla_\lambda R_{uu}) - 2\Gamma^\lambda_{uu}\nabla_v R_{\lambda u} = \frac{1}{2}\frac{\partial^3 F}{y} + \frac{1}{2}\frac{\partial^2 F}{y^2} - \frac{1}{2}\frac{\partial F}{y^3}$,

Finally we find:

$$\Box R_{uu} = \frac{1}{l^2}\left[\frac{1}{2}y^2\partial^4 F + y\partial^3 F - \frac{3}{2}\partial^2 F + \frac{3}{2}\frac{\partial F}{y}\right].$$

We also have

$$R = -\frac{6}{l^2} \quad \Rightarrow \quad \nabla_u \nabla_u R = 0 \quad and \quad \Box R = 0,$$

$$R^{\mu\nu} = g^{\mu\sigma} g^{\nu\rho} R_{\sigma\rho}.$$

The non-zero components of $R^{\mu\nu}$ are:

$$R^{vv} = \frac{y^4}{l^4}\left[\frac{1}{2}\partial^2 F - \frac{1}{2}\frac{\partial F}{y} - \frac{2F}{y^2}\right]$$

$$R^{yy} = \frac{-2y^2}{l^4}$$

$$R^{uv} = \frac{2y^2}{l^4}$$

$$g_{\mu\nu}R_{\rho\sigma}R^{\rho\sigma} = -12\frac{F}{y^2}$$

- $R_{u\alpha u\beta}R^{\alpha\beta} = R_{uvuv}R^{vv} + R_{uyuy}R^{yy} + 0 + 0$

So

$$R_{uvuv} = \frac{l^2}{y^4} \quad \text{and} \quad R_{uyuy} = \frac{l^2}{y^4}\left[\frac{1}{2}\partial^2 F - \frac{1}{2}\frac{\partial F}{y} + F\right]$$

Then we have:

- $$R_{u\alpha u\beta}R^{\alpha\beta} = \frac{1}{l^2}\left[-\frac{1}{2}\partial^2 F + \frac{1}{2}\frac{\partial F}{y} - \frac{4F}{y^2}\right]$$

Combining these results, and knowing that the AdS radius l and the bare cosmological constant $|\Lambda|$ together with the mass scale m satisfy the following relationship: $1 + 4m^2(l^2 + l^4|\Lambda|) = 0$, we find the equations of motion to be:

$$\left[y^4\partial_y^4 F + 2y^3\partial_y^3 F - \frac{(1+2l^2m^2)}{2}\left(y^2\partial_y^2 F - y\partial_y F\right)\right]\frac{1}{2l^2m^2y^2} = |p|\frac{l}{y_0}\delta(u)\delta(y-y_0). \quad (4.15)$$

In [12] the final solution to this equation was found to be:

$$F(y) = \left(\frac{m^2|p|l^3}{\beta(1-\beta^2)}\right)\left[\theta(y-y_0)\left(\frac{y}{y_0}\right)^{1-\beta} + \theta(y-y_0)\left(\left(\frac{y}{y_0}\right)^{1+\beta} + \beta\left(\left(\frac{y}{y_0}\right)^2 - 1\right)\right)\right]. \quad (4.16)$$

where $\beta = \sqrt{\frac{1}{2} + l^2m^2}$.

Similar to the TMG case, the time-delay can be found as:

$$\Delta v = \int_{0^-}^{0^+} du F(u,y) = \left(\frac{2ml^2}{2m^2l^2 - 1}\right)\frac{|p|}{\sqrt{1 + \frac{1}{2m^2l^2}}} \times \left(\frac{y}{y_0}\right)^{\overline{\frac{l}{l - m\sqrt{1 + \frac{1}{2l^2m^2}}}}} . \quad (4.17)$$

This shift turns out to be positive which gives us a time delay. Meaning that there is no causality violation for this case in NMG.

CHAPTER 5

CONCLUSIONS

In this work we have studied local causality in flat and AdS_3 spacetime especially in the (2+1) dimensional case. The apparent causality violation when we studied the scattering of a massless particle in 4-dimensional Einstein gravity was just a mathematical artifact of our coordinates and units choices.

But when we moved to study the case of non-minimally coupled photon there was a possibility to have a time advance. But the fact that the used action breaks the Strong Equivalence Principle is a game changer here which reminds us that before we mark any case as a possible causality violation we should check that the possibility to have a local inertial reference frame can still hold.

In the case of (2+1) gravity it was remarkable to find that in TMG and NMG the same requirement for the theory to be unitary was also necessary to restore causality, namely: setting the Newton's constant sign to be negative. It can be seen from (2.11) that this situation can happen as well in other dimensions and the case for $D = 4$ is in fact the peculiar one due to the difference that the integration of (2.11) in 4-dimensions does not involve a sign change since it is the derivative of a logarithmic function. As a further step it is better to further check this speculation about unitarity and causality requirements in $D > 4$ cases.

It should be mentioned as well that the same results in calculating the time shift due to the scattering can be obtained calculating the scattering amplitudes in the Eikonal approximation [12].

It would seem logical to think of the method used in this work as a way to probe suggested modifications to GR and check whether or not they provide a physically meaningful candidate to a Quantum Gravity theory. But a further analysis shows that even if a model exhibited a possible causality violation in the IR range in curved spacetime, these problems can be lifted over if the theory is immersed in a causal UV completion. It was shown in [18] that the scattering amplitude are energy-dependent and the phase shift goes away as the energy scale goes up. This is not the case when we consider actions in flat spacetime. It was indicated clearly in the mentioned analysis that the key factor here is gravity itself.

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