EMPIRICAL COMPARISON OF PORTFOLIO RISK DIVERSIFICATION ALGORITHMS

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF APPLIED MATHEMATICS
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

ÇİĞDEM YERLİ

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF MASTER OF SCIENCE
IN
FINANCIAL MATHEMATICS

SEPTEMBER 2018

Approval of the thesis:

EMPIRICAL COMPARISON OF PORTFOLIO RISK DIVERSIFICATION ALGORITHMS

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ABSTRACT

EMPIRICAL COMPARISON OF PORTFOLIO RISK DIVERSIFICATION ALGORITHMS

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September 2018, 63 pages

The enhanced correlations during global financial crisis has revealed that simple asset allocation portfolios prove to be not well-diversified across different risk factors, which makes the risk based asset allocation strategies popular. However, the strategies still construct the risk concentrated portfolios due to the correlation among the asset classes. As a result, risk allocation among uncorrelated risk factors instead of risk allocation among asset classes have become widely used. This thesis aims to distribute the risk among uncorrelated risk factors in a portfolio to prevent constructing risk concentrated portfolio. We employ “diversified risk parity strategy”. The first step in this approach is the construction of the uncorrelated portfolios. To construct uncorrelated portfolios, we follow two different approaches: principal component analysis and minimum linear torsion model. These uncorrelated portfolios are also known as uncorrelated risk factors in the literature. In the second step, we apply the risk parity strategy to these uncorrelated risk factors to obtain equal risk budget from each risk source. While the literature evaluates each uncorrelated portfolio as one kind of risk factor, we focus on three main risk sources, namely equity risk, inflation rate risk and inflation risk. In this work, we give the background of diversified risk parity strategy and traditional risk based asset allocation strategies and explain how uncorrelated portfolios constructed based on principal component analysis and minimum linear torsion model with examining their return and risk properties. Then we provide an application of the strategies to selected asset classes. The poor performance of mean-variance strategy due to large estimation errors in estimated mean leads to risk-based strategies popular. Therefore, to make clear comparison, we also include mean-variance optimization and compare the out-of-sample performance with both risk-based strategies and diversified risk parity strategies in the empirical analysis.

Keywords : Diversified risk parity strategy, principal component analysis, minimum linear torsion model, risk based asset allocations, risk diversification.

ÖZ

PORTFÖY RİSK ÇEŞİTLENDİRME ALGORİTMALARININ AMPİRİK KARŞILAŞTIRMASI

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Eylül 2018, 63 sayfa

Son yaşanan finansal kriz, riski daha çeşitlendirilmiş portföylere olan ihtiyacı ortaya koymuş ve riske dayalı varlık dağıtımı yapan stratejileri popüler yapmıştır. Fakat bu stratejiler, varlık sınıfları arasındaki korelasyondan dolayı hala riski yoğun portföyler oluşturmaktadır. Bu nedenle yeni çalışmalar risk dağıtımında varlık sınıfları yerine bağımsız risk faktöleri üzerine odaklandılar. Bu çalışmanın amacı riski bağımsız risk faktörlerine dağıtarak, riski tek faktöre yoğunlaşmış portföy elde edilmesini engellemek. Bu çalışma, "çeşitlendirilmiş risk paritesi stratejini" kullanmaktadır. İlk adımda bağımsız portföyleri oluşturulacaktır. Bağımsız portföyleri oluşturmak için iki farklı model, temel bileşenler analizi veya minimum doğrusal torsiyon modeli kullanılmaktadır. Bu bağımsız portföyler aynı zamanda bağımsız risk faktörleri olarak da bilinir. Elde edilen bağımsız porföylere dolayısıyla, risk faktörlerine risk paritesi stratejisi uygulanarak her bir risk faktöründen eşit risk bütçesi elde edilmektedir. Her bir portföyü ayrı bir risk faktörü olarak değerlendiren literatürden farklı olarak üç temel risk faktörüne (piyasa riski, faiz riski ve enflasyon riski) odaklanılmıştır. Çeşitlendirilmiş risk paritesi stratejisinin ve geleneksel risk bazlı varlık tahsis stratejilerinin arka planını verilerek temel bileşen analizi ve minimum doğrusal torsiyon modeline dayanarak bağımsız portföylerin nasıl oluşturulduğu, kar ve risk özellikleri de incelenerek açıklanmıştır ve seçilen varlık sınıflarına stratejilerin bir uygulaması sunulmaktadır. Tahmini ortalamadaki büyük tahmin hatalarından dolayı ortalama varyans stratejisinin zayıf performansı, risk temelli stratejilerin popüler olmasına neden olmaktadır. Bu nedenle, net bir karşılaştırma yapmak için, örneklem dışı performansı testi ortalama varyans optimizasyonunu da içermekte ve bu tezde bahsedilen stratejilerle karşılaştırılmaktadır.

Anahtar Kelimeler : Çeşitlendirilmiş risk paritesi stratejisi, temel bileşenler analizi, minimum doğrusal torsiyon modeli, risk bazlı varlık dağıtımı, risk çeşitlendirme.

To My Family

ACKNOWLEDGMENTS

I would like to express my sincere gratitude to Prof. Dr. Sevtap Kestel, Dr. Nilüfer Schindler for their guidance, encouragement and valuable advices during my study. Their willingness to share their valuable time and experiences has enlightened my path. It is my honor to be a student of such an excellent and productive academicians and great people.

I want to special thank to my committee members Assoc. Prof. Kasirga Yıldırak and Assoc. Prof. Dr. Ali Devin Sezer for their support and guidance.

I also thank institute academic and administrative staff of the Institute for their worthwhile support and friendly environment. I express my great appreciation to Prof. Dr. Gerhard-Wilhelm Weber for his continuous support during my graduate study. He always has time for his students and his heart is full of goodness. His immense knowledge and enthusiastic encouragement are priceless.

Last but not least, I wish to express my deepest gratitude to my family for their precious support and guidance. I am honored to have you as my parents.

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LIST OF ABBREVIATIONS

DRP	Diversified risk parity strategy
DRP_{MTP}	Diversified risk parity strategy from minimum linear torsion portfolios
DRP_{PP}	Diversified risk parity strategy from principal portfolios
ERC	Equal risk contribution strategy
EW	Equally weighted strategy
GMV	Global minimum variance strategy
IV	Inverse volatility strategy
MDD	Maximum drawdown
MLT	Minimum linear torsion transformation
MRC	Marginal risk contribution
MTP	Minimum linear torsion portfolio
PCA	Principal component analysis
PC	Principal component
PP	Principal portfolio
RC	Risk contribution (Total risk contribution)
RP	Risk parity strategy

CHAPTER 1

INTRODUCTION

Asset allocation plays an essential role in investment management. Investors try to understand how they should invest their capital among different asset classes. For this, Markowitz mean-variance optimization that is one of the quantitative techniques aims to find the best asset allocation based on risk-return trade-off. Although this approach has been used widely due to its theoretical rationality, it has some obstacles due to large estimation errors [6, 5]. To avoid large estimation errors due to estimated mean, recent works have been focused on generating more diversified portfolios excluding the mean called "risk-based asset allocation strategies". Since these strategies only use covariance matrix, the strategies are also known as "$\mu-$free strategies". Due to their acceptance being as robust in the literature and good performance during the 2008 financial crisis, risk based asset allocation strategies have been popular [21, 32]. The aim of risk based strategies is to allocate the risk among asset classes instead of the capital and construct well balanced portfolios in terms of risk. However, the ability of diversification of risk based strategies limitation depends on the characteristics of the underlying assets. If chosen assets are highly correlated and dependent on the same underlying risk factors, the aim of diversification may not be achieved and the portfolio may have a concentrated risk structure. Especially, this problem can arise during the financial crisis times, since the correlations generally increase when economy goes bad.

The high correlation among asset classes results in having more than one risk source. [29] gives an example to demonstrate how a portfolio is concentrated on one kind of risk factor. In his example, the portfolio consists of four equity classes and four fixed income asset classes. The fixed income asset classes are high yield, emerging-market debt, inflation-linked bonds, and investment-grade bonds. The expectation from the portfolio to have two different risk sources that are equity risk and bond risk. Since majority of the bond classes have different degree of equity risk, the portfolio is indeed skewed to equity risk and demonstrates a concentrated risk structure. Generally, many asset classes are affected by equity risk. As mentioned recently, some of fixed income classes consist of different degree of equity risk although bonds and equities generally move opposite directions. It is known that high yield is one kind of bond, however there is generally a high correlation between the equity and high yield. The high correlation between equity and other classes generally leads that portfolios can heavily skewed to equity risk. Hence, the underlying risk sources in the asset classes do not make risk allocation easier. To overcome this problem, the portfolio decision puts importance on not only risk contributions from asset classes but also risk contributions from risk factors. The key property is that the desired risk factors should be less correlated, if possible, uncorrelated.

To obtain uncorrelated risk sources, there are some quantitative methods. One of them is principal component analysis (PCA), that is a statistical dimension reduction technique using orthogonal transformation of variables into the linearly uncorrelated new variables. Partovi and Caputo (2004) [26] employs PCA approach to construct uncorrelated portfolios that are also called principal portfolios. Furthermore, these portfolios can be considered as uncorrelated risk sources. This approach is criticized for being unstable over time, lacking of economic interpretation and not having unique eigenvectors [25]. Upon these claims, Meucci et al. (2014) [25] put forward another methodology, namely minimum linear torsion (MLT), that that extracts uncorrelated variables that closely follow the original variables. Hence, this methodology is expected to be more robust than the PCA approach. [26, 24, 25] distribute whole portfolio risk among these uncorrelated portfolios. In this case, the portfolio risk is already known and the total risk is distributed among risk factors to prevent risk concentration. On the other hand, Lohre et al. [20] directly construct the portfolio with applying risk parity strategy to uncorrelated portfolios from PCA and MLT. This strategy is called “diversified risk parity” [20]. Following [20], we use the diversified risk parity strategy in this work.

One of the major questions in the asset management is how many underlying risk factors drive the asset returns. There are two key risk dimensions: growth risk and inflation risk [29]. Growth risk is divided into two parts: equity risk and interest rate risk that move oppositely when economic growth change. Along with the inflation risk, there are real-return premium and nominal return premium, which move oppositely based on inflation structure. To create a well balanced portfolio that is exposed to growth and inflation risks, risk based portfolios should be constructed with a balanced risk budget from three main risk factors: equity risk, interest rate risk and inflation risk [29]. From this point of view, there are three main risk premiums that are equity risk, interest rate risk and inflation risk [29]. Other risks are mixture of these three main risk sources. For example, credit risk is a mixed of interest rate risk and equity risk [29]. An investor who wants a well diversified portfolio should distribute the overall portfolio risk among these risk drivers. Therefore, in this thesis, we focus on only these risk drivers.

This thesis aims to distribute the portfolio risk among three main risk sources with employing diversified risk parity strategy based on MLT approach and PCA approach with following the approach by [20, 2]. We expect that a portfolio based on MLT approach should be more balanced in terms of risk compared to diversified risk parity on PCA and other risk parity strategies. Comparison of these methods is performed on real life data collected from Bloomberg between years 1988 and 2017. This thesis contributes to the literature in several ways. First, we give a through comparison of risk based asset allocation strategies including PCA and MLT to empirical price data between January 1988 to December 2017. This period is long to contain two financial crisis and great variation, which allows a fine comparison of these algorithms. Second, rather than using all risk factors in PCA and MLT, we focus on only three risk factors, equity risk, inflation risk and interest rate risk. Third, to the best of our knowledge, MLT approach has so far been applied only to commodity prices; we extend this application to more general asset classes. Finally, we present out-of-sample performances of the strategies for different time intervals that help us to capture the economic changes in the market. In the out-of-sample testing, we include the mean-variance strategy as well to make a clear comparison of estimation error across the strategies.

The organization of the thesis is as following. Chapter 1 presents the theoretical background of risk based asset allocation strategies, namely equally weighted, global minimum variance and risk parity strategies such as inverse volatility strategy and equal risk contribution. Chapter 2 gives the information about risk-based asset allocation strategies. Chapter 3 presents the diversified risk parity with explaining theoretical framework of PCA and MLT. We explain how diversified risk parity strategy can be applied to these methodologies and can be generated uncorrelated portfolios based on these approaches. Their return and risk properties are also given. Chapter 4 presents the empirical analysis results. In this part, we present the performance and risk characteristics of mentioned strategies. Then, we check the robustness of strategies with rolling window approach. The poor performance of mean-variance strategy due to large estimation errors in estimated mean has become popular risk-based strategies as mentioned previously. Therefore, to make clear comparison, we also include mean-variance optimization and compare the out-of-sample performance with both risk-based strategies and diversified risk parity strategies. The conclusion further comments on our results and points out directions for future research.

1.1 Literature Review

The need of diversification of portfolio risk among underlying factors leads to focus on constructing risk factors that drive the asset returns. Therefore, capital asset pricing model [23], arbitrage pricing model [31] and factor models focus on the risk factors to explain the asset returns. Capital allocation among risk factors instead of individual stocks have better advantages such as better risk management [14]. Although factor investing seems attractive, there is still need for better allocation methods since mentioned factor models do not hold their assumptions in the financial markets and overlook the correlations among the assets, which cause inefficient allocation [13]. Specifically, the disregard generates portfolios that concentrate on few risk sources.

To overcome previous problems, the portfolio literature has emphasized the risk budgeting and risk strategies with special focus on risk parity portfolios [30, 21, 32]. The “risk parity” term was introduced by [30] who demonstrates that a portfolio consisting of 60% of stocks and 40% of bonds might be balanced in terms of asset allocation but not be balanced in terms of risk as stocks generate more than 90% of portfolio risk and the risk contribution by bonds is less than 10%. Therefore, the author suggests equal risk allocated portfolios with distributing the same risk across asset classes. [29] says that a well-diversified portfolio should be balanced in equity risk, inflation risk and interest rate risk. Risk parity strategies have been popular and obtained many attention among both researchers and investors since 2008 financial crisis. Thus, there are different kinds of risk parity in the literature. The first implementation of the risk parity is inverse volatility strategy (IV) that distributes the weights inversely proportional to variance of assets [7]. This strategy penalizes the assets whose variance is high. The flaw of IV strategy is not to consider the correlation among asset classes. This drawback is corrected by equal risk contribution strategy (ERC) which distributes the portfolio risk according to both variances and correlations of assets, thus it presents better risk-adjusted returns than the IV strategy [32, 21]. Also these strategies are known as risk-based asset allocation or "μ-free" strategies since they only consider the covariance matrix as an input parameter. [3] demonstrates that these strategies fail to distribute the risk among underlying risk factors such that these strategies still generate risk concentrated portfolios due to hidden risk drivers among asset classes. Then, the new kind of risk parity is generated by principal component analysis and minimum linear torsion approach that generate uncorrelated risk factors.

To avoid the risk concentrated portfolios, the uncorrelated risk sources have been introduced by [26] who uses principal component analysis to construct the uncorrelated portfolios that generate the efficient frontier. The uncorrelated portfolios or principal portfolios represent also uncorrelated risk factors. [24] contributes to the literature with a comprehensive framework that measures and manages diversification in a stock investment universe from Russel 3000 index. His work demonstrates how to extract the main drivers of the asset returns with PCA method. He claims that the portfolio risk should be distributed among these risk factors to achieve a well-diversified portfolio with exponential entropy approach. [20] follows the approach by [24] to determine the maximum diversification in a portfolio that consists of various asset classes. Their investment strategy employs risk budget by principal portfolios instead of individual assets. This strategy is called as "diversified risk parity strategy". The authors show that the diversified risk parity strategy provides better risk-adjusted performance in the multi-asset class set than risk-based allocation models. [2] applies the same strategy by [20] to equity domain and reached the same conclusion. [16] also adopts the strategy of [24] and demonstrates that [24]'s approach equals the application of risk parity to risk factors. According to theoretical background, diversified portfolio among uncorrelated risk factors should should outperform the nominal diversified portfolio. However, according to backtests by [16], diversification strategies based on principal portfolios performs worse than nominal strategies. Similar results are also found by [27]. These papers present that PCA may not be the appropriate approach to extract the risk factors since PCA has some drawbacks such that the eigenvectors are not unique, the factors may not be interpretable economically [25].

[25] proposes a new model, namely minimum torsion transformation, that extract the uncorrelated variables with closely following the original data. Hence, this methodology is expected to be more robust than the PCA approach. [2] applies the risk parity strategy with using minimum torsion transformation to commodity data and found that minimum torsion approach extracts interpretable risk factors and constructs stable portfolios compared to PCA method. [17] compares the diversified portfolios generated by PCA and minimum torsion transformation and concludes that minimum torsion model solves the problems related with PCA and extracts risk factors better than PCA. However, the portfolios generated by minimum torsion may not outperform the nominal strategies, such as minimum variance and maximum diversification, in terms of Sharpe ratio. Despite this low performance, minimum torsion portfolios significantly

reduce downside risk and provide low turnover ratio [17].

CHAPTER 2

RISK BASED ASSET ALLOCATION STRATEGIES

This chapter presents the theoretical background of risk based allocation strategies, namely, equally weighted (EW), global minimum variance (GMV), and risk parity (RP) strategies. These strategies are also known as μ-free strategies and have attracted investment area especially after the 2008 financial crisis. The failure of mean-variance strategy due to estimation errors in the estimated mean lead to researchers and investors to use risk based strategies that do not have to estimate the expected mean, but only the covariance structure. Therefore, these strategies are generally accepted as robust in the literature due to their good performance over the 2008 financial crisis period.

The aim of risk based strategies is to allocate the risk instead of to allocate the capital among asset classes.

2.1 Properties of Risk Based Strategies

The goal of the investment is to obtain positive returns, but the gains are subjected to risk. Risk plays a significant role in portfolio management.

2.1.1 Risk Measure

Let $V^2(\mathbf{w})$ denote the portfolio risk, which is a positive and increasing function and bounded by below zero. That is for i.e. $\epsilon > 0$, which is

$$V^2(\mathbf{w}) \geq \epsilon \parallel \mathbf{w} \parallel^2 .$$

An investor aims to reduce the risk, but it is relevant with the market conditions. A portfolio with more risk will have more earnings in a favorable market, and will have losses in a unfavorable market conditions.

There are many risk measures such as variance, semi variance, VaR and CVaR. In this study, we focus on the variance as a risk measure. Variance (σ^2) measures the dispersion of the data from the mean. Its square root is known as volatility. Variance is the most popular risk measure due to its computational simplicity and easy interpretation.

Given $\sigma^2(w)$ is the portfolio variance, then

$$\sigma^2(w) = w'\Sigma w \tag{2.1}$$

where *w* is the security weight in the portfolio and Σ denotes the covariance matrix of security returns.

2.1.2 Marginal risk contribution

The marginal risk contribution (MRC) of the components in a portfolio is calculated by taking the partial derivative of the whole portfolio risk to each component. The MRC of asset i is given as following.

Definition 2.1. Let **w** be vector of asset weights and $\sigma(\mathbf{w})$ be the portfolio risk measure, then the marginal risk contribution of the i^{th} asset is the first derivative of the risk measure with respect to its weight w_i such that

$$MRC_i(\mathbf{w}) = \frac{\partial \sigma(\mathbf{w})}{\partial w_i}.$$

MRC gives an infinitesimal change in the whole portfolio risk caused by the i^{th} component. Let denote $MRC(\mathbf{w})$ as a vector representation, i.e.

$$MRC(\mathbf{w}) = \frac{\partial R(\mathbf{w})}{\partial w_i}. \tag{2.2}$$

The decomposition in Equation 2.2 is possible if the risk of the portfolio is a homogeneous function.

Definition 2.2. Let $f : x \subset R^n \to R$ be a function. Then f is a homogenous function of degree $d \in R$ if

$$f(\gamma x) = \gamma^d f(x)$$

for $\gamma \in R$ and $x \in R^n$.

Proposition 2.1. *The portfolio volatility, $\sigma(\boldsymbol{w})$, is a homogeneous function of degree one, $d = 1$. Then the marginal risk contributions of asset returns presented by*

$$\frac{\partial \sigma(\boldsymbol{w})}{\partial \boldsymbol{w}} = \frac{\Sigma \boldsymbol{w}}{\sqrt{\mathbf{w}^T \Sigma \boldsymbol{w}}}, \quad for \quad \boldsymbol{w} \in R^n. \tag{2.3}$$

Proof. We demonstrate the first statement that the portfolio volatility $\sigma(\mathbf{w})$ is a homogeneous function of degree one. Consider the portfolio volatility as in Equation 2.1 and let $a \in R$, then

$$0 \leq \sigma(a\mathbf{w}) = ((a\mathbf{w})^T \Sigma (a\mathbf{w}))^{\frac{1}{2}} = (a^2 \mathbf{w}^T \Sigma \mathbf{w})^{\frac{1}{2}} = \mid a \mid (\mathbf{w}^T \Sigma \mathbf{w})^{\frac{1}{2}} = \mid a \mid \sigma(\mathbf{w}) = a\sigma(\mathbf{w}).$$

The Equation 2.3 can be shown simply taking the partial derivative of the volatility as following

$$\frac{\partial \sigma(\mathbf{w})}{\partial \mathbf{w}} = \frac{\partial (\mathbf{w}^T \Sigma \mathbf{w})^{\frac{1}{2}}}{\partial \mathbf{w}} = \frac{1}{2} (\mathbf{w}^T \Sigma \mathbf{w})^{\frac{-1}{2}} 2 \Sigma \mathbf{w} = \frac{\Sigma \mathbf{w}}{(\mathbf{w}^T \Sigma \mathbf{w})^{\frac{1}{2}}}. \tag{2.4}$$

□

MRC of a specific asset i is proportional to the i^{th} row of the $\Sigma\mathbf{w}$ product matrices such that

$$\frac{\partial\sigma(\mathbf{w})}{\partial w_i} \propto (\Sigma\mathbf{w}) = w_i\sigma_i^2 + \sigma_i \sum_{i\neq j}^{n} w_j\sigma_j\rho_{ij} \tag{2.5}$$

where ρ_{ij} is the correlation of the i^{th} and j^{th} assets.

Then, normalization of the Equation 2.5 by portfolio risk gives that

$$\frac{\partial(\mathbf{w})}{\partial w_i} = \frac{(\Sigma\mathbf{w})_i}{\sigma(\mathbf{w})}. \tag{2.6}$$

2.1.3 Total risk contribution

Risk contribution (RC) is a weighted marginal contribution of a component.

Definition 2.3. Let $\sigma(\mathbf{w})$ be the portfolio's risk measure. Then the risk contribution of the i^{th} component, $RC_i(\mathbf{w})$, is

$$RC_i(\mathbf{w}) = w_i MRC_i(\mathbf{w})$$

where MRC_i is given as in Equation 2.2.

Total risk contribution requires the equalization of risk contributions' sum to the total portfolio risk. Euler's decomposition demonstrates the relationship between the portfolio risk measure $\sigma(\mathbf{w})$ and the risk contribution of asset i $RC_i(\mathbf{w})$. With Euler theorem, we obtain the risk measure $\sigma(\mathbf{w})$ as the sum of the risk contributions of the components.

Theorem 2.2. *(Euler's Theorem) Let $R^n \to R$ be a continuous differentiable function. Then f is homogenous of degree d if and only if for $\forall\mathbf{w} \in R^n$, it satisfies the following*

$$df(\sigma(\mathbf{w})) = \sum_{i=1}^{n} w_i \frac{\partial f(\mathbf{w})}{\partial w_i}.$$

Proof. The proof is omitted. One can find detailed proof in [10].

The portfolio volatility can be written as a linear combination of relative risk contributions of assets such that

$$\begin{aligned} \sigma(\mathbf{w}) &= w_1\tfrac{\partial\sigma(\mathbf{w})}{\partial w_1} + w_2\tfrac{\partial\sigma(\mathbf{w})}{\partial w_2}\cdots + w_n\tfrac{\partial\sigma(\mathbf{w})}{\partial w_n} \\ &= \textstyle\sum_{i=1}^{n} w_i MRC_i(\mathbf{w}) \\ &= \mathbf{w}^T MRC(\mathbf{w}) \\ &= \mathbb{1}^T RC(\mathbf{w}) \end{aligned} \tag{2.7}$$

where $MRC(\mathbf{w})$ and $RC(\mathbf{w})$ are $n \times 1$ vectors that represent the marginal and total risk contributions, respectively. $\mathbb{1}$ shows the indicator function. □

The percentage risk contribution of the i^{th} component is the ratio of respective component risk contribution to the portfolio risk, i.e.

$$\%RC_i(\mathbf{w}) = \frac{RC_i(\mathbf{w})}{\sigma(\mathbf{w})}.$$

2.1.4 Diversification Index

Diversification is another important property in the risk based strategies. There are different approaches to measure diversification such as Gini coefficient, Shannon entropy, diversification ratio and Herfindall index.

Chaves et. al. (2012) employs the Gini coefficient to measure the diversification level regarding to both risk contributions and weights of the assets. For each strategy k, the **Gini coefficient** based on risk contributions and weights of assets given as

$$\begin{aligned} \text{Gini}_k(risk) &= \frac{2}{N}\sum_{i=1}^{N}(\sigma_{i,k} - \bar{\sigma}_k) \qquad and \\ \text{Gini}_k(w) &= \frac{2}{N}\sum_{i=1}^{N}(w_{i,k} - \bar{w}_k), \end{aligned} \tag{2.8}$$

respectively. Here, $\sigma_{i,k}$ denotes the i^{th} asset volatility of the k^{th} strategy and $\bar{\sigma}_k$ represents the volatility of the whole portfolio constructed based on the k strategy. $w_{i,k}$ is the weight of the i^{th} asset in the k strategy and $\bar{w}_k$ is the weight of the k strategy.

Gini coefficient takes values between zero and one. If the value is zero, the portfolio is equally weighted in terms of risk or weights. If the value is one, the portfolio is concentrated and not well diversified in terms of risk or weights.

Another approach of measuring diversification is **Shannon entropy** that is given as

$$H = -\sum_{i=1}^{n} RC_{i,k} ln RC_{i,k}$$

where $RC_{i,k}$ is the risk contribution of i^{th} asset in the k strategy. If the risk contributions from assets are identical, the Shannon entropy has the maximum value (*n*) that is the number of value included in the portfolio. If the portfolio is concentrated in one risk source, then the Shannon entropy measure is one. This measure will be re-examined in next chapter.

Diversification ratio and Herfindall Index can be seen in detail in [5], which are not taken into consideration in this study.

2.2 Risk Based Asset Allocation Strategies

This section presents commonly used risk based strategies, namely, equally weighted, global minimum variance, and risk parity strategies. The aim of these strategies is to minimize the portfolio's risk with balancing the risk among assets. Their common characteristic is to exclude the estimation of the expected mean as an input parameter.

2.2.1 Equally weighted strategy

In equally weighted (EW) strategy, investors hold equal weights from each asset in their portfolio. It is also known as "naive strategy" since it does not require any optimization methods and it is easy to implement. EW strategy does not have any estimated parameters contrary to classical mean-variance approach for asset allocation. The estimation of the input parameters are not easy and generally leads to errors. Therefore, EW strategy is considered as robust by excluding estimation parameters. It ignores the risk, return and correlation information of assets. Briefly, EW strategy is not dependent on any moments of returns and optimization construction. Moreover, EW approach is well diversified in terms of weight allocation of assets.

The reason why the EW strategy is counted as a risk-based strategy is it is a kind of risk management tool. Furthermore, the motivation behind EW is not neither a target return nor the generation of complex performance skills [5]

The number of assets included in a portfolio determines the weights. In the case of n number of securities in a portfolio, each asset weight is

$$w_i = \frac{1}{n}, \qquad i = 1, \dots, n.$$

The more assets hold in a portfolio, the lower is the weight allocation. The key role in this strategy belongs to the number of securities in a portfolio. Therefore, this simple strategy takes advantage from the law of large numbers and performs better in the long run for several reasons [5]. First, EW strategy takes benefit from the small-cap bias. Second, it is also interested in smoothing of the asset price volatilities effectively.

For an n securities portfolio, the return (average return) and the risk are

$$\begin{aligned} R &= \frac{1}{n}\sum_{i=1}^{n} R_i, \qquad and \\ \sigma &= \sqrt{(\frac{1}{n})^T \Sigma \frac{1}{n}} = \frac{1}{n}\sqrt{\mathbb{1}^T \Sigma \mathbb{1}}, \end{aligned} \tag{2.9}$$

respectively.

The marginal and total risk contributions of the asset i become

$$\begin{aligned} MRC_i &= \frac{w_i}{\sqrt{w_i^T \Sigma w_i}} \qquad and \\ TRC_i &= w_i \frac{w_i}{\sqrt{w_i^T \Sigma w_i}}, \end{aligned} \tag{2.10}$$

respectively. Despite of its simplicity and equal capital allocation advantages, this approach has some drawbacks such as being illiquid and lack of economic interpretations. However, some researchers claim that EW strategy outperforms the mean-variance model and its sophisticated extended versions based on the Sharpe ratio, and it demonstrates better out of sample results than advanced models [1, 8]. On the other hand, in long term estimations EW strategy is not a reasonable option compared to optimized portfolios [18].

2.2.2 Global minimum variance

Global minimum variance (GMV) strategy aims to construct a portfolio with a lowest possible variance that lies on the the most left of the efficient frontier that is introduced by [22]. Despite of being on the efficient frontier, it does not rely on the expected mean and the covariance matrix is the only input parameter.

The quadratic optimization problem of the strategy to obtain the optimal asset weights goals to have the portfolio with the minimum risk. The input parameters are correlations and volatilities of assets. The unconstrained GMV optimization problem is

$$w = \underset{\mathbf{w} \in R^n}{\text{argmin}} \ \frac{1}{2} \mathbf{w}^T \Sigma \mathbf{w} \tag{2.11}$$

The solution of Equation 2.11 is

$$w = \frac{\Sigma^{-1} \mathbb{1}}{\mathbb{1}^T \Sigma^{-1} \mathbb{1}}.$$

where $\mathbb{1}$ shows indicator function.

If the GMV portfolio is subject to long only and budget constraints, then the optimization problem is

$$\begin{gathered} w = \underset{\mathbf{w} \in R^n}{\text{argmin}} \ \tfrac{1}{2} \mathbf{w}^T \Sigma \mathbf{w} \\ \mathbf{w}^T \mathbb{1} = 1 \\ 0 \leq \mathbf{w} \leq \mathbb{1}. \end{gathered} \tag{2.12}$$

The marginal and total risk contribution of asset i are given as respectively

$$\begin{aligned} MRC_i &= \frac{w_i}{\sqrt{w_i^T \Sigma w_i}} \\ TRC_i &= w_i \frac{w_i}{\sqrt{w_i^T \Sigma w_i}}. \end{aligned} \tag{2.13}$$

GMV strategy equalizes the marginal risk contributions by minimizing the assets' volatilities and correlations. It can be thought that weights are obtained by equalizing the marginal risk contributions.

Some researchers find that GMV strategy outperforms the market-weighted portfolio [5, 13] since its low volatility and high return result in high Sharpe ratio as opposed to market-weighted portfolio. It can be result of the low volatility anomaly such that low risky assets outperform the high risky assets in terms of returns in the long period.

GMV strategy only pursues the reduction of the portfolio risk therefore it does not provide a well diversified portfolio as one expects since GMV strategy concentrates on the low volatility stocks. [8] finds that GMV approach actually generates an undiversified portfolio. GMV portfolio is not a well diversified in both capital allocation and risk allocation.

2.2.3 Risk Parity

Risk parity (RP) is an asset allocation strategy that allocates the weights according to risk characteristics of asset classes. The main concept behind this method is to diminish the concentrated risk from one market regime by obtaining assets based on their respective amount of risk. Covariance matrix is the only input parameter to eliminate the estimation error in the expected return. The aim of the approach is distributing the whole portfolio risk equally based on the volatility of included asset classes. The risk contribution of asset class i to overall risk is the center interest of the RP.

Two different approaches in RP strategy are inverse volatility and equal risk contribution strategies.

2.2.3.1 Inverse Volatility Strategy

Inverse volatility (IV) strategy that is also known as naive risk parity allocates the weights of assets inversely to their risk. The volatilities of the components determine the component weights. Investors apply this method assuming the uniform correlations among all asset classes. In other words, there is no role of the assets' correlations in this strategy. The optimal weights of the components are given by

$$w_i = \frac{\sigma_i^{-1}}{\sum_{j=1}^{n} \sigma_j^{-1}}, \qquad i, j = 1, 2, \ldots, n. \tag{2.14}$$

The asset class with higher volatility has low weight in IV strategy. Note that if asset classes have equal volatilities, i.e. $\sigma_i = \sigma_j$, then strategy would be equally weighted strategy, i.e. $w_i = n^{-1}$ for n assets.

If the portfolio consists of only two asset classes, it provides the same results with the equal risk contribution approach. Furthermore, IV strategy generates an optimal portfolio when assets have the same Sharpe ratios and identical correlations between two assets. However, if there are more than two asset classes, the portfolio becomes very sensitive to the correlation of the assets. The ignorance of the relationship between different securities will thus lead to the potential undiversible portfolio risk problem. The whole portfolio risk is not indeed diminished than one's expectations. Therefore, one who generates an IV portfolio should carefully select the assets since the high correlation among the assets will still cause the same risk and hence the portfolio will even be dominated by one specific risk.

2.2.3.2 Equal Risk Contribution Strategy

From the point of risk budgeting, asset allocation should be in terms of risk contributions of asset classes rather than in terms of asset weights. This means that specifying the preferred risk contributions become starting point to construct the appropriate portfolio. Then the obtained portfolio has asset weights determined by the desired risk allocation.

Risk contribution is redefined by [32] that introduce the risk contribution constraint of each asset class, called risk budget. Suppose, there are n asset classes and set the risk budgets ($b_1, b_2, \ldots, b_n$) and targeted risk contributions are ($TRC_1, TRC_2, \ldots, TRC_n$) to general risk measure R.

Then the risk budgeting portfolio is given as

$$\begin{gathered} TRC_1(w_1, w_2, \ldots, w_n) = b_1 \\ TRC_2(w_1, w_2, \ldots, w_n) = b_2 \\ \vdots \\ TRC_i(w_1, w_2, \ldots, w_n) = b_i \\ \vdots \\ TRC_n(w_1, w_2, \ldots, w_n) = b_n \end{gathered} \tag{2.15}$$

Risk budgeting portfolio does not require the optimization technique and expected return estimation. However, there are some drawbacks. First, the component exposures are not clear. Second, if the assets have negative risk budgets, the portfolio risk is concentrated on other assets, which is not consistent with the diversification aim. Thus, to overcome these problems above system can be applied as a nonlinear system, i.e.

$$w_i(\Sigma w_i) = b_i(\mathbf{w}^T \Sigma \mathbf{w})$$

then the optimization problem is

$$\begin{aligned} w_{RB} = \text{argmin} \quad & \textstyle\sum_{i=1}^{n} (w_i(\Sigma w_i)) - b_i w^T \Sigma w)^2 \\ & \textstyle\sum_{i=1}^{n} w_i = 1 \\ & \textstyle\sum_{i=1}^{n} b_i = 1 \\ & w_i, b_i \geq 0 \end{aligned} \tag{2.16}$$

where w_{RB} is the risk budgeting portfolio weight matrix, w_i represents each asset weights in the portfolio, b_i denotes the risk budget vector. However, the analytical solution of the system in optimization in Equation 2.16 is not possible since it provides the more than one optimal solution. Then, [21] proposes an optimal solution that is also known as “equal risk contribution” strategy (ERC) such that the risk contributions of each asset are equal. The aim of the approach is to assure that any component does not have a dominant role on the whole portfolio risk so that the same risk budget or contribution should be evenly distributed to each component. This can be shown as following

$$w_i \frac{\partial(\mathbf{w})}{\partial w_i} = w_j \frac{\partial(\mathbf{w})}{\partial w_i}, \quad \forall i, j. \tag{2.17}$$

Then the objective function is the minimization of the square of the difference between risk contributions of all pairs of components, i.e.

$$f(\mathbf{w}) = \sum_{i=1}^{n} \sum_{j=1}^{n} (w_i \frac{\partial(\mathbf{w})}{\partial w_i} - w_j \frac{\partial(\mathbf{w})}{\partial w_i})^2 \tag{2.18}$$

$$\begin{gathered} \mathbf{w}_{ERC} = \underset{\mathbf{w} \in R^n}{\text{argmin}} \;\; f(\mathbf{w}) \\ \mathbf{w}^T \mathbb{1} = 1 \\ 0 \leq \mathbf{w} \leq \mathbb{1}. \end{gathered} \tag{2.19}$$

The minimization of objective function $f(\mathbf{w})$ is required. The optimization problem to solve for $\mathbf{w}_{ERC}$ is

$$\begin{gathered} \underset{\mathbf{w}_{ERC}}{\mathbf{min}} \;\; \textstyle\sum_{i=1}^{n} \sum_{j=1}^{n} (w_i \frac{\partial(\mathbf{w})}{\partial w_i} - w_j \frac{\partial(\mathbf{w})}{\partial w_i})^2 \\ s.t. \textstyle\sum_{i=1}^{n} = 1 \\ 0 \leq w_i \leq 1. \end{gathered} \tag{2.20}$$

The volatility of ERC portfolio is higher than the volatility of GMV portfolio and smaller than the volatility of EW portfolio [21], i.e.

$$\sigma^2_{GMV} \leq \sigma^2_{ERC} \leq \sigma^2_{EW}.$$

Contrary to the IV strategy, ERC strategy considers the correlation among asset classes. However, it still underestimates the well-diversified portfolio with considering risk from asset classes and ignoring the underlying risk factors. Likewise IV strategy, ERC approach may be heavily exposed to few number of risk sources and may have concentrated risk structure as opposed to risk diversification goal.

CHAPTER 3

DIVERSIFIED RISK PARITY

A general information about diversified risk parity (DRP) based on the works by Meucci (2010) [24] and Lohre et al. (2011) [20] is presented. This is a special case of risk parity (RP) with employing uncorrelated portfolios as risk sources. Applying risk parity strategy to uncorrelated risk sources and maximizing the number of risk sources in a portfolio is known as "diversified risk parity strategy". The theory of portfolio construction based on the DRP is given with its terminologies. For detailed information and proofs, one can see [20, 26, 24, 25].

In the previous chapter, we examine the RP strategy that aims to distribute the portfolio risk equally among the included asset classes in the portfolio. To accomplish this goal, the asset classes are required to be minimum correlated or if possible uncorrelated to each other.

As mentioned before, correlation among asset classes have become more unstable and higher than expected, especially in market drops. Bhansali et al. [3] demonstrates that RP portfolio constructed with asset classes is not a well-diversified portfolio. The overlap of correlations between asset classes lead to poor diversification of RP strategy. Specifically, during the financial crisis, correlations increase significantly exceeding 90%.

Even if the portfolio is constructed from different asset classes, the portfolio is generally equity risk concentrated since almost all asset classes have a correlation with equity market [3]. Therefore, each asset class is exposed to more than one risk and investing in different asset classes do not guarantee a diversified portfolio in terms of risk. The hidden risk concentration problem leads to academics and investors to focus on independent and underlying drivers of asset returns for a well-diversified portfolio. These drivers are called risk factors.

For a well-diversified portfolio, the key point is that the risk sources or risk factors should be uncorrelated. Partovi and Caputo (2004) [26] use the principal component analysis (PCA) to generate uncorrelated portfolios that are also called as uncorrelated risk sources. However, PCA has some drawbacks and does not provide the robust results in backtesting [27]. Therefore, Meucci et al. [25] propose a new approach namely minimum linear torsion (MLT) model to extract uncorrelated risk factors. Lohre et al. [20] apply RP strategy to uncorrelated portfolios using PCA and MLT and maximizes the number of risk sources in a portfolio. This strategy is called “diversified risk parity” [20].

Contrary to the RP strategy based on asset classes in the previous part, we focus on the RP approach that aims diversification based on the main risk sources driving the asset returns. In this part, we apply DRP strategy to both PCA and MLT approaches. First, we give some theoretical background of PCA and MLT transformations, then we introduce the synthetic portfolios or risk sources generated from these strategies.

3.1 Principal Component Analysis

Principal component analysis (PCA) is a multivariate statistical dimension reduction technique that uses an orthogonal transformation of variables into the linearly uncorrelated synthetic variables, namely principal components (PCs). The general aim of the PCA approach is to find another basis that denotes a linear combination of the original basis and furthermore, this new basis redefines the data optimally.

For calculating PCA, the key point is to determine the new basis that re-expresses the data in a best way. Thus, it is important to construct the new basis based on the independence between principal components. The variance of the original data is taken into consideration to define the independence from the point of PCA. PCA decorrelates the original data by obtaining the directions that have maximum variance and the found directions are utilized to determine the new basis.

Consider a multivariate data matrix X. This matrix is centered without loss of generality, i.e. $X = X - m^{-1}\mathbb{1}\mathbb{1}X$, where m denotes the number of observations. The $n \times n$ covariance matrix S of matrix X is

$$S = n^{-1}X^TX.$$

Then, the linear combination of new variables are written as following

$$p_i = \sum_{j=1}^{n} \alpha_j x_{ij} = \alpha^T \mathbf{x}_i, \quad i = 1, \ldots, n, \tag{3.1}$$

where α is a weighting vector $(\alpha_1, \ldots, \alpha_n)^T$ and $\mathbf{x}_i$ represents the vector of observations, i.e. $(x_{i1}, \ldots, x_{ip})^T$. To define sets of normalized weights, the maximization of variation in the p_i's is required with using PCA. The first step is to find a vector $\mathbf{e}_1 = (e_{11}, \ldots, e_{n1})^T$ that decomposes the covariance matrix S as large as possible such that

$$\begin{aligned} Cov[X\mathbf{e}_1] &= E[(X\mathbf{e}_1)^T X\mathbf{e}_1] - E[X\mathbf{e}_1]^T E[X\mathbf{e}_1] & (3.2)\\ &= E[\mathbf{e}_1^T X^T X\mathbf{e}_1] - X\mathbf{e}_1^T E[X]^T E[X]\mathbf{e}_1^T & (3.3)\\ &= \mathbf{e}_1^T Cov[X]\mathbf{e}_1 = \mathbf{e}_1^T S\mathbf{e}_1. & (3.4) \end{aligned}$$

To guarantee not to obtain arbitrary large values, the next constraint is applied

$$\| \mathbf{e}_1 \|^2 = \mathbf{e}_1^T \mathbf{e}_1 = 1. \tag{3.5}$$

Rearranging formula in Equation 3.1 with the constraint 3.5 gives a maximizing problem of mean square as following

$$\frac{1}{n}\sum_{i=1}^{n} p_{i1}^2 \qquad \text{subject to} \quad || \mathbf{e}_1 ||^2 = 1,$$

where $p_{j1} = \sum_{i=1}^{n} e_{j1}x_{ij} = \mathbf{e}_1\mathbf{x}_i$. The linear combination p_{i1} is known as a "principal component score". To continue to define the basis of new variables, each projection must be uncorrelated from previous ones, i.e.

$$\sum_{j=1}^{n} e_{jk}e_{jl} = \mathbf{e}_k^T\mathbf{e}_l = 0, \qquad k < t \leq n,$$

where t is taken number of steps that are restricted by the number of n variables. This ensures orthogonal projections of the new basis.

Above problem with constraints can be defined as an optimization problem

$$\max_{\mathbf{e}_1} \quad \mathbf{e}_1^T S\mathbf{e}_1 - \lambda(\mathbf{e}_1^T\mathbf{e}_1 - 1), \tag{3.6}$$

that has a solution of

$$\begin{aligned} (S - \lambda\mathbf{I})\mathbf{e}_1 &= 0 \\ S\mathbf{e}_1 &= \lambda\mathbf{e}_1. \end{aligned} \tag{3.7}$$

Above optimization problem finds the eigenvalues λ and corresponding eigenvectors $\mathbf{e}_j$ of the covariance matrix S. An eigenvector demonstrates the component direction in the new space and its eigenvalue explains how much variance there is in the component direction.

The semi-definite covariance matrix S can be decomposed into an orthogonal matrix U and a diagonal matrix Λ, whose entries are eigenvalues $\lambda_1 > \cdots > \lambda_n$. Then the decomposition is

$$S = E^T\Lambda E = \sum_{j=1}^{n} \lambda_j\mathbf{e}_j\mathbf{e}_j^T. \tag{3.8}$$

Using Equation 3.6 and let define $\tilde{\mathbf{e}_1} = U\mathbf{e}_1$, where

$$|| \tilde{\mathbf{e}_1} ||^2 = \tilde{\mathbf{e}_1}^T\tilde{\mathbf{e}_1} = (U\mathbf{e}_1)^T U\mathbf{e}_1 = \mathbf{e}_1^T U^T U\mathbf{e}_1 = e^T e_1 = || e_1 ||^2,$$

then the optimization problem in Equation 3.6 is rearranged as

$$\max_{\tilde{\mathbf{e}}_1 : \tilde{\mathbf{e}}_1^T\tilde{\mathbf{e}}_1 = 1} \tilde{\mathbf{e}}_1^T\Lambda\tilde{\mathbf{e}}_1 = \max_{\tilde{\mathbf{e}}_1 : \tilde{\mathbf{e}}_1^T\tilde{\mathbf{e}}_1 = 1} \sum_{j=1}^{n} \tilde{e}_{1j}^2\lambda_j.$$

This reaches its maximum value when $\tilde{e}_{11} = 1$ and $\tilde{e}_1 = \mathbf{a}_1$. $\mathbf{a}_1$ is the unit vector that has one in the first entry. We know that each component is uncorrelated from each other such that

$$0 = Cov[X\mathbf{e}_1, X\mathbf{e}_2] = e_1^T Cov[X, X]\mathbf{e}_2 = \mathbf{e}_1^T U^T\Lambda U\mathbf{e}_2 = \tilde{\mathbf{e}}_1^T\Lambda\tilde{\mathbf{e}}_2. \tag{3.9}$$

Since $\tilde{e}_1 = \mathbf{a}_1, 0 = \lambda_1 \tilde{e}_{21} \Leftrightarrow \tilde{e}_{21} = 0$. Then, the new optimization problem is

$$\max_{\tilde{\mathbf{e}}_2 : \tilde{\mathbf{e}}_2^T \tilde{\mathbf{e}}_2 = 1; \tilde{\mathbf{e}}_{21} = 0} \tilde{e}_2^T \Lambda \tilde{\mathbf{e}}_2 = \max_{\tilde{\mathbf{e}}_2 : \tilde{e}_2^T \tilde{\mathbf{e}}_2 = 1; \tilde{e}_{21} = 0} \sum_{j=1}^{n} \tilde{e}_{2j}^2 \lambda_j .$$

This reaches its maximum value when $\tilde{e}_{22} = 1$ and $\tilde{\mathbf{e}}_2 = \mathbf{a}_2$. $\mathbf{a}_2$ is the unit vector that has one in the second entry.

This optimization is repeated until p principal components are found.

3.1.1 Principal Portfolios

In real-life applications, portfolios consist of securities with non-zero covariances. Partovi and Caputo (2004) [26] employs PCA to construct the uncorrelated portfolios, called principal portfolios. The definitions and theorems in this section are based on [26, 24]. These portfolios are realizable whenever there is no constraint on short-selling. Furthermore, these portfolios can be evaluated as uncorrelated risk sources.

Definition 3.1. [Principal Portfolios (PP)] Let Σ be an $n \times n$ covariance matrix. Applying principal component decomposition to Σ as in Equation 3.8, we obtain that $E^T \Sigma E = \Lambda = diag(\lambda_1, \ldots, \lambda_n)$. It is equivalent to $E^{-T} \Lambda E^{-1} = \Sigma$. The columns of E are called principal portfolios.

Definition 3.2. Let w be an $n \times 1$ weight vector of original portfolio, and E is an eigenvector matrix of covariance matrix Σ of original data. Then, unique vectors $\tilde{w}_{PP}$ satisfying

$$w = E \tilde{w}_{PP}$$

and equivalently

$$\tilde{w}_{PP} = E^{-1} w = E^T w$$

are called principal portfolio weights.

Remark 3.1. Let R be a matrix consisting of returns of original securities, then $\tilde{R}_{PP}$ is a vector of combination of asset returns that are represented in the principal component space. Then $\tilde{R}_{PP}$ satisfies that

$$E \tilde{R}_{PP} = R$$

which implies to

$$\tilde{R}_{PP} = E^{-1} R.$$

Using Definition 3.1, next proposition is presented.

Proposition 3.1. *The return of i^{th} principal portfolio $\tilde{r}_{PP,i}$ is linear combination of original return matrix $R = (r_1, r_2, \ldots, r_n)$. $\tilde{r}_{PP,i}$ is given as*

$$\tilde{r}_{PP,i} = e_i^T R, \quad i = 1, 2, \ldots, n.$$

The variance of $\tilde{r}_{PP,i}$ is

$$\sigma^2(\tilde{r}_{PP,i}) = e_i^T \Sigma e_i = \lambda_i$$

with the covariance between different principal portfolios i^{th} and j^{th}

$$Cov(\tilde{r}_{PP,i}, \tilde{r}_j) = e_i^T \Sigma e_j = 0.$$

This refers that the i^{th} and j^{th} principal portfolios are mutually uncorrelated, thus an investor is able to invest in uncorrelated principal portfolios or risk sources. Furthermore, these principal portfolios are constructed with the variances as large as possible via the constraint $e^T e = 1$. Hence the first principal portfolio has the largest variance λ_1, second portfolio has the second largest variance λ_2. This continues n principal portfolios with n variances and n^{th} principal portfolio has the smallest variance λ_1.

Total variance of principal portfolios is established by next proposition.

Proposition 3.2. *Let R be return of original assets and $\sigma^2(R)$ denotes the variance of original securities, then total variance of principal portfolios $\sigma^2(\tilde{R}_{PP})$ is represented as*

$$\sigma^2(R) = \sigma^2(\tilde{R}_{PP}) = tr(\Sigma) = \sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} \sigma^2(\tilde{r}_{PP,i})$$

where Σ is a covariance matrix of original data, λ_i represents eigenvalues of Σ and $\sigma^2(\tilde{r}_{PP,i})$ is a variance of each principal portfolio.

As the synthetic portfolios are uncorrelated, we can add the variances directly.

Remark 3.2. Since the sum of the each principal portfolio variance is equal to total variance, the risk contribution of each principal portfolio to total variance can be written as

$$\frac{\sigma^2(\tilde{r}_{PP,i})}{\sigma^2(R)} = \frac{\lambda_i}{\lambda_1 + \lambda_2 + \cdots + \lambda_n}.$$

As seen from the Proposition 3.2, total variances of original data return (R) and total variance of artificial principal portfolio returns ($\tilde{R}_{PP}$) are the same. However, total variance of uncorrelated principal portfolios are directly additive. The Remark (3.2) indicates the variance contribution of each individual principal portfolios to total risk.

Remark 3.3. The marginal risk contribution of each principal portfolio is equal to

$$MRC_{PP} = \frac{\partial \sigma(\tilde{R}_{PP})}{\partial w_i} = \frac{1}{2\sqrt{\sum_{i=1}^{n} \tilde{w}_{PP,i}^2 \lambda_i}} 2\tilde{w}_{PP,i}\lambda_i = \frac{\tilde{w}_{PP,i}\lambda_i}{\sigma(\tilde{R}_{PP})}. \tag{3.10}$$

Since covariances in the principal space are equal to zero, the risk contribution of each principal portfolio is given by

$$\tilde{RC}_{PP,i} = \frac{\tilde{w}_{PP,i}^2 \lambda_i}{\sqrt{\sum_{i=1}^{n} \tilde{w}_{PP,i}^2 \lambda_i}} = \frac{\tilde{w}_{PP,i}^2 \lambda_i}{\sigma(\tilde{R}_{PP})}. \tag{3.11}$$

3.1.2 Diversified risk parity using Principal Component Analysis

Lohre et al. (2011) [20] apply risk parity strategy to uncorrelated portfolios and distribute the portfolio risk among uncorrelated risk sources. To prevent the risk concentration, a portfolio should have maximum number of risk sources and the risk should be allocated among these risk sources uniformly [24]. To achieve this goal, exponential of Shannon entropy should reach its maximum value. The definitions and theorems in this section are based on [24, 20].

First, we examine how Meucci (2010) [24] utilizes Shannon Entropy as a risk diversification measure, and then we explain how risk parity strategy is applied to uncorrelated risk factors.

It starts with introducing some terminologies which are used by Meucci (2010) for risk diversification in principal space. Following Meucci's (2010) path, firstly the variance concentration curve is introduced as

$$v_{PP,i} = \tilde{w}^2_{PP,i}\lambda_i, \qquad i = 1, 2, \ldots, n \tag{3.12}$$

where $v_{PP,i}$ denotes the variance of the i^{th} principal portfolio, whose weight is $\tilde{w}_{PP,i}$ and variance is λ_i. Due to uncorrelated link among weighted principal portfolios, the total variance is

$$\begin{aligned} \sigma^2_P(w) &= w'\Sigma w \\ &= w'E\Lambda E'w \\ &= \tilde{w}'_{PP}\Lambda\tilde{w}_{PP} \\ &= \sigma^2_P(\tilde{w}_{PP}) \\ &= \textstyle\sum_{i=1}^{n} \tilde{w}^2_{PP,i}\lambda_i \\ &= \textstyle\sum_{i=1}^{n} v_{PP,i}. \end{aligned} \tag{3.13}$$

Then, the standard deviation of portfolio $P(\tilde{w}_{PP})$ is $\sigma(\tilde{w}_{PP})$ that is utilized in volatility concentration curve given as

$$s_{PP,i} = \frac{v_{PP,i}}{\sigma_P(\tilde{w}_{PP})} = \frac{\tilde{w}^2_{PP,i}\lambda_i}{\sqrt{\sum_{i=1}^{n} \tilde{w}^2_{PP,i}\lambda_i}}, \qquad i = 1, 2, \ldots, n. \tag{3.14}$$

In fact, Equation 3.14 also demonstrates the sensitivity of variance contribution from each principal portfolio to changes in portfolio weights. In other words, the volatility concentration curve is a decomposition of portfolio volatility regarding the corresponding weighted principal portfolios [19].

After that, the diversification distribution p is given as following [24]

$$p_{PP,i} = \frac{v_{PP,i}}{\sigma^2_P(\tilde{w}_{PP})} = \frac{\tilde{w}^2_{PP,i}\lambda_i}{\sum_{i=1}^{n} \tilde{w}^2_{PP,i}\lambda_i}, \qquad i = 1, 2, \ldots, n. \tag{3.15}$$

Considering the given expressions, it can be intuitively said that each principal portfolio should affect the portfolio risk equally. Since the principal portfolios are uncorrelated, the manager who wants the well-diversified portfolio should invest the principal portfolios to achieve the uniform diversification distribution. This uniform diversification distribution leads the principal portfolios to have the same exposures to the shocks.

When the variance concentration curve ($v_{PP,i}$) is normalized, the following property is obtained

$$\sum_{i=1}^{n} p_{PP,i} = 1, \qquad 0 \leq p_{PP,i} \leq 1.$$

Then, if probability masses $p_{PP,i}$ are almost equal for each i, a well-diversified portfolio is constructed. An investor should avoid concentrated probability masses. With utilizing the probability masses $p_{PP,i}$s, this challenge is achieved by maximum entropy.

Entropy, a concept used in Physics to measure the level of uncertainty of a system, has also a relationship with portfolio diversification. The relation between portfolio diversification and entropy is based on the notion of uncertainty. Also, entropy is utilized as the level of predictability of a stochastic system in Information Theory. Higher entropy means less predictable system. If additional information enters the system, this decreases entropy value. From the same point of view, investors use risk diversification in the case of lack of information or uncertain financial markets. Thus, especially the Principle of Maximum Entropy plays an important role in measuring diversification. The principle is based on the foundation that estimation of the probability distribution requires the selection of distribution which leaves the largest uncertainty (i.e. maximum entropy) consistent with the applied constraints. Thus, any additional assumptions are not needed in the calculation. More details and proofs for entropy can be found in [34, 12, 35].

Definition 3.3. Let p be a discrete probability function on given set $z_1, z_2, \ldots, z_n$ with $p_i = p(z_i)$, the *entropy* of p is given as

$$H = -\sum_{i=1}^{n} p_i log p_i. \tag{3.16}$$

The link between risk diversification and exponential of Shannon entropy is given in next definition.

Definition 3.4. Consider Shannon Entropy given in Definition 3.3 and diversification distribution of principal portfolios in Equation 3.15. The number of uncorrelated factors, N_{Ent}, that are the exponential entropy of the diversification distribution is defined as

$$N_{PP,Ent} = exp(-\sum_{i=1}^{n} p_{PP,i} log p_{PP,i}). \tag{3.17}$$

Remark 3.4. From Definition 3.3 and 3.4, the following results can be reached:

(i) $N_{PP,Ent}$ reaches its maximal value (i.e. $N_{PP,Ent} = n$) if the system is completely unpredictable, where all principal portfolios are equally likely i.e, $p_{PP,i} = \dfrac{1}{n}$ for all i. This case represents the well-diversified portfolio.

(ii) $N_{PP,Ent}$ has the minimum value, $N_{PP,Ent} = 1$, if the system is completely deterministic where the probability of one principal portfolio is one. In this case, a sharp peak happens in the diversification distribution and leads ill-diversified portfolio.

(iii) If probability masses $p_{PP,i}$s are uniform on k (such that $k < n$) principal portfolios then $N_{PP,Ent} = m$.

$N_{PP,Ent}$ ranges from one (highest concentration) to n (highest diversification). The exponential of entropy measure is defined for long-only portfolios whose sum of weights is one. In fact, the maximal value of measure is achieved when the naive risk parity strategy (i.e. weights are inversely proportional to assets' volatility) is applied [28]. Also the number of risk sources in the risk space are denoted by $N_{PP,Ent}$.

Given a portfolio without constraints, maximum entropy is achieved if $p_{PP,i}$ has uniform distribution. This achievement is a sign that each risk equally contribute to total risk.

Definition 3.5 (Diversified risk parity)**.** If the diversification distribution in Equation 3.15 is close to uniform, the strategy is called diversified risk parity.

The optimization problem is

$$\begin{gathered} \underset{w_{PP}}{\text{argmax}} \;\; N_{PP,Ent} \\ \text{subject to } \; \mathbb{1}^T \mathbf{w}_{PP} = 1 \\ -1 \leq \mathbf{w}_{PP} \leq 1. \end{gathered} \tag{3.18}$$

DRP is also obtained by applying ERC optimization in Equation 2.20 to principal portfolios with using Remark 3.3 gives that

$$\begin{gathered} \tilde{\mathbf{w}}^*_{PP} = \text{argmin} \quad f(\tilde{\mathbf{w}}_{PP}) \\ \text{subject to } \; \mathbb{1}^T \tilde{\mathbf{w}}_{PP} = 1 \\ 0 \leq \tilde{\mathbf{w}}_{PP} \leq 1 \\ \text{where } \; f(\tilde{\mathbf{w}}_{PP}) = \textstyle\sum_{i=1}^n \sum_{j=1}^n (\tilde{RC}_i - \tilde{RC}_j)^2 \end{gathered} \tag{3.19}$$

Due to zero covariances in the principal space, the weights can be calculated from a closed-form solution as in Equation 2.14. Then the optimal weights of the principal portfolios are given by

$$\tilde{w}^*_{PP,i} = \frac{(\sqrt{\lambda_i})^{-1}}{\sum_{i=1}^n (\sqrt{\lambda_i})^{-1}}. \tag{3.20}$$

Therefore, the optimal weights $w^*_{PP,i}$ provide the equal risk contribution from each risk factor.

ERC to principal portfolios and DRP strategies are based on different risk measures. While ERC strategy aims equal risk contributions, DRP approach targets the uniform diversification distribution. However, the optimization solutions in Equation 3.19 and Equation 3.18 give the same results without any further constraints since the risk contributions $\tilde{RC}_{PP,i}$ in principal space almost equal the diversification distribution $p_{PP,i}$.

3.2 Minimum Linear Torsion Transformation

PCA approach has some drawbacks such that the factors from PCA cannot be economically interpretable and the factors are presented by low eigenvalues may be unstable over the time. Furthermore, it does not provide robust results in backtests [27, 16]. Minimum linear torsion (MLT) model is a way to extract uncorrelated risk factors. The definitions and theorems in this section are based on [25, 2].

Let $X \quad (= X_1, X_2, \ldots, X_n)^T$ be a random vector that gives the return of n number of assets. represent the $n \times m$ random matrix with n variables and m observations. Then, the orthogonal variables, X_t, can be obtained as

$$X_t = t \times X,$$

where t is an $n \times n$ transformation matrix that is generated by MLT of original data.

The MLT approach guarantees that synthetic variables represent the nearest uncorrelated representation of original data. Furthermore, new variables have the same volatility with the original variables. Thus, the covariance matrix of new data D^2 can also rewritten as

$$D^2 = diag(\Sigma)$$

where Σ denotes the covariance matrix of original data.

Diagonalizing covariance matrix Σ is equivalent to diagonalizing the correlation matrix C, i.e.

$$C = \Sigma = \text{diag}(\Sigma)^{1/2} C \text{diag}(\Sigma)^{1/2}$$

where $diag(.)$ presents the diagonal elements of a square matrix and its square root is the square root of the diagonal elements.

Let us consider Cholesky decomposition of correlation matrix C as below

$$\begin{aligned} C &= LDL' && (3.21) \\ &= LD^{1/2}D^{1/2}L' && (3.22) \\ &= (LD^{1/2}L')(LD^{1/2}L'), && (3.23) \end{aligned}$$

where L is a lower triangular matrix and D is a diagonal matrix. Define

$$c = LD^{1/2}L' \quad \text{and} \quad c = c', \tag{3.24}$$

then

$$C = cc' = c^2. \tag{3.25}$$

The minimum linear torsion transformation requires the minimization of the net tracking errors between the generated variables and original variables. The optimization problem is

$$t = \underset{Corr(tX)=I_{n\times n}}{\text{argmin}} \sqrt{\frac{1}{n}\sum_{i=1}^{n} Var(\frac{(tX)_i - X_i}{\sigma_i})} \tag{3.26}$$

where σ represents the volatility of original data. Transformation matrix t ensures new variables are uncorrelated.

The minimum linear torsion transformation is obtained by minimizing the squared net tracking errors between the original data and new synthetic data. Then, Equation 3.26 can be rearranged as

$$t = \underset{Corr(tX)=I_{n\times n}}{\text{argmin}} [NTE\{x_{t_1}, x_1\}^2 + \cdots + \{x_{t_n}, x_n\}^2] \tag{3.27}$$

where n is the number of original variables and $NTE\{.\}$ represents the net tracking error (NTE) function. Then the solution for the NTE becomes

$$\begin{aligned} \sum_{i=1}^{n} \text{NTE}\{x_{t_i}, x_i\}^2 &= \sum_{i=1}^{n} \text{Var}(x_{t_i} - x_i) = \sum_{i=1}^{n} \text{Var}(a_i'x - b_i'x) && (3.28) \\ &= \sum_{i=1}^{n} \text{Var}([a_i - b_i]'x) = \sum_{i=1}^{n} \text{Var}[a_i - b_i]'\Sigma[a_i - b_i] && (3.29) \\ &= tr([t - I_n]'\Sigma[t - I_n]) = tr(t'\Sigma t - t'\Sigma - \Sigma t + \Sigma) && (3.30) \\ &= tr(D^2) + tr(\Sigma) - 2tr(t'\Sigma), && (3.31) \end{aligned}$$

where a_i is the i^{th} column of matrix t and b_i is the i^{th} elementary factor. The minimization the NTE can be obtained by maximizing $tr(t'\Sigma)$ (for details see [25]).

First, let decompose the matrix Σ by PCA, i.e. $\Sigma = E\Lambda^2 E$, E is an eigenvector matrix and Λ^2 represent the eigenvalue matrix. Then, $tr(t'\Sigma)$ can be expanded as

$$tr(t'\Sigma) = tr(DD^{-1}t'E\Lambda\Lambda E'). \tag{3.32}$$

Let $P' = D^{-1}t'E\Lambda$, then Equation 3.32 can be rewritten as

$$tr(t'\Sigma) = tr(P'\Lambda E'D) \tag{3.33}$$

where P holds the property $PP' = I_n$ and I_n denotes the $n \times n$ identity matrix. Next, we apply the singular value decomposition to $\Lambda E'D$ as

$$\Lambda E'D = USV' \tag{3.34}$$

where U and V are orthogonal matrices to each other. The diagonal matrix S has singular values of $\Lambda E'D$. The substitution of $\Lambda E'D$ to Equation 3.34 gives

$$tr(P'\Lambda E'D) = tr(P'USV') = tr(V'P'US). \tag{3.35}$$

Let $V'P'US$ denote as Z that satisfies $ZZ' = I_n$. Then substituting the Z to Equation 3.35 gives

$$tr(V'P'US) = tr(ZS) = \sum_{i=1}^{n} z_{ii}s_{ii} \leq \sum_{i=1}^{n} s_{ii}. \tag{3.36}$$

Equation 3.36 is maximized when $z_{ii} = 1$ for all i, i.e. $Z = I_n$. Solving the Equation 3.36 for P', we obtain

$$Z = V'P'U = I_n$$

then,

$$P' = UV'.$$

Since $P' = D^{-1}t'E\Lambda$, the transformation matrix can be solved as following

$$P' = D^{-1}t'E\Lambda \tag{3.37}$$

$$UV' = D^{-1}t'E\Lambda \tag{3.38}$$

$$t = E\Lambda^{-1}UV'D. \tag{3.39}$$

Then the transformation matrix is

$$t = E\Lambda^{-1}UV'D. \tag{3.40}$$

Definition 3.6. [Minimum-Torsion Transformation] Consider a random vector $X(= X_1, X_2, \ldots, X_n)^T$ which represents asset returns. Minimum-torsion transformation minimizes the tracking error between vector X and new vector tX and ensures that the vectors X and tX are uncorrelated. Torsion matrix t is given below

$$t = \underset{Corr(tX)=I_{n\times n}}{\text{argmin}} \sqrt{\frac{1}{n}\sum_{i=1}^{n} Var(\frac{(tX)_i - X_i}{\sigma_i})} \tag{3.41}$$

where σ_i represents the volatility of the vector X_i.

Theorem 3.3. *Let $X(= X_1, X_2, \ldots, X_n)$ be a random vector, t is an $n \times n$ rotation or torsion matrix, then minimum linear torsion transformation exists as described in Definition 3.6. The transformation matrix t in Definition 3.6 is equivalent to*

$$t = \textit{diag}(\sigma)\alpha c^{-1}\textit{diag}(\sigma)^{-1} \tag{3.42}$$

where $diag(.)$ takes the diagonal entries by making the diagonal matrix whose non-diagonal entries are zero and diagonal entries consist of $n \times 1$ vector v, c is given in Equation 3.25, α is a perturbation matrix computed with using a recursive algorithm as given in Table 3.1.

The Equation 3.41 is equivalent in solving below optimization

$$\begin{aligned} t^* = \text{argmin} \ & tr(Cov\{\text{diag}(\sigma)^{-1}t\mu - \text{diag}(\sigma)^{-1})\mu\}) \\ \text{subject to} \ & Corr\{t\mu\} = I \end{aligned} \tag{3.43}$$

where Cov$\{.\}$ is the $n \times n$ covariance matrix of the inside subject, μ denotes the mean of the variables, *Corr*$\{.\}$ is the $n \times n$ correlation matrix of the inside subject and *I* is the $n \times n$ identity matrix. *diag*(.) extracts the diagonal entries of the covariance matrix.

Let $Q = diag(\sigma)^{-1}\mu$ denote as the normalized means. Then

$$\Sigma_Q = \Omega_\mu = \text{diag}(\sigma)^{-1}\Sigma_\mu \text{diag}(\sigma)^{-1} \tag{3.44}$$

where Ω is the correlation matrix of means of the variables.

$$t^* = \text{argmin} \;\; tr(Cov\{\text{diag}(\sigma)^{-1} t \text{diag}(\sigma)Q - Q\}). \tag{3.45}$$

Rewriting Equation 3.45 in a different way gives as

$$t^* = \text{diag}(\sigma)\alpha\text{diag}(\sigma)^{-1} \tag{3.46}$$

where α solves that

$$\begin{aligned}
\alpha^* &= \text{argmin} \;\; tr(Cov\{(\alpha - I)Q\}) & (3.47)\\
&= \text{argmin} \;\; tr((\alpha - I)\Omega(\alpha' - I)) & (3.48)\\
&= \text{argmin} \;\; tr(\alpha\Omega\alpha' - \alpha\Omega - \Omega\alpha' + \Omega) & (3.49)\\
&= \text{argmin} \;\; tr(\alpha\Omega\alpha' - 2\alpha\Omega) + \bar{a}. & (3.50)
\end{aligned}$$

We can decompose the correlation matrix Ω with Cholesky decomposition as in Equation **??**. Then Equation 3.50 becomes

$$\alpha^* = \text{argmin} \;\; tr(\alpha cc'\alpha' - 2\alpha cc) \tag{3.51}$$

We denote $\tau = \alpha c$, then

$$\alpha^* = \tau^* c^{-1} \tag{3.52}$$

where

$$\tau^* = \text{argmin} \;\; tr(\tau\tau' - 2\tau c). \tag{3.53}$$

Meucci (2014) uses the solution from [9], the optimization in Equation 3.53 has an iterative process with two steps.

First, denote

$$\tau = xy$$

where x is a diagonal matrix and y is orthonormal matrix. In step 1, we assume the diagonal matrix x written as $\tau\tau' = x^2$. Then the optimization in Equation 3.53 is re-written with using the symmetry of Cholesky decomposition as following

$$\tau^* = \text{argmin} \;\; tr(\tau\tau' - 2\tau c) = \text{argmax} \;\; tr(c\tau) \tag{3.54}$$

The Procrustes problem seeks an orthogonal matrix that closely transforms a matrix into a second matrix. The optimization in Equation 3.54 requires employing the solution of orthogonal Procrustes problem from [33] whose solution is given by [25]

$$\begin{aligned}
\tau^* &= x(\sqrt{(xc^2x)})^{-1}xc & (3.55)\\
y^* &= x^{-1}\tau^* = (\sqrt{(xc^2x)})^{-1}xc & (3.56)\\
x^* &= \text{diag}(\text{diag}^{-1}(yc)). & (3.57)
\end{aligned}$$

The solution of τ^* is obtained by alternating the x^* and y^* in Equations 3.56 and 3.57. The algorithm of τ is given as

Goal: Perturb (c)

Table 3.1: The algorithm of perturbation matrix τ [25, p. 7]

Step 1	Fix diagonal matrix x as an identity matrix, $x = I$.
Step 2	We know that $c = e\sqrt{\lambda}e'$. Then, using the solution of orthogonal Procrustes problem, u comes from the Cholesky decomposition in 3.24 $u = (xc^2x)^{1/2}$.
Step 3	Rotation property of the transformation matrix t comes from $y = u^{-1}xc$.
Step 4	Stretching property of the torsion matrix t comes from $x = \text{diag}(diag^{-1}(yc))$ where $diag^{-1}(.)$ extracts an $n \times 1$ vector on the principal diagonal.
Step 5	Finally, perturbation is $\tau = xy$
Step 6	If the outcomes of τ converges, the algorithm stops, otherwise it goes to 1.

Proposition 3.4. *Let Σ be an $n \times n$ matrix. It can be decomposed by torsion matrix t as following*

$$\Sigma = (t')^{-1}\Sigma_t t^{-1} \tag{3.58}$$

where Σ_t consists of only diagonal entries, i.e. $\sigma_t = \sigma_{t,1}, \sigma_{t,2}, \ldots, \sigma_{t,n}$.

3.2.1 Minimum Linear Torsion Portfolios

The design of the principal portfolios is based on capturing the most of the original assets' volatility. They are statistical factors that do not exhibit clear economic interpretation and depict unstable character over period as mentioned before. To overcome these problems, Meucci et al. (2015) propose a factor model that produces uncorrelated portfolios. We design minimum linear torsion portfolios as following. The definitions and theorems in this section are based on [25, 2].

Definition 3.7. [Minimum linear torsion portfolios (MTP)] Let t is an $n \times n$ minimum torsion transformation matrix as given is Theorem 3.3. Each column of t matrix, $t_1, t_2, \ldots, t_n$ for *n* number of assets, is called "minimum linear torsion portfolio" (MTP).

Definition 3.8. Let w be an $n \times 1$ weight vectors of original portfolio, and t is a minimum torsion matrix of covariance matrix Σ of original data. Then, unique vectors $\tilde{w}_{MTP}$ satisfying

$$\tilde{w}_{MTP} = t'^{-1}w$$

are called minimum torsion portfolio weights.

Remark 3.5. Let R be a matrix consisting of returns of original securities, then $\tilde{R}_{MTP,i}$ is a vector of combination of asset returns that are represented in the minimum torsion space. Then $\tilde{R}_{MTP}$ satisfies that

$$R = t'\tilde{R}_{MTP}$$

which implies to

$$\tilde{R}_{MTP} = t'^{-1}R.$$

Using Definition 3.7, next proposition is presented.

Proposition 3.5. *The return of i^{th} minimum torsion portfolio $\tilde{r}_{MTP,i}$ is linear combination of original return matrix $R = (r_1, r_2, \ldots, r_n)$. $\tilde{r}_{MTP,i}$ is given as*

$$\tilde{r}_{MTP,i} = t_i'^{-1} R, \quad i = 1, 2, \ldots, n.$$

The variance of $\tilde{r}_i$ is

$$\sigma^2(\tilde{r}_{MTP,i}) = t_i'^{-1} \Sigma_t t_i^{-1}$$

with the covariance between different principal portfolios i^{th} and j^{th}

$$Cov(\tilde{r}_{MTP,i}, \tilde{r}_{MTP,j}) = 0.$$

This refers that the i^{th} and j^{th} minimum portfolios are mutually uncorrelated, thus an investor is able to invest in uncorrelated principal portfolios or risk sources.

Total variance of minimum torsion portfolios is established by next proposition.

Proposition 3.6. *Let R be return of original assets and $\sigma^2(R)$ denotes the variance of original securities, then total variance of principal portfolios $\sigma^2(\tilde{R}_{MTP})$ is represented as*

$$\sigma^2(R) = \sigma^2(\tilde{R}_{MTP}) = \sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} \sigma_{t,i}^2(\tilde{r}_{MTP,i})$$

where Σ is a covariance matrix of original data, $\sigma_{t,i}$ is the variance of each minimum torsion portfolio.

As the synthetic portfolios are uncorrelated, we can add the variances directly.

Remark 3.6. Since the sum of the each minimum torsion portfolio variance is equal to total variance, the risk contribution of each minimum torsion portfolio to total variance can be written as

$$\frac{\sigma^2(\tilde{r}_i)}{\sigma^2(R)} = \frac{\sigma_{t,i}^2}{\sigma_{t,1}^2 + \sigma_{t,2}^2 + \cdots + \sigma_{t,n}^2}.$$

As seen from the Proposition 3.6, total variances of original data return (R) and total variance of artificial minimum linear torsion (MT) portfolio returns ($\underset{\sim}{R}_{MTP}$) are the same. However, total variance of uncorrelated MT portfolios are directly additive. The Remark (3.6) indicates the variance contribution of individual MT portfolios to total risk.

Remark 3.7. The marginal risk contribution of each minimum torsion portfolio is equal to

$$MRC_{MTP} = \frac{\partial \sigma(\tilde{R})}{\partial w_i} = \frac{1}{2\sqrt{\sum_{i=1}^{n} \tilde{w}_{MTP,i}^2 \sigma_{t,i}}} 2\tilde{w}_{MTP,i}\sigma_{t,i} = \frac{\tilde{w}_{MTP,i}\sigma_{t,i}}{\sigma(\tilde{R}_{MTP})}. \qquad (3.59)$$

Since covariances in the minimum torsion space are equal to zero, the risk contribution of each minimum torsion portfolio is given by

$$\tilde{RC}_{MTP,i} = \frac{\tilde{w}^2_{MTP,i}\lambda_i}{\sqrt{\sum_{i=1}^{n} \tilde{w}^2_{MTP,i}\sigma_{t,i}}} = \frac{\tilde{w}^2_{MTP,i}\sigma_{t,i}}{\sigma(\tilde{R}_{MTP})}. \tag{3.60}$$

3.2.2 Diversified Risk Parity using Minimum Torsion Transformation

The derivation of diversified risk parity from minimum torsion portfolios (DRP_{MTP}) follows the similar path as in 3.1.2. Firstly the variance concentration curve is introduced as

$$v_{MTP,i} = \tilde{w}^2_{MTP,i}\sigma^2_{t,i}, \qquad i = 1, 2, \ldots, n \tag{3.61}$$

where $v_{MTP,i}$ denotes the variance of the i^{th} minimum torsion portfolio, whose weight is $\tilde{w}_{MTP,i}$ and variance is $\sigma_{t,i}$. Due to uncorrelated link among weighted minimum portfolios, the total variance is

$$\begin{aligned} \sigma^2_P(w_{MTP}) = w'\Sigma w = w't'^{-1}\Sigma_t t^{-1} w = \tilde{w}'_{MTP}\Sigma_t \tilde{w}_{MTP} = \sigma^2_P(\tilde{w}_{MTP}) \\ = \sum_{i=1}^{n} \tilde{w}^2_{MTP,i}\sigma^2_{t,i} = \sum_{i=1}^{n} v_{MTP,i}. \end{aligned} \tag{3.62}$$

Then, the standard deviation of $P(\tilde{w}_{MTP})$ is $\sigma(\tilde{w}_{MTP})$ that utilized in volatility concentration curve given as

$$s_{MTP,i} = \frac{v_{MTP,i}}{\sigma_P(\tilde{w}_{MTP})} = \frac{\tilde{w}^2_{MTP,i}\sigma^2_{t,i}}{\sqrt{\sum_{i=1}^{n} \tilde{w}^2_{MTP,i}\sigma^2_{t,i}}}, \qquad i = 1, 2, \ldots, n. \tag{3.63}$$

In fact, Equation 3.63 also demonstrates the sensitivity of variance contribution from each MT portfolio to changes in portfolio weights. In other words, the volatility concentration curve is a decomposition of portfolio volatility regarding the corresponding weighted MT portfolios [19].

After that, the diversification distribution is given as following

$$p_{MTP,i} = \frac{v_{MTP,i}}{\sigma^2_P(\tilde{w})} = \frac{\tilde{w}^2_{MTP,i}\sigma^2_{t,i}}{\sum_{i=1}^{n} \tilde{w}^2_{MTP,i}\sigma^2_{t,i}}, \qquad i = 1, 2, \ldots, n. \tag{3.64}$$

Considering the given expressions, it can be intuitively said that each minimum torsion portfolio should affect the portfolio risk equally. Since the MT portfolios are uncorrelated, the manager who wants the well-diversified portfolio should invest these portfolios to achieve the uniform diversification distribution. This uniform diversification distribution leads the minimum torsion portfolios to have the same exposures to the shocks.

When the variance concentration curve ($v_{MTP,i}$) is normalized, the following property is obtained

$$\sum_{i=1}^{n} p_{MTP,i} = 1, \qquad 0 \leq p_i \leq 1.$$

Then, if probability masses $p_{MTP,i}$s are almost equal, a well-diversified portfolio is constructed. An investor should avoid concentrated probability masses. With utilizing the probability masses $p_{MTP,i}$s, this challenge is achieved by maximum entropy. Since the entropy is introduced before, we give directly the definition the number of uncorrelated risk factors based on minimum torsion transformation.

Definition 3.9. Consider Shannon Entropy given in Definition 3.3 and diversification distribution of MT portfolios in Equation 3.64. The number of uncorrelated risk factors, $N_{MTP,Ent}$, that are the exponential entropy of the diversification distribution is defined as

$$N_{MTP,Ent} = exp(-\sum_{i=1}^{n} p_{MTP,i} log p_{MTP,i}). \tag{3.65}$$

$N_{MTP,Ent}$ ranges from one (highest concentration) to n (highest diversification). The exponential of entropy measure is well-designed for long-only portfolios whose sum of weights is one. In fact, the maximal value of measure is achieved when the naive risk parity strategy, i.e. weights are inversely proportional to assets' volatility, is applied [28]. Also average number of relevant assets in the risk space are denoted by $N_{MTP,Ent}$.

Given a portfolio without constraints, maximum entropy is achieved if $p_{MTP,i}$ has uniform distribution. This achievement is a sign that each risk equally contribute to total risk.

Definition 3.10 (Diversified risk parity using minimum torsion transformation)**.** If the diversification distribution in Equation 3.64 is close to uniform, the strategy is called diversified risk parity.

The optimization problem becomes

$$\begin{array}{c} \underset{w_{MTP}}{\text{argmax}} \;\; N_{MTP,Ent} \\ \text{subject to } \; \mathbb{1}^T \mathbf{w}_{MTP} = 1 \\ -1 \leq \mathbf{w}_{MTP} \leq 1. \end{array} \tag{3.66}$$

DRP is also obtained by applying ERC optimization in Equation 2.20 to MT portfolios with using Remark 3.7 gives that

$$\begin{array}{c} \tilde{\mathbf{w}}^{*}_{MTP} = \text{argmin} \;\; f(\tilde{\mathbf{w}}_{MTP}) \\ \text{subject to } \; \mathbb{1}^T \tilde{\mathbf{w}}_{MTP} = 1 \\ 0 \leq \tilde{\mathbf{w}}_{MTP} \leq 1 \\ \text{where } \; f(\tilde{\mathbf{w}}_{MTP}) = \sum_{i=1}^{n} \sum_{j=1}^{n} (\tilde{RC}_i - \tilde{RC}_j)^2 \end{array} \tag{3.67}$$

Due to zero covariances in the minimum torsion space, the weights can be calculated from a closed-form solution as in Equation 2.14. Then the optimal weights of the principal portfolios are given by

$$\tilde{w}^*_{MTP,i} = \frac{(\sqrt{\lambda_i})^{-1}}{\sum_{i=1}^{n}(\sqrt{\lambda_i})^{-1}}. \tag{3.68}$$

Therefore, the optimal weights $\mathbf{w}^*$ provide the equal risk contribution from each risk factor.

ERC to principal portfolios and DRP strategies are based different risk measures. While ERC strategy aims equal risk contributions, DRP approach targets the uniform diversification distribution. However, the optimization solutions in Equation 3.67 and Equation 3.66 give the same results without any further constraints since the risk contributions $\tilde{RC}_i$ in principal space almost equal the diversification distribution $p_{MTP,i}$.

To sum up the methodologies are mentioned in this study, Table 3.2 and Table 3.3 are constructed.

Table 3.2: Summary table of portfolio weights with respect to strategies

Portfolio Strategy	Weights
EW	$w_i = n^{-1}$
GMV	$w_i = \underset{\mathbf{w}}{min} \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{i,j}$
IV	$w_i = \frac{\sigma_i^{-1}}{\sum_{j=1}^{n}\sigma_j^{-1}}$
ERC	$w_i = \underset{\mathbf{w}}{argmin} \sum_{i=1}^{n}\sum_{j=1}^{n}(w_i(\Sigma w)_i - w_j(\Sigma w)_j)^2$
DRP_{PP}	$\tilde{w}_i = \frac{\lambda_i^{-1}}{\sum_{j=1}^{n}\lambda_j^{-1}}$
DRP_{MTP}	$\tilde{w}_i = \frac{\sigma_{MTP,i}^{-1}}{\sum_{j=1}^{n}\sigma_{MTP,j}^{-1}}$

Table 3.3: Summary of definitions used in asset, principal and minimum torsion spaces

	Asset Space [[16]]	Principal Space [[16]]	Minimum Torsion Space
Weights	w_i	$\tilde{w}_i$	$\tilde{w}_i$
Portfolio Variance	$\sigma_P^2(w) = \sum_{i=1}^n w_i^2 \sigma_i^2 + \sum_{i \neq j}^n w_i w_j \sigma_{i,j}$	$\sigma_P^2(\tilde{\mathbf{w}}) = \sum_{i=1}^n \tilde{w}_i^2 \lambda_i$	$\sigma_P^2(\tilde{\mathbf{w}}) = \sum_{i=1}^n \tilde{w}_i^2 \sigma_{MTP,i}$
Marginal Risk Contribution	$\partial_{w_i} \sigma_P^2(\mathbf{w}) = \frac{w_i \sigma_i^2 + \sum_{i \neq j}^n w_j \sigma_{i,j}}{\sigma_P(\mathbf{w})}$	$\partial_{\tilde{w}_i} \sigma_P^2(\tilde{\mathbf{w}}) = \frac{\tilde{w}_i \lambda_i}{\sigma_P(\tilde{\mathbf{w}})}$	$\partial_{\tilde{w}_i} \sigma_P^2(\tilde{\mathbf{w}}) = \frac{\tilde{w}_i \sigma_{MTP,i}^2}{\sigma_P(\tilde{\mathbf{w}})}$
Risk Contribution	RC_i= $\frac{w_i^2 \sigma_i^2 + \sum_{i \neq j}^n w_i w_j \sigma_{i,j}}{\sigma_P(\mathbf{w})}$	$\tilde{RC}_i = \frac{\tilde{w}_i^2 \lambda_i}{\sigma_P(\tilde{\mathbf{w}})}$	$\tilde{RC}_i = \frac{\tilde{w}_i^2 \sigma_{MTP,i}^2}{\sigma_P(\tilde{\mathbf{w}})}$

CHAPTER 4

EMPRICAL ANALYSIS

In this chapter, we provide an empirical analysis to examine the strategies that are discussed in the previous chapters and to see the difference of risk allocation among asset classes versus uncorrelated risk factors. The methodology corresponds to the related work of [24, 25, 20] to generate the uncorrelated risk factors. We also follow [29]'s work in defining the set of risk factors. Then, we present the performance of both diversified risk parity strategies and risk based asset allocation strategies. The poor performance of mean-variance strategy due to large estimation errors in estimated mean has become popular risk-based strategies as mentioned previously. Therefore, to make clear comparison, we also include mean-variance optimization and compare the out-of-sample performance with both risk-based strategies and DRP strategies.

4.1 Data

This work focuses on seven broad asset classes representing equity, bond and commodity indices to construct the asset allocation strategies. Monthly prices of the assets between January 1988 and December 2017 are retrieved from Bloomberg [4]. The logarithmic return is calculated based on the closed prices at the end of each month. The analyses are run in Matlab 2014. Two equities, four bonds and one commodity indices are chosen to set up the portfolios as shown in Table 4.1. The reason of working with these asset classes is that they are enough broad and widely used to represent the corresponding risk factors.

The data used in this thesis are summarized in Table 4.1. We employ the MSCI World Total Return Index for developed equities and MSCI Emerging Markets Total Return Index for emerging countries, which are denoted by M1WO and M1EF, respectively. High yield index that is a kind of fixed income tracks the performance of US dollar denominated below investment grade rated corporate debt publicly issued in the US domestic market, which is represented by H0A0. The bond indices are Citi WGBI Currency Hedged USD for world government bonds hedged, Citi WBGI USD for world government bonds and Barclay's U.S. Aggregate for U.S. aggregate bonds, which are represented by SBWGC, SBWGU and LBUSTRUU, respectively. The commodity is given by S&P GSCI index that is denoted by SPGSCITR. The abbreviations are kept as in Bloomberg tickers. Since we focus on the risk distribution among risk factors,

Table 4.1: Summary of asset classes

Bloomberg ticker	Index name	Index definition	Asset type	Representation of risk source
M1WO	MSCI World Total Return Index for developed equities	Developed equity	Equity	Equity risk
M1EF	MSCI World Total Return Index for emerging equities	Emerging equity	Equity	Equity risk
H0A0	ICE BofAML US High Yield Master II Index value	High Yield	Fixed income	Equity risk*
SBWGC	Citi WGBI Currency-Hedged USD	World government bonds hedged	Fixed income	Interest rate risk
SBWGU	Citi WGBI USD	World government bonds	Fixed income	Interest rate risk
LBUSTRUU	U.S. Aggregate bonds	U.S. Aggregate	Fixed income	Interest rate risk
SPGSCITR	S&P GSCI Total Return CME	The measure of general commodity price movements and inflation	Commodity	Inflation risk

* Although high yield is a kind of bond, it represents the equity risk since it has generally high correlation with equity.

we also give which data represents what kind of risk. Two equity indices represent the equity risk factor. High yield demonstrates a different structure from other bonds. Although it is a kind of bond, it also represents equity risk factor since it has generally high correlation with equity [29]. Remaining bonds represent the interest rate risk. Commodity index represents the inflation risk [29, 20].

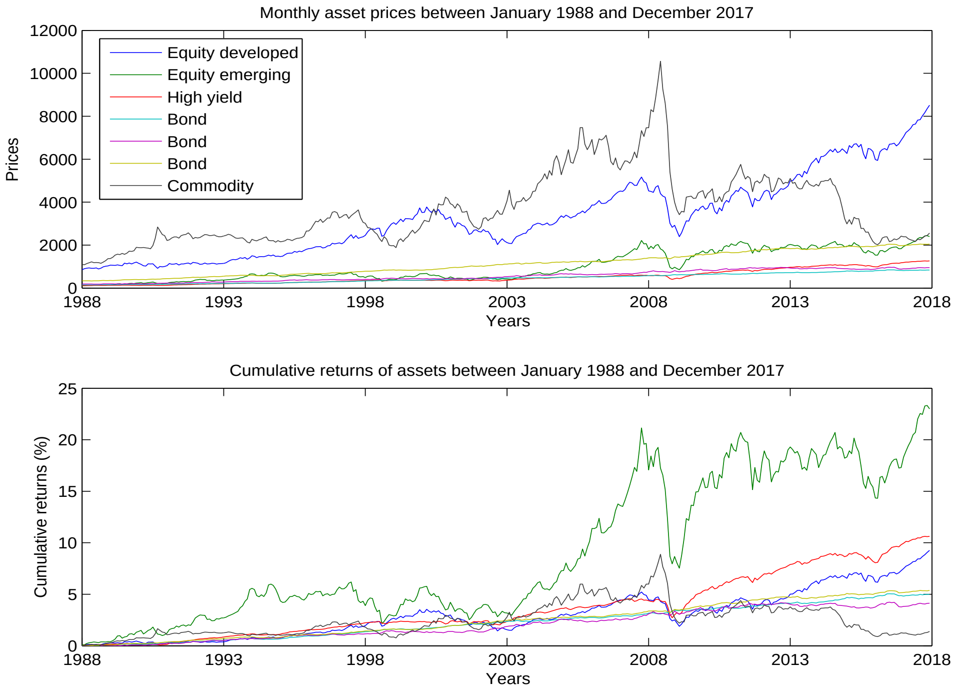


Figure 4.1: Monthly asset prices and their cumulative returns

The monthly asset prices and their cumulative returns can be seen in Figure 4.1. Commodity and developed equity demonstrate a volatile price pattern over the period. The remaining asset prices present more stable pattern. As for the cumulative returns, emerging equities give the highest return, which is followed by high yield index. Their reaction to 2008 financial crisis can be seen in the same figure. Most of the prices are declined sharply except bonds. The returns of the emerging are decreased mostly among all asset classes. The returns of high yield, commodity and developed equity indices are also decreased. There is not remarkable decrease in the returns of bonds.

Table 4.2: Descriptive statistics of the selected assets

	Return	Risk	Sharpe Ratio	MDD
M1WO	6.83%	15.63%	0.22	14.00%
M1EF	12.43%	24.70%	0.36	16.03%
H0A0	8.78%	8.79%	0.61	8.67%
SBWGC	6.84%	3.17%	1.09	1.33%
SBWGU	6.93%	6.79%	0.52	2.34%
LBUSTRUU	7.34%	3.87%	1.02	1.34%
SPGSCITR	6.52%	21.27%	0.15	17.65%

Table 4.2 reports the descriptive statistics of the data. The developed equity has annualized return of 6.83% at a risk of 15.63%. On the other hand, emerging equity demonstrates higher return and volatility compared to developed equity, which are 12.43% and 24.70%, respectively. High yield index is a bond index but it has similar return and volatility as equity indices. High yield index has higher return and volatility compared to bonds. Bonds indices have the lowest volatility and return in all asset classes. The final asset class commodity draws a similar figure with developed countries with respect to return and risk. In evaluating the performances of the asset classes with regarding to Sharpe Ratio, the lowest ratio belongs to commodity index (0.15). This can be the result of the high volatility of the oil prices with low return. The unexpected results come from the bond indices having the highest Sharpe ratio. This may be the result of 2008 and subsequent financial crises, since bonds generally give the high performance during the bad times [5]. High yield also performs well with the Sharpe ratio of 0.61. Developed equity presents the poor Sharpe ratio (0.22) compared to emerging equity that has the ratio of 0.36. World government bonds hedged and world government bonds have the highest Sharpe ratios of 1.09 and 1.02, respectively. Aggregate bond also has a good Sharpe ratio of 0.52 compared to equity asset classes and high yield asset class. The highest drawdowns belong to commodity and equity indices. High yield also have high drawdown compared to bond indices. The drawdowns of bonds are around 1% and 2%.

It is seen in Table 4.3 that there is a high correlation between equity indices and high yield index. This leads to high yield being categorized as an equity index although it is a kind of bond index. Bond indices have a low correlation with the remaining asset classes. The correlation structure of the asset classes is also evaluated for the crisis period between August 2007 and February 2008 as in Table 4.4.

A general positive increase of the correlation among asset classes is observed. The correlation between equity indices and high yield index is over 90%. The increase

Table 4.3: Correlation matrix of asset classes between January 1988 and December 2017

Asset type	Equity M1WO	Equity M1EF	High yield H0A0	Bond SBWGC	Bond SBWGU	Bond LBUSTRUU	Commodity SPGSCITR
M1WO	1						
M1EF	**0.74**	1					
H0A0	**0.61**	**0.58**					
SBWGC	0.01	-0.11	0.01	1			
SBWGU	0.26	0.06	0.09	**0.55**	1		
LBUSTRUU	0.12	0.01	0.24	**0.85**	**0.57**	1	
SPGSCITR	0.24	0.28	0.22	-0.16	0.12	-0.02	1

Table 4.4: Correlation matrix of asset classes between August 2008 and February 2008

Asset type	Equity M1WO	Equity M1EF	High yield H0A0	Bond SBWGC	Bond SBWGU	Bond LBUSTRUU	Commodity SPGSCITR
M1WO	1						
M1EF	**0.95**	1					
H0A0	**0.92**	0.87					
SBWGC	-0.20	-0.15	-0.14	1			
SBWGU	0.19	0.28	0.29	**0.68**	1		
LBUSTRUU	**0.45**	**0.46**	**0.50**	**0.72**	**0.79**	1	
SPGSCITR	**0.62**	**0.64**	**0.62**	-0.54	-0.04	0.02	1

happens from 0.61 to 0.92 and from 0.58 to 0.87 for developed and emerging equities respectively. Furthermore, the correlation increases between both equities and commodity, which is from 0.24 to 0.62 and from 0.28 to 0.64 for developed and emerging equities respectively. As for the bonds, only SBWGC index that shows increased negative correlation with other asset classes is separated from other bond indices. Other bond indices have increased positive correlations with other asset classes. The correlation between high yield index and commodity index also increases from 0.22 to 0.62. Briefly, asset classes have correlations among themselves, specifically over the bad economic times they demonstrate highly correlated behavior contrary to the expectation and desire of the uncorrelated structure.

4.2 Constructing Uncorrelated Portfolios

To extract the uncorrelated risk factors hidden in the multi-asset classes, we apply two methods: PCA ad MTP.

4.2.1 Principal portfolios

The economic interpretation of the principal portfolios is based on the coefficients of the asset classes in the eigenvectors. The asset that has high coefficients in absolute value drives the volatility of the eigenvector. We know that each eigenvector represent uncorrelated principal portfolios (PPs) as given in Definition 3.1. There exists seven PPs as shown in Table 4.5 which presents the eigenvector matrix of asset classes' monthly returns based on the sample period from January 1988 to December 2017. The economic interpretation of each eigenvector is presented as well as with their variances. Bold numbers demonstrate high coefficients.

Table 4.5: Eigenvector matrix

		PP1	PP2	PP3	PP4	PP5	PP6	PP7
Equity	M1WO	**0.44**	-0.18	**-0.74**	0.22	**0.41**	-0.08	-0.01
Equity	M1EF	**0.79**	-0.33	**0.48**	-0.20	-0.03	-0.01	0.00
High Yield	H0A0	0.19	-0.06	-0.23	**0.42**	**-0.84**	0.17	0.07
Bond	SBWGC	-0.01	-0.02	-0.12	-0.27	-0.15	**-0.53**	**0.78**
Bond	SBWGU	0.03	0.02	-0.26	**-0.76**	-0.17	**0.50**	-0.01
Bond	LBUSTRUU	0.01	-0.01	-0.16	-0.28	-0.28	**-0.65**	**-0.62**
Commodity	SPGSCITR	**0.38**	**0.92**	0.01	0.01	0.02	-0.03	0.01
	Risk type	Equity + Commodity risk	Inflation risk	Equity risk	Not defined	Not defined	Interest rate risk	Not defined
	Variance	4.22%	2.59%	1.19%	1.04%	0.81%	0.38%	0.17%

The first principal portfolio (PP1) is dominated by both equity and commodity risks with the weights of 0.44, 0.79 and 0.38 for developed equity, emerging equity and commodity indices, respectively. We know that first eigenvector has the highest variance, besides the first eigenvector is driven by high volatile assets, i.e. equities and commodity, yielding the highest variance with 4.22%. Second principal portfolio (PP2) is purely driven by commodity index whose weight is 0.92. Therefore, PP2 is inflation risk with the variance of 2.60%. In the third principal portfolio (PP3), developed and emerging equities have the highest weights, hence PP3 represents the equity risk that accounts for 1.19% of the total variance. The fourth principal portfolio (PP4) and the fifth principal portfolio (PP5) cannot have economic interpretation since high weights do not belong to one asset class. The sixth principal portfolio is dominated by bonds, and therefore it represents the interest rate risk with a volatility of 0.38%. Again, the seventh principal portfolio (PP7) is not defined since there is no one type of asset class having high coefficients to dominate this eigenvector. In PP7, two bonds have high coefficients in absolute value of 0.78 and 0.62 but the other bond has quite low coefficient, which prevents the domination of bonds in the seventh principal portfolio. Therefore, we extract three main uncorrelated risk sources with using PCA. PP2, PP3 and PP6 represent the inflation risk, equity risk and interest rate risk respectively. Based on the selection made using PCA, the variance of each risk is given as

$$\begin{aligned}
\sigma^2_{PP,equity} &= 1.19\% \\
\sigma^2_{PP,inflation} &= 2.59\% \\
\sigma^2_{PP,interestrate} &= 0.38\%.
\end{aligned}$$

4.2.2 Minimum torsion portfolios

MLT model is another approach to obtain the uncorrelated risk sources. On the contrary to PCA, MLT gives the uncorrelated risk factors that closely track the original factors. This property helps to extract and interpret the torsion portfolios straightforwardly. Seven portfolios represented by MTP are presented in Table 4.6 that presents the torsion matrix of asset classes' monthly returns based on the sample period from January 1988 to December 2017. The economic interpretation of each column of torsion matrix is presented as well as with their variances. Bold numbers in the table demonstrate the high coefficients.

Table 4.6: Torsion matrix

		MTP1	MTP2	MTP3	MTP4	MTP5	MTP6	MTP7
Equity	M1WO	**1.32**	-0.03	-0.50	-0.02	-0.45	0.20	-0.02
Equity	M1EF	-0.75	**1.28**	-0.61	0.27	0.15	0.30	-0.11
High Yield	H0A0	-0.16	-0.08	**1.20**	0.38	0.10	-0.64	-0.02
Bond	SBWGC	0.00	0.00	0.05	**1.55**	-0.10	-0.68	0.02
Bond	SBWGU	-0.08	0.01	0.58	-0.47	**1.16**	-0.38	-0.03
Bond	LBUSTRUU	0.01	0.00	-0.12	-1.01	-0.12	**1.57**	-0.01
Commodity	SPGSCITR	-0.04	-0.08	-0.12	0.96	-0.30	-0.21	**1.04**
	Risk type	Equity risk	Equity risk	Equity risk	Interest rate risk	Interest rate risk	Interest rate risk	Inflation risk
	Variance	0.14%	0.35%	0.05%	0.06%	0.03%	0.07%	0.34%

In Table 4.6, each column has the highest score for only one asset. Therefore, it is easy to match the risk sources. First two columns represent the equity indices with the variances 0.14% and 0.35% respectively. Third column has the highest score for the high yield that is assessed as the equity. Therefore, first three columns represent the equity risk. Forth, fifth and sixth columns are for the bond indices with the volatility of 0.06%, 0.03% and 0.07% respectively. These columns denote the inflation rate risk. The last column with the volatility of 0.34% presents the commodity risk. Therefore, we obtain three main uncorrelated risk sources. The variance of each risk is the sum of variances of the corresponding representative columns' variances since they are uncorrelated. For example, the variance of equity risk is the sum of the variances of first, second and third columns.

Therefore, the variance of each risk is stated as

$$\begin{aligned}\sigma^2_{MTP,equity} &= \sigma^2_{MTP1} + \sigma^2_{MTP2} + \sigma^2_{MTP3} = 0.54\% \\ \sigma^2_{MTP,inflation} &= \sigma^2_{MTP4} + \sigma^2_{MTP5} + \sigma^2_{MTP6} = 0.16\% \\ \sigma^2_{MTP,interestrate} &= \sigma^2_{MTP7} = 0.34\%.\end{aligned}$$

4.3 Portfolio performances based on strategies

According to the risk and asset categories, the performance of portfolios with respect to diversified risk parity (PP and MTP), equally weighted, global minimum variance, inverse volatiliy and equal risk contribution strategies are evaluated. The performance measures are selected as annualized return, risk, Sharpe ratio, maximum drawdown (MDD), Gini coefficient.

Table 4.7 presents the performance and risk results of two DRP strategies with risk-based benchmark strategies. The table shows both the performance and risk characteristics results of chosen asset allocation strategies according to the period from January 1988 to December 2017. Return, risk and Sharpe ratio are annualized results. Sharpe ratio is computed with the monthly risk-free rate that is taken from Fama-French website [11]. MDD is reported over one year during the whole sample period. $Gini_{weight}$ is calculated with portfolio weights and $Gini_{risk}$ is calculated with risk decompositions of asset classes for asset allocation strategies and risk decompositions of uncorrelated risk sources for diversified risk parity strategies. The number of uncorrelated risks gives the result of the uncorrelated risk sources with using the exponential entropy of risk decompositions.

DRP_{MTP} has the return of 6.3% at 5.8% volatility. Given that DRP_{MTP} has the highest Sharpe ratio of 0.63. DRP_{PP} approach gains 5.9% return with 5.7% risk, which gives the Sharpe ratio of 0.37. DRP_{PP} portfolio has the second lowest risk but the return is also relatively low and this results in the lowest Sharpe ratio among all strategies. This meets the expectation of the low risk low return case.

As for the benchmark strategies, the highest return (7.9%) belongs to the EW strategy with the highest volatility (8.3%). Furthermore, this approach demonstrates the highest drawdown among all strategies. In contrast, GMV strategy has the lowest return of 5.2%, yielding the benefit of the lowest volatility (2.3%). Moreover, its Sharpe ratio is 0.61, which is a favorable performance and it has the lowest drawdown among all strategies. The good performance of GMV portfolio is also found in several works [5, 13]. DRP strategies have slightly higher drawdown ratio than the GMV has.

Table 4.7: Performance results of asset allocation strategies

	Return (%)	Risk (%)	Sharpe Ratio	MDD (%)	$Gini_{weight}$	$Gini_{risk}$	Number of uncorrelated risks
DRP_{MTP}	6.3	5.8	0.63	44.5%	0.39	0.00	3.00
DRP_{PP}	5.9	5.7	0.37	48.5%	0.56	0.00	3.00
EW	7.9	8.3	0.49	50.7	0.00	0.57	1.08
GMV	5.2	2.9	0.48	22.5	0.91	0.90	1.01
IV	6.5	6.9	0.39	31.7	0.45	0.10	1.90
ERC	6.4	6.3	0.41	34.8	0.47	0.00	2.10

Like Maillard et al. (2010) [21], two risk parity strategies, namely IV and ERC, place between EW and GMW strategies in terms of both return and risk. ERC strategy has the return of 6.9% with 5.6% risk, which gives the Sharpe ratio as 0.55. IV strategy earns higher return of 7.2% with the cost of higher risk of 6.2%. This results slightly lower Sharpe ratio of 0.54. However, IV strategy has higher drawdown (15.7%) than the ERC strategy has (13.4%). The drawdown results are low when compared to the EW approach.

When considering the aim of the risk based asset allocation strategies, Sharpe ratio is not enough to evaluate the performance of the strategies. Therefore, their risk characteristics are examined as well. The risk contributions of the asset classes for each strategy can be found in the Table 4.8.

First, both DRP strategies are examined. DRP_{MTP}has positive weights in all asset classes except high yield (H0A0), which leads to negative risk contribution from high yield (Table 4.8). The weights are not well distributed and hence this unbalanced weight allocation is also demonstrated by Gini coeeficient of 0.39 (Table 4.7). The highest risk contribution comes from the commodity with 34.7% that is almost one third of the portfolio risk. Then the emerging equity with 22% and world government bond with 13.9% contribute to the total portfolio risk. The risk contribution of the remaining asset classes to the whole risk is relatively low. The risk is well diversified among three main risk sources and coefficient of $Gini_{risk}$ is zero (Table 4.7). Furthermore, this is also supported by the number of uncorrelated risk factors, which is three. As for the other DRP strategy, DRP_{PP} has four shorted asset classes, which are emerging equity, high yield, world government bond and commodity. The skewed weight distribution is also supported by $Gini_{weight}$ with 0.56 (Table 4.7). The highest risk contribution comes from the developed equity index with 41%. The U.S. aggregate bond has the second highest risk contribution (26%), almost equal to 1/3 of the whole risk. Then, the commodity and world government bond risk contributions are high. The remaining asset classes risk contributions are relatively low (Table 4.8). However, the risk contributions among three risk risk sources are equal and $Gini_{risk}$ with zero and the number of uncorrelated factors with three supports the equal risk sources distribution (Table 4.7).

Next, we examine the benchmark strategies. First examining EW strategy, we observe that 70% of the all portfolio risk is driven by equities including high yield. The highly volatile emerging equity contributes the greatest proportion of the risk budget (36.2%). The commodity index consist of the most of the remaining portfolio risk (24.1%) and the bonds are close being irrelevant due to their low risk contributions (Table 4.8). EW is well balanced in terms of weights, which is demonstrated by $Gini_{weight}$ with zero. On the other hand, the distortion of risk contribution by asset classes is supported by $Gini_{risk}$ of 0.78, which is highly unfavorable. Also, the number of uncorrelated risk factors is 1.08 (Table 4.7). This portfolio is concentrated in one risk source. We know that almost 70% of the risk comes from the equities, then the portfolio risk is concentrated in equity risk. Briefly, the risk of the portfolio is not distributed well among three risk sources.

Second, GMV strategy exhibits heavily concentrated portfolio risk structure. In details, the low risk asset class, i.e bonds, has the major share of the portfolio as a total almost 88%. Therefore, their risk contributions are over 90%. The distorted weight structure is also shown by $Gini_{weight}$ of 0.91, which is the highest unbalanced weight allocation

Table 4.8: Weights and risk contributions of asset classes based on the strategies (in %)

Asset	Asset		EW			GMV			IV	
type	name	Weights	MRC	RC	Weights	MRC	RC	Weights	MRC	RC
Equity	M1WO	14.3	156.5	22.4	0.8	131.5	1.0	6.6	248.4	16.5
Equity	M1EF	14.3	253.2	36.2	0.5	147.7	0.7	4.2	353.7	14.9
High yield	H0A0	14.3	74.8	10.7	7.8	100.9	7.8	11.9	133.7	15.9
Bond	SBWGC	14.3	3.2	0.5	85.6	98.6	84.4	31.3	38.6	12.1
Bond	SBWGU	14.3	31.4	4.5	0.6	139.2	0.8	14.9	102.5	15.3
Bond	LBUSTRUU	14.3	12.3	1.8	1.7	112.5	1.9	26.3	59.4	15.6
Commodity	SPGSCITR	14.3	168.6	24.1	3.2	106.1	3.4	4.7	206.3	9.7

Asset		ERC			DRP_{PP}			DRP_{MTP}	
name	Weights	MRC	RC	Weights	MRC	RC	Weights	MRC	RC
M1WO	5.8	244.7	14.3	39.5	113.4	44.8	7.8	178.3	13.9
M1EF	4.1	352.0	14.3	-10.0	25.5	-2.5	8.8	250.5	22.0
H0A0	10.8	132.2	14.3	-10.6	28.3	-3.0	-4.3	53.5	-2.3
SBWGC	36.0	39.7	14.3	62.9	31.0	19.5	34.1	34.6	11.8
SBWGU	13.7	104.4	14.3	-43.2	11.9	-5.2	25.8	59.4	15.3
LBUSTRUU	23.6	60.4	14.3	73.4	35.5	26.0	15.5	30.0	4.6
SPGSCITR	6.0	237.9	14.3	-11.9	-170.9	20.4	12.4	279.5	34.7

ratio among all strategies. Then, the $Gini_{risk}$ demonstrates the concentrated risk with 0.90. Also, the number of uncorrelated factors of 1.01 demonstrates the portfolio risk is concentrated in one risk source (Table 4.7). It is obvious that the risk source of this portfolio is interest rate risk.

Finally, we examine the risk parity strategies. IV portfolio distributes the weights mostly low risk asset class (i.e. bonds) and lower weights go to high volatile asset classes as expected. The high volatile assets (equities and commodity) have weights of 6.6%, 4.2% and 4.7%, respectively. The bonds have more weights, i.e. 31.3%, 14.9% and 26.3% (Table 4.8). Therefore, this leads to almost equal risk contribution among asset classes. The concentration of weights is shown by $Gini_{weight}$ of 0.45. However, the balanced risk distribution is supported by $Gini_{risk}$ of 0.10, which is highly favorable (Table 4.7). As for the ERC approach, the weights of the asset classes exhibit the similar pattern with the weights of IV strategy. Therefore, $Gini_{weight}$ is 0.47. Contrary to IV strategy, this approach distributes the risk among asset classes equally, then the $Gini_{risk}$ is zero. Although these strategies distribute the risk well among asset classes, they do not show the same performance for the risk distribution among risk sources. The number of uncorrelated risk factors are 1.90 and 2.10 for IV and ERC strategies respectively (Table 4.7). These strategies are concentrated on almost two risk sources.

As summary, diversified risk parity strategy based on minimum torsion approach exhibits the best performance among all strategies in terms of both risk/return tradeoff and risk distribution. The diversified risk parity based on principal portfolios has the lowest Sharpe ratio contrary to work by [20]. Among the benchmark strategies, EW and GMV demonstrates good reward to volatility ratio but they have concentrated risk

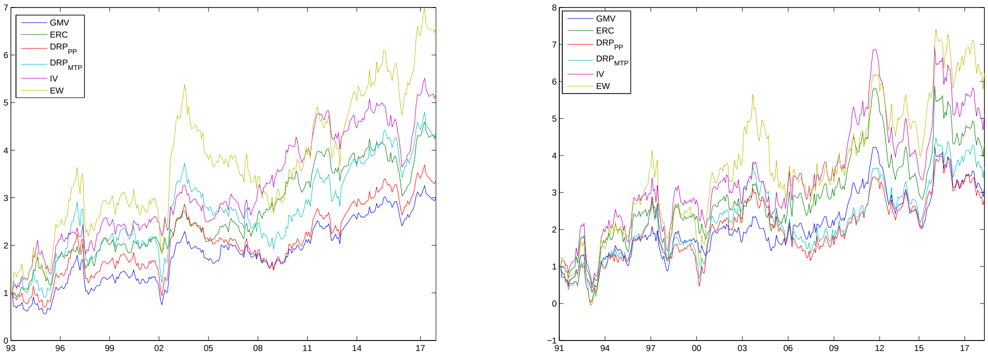


Figure 4.2: Scaled Return of Asset Allocation Strategies

The y-axis represents the percentage returns while the x-axis represents the years [1]

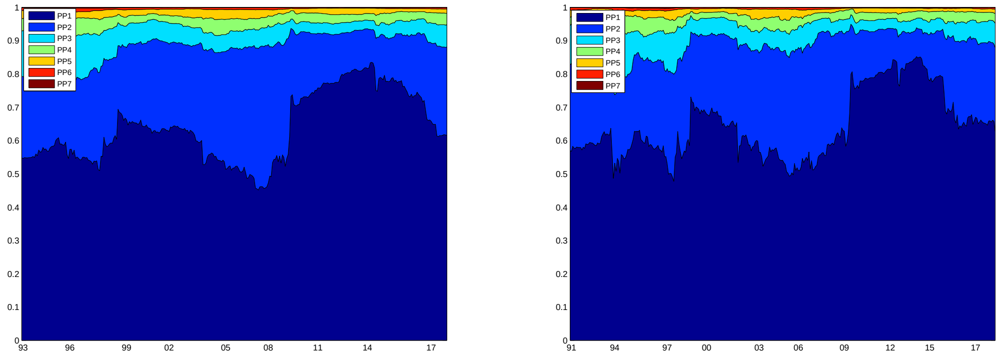
structure. Risk parity strategies are well balanced in terms of risk from asset allocation but they are actually driven by few risk sources, hence they do not meet the expectations. The findings are also supported by the results of [20, 32, 16].

4.3.1 Time impact on portfolio strategies

In this part, we test the performance and risk characteristics of the strategies with applying rolling window approach. We use three-year and five-year rolling window estimation. Initial backtest period uses first three years of the data for three-year rolling window and first five years data for five-year rolling window estimation. Thus, the estimation results start from 1991 and 1993 for three-year and five-year rolling windows, respectively. The rolling window approach is used for the out-of-sample testing over the period. Furthermore, we examine the stability of the results in terms of weights and risk contribution with respect to each strategy. Three-year and five-year rolling window estimations are compared.

Rolling window estimation is applied as following. Let T and M give the length of the data set and the size of the estimation window, respectively. In this work, the estimation window sizes are chosen $M = 60$ months (5 years) and $M = 36$ months (3 years). Each month, t, starts from t=M+1. The required parameters for each portfolio strategy over the M previous months are estimated. The estimated parameters are used to calculate the asset weights in each portfolio strategy. The calculated asset weights are employed for the calculation of return and covariance matrix in month $t+1$. Finally, we compute the portfolio return in period t from the weights of M previous months. To keep the estimation window size is fixed, we drop the earliest return and add the return to the next month. The steps are repeated until the data is ended. Given a T-month length data set, the rolling window estimation produces $T - M$ monthly out-of-sample returns.

The scaled returns of different strategies are shown in Figure 4.2 for three-year and

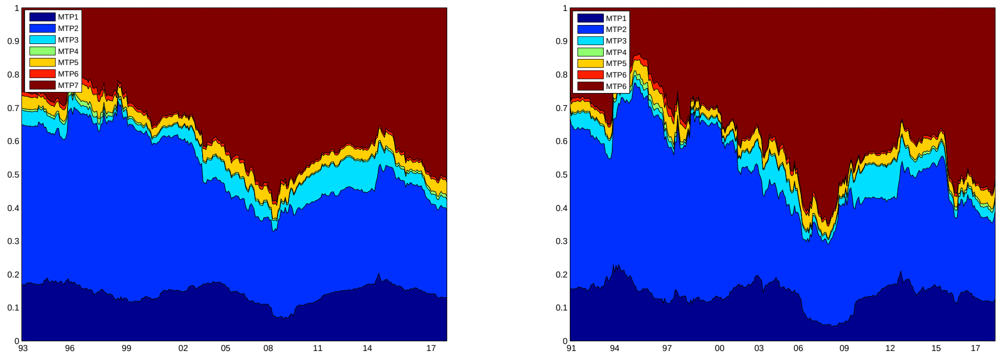


(a) 5 years rolling window

(b) 3 years rolling window

Figure 4.3: Variances of the Principal Portfolios

The x-axis represents the years while y-axis represents the scaled variance. [2]



(a) 5 years rolling window

(b) 3 years rolling window

Figure 4.4: Variances of the Torsion Portfolios

The x-axis represents the years while y-axis represents the scaled variance. [3]

five-year rolling windows. In both figures, EW portfolio generally outperforms other strategies. It is followed by risk parity strategies, namely IV and ERC. Sometimes, ERC strategy outperforms the EW strategy. The return of DRP_{MTP} follows the risk parity strategies' gains. DRP_{PP} and GMV portfolios demonstrate similar pattern over the period. The return performance of the strategies follow consistent pattern and they do not have sharp behaviors.

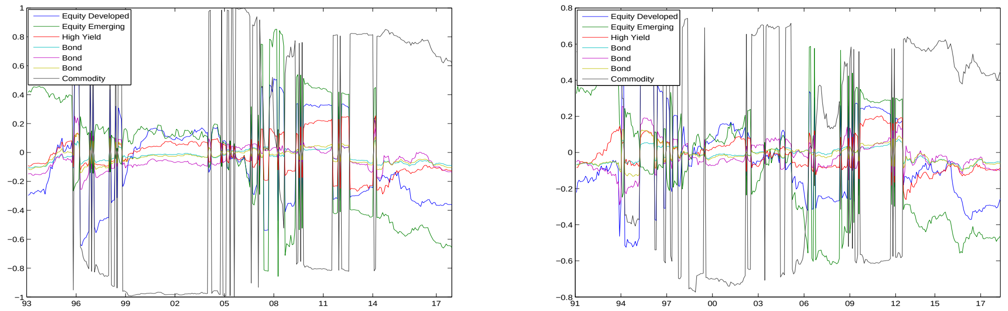
4.3.2 The analyses on portfolio variances

We examine how the variances of each uncorrelated portfolio change over the period as presented in Figures 4.3 and 4.4. The first and second principal portfolios' variances demonstrate a volatile pattern over the periods, so they are not stable. the variances of third, fourth and fifth principal portfolios are relatively stable. The final principal portfolio's variance is almost irrelevant. For the torsion portfolios, their variances are not stable and seem more volatile than the variances of principal portfolios as can be seen in Figure 4.4. The change of variances during the period affects the portfolio construction since the diversified risk parity strategy is constructed based on the variances of uncorrelated portfolios.

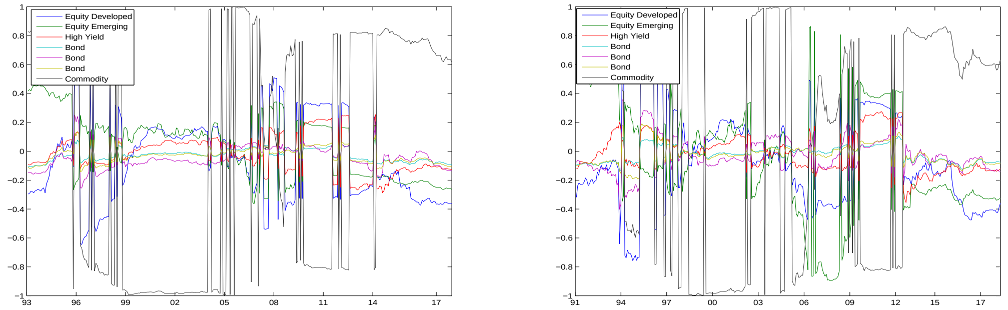
Then, we check whether principal components and minimum torsions give the same economic interpretations over the time. Figures 4.5 and 4.6 demonstrate the weights of each principal portfolio for five-year and three-year rolling window estimations. First principal portfolio (PP1) is mostly dominated by commodity and equity risks over the period. However, between 1999-2005 equity risk seems less effective. Therefore, there is no consistency. Second principal portfolio (PP2) is dominated by the commodity risk, in short times equity risk demonstrates itself, but it is not very effective. In general, it can be said that PP2 represents the commodity risk but it is not strictly stable over the time. Third principal portfolio is obviously equity risk, however commodity risk is shown only over 2008 for a short time. Forth and fifth principal portfolios do not demonstrate a clear pattern, thus they are not defined. Sixth principal portfolio is more robust than the others. Over the period, it exhibits the interest rate risk. The last portfolio is not defined as well. In general, principal portfolios do not demonstrate consistent pattern according to the three-year and five-year rolling windows.

As for the torsion portfolios, they have the most robust results and each torsion portfolio clearly tracks the original corresponding factor. Therefore, it makes the easy for economic interpretation. First three torsion portfolios present the equity risk, following three torsion portfolios represent the interest rate risk and the remaining exhibit the commodity risk.

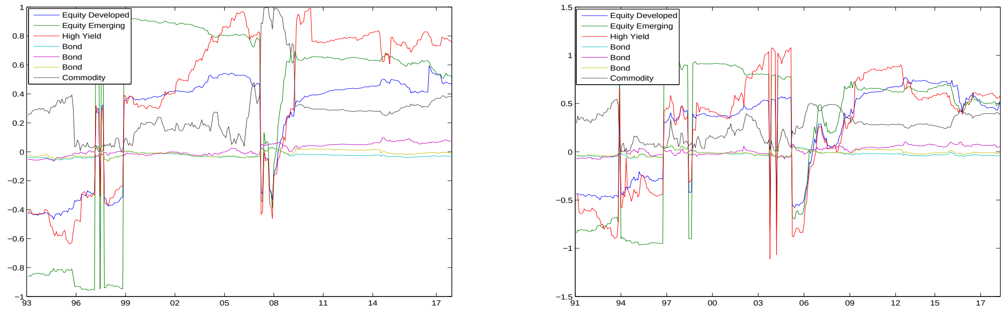
We explain how the asset classes' weights in chosen portfolio and their risk contributions change over time by examining Figures 4.9 and 4.11. The weights of asset classes in EW strategy do not change over time. EW portfolio is dominated by high risky stocks i.e. equities and commodity. Mostly, EW portfolio is concentrated on equity risk. Furthermore, the risk contribution by commodity increases over the period while the risk contribution by emerging equity decreases. The other assets almost contribute the same risk budget over the period and do not show volatile pattern. GMV



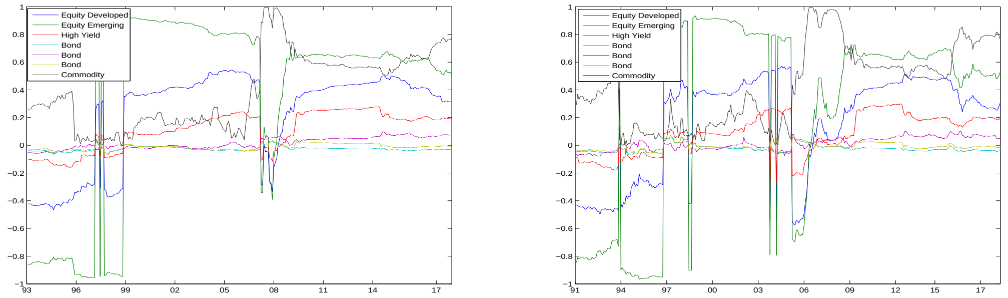
(a) PP1 (b) PP1



(c) PP2 (d) PP2



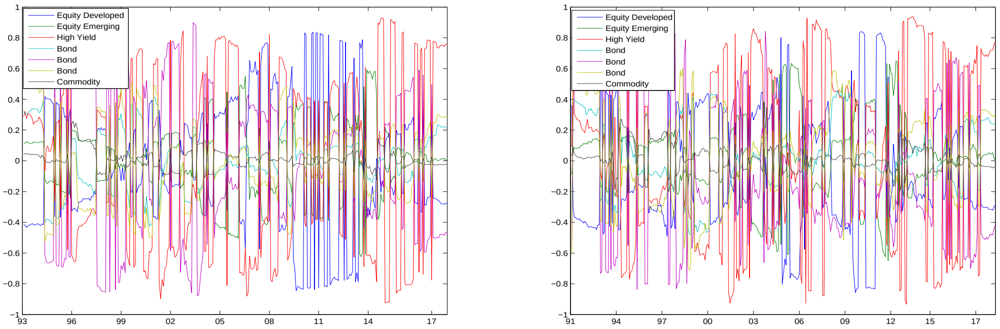
(e) PP3 (f) PP3



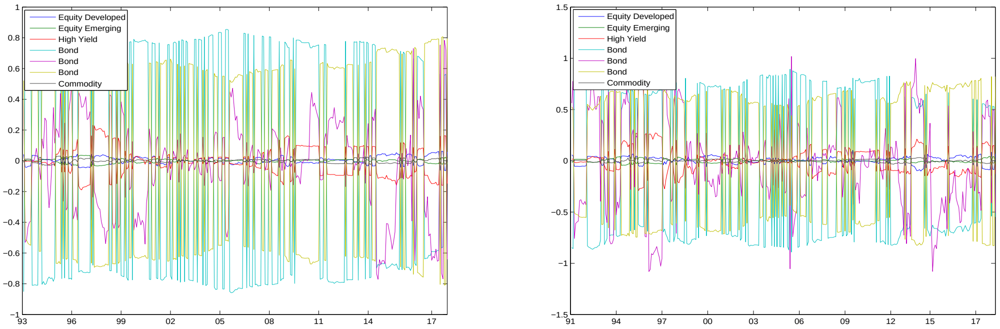
(g) PP4 (h) PP4

Figure 4.5: Weights of Principal Portfolios-a

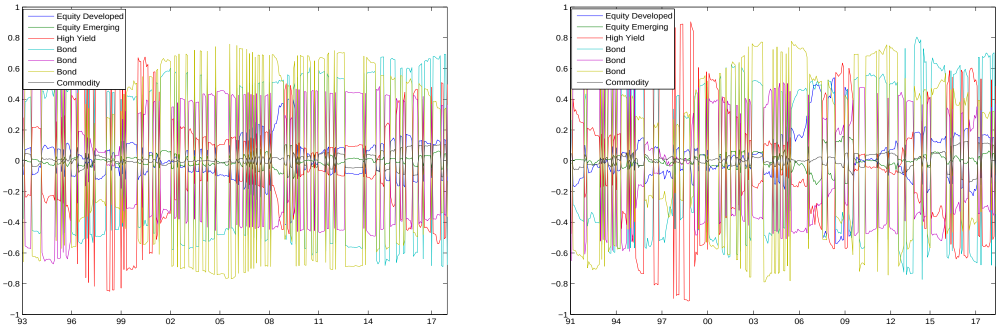
The individual asset weights of each principal portfolio are calculated with using rolling window estimation. Left-hand side presents the five-year rolling window results, while the right-hand side presents the three-year rolling window results.[4]



(a) PP5 (b) PP5



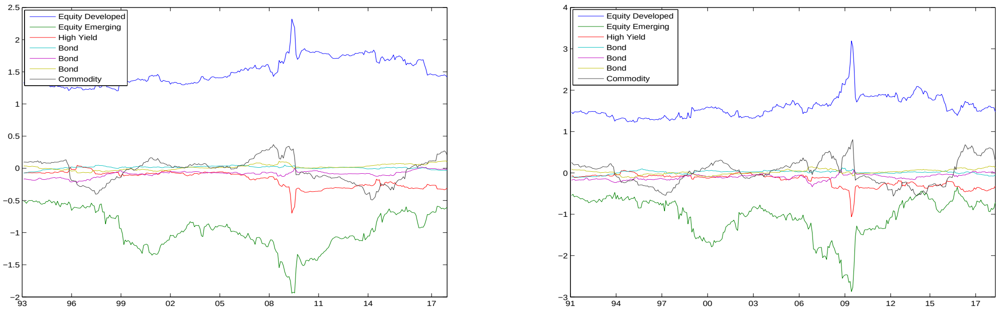
(c) PP6 (d) PP6



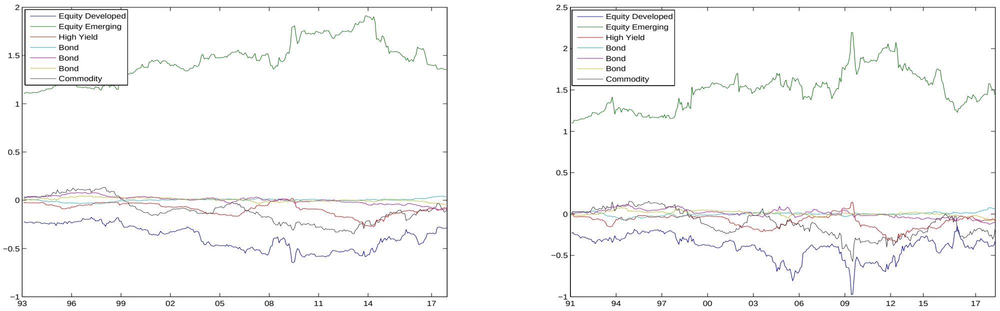
(e) PP7 (f) PP7

Figure 4.6: Weights of Principal Portfolios-b

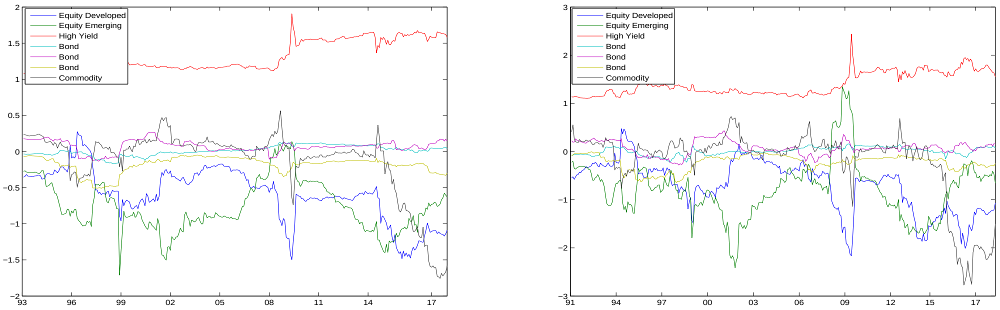
The individual asset weights of each principal portfolio are calculated with using rolling window estimation. Left-hand side presents the five-year rolling window results, while the right-hand side presents the three-year rolling window results.[5]



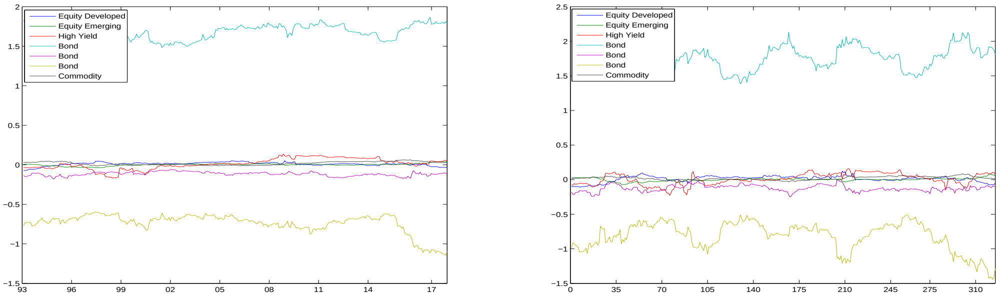
(a) MTP1 (b) MTP1



(c) MTP2 (d) MTP2



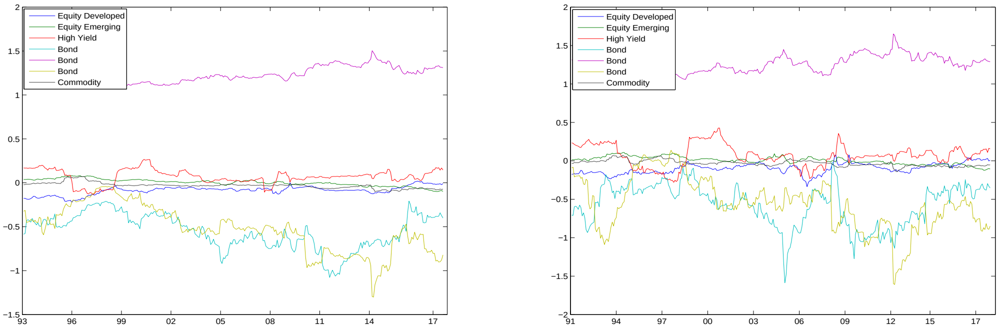
(e) MTP3 (f) MTP3



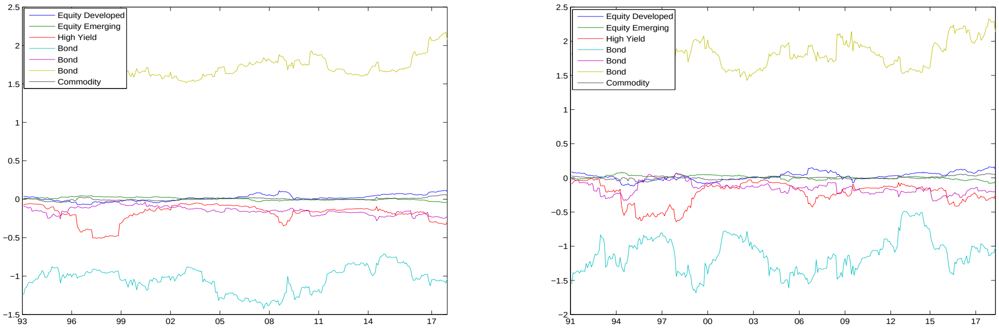
(g) MTP4 (h) MTP4

Figure 4.7: Weights of Torsion Portfolios-a

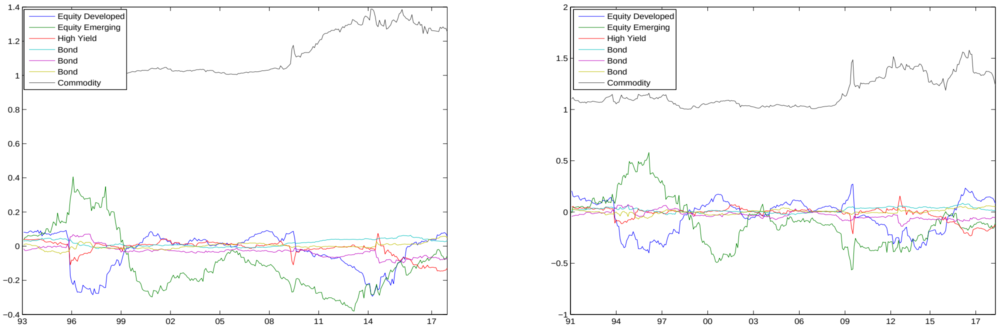
The individual asset weights of each torsion portfolio are calculated with using rolling window estimation. Left-hand side presents the five-year rolling window results, while the right-hand side presents the three-year rolling window results.[6]



(a) MTP5 (b) MTP5



(c) MTP6 (d) MTP6



(e) MTP7 (f) MTP7

Figure 4.8: Weights of Torsion Portfolios-b

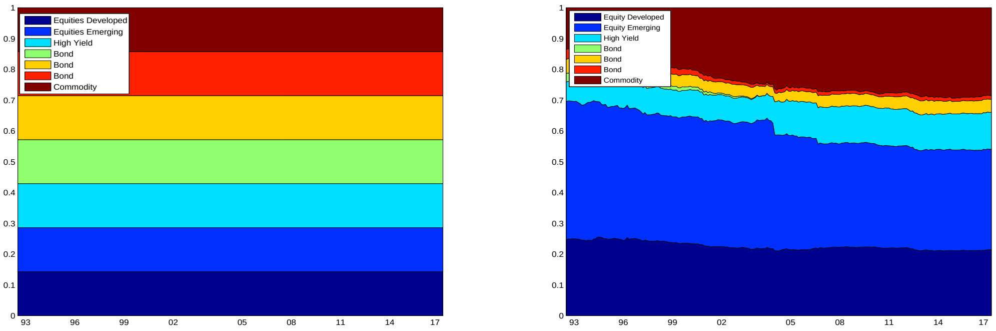
The individual asset weights of each principal portfolio are calculated with using rolling window estimation. Left-hand side presents the five-year rolling window results, while the right-hand side presents the three-year rolling window results.[7]

strategy gives the most of the weight to world government bond and the most of the contribution comes from this asset class. The remaining assets' weights and risk contributions are relatively low. Over the time, the result is consistent. For IV and ERC, we know that they allocate the weights inversely proportional to the volatility of asset classes. ERC approach considers the correlations among the asset classes while IV strategy does not. However, both strategies give more weights to low volatility assets and lessen the domination of high volatile assets. As for the IV strategy, the weights of all asset classes follow the smooth path and do not change sharply over the period. Also, the same behavior is observed for risk contributions. It presents the almost stable structure during whole period in both 3-year and 5-year rolling window estimations. As for the ERC strategy, the portfolio is mostly dominated by bonds in terms of weight allocation. Contrary to IV strategy that presents smooth path in risk contribution, the risk contributions from asset classes are exactly equal in ERC portfolio.

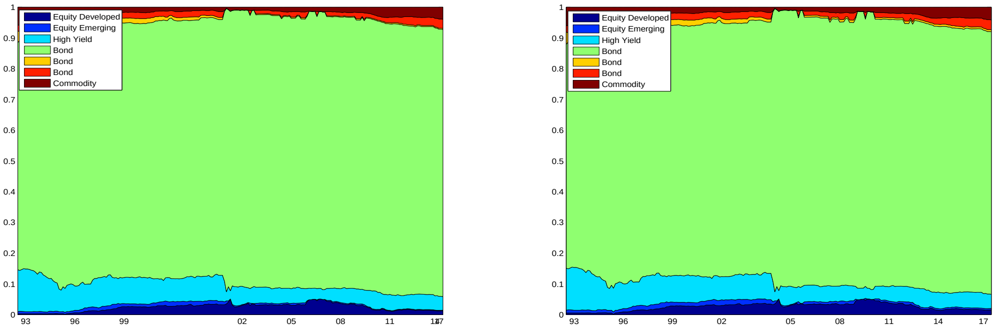
4.3.3 The analyses of weights and risk contribution

Finally, we investigate the DRP strategies as exhibited in Figures 4.10 and 4.12. In these strategies, we mainly focus on the risk distribution among uncorrelated risk sources instead of asset classes. There are three main risk sources considered in this study: equity risk, inflation risk and interest rate risk. First we give details of DRP_{PP} strategy results. In this strategy, the risk is equally distributed among three risk sources over the period while majority of weights consists of bond, which is almost equal to 70% of the portfolio. 20% of the weights come from the equity and the remaining belongs to commodity. However, we know that principal portfolios are not stable over the time. Sometimes, principal portfolios are not dominated by the same asset classes. Thus even in a short time the principal portfolio may reflect different kind of risk source. Although the risk is distributed among three uncorrelated portfolios we cannot be sure that the risk is distributed among three main risk sources. This is the one of the major drawbacks of the PCA strategy. In details, the risk contributions from the assets exhibit a volatile structure. The risk contribution from equities and commodity seem almost irrelevant. The risk contributions from the bonds have the highest proportion. During the crisis periods such as 1999-2000 and 2007-2008, the portfolio is driven by the risky assets. However, it is expected that the portfolio should be driven by low risky assets especially during the crisis periods. The reason may be that the principal portfolios sometimes can represent the different type of risk and therefore DRP_{PP} does not distribute the risk among the uncorrelated risk sources. The risk contributions from risk factors vary over the time for DRP_{PP} portfolio. DRP_{MTP} approach also demonstrates the equal risk distribution among three main risk sources. In details, the most of the risk comes from the commodity while equities and bonds have less risk contribution. Both diversified risk parity strategies do not demonstrate a consistent pattern when we consider the risk contributions from the asset classes.

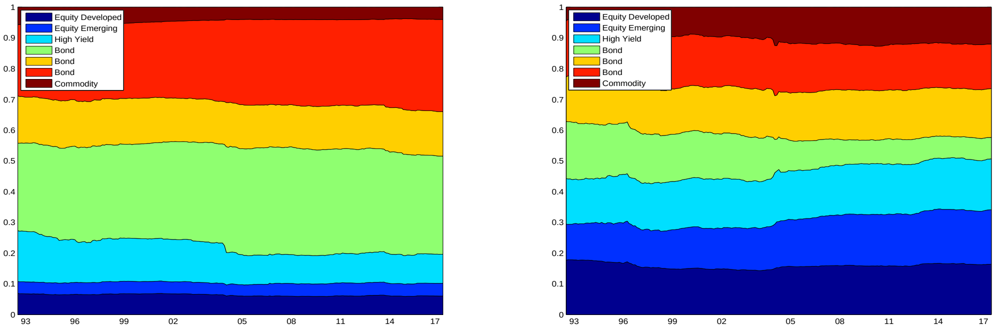
As a conclusion, three-year and five-year rolling window estimation supports the results we reach in previous part. The strategies demonstrate a consistent results in both weights and risk contributions except the principal portfolios. Principal portfolios do not represent the same economic interpretations all the time. Diversified risk parity



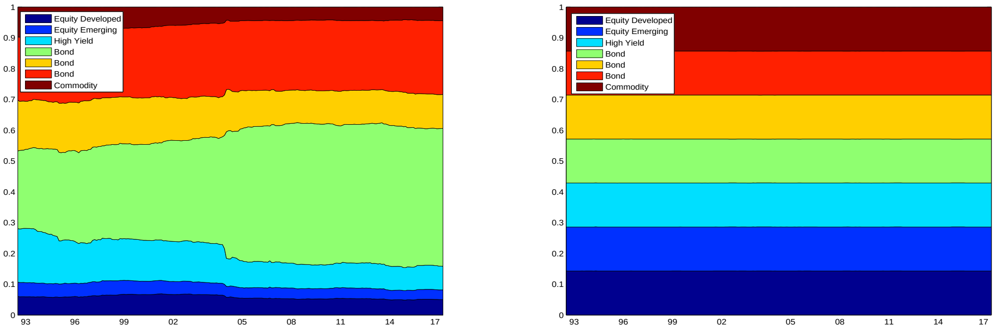
(a) EW Weights (b) EW Risk contributions by assets



(c) GMV Weights (d) GMV Risk contributions by assets

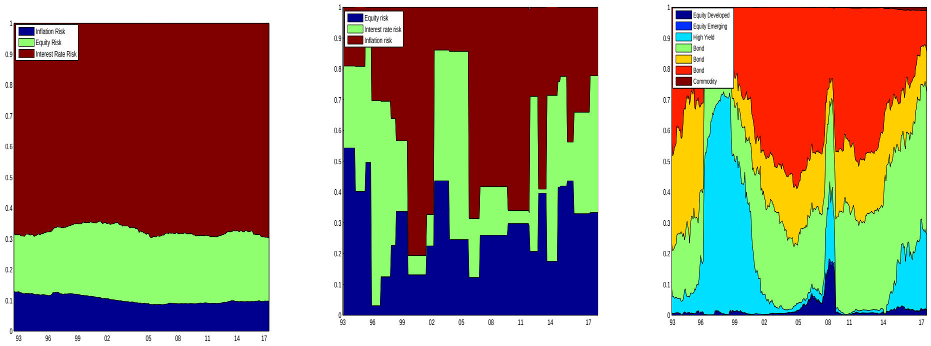


(e) IV Weights (f) IV Risk contributions by assets



(g) ERC Weights (h) ERC Risk contributions by assets

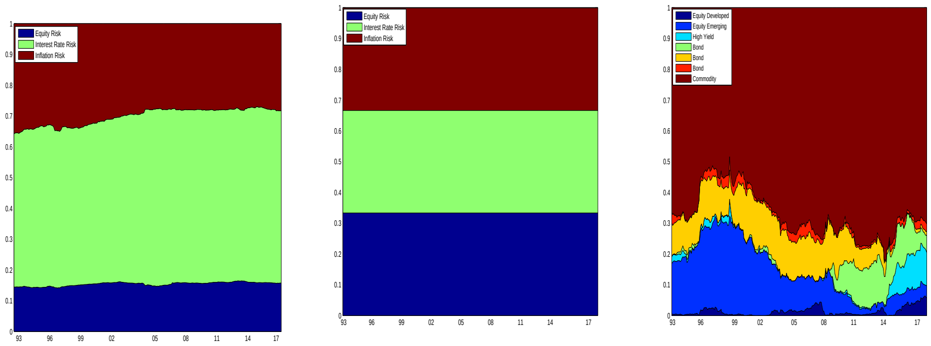
Figure 4.9: Weights and Risk Contributions of Risk-based Strategies for 5-year rolling window



(a) DRP_{PP} Weights

(b) DRP_{PP} Risk contributions by risk sources

(c) DRP_{PP} Risk contributions by assets

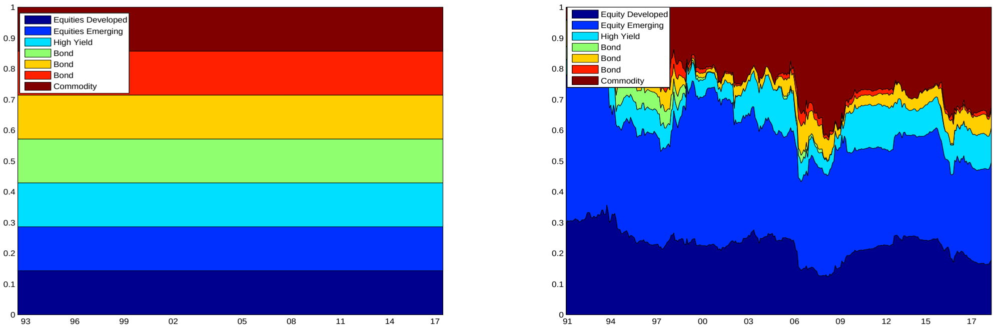


(d) DRP_{MTP} Weights

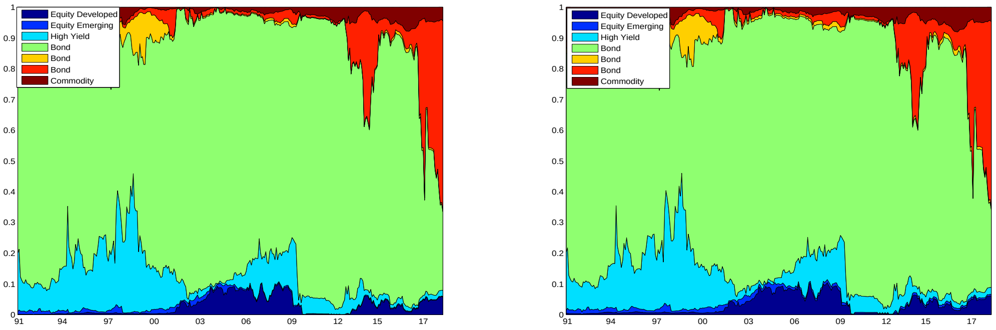
(e) DRP_{MTP} Risk contributions by risk sources

(f) DRP_{MTP} Risk contributions by assets

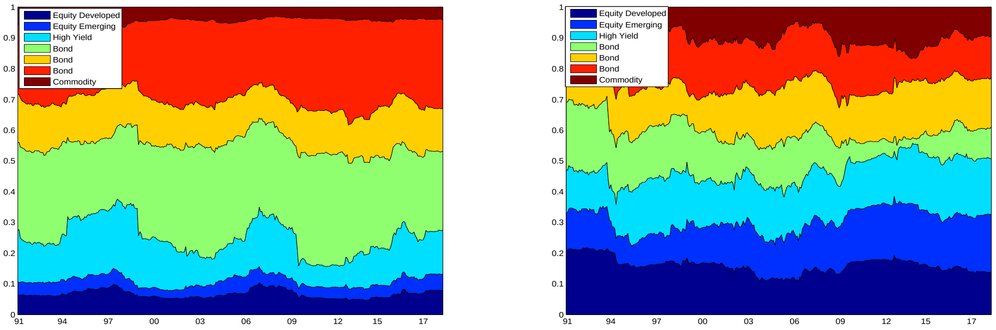
Figure 4.10: Weights and Risk Contributions of DRP Strategies based on PCA and MTP for 5-year rolling window



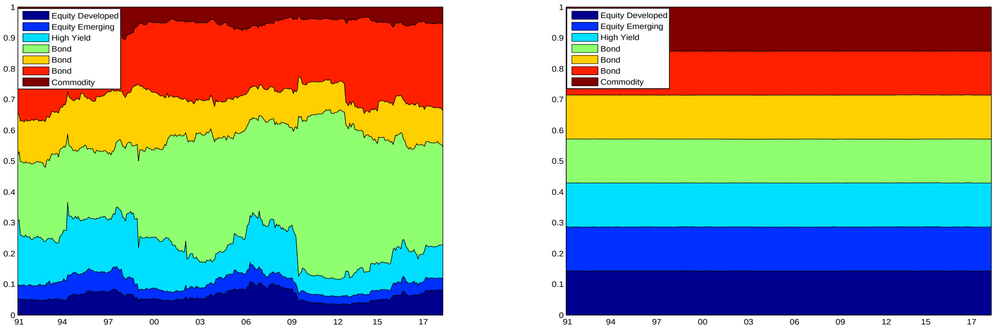
(a) EW Weights (b) EW Risk contributions by assets



(c) GMV Weights (d) GMV Risk contributions by assets

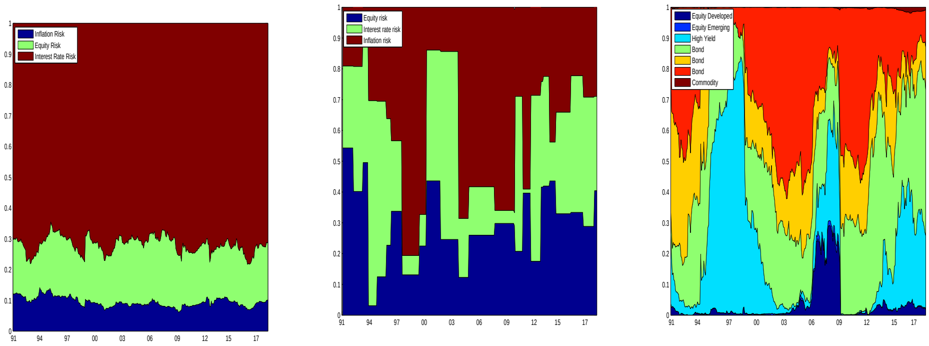


(e) IV Weights (f) IV Risk contributions by assets



(g) ERC Weights (h) ERC Risk contributions by assets

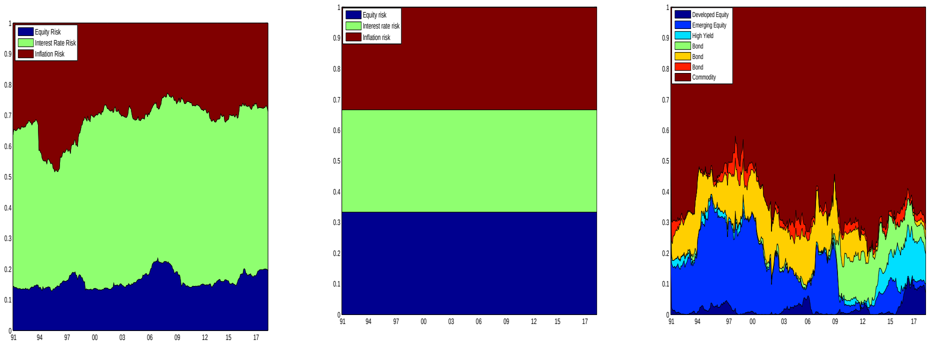
Figure 4.11: Weights and Risk Contributions of Risk-based Strategies for 3-year rolling window



(a) DRP_{PP} Weights

(b) DRP_{PP} Risk contributions by risk sources

(c) DRP_{PP} Risk contributions by assets



(d) DRP_{MTP} Weights

(e) DRP_{MTP} Risk contributions by risk sources

(f) DRP_{MTP} Risk contributions by assets

Figure 4.12: Weights and Risk Contributions of DRP Strategies based on PCA and MTP for 3-year rolling window

portfolio based on the minimum torsion demonstrates a stable risk distribution among the uncorrelated risk factors however the risk contributions from the asset classes are more volatile.

4.3.4 Out-of sample testing

This part presents the result of out-of-sample testing. One of the reasons why risk-based strategies have become popular is that mean-variance (MV) optimization demonstrates poor performance in out-of-sample testing due to large estimation errors. Thus, we include MV strategy to make a clear comparison among strategies. We have four different periods. The reason of employing different periods is to capture the changes in the economy. The Table 4.9 presents the results. The left-hand side of the table provides the weights to be used in the right-hand side out-of-sample periods. Percent estimation error is calculated based on Sharpe ratios by following [27]. Besides Sharpe ratio, we also provide the results of return, risk, the gini coefficient of risk and number of uncorrelated risks. The returns, risks and Sharpe ratios are annualized.

According to first out-of-sample results (2003-2007), MV strategy has the highest estimation error (91.7%) in its Sharpe ratio. The second highest estimation error (45.7%) belongs to the GMV strategy. EW portfolio shoes the lowest estimation error of 19.8%. Risk parity strategies and DRP strategies demonstrate close results about 25%. Then, we examine the risk characteristics of the portfolios. In sample period, DRP strategies distribute the risk among three risk factors. In the out-of sample period, DRP_{MTP} still spreads the risk among almost three factors while DRP_{PP} distributes the risk almost two risk factors. The remaining portfolios demonstrate risk concentrated risk structure in the out-of-sample results.

The second out-of-sample period (2004-2008) shows that MV portfolio has the highest estimation error of 104.4%. GMV strategy has th second highest estimation error of 70.2%. The lowest estimation error (52.6%) belongs to the DRP_{MTP} but there is not remarkable difference with the remaining strategies. Compared to the first period, the estimation errors in this period are high. The reason of this increase might be the result of 2008 financial crisis. As for the risk characteristics, the portfolios show the risk concentration on one risk factor except DRP_{MTP} strategy. DRP_{MTP} portfolio distributes the risk among almost three risk factors.

According third out-of-sample period (2009-2013) results, the highest estimation error (189%) is shown by the MV strategy. GMV has the second highest estimation error of 49.2%. The lowest error (9%) belongs to the EW portfolio. DRP strategies (PP and MTP) also demonstrate favorably low estimation errors (10% and 12.3%, respectively). In this period, there is large decrease in returns. Despite of these decreases, the Sharpe ratios are not such low and there are not large estimation errors (except MV startegy) compared to previous periods. After financial crisis, interest rates are reduced to almost zero. Therefore, in the period, the risk-free rate is very low so that the excess returns of the portfolios remain high, which leads to high Sharpe ratios. Except DRP_{MTP} strategy, all strategies are concentrated on one risk source. DRP_{MTP} strategy again distributes the risk almost three risk factors.

In the last out-of-sample testing, MV strategy has the highest estimation error of 144.3%. Different from other periods, ERC portfolio has the second highest estimation error of 71%. EW shows the lowest estimation error of 18.3%. DRP strategies also have low estimation errors. As for the risk structure, ERC and DRP_{PP} distribute the risk among almost two risk sources. DRP_{MTP} spreads the risk across almost three risk factors. The remaining portfolios have risk concentrated structure.

To sum up, MV strategy has the highest estimation errors in all out-of-sample results. This drawback of MV optimization is also shown by different researches such as [8, 15]. The reason of the poor performance is that MV strategy includes the expected mean estimation which leads to large estimation errors. EW portfolio has the lowest estimation errors in all results except in one period. DRP strategies also demonstrate much lower estimation errors than MV strategy. Contrary to the works by [27, 16], we obtain good out-of-sample results based on Sharpe ratio for the DRP strategies. As for the risk structures, all strategies have risk concentrated on risk source except DRP_{MTP} portfolio. The reason of well-diversified structure of DRP_{MTP}, it defines each risk properly as shown in Figures 4.7 and 4.8.

Table 4.9: Out-of-sample Testing

		1998	-	**2002**			**2003**	-	**2007**		
	Return (%)	Risk (%)	Sharpe Ratio	$Gini_r$	Number of Uncorrelated risks	Return (%)	Risk (%)	Sharpe Ratio	$Gini_r$	Number of Uncorrelated risks	% Estimation Error
MV	10.37	7.20	0.75	0.42	1.73	4.72	4.60	0.39	0.80	1.02	91.7
EW	8.22	9.51	0.34	0.60	1.44	6.59	8.65	0.42	0.76	1.05	19.8
GMV	6.33	3.01	0.45	0.85	1.02	5.51	3.15	0.82	0.82	1.01	45.7
IV	8.17	5.58	0.57	0.46	1.71	7.05	5.49	0.75	0.42	1.33	24.1
ERC	8.19	4.92	0.65	0.30	2.10	8.08	5.86	0.88	0.34	1.44	25.9
DRP_{PP}	8.02	4.08	0.74	0.00	3.00	7.52	6.96	0.66	0.26	1.32	12.4
DRP_{MTP}	8.24	4.15	0.79	0.00	3.00	8.65	6.81	0.84	0.05	2.79	6.7
		1988	-	**2003**			**2004**	-	**2008**		
	Return (%)	Risk (%)	Sharpe Ratio	$Gini_r$	Number of Uncorrelated risks	Return (%)	Risk (%)	Sharpe Ratio	$Gini_r$	Number of Uncorrelated risks	% Estimation Error
MV	10.92	5.10	1.29	0.34	2.47	4.15	6.47	0.63	0.36	1.64	104.4
EW	7.54	8.60	0.37	0.47	1.56	7.58	8.92	0.84	0.64	1.52	55.7
GMV	5.32	3.98	0.25	0.95	1.00	2.07	2.38	0.85	0.95	1.04	70.2
IV	7.33	5.18	0.58	0.11	1.87	8.34	6.49	1.28	0.81	1.39	54.5
ERC	7.33	5.92	0.51	0.10	1.99	8.05	6.82	1.17	0.83	1.35	56.6
DRP_{PP}	6.32	5.57	0.36	0.00	3.00	3.57	4.51	0.78	0.10	2.59	53.8
DRP_{MTP}	7.51	5.61	0.57	0.00	3.00	7.35	6.07	1.20	0.04	2.94	52.6
		1988	-	**2008**			**2009**	-	**2013**		
	Return (%)	Risk (%)	Sharpe Ratio	$Gini_r$	Number of Uncorrelated risks	Return (%)	Risk (%)	Sharpe Ratio	$Gini_r$	Number of Uncorrelated risks	% Estimation Error
MV	7.16	2.97	1.58	0.49	2.11	2.80	4.76	0.55	0.78	1.26	189.0
EW	5.96	7.00	0.50	0.67	1.10	2.77	4.69	0.57	0.54	1.16	9.00
GMV	3.95	2.90	0.51	0.90	1.01	2.81	2.70	0.96	0.76	1.03	47.2
IV	7.37	5.19	0.94	0.40	1.90	2.54	3.69	0.78	0.33	1.19	49
ERC	7.32	5.10	0.95	0.39	2.00	2.81	3.34	0.78	0.19	1.27	21.6
DRP_{PP}	6.56	5.98	0.68	0.00	3.00	2.56	4.31	0.55	0.13	1.35	10.0
DRP_{MTP}	7.14	6.22	0.75	0.00	3.00	2.94	4.13	0.66	0.07	2.66	13.6
		1988	-	**2012**			**2013**	-	**2017**		
	Return (%)	Risk (%)	Sharpe Ratio	$Gini_r$	Number of Uncorrelated risks	Return (%)	Risk (%)	Sharpe Ratio	$Gini_r$	Number of Uncorrelated risks	% Estimation Error
MV	9.51	7.07	0.67	0.56	1.47	4.37	3.57	0.27	0.57	1.41	144.3
EW	9.01	10.52	0.40	0.85	1.32	7.88	9.10	0.49	0.77	1.10	18.3
GMV	5.68	3.05	0.30	0.86	1.21	3.95	2.94	0.19	0.87	1.01	57.6
IV	8.45	3.87	0.95	0.55	1.73	6.73	5.22	0.64	0.58	1.30	48.7
ERC	8.51	3.99	0.94	0.20	2.30	6.59	5.82	0.55	0.29	1.79	71.0
DRP_{PP}	7.85	4.99	0.62	0.00	3.00	6.41	6.08	0.50	0.15	2.06	24.5
DRP_{MTP}	8.74	5.17	0.77	0.00	3.00	6.95	5.67	0.63	0.08	2.94	22.4

CHAPTER 5

CONCLUSION

In this thesis, we aim to maximize risk diversification of a portfolio with distributing the risk among the uncorrelated risk factors. To achieve our goal, we examine the diversified risk parity strategies compared to risk based asset allocation strategies.

We apply the proposed strategies to different asset class indices consisting of equities, bonds and commodity. DRP strategy is determined based on two approaches: PCA and MTP. The DRP_{MTP} strategy has a well balanced risk structure with distributing the whole risk among three main risk sources. The result is consistent according to the three-year and five-year rolling window estimations. The other diversified risk parity strategy based on principal component analysis also demonstrates the similar result. However, compared to MTP, we observe that principal portfolios are not stable over the time and thus do not give the same economic interpretations. The portfolio may actually concentrate on one or few risk sources.

The benchmark strategies created ill-diversified portfolios in terms of risk. Contrary to diversified risk parity strategies, the risk contribution of these strategies comes from the asset classes instead of the risk factors. The risk contributions from GMV and EW strategies exhibit the most unbalanced structure. On the other hand, ERC and IV strategies distribute the risk quite balanced among the asset classes. However they do not demonstrate the same performance distributing the risk among the risk factors. As for the return performance, risk based strategies except GMV outperformed the diversified risk parity strategies. However, DRP_{MTP} has the best reward to volatility ratio. Our results are consistent with [16, 27] but contradict with the results by [20, 2].

One of the reasons why risk-based strategies have become popular is that mean-variance (MV) optimization demonstrates poor performance in out-of-sample testing due to large estimation errors. Thus, we include MV strategy to make a clear comparison among strategies in out-of-sample testing. MV strategy has the highest estimation errors in all out-of-sample results. This drawback of MV optimization is also shown by different researches such as [8, 15]. The reason of the poor performance is that MV strategy includes the expected mean estimation which leads to large estimation errors. EW portfolio has the lowest estimation errors in all results except in one period. DRP strategies also demonstrate much lower estimation errors than MV strategy. Contrary to the works by [27, 16], we obtain good out-of-sample results based on Sharpe ratio for the DRP strategies. As for the risk structures, all strategies have risk concentrated on risk source except DRP_{MTP} portfolio.

As a conclusion, to construct well-diversified portfolio for distributing the risk among three factors, DRP strategies demonstrate good performance in both Sharpe ratio and risk diversification in out-of-sample testing. Specifically, DRP_{MTP} constructs a well-diversified portfolio spreading the risk among three factors. The remarkable result of this strategy is good performance in out-of-sample testing. This strategy may help the investors to construct risk concentrated portfolios, even in financial crisis.

The future work may examine the long-short constraint since we only focused on the long only case. The effect of the transaction cost also has a key role when evaluating the performance of the portfolios thus it may be examined for future research. In this work, we used the variance as risk measure however, there are other kind of risk measures such as value at risk, expected shortfall. The DRP strategies may be examined when employing other risk measures.

REFERENCES

[1] P. Behr, A. Güttler, and F. Miebs, Is minimum-variance investing really worth the while? an analysis with robust performance inference, EDHEC-Risk working paper, 2008.

[2] S. Bernardi, M. Leippold, and H. Lohre, Maximum diversification strategies along commodity risk factors, European Financial Management, 24(1), 53-78, 2018.

[3] V. Bhansali, J. Davis, G. Rennison, J. Hsu, and F. Li, The risk in risk parity: A factor-based analysis of asset-based risk parity, Journal of investing, 21(3), 102-110, 2012.

[4] Bloomberg, Bloomberg terminal, [Online; accessed 4-January-2018].

[5] M. D. Braga, *Risk-Based Approaches to Asset Allocation: Concepts and Practical Applications*, Springer, 2016.

[6] V. K. Chopra and W. T. Ziemba, The effect of errors in means, variances, and covariances on optimal portfolio choice, The Kelly Capital Growth Investment Criterion: Theory and Practice, 3, 6-11, 2011.

[7] Y. Choueifaty and Y. Coignard, Toward maximum diversification, The Journal of Portfolio Management, 35(1), 40-51, 2008.

[8] V. DeMiguel, L. Garlappi, and R. Uppal, Optimal versus naive diversification: How inefficient is the $1/n$ portfolio strategy?, The review of Financial studies, 22(5), 1915-1953, 2007.

[9] R. Everson, Orthogonal, but not orthonormal, procrustes problems, Advances in computational Mathematics, 3(4), 1998.

[10] W. Fleming, *Functions of several variables*, Springer Science & Business Media, 2012.

[11] French, Kenneth R., Risk free rate, [Online; accessed 4-January-2018].

[12] R. Gray, *Entropy and Information Theory*, SpringerLink : Bücher, Springer US, 2011, ISBN 9781441979704.

[13] R. A. Haugen and N. L. Baker, The efficient market inefficiency of capitalization–weighted stock portfolios, The Journal of Portfolio Management, 17(3), 35-40, 1991.

[14] A. Ilmanen and J. Kizer, The death of diversification has been greatly exaggerated, The Journal of Portfolio Management, 38(3), 15-27, 2012.

[15] J. D. Jobson and R. M. Korkie, Putting markowitz theory to work, The Journal of Portfolio Management, 7(4), 70-74, 1981.

[16] C. Kind, Risk-based allocation of principal portfolios, 2013.

[17] C. Kind and M. Poonia, Diversification management of a multi-asset portfolio, 2014.

[18] M. Kritzman, S. Page, and D. Turkington, In defense of optimization: the fallacy of 1/n, Financial Analysts Journal, 66(2), 31-39, 2010.

[19] R. Litterman, Hot spots and hedges, The Journal of Portfolio Management, 23(5), 52-75, 1996.

[20] H. Lohre, H. Opfer, and G. Orszag, Diversifying risk parity, Journal of Risk, 16(5), 53-79, 2014.

[21] S. Maillard, T. Roncalli, and J. Teïletche, The properties of equally weighted risk contribution portfolios, The Journal of Portfolio Management, 36(4), 60-70, 2010.

[22] H. Markowitz, Portfolio selection, The journal of finance, 7(1), 77-91, 1952.

[23] R. C. Merton, An intertemporal capital asset pricing model, Econometrica: Journal of the Econometric Society, 867-887, 1973.

[24] A. Meucci, Managing diversification, Risk, 22(5), 74-79, 2010.

[25] A. Meucci, A. Santangelo, and R. Deguest, Risk budgeting and diversification based on optimized uncorrelated factors, 2015.

[26] M. H. Partovi and M. Caputo, Principal portfolios: Recasting the efficient frontier, Economics Bulletin, 7(3), 1-10, 2004.

[27] T. Poddig and A. Unger, On the robustness of risk-based asset allocations, Financial Markets and Portfolio Management, 26(3), 369-401, 2012.

[28] G. Pola, On entropy and portfolio diversification, Journal of Asset Management, 17(4), 218-228, 2016.

[29] E. Qian, Are risk-parity managers at risk parity?, Journal of Portfolio Management, 40(1), 20-26, 2013.

[30] E. E. Qian, On the financial interpretation of risk contribution: Risk budgets do add up, Journal of Investment Management, 4, 1-11, 2005.

[31] R. Roll and S. A. Ross, An empirical investigation of the arbitrage pricing theory, The Journal of Finance, 35(5), 1073-1103, 1980.

[32] T. Roncalli, *Introduction to risk parity and budgeting*, CRC Press, 2013.

[33] P. H. Schönemann, A generalized solution of the orthogonal procrustes problem, Psychometrika, 31(1), 1-10, 1966.

[34] G. Xianguang, Application of improved entropy method in evaluation of economic result, Systems Engineering-Theory & Practice, 12, 1998.

[35] R. Zhou, R. Cai, and G. Tong, Applications of entropy in finance: A review, Entropy, 15(11), 4909-4931, 2013.