GENERALIZED CHILLINGWORTH CLASSES ON SUBSURFACE TORELLI GROUPS

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

HATİCE ÜNLÜ EROĞLU

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY
IN
MATHEMATICS

AUGUST 2018

Approval of the thesis:

GENERALIZED CHILLINGWORTH CLASSES ON SUBSURFACE TORELLI GROUPS

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ABSTRACT

GENERALIZED CHILLINGWORTH CLASSES ON SUBSURFACE TORELLI GROUPS

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August 2018, 61 pages

The Torelli group is the subgroup of the mapping class group that acts trivially on homology. Putman's subsurface Torelli groups are an important construction for working with the Torelli group, as they restore the functoriality essential for the inductive arguments on which mapping class group arguments are invariably based. The other important structure on the Torelli group is the Johnson homomorphism. The contraction of the image of the Johnson homomorphism is the Chillingworth class. In this thesis, a combinatorial description of the Chillingworth class is derived for the subsurface Torelli groups. This thesis also brings in the naturality and uniqueness properties on the map whose image is the dual of the Chillingworth classes of the subsurface Torelli groups. Moreover, a relation between the Chillingworth classes of the subsurface Torelli groups and the partitioned Johnson homomorphism is presented.

Keywords: the Torelli group, the Johnson homomorphism, the Chillingworth class

ÖZ

ALTYÜZEY TORELLİ GRUPLARI ÜZERİNDE GENELLEŞTİRİLMİŞ CHILLINGWORTH SINIFLARI

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Ağustos 2018 , 61 sayfa

Torelli grubu, gönderim sınıfları grubunun homoloji üzerinde aşikar şekilde etki eden alt grubudur. Gönderim sınıfları argümanlarının zaman zaman dayandırıldığı tümevarım argümanları için funktörlük özelliğini yeniden kurduğundan dolayı Putman'ın altyüzey Torelli grupları Torelli gruplarıyla çalışmada önemli bir inşadır. Torelli grupları üzerinde diğer önemli bir yapı Johnson homomorfizmasıdır. Johnson homomorfizmasının görüntüsünün büzülmesi Chillingworth sınıfını verir. Bu tezde, altyüzey Torelli grupları için Chillingworth sınıfının kombinatoryal bir tanımı türetilmektedir. Bu tez, görüntüsü altyüzey Torelli gruplarının Chillingworth sınıflarının duali olan dönüşüme doğallık ve teklik özelliklerini de kazandırır. Ayrıca, altyüzey Torelli gruplarının Chillingworth sınıfları ve bölüntülü Johnson homomorfizması arasındaki bağlantı sunulmaktadır.

Anahtar Kelimeler: Torelli grubu, Johnson homomorfizması, Chillingworth sınıfı

To my family

ACKNOWLEDGMENTS

I would like to express my sincere gratitude to my advisor, Prof. Dr. Mustafa Korkmaz, for his support, patience and vast knowledge. I am thankful for his comments and advice that he offered so generously during this thesis. His thoughts and comments have always forced me to think things through more carefully.

Many special thanks go to Dr. Ingrid Irmer for sharing her ideas, her enthusiasm for this thesis. I feel indebted to her for her sincere and continuous support, encouragement, friendly attitude and kind guidance. I doubt that I will ever be able to convey my appreciation fully, but I owe her my eternal gratitude.

I would like to thank the examining committee members, Prof. Dr. Turgut Önder, Assoc. Prof. Dr. Ferihe Atalan Ozan, Assoc. Prof. Dr. M. Fırat Arıkan, Assoc. Prof. Dr. Özgün Ünlü for their comments.

I would like to thank to my colleagues and my dear friends Elçin Çalişkan, Dr. Hanife Varlı, Dr. Adalet Çengel, Dr. Tülin Altunöz, Dr. Hatice Çoban, Dr. Sabahattin Ilbıra, Merve Seçgin. I appreciate them for inspiring study environment and their lovely friendship. I also would like to thank Sadenur Doğan Aslantatar for her friendship and moral support especially in my early years in PhD.

I want to thank all members of the Mathematics Department, especially Adem Bulat and Saadettin Selvitop.

This work is financially supported by TÜBİTAK-BİDEB National Graduate Scholarship Programme for PhD (2211).

I owe my deepest gratitude to my mother Nazlı, my father Tekin, my sisters Safiye and Bahar, my grandmother Hatice for their endless love, support and encouragement.

Last but definitely not the least, I wish to deeply thank to my husband Alperen for his support, understanding and love.

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CHAPTER 1

INTRODUCTION

Let $\Sigma_{g,n}$ be a compact connected oriented smooth surface of genus g with n boundary components. For closed surfaces, we prefer to use the notation Σ_g. For $g \geq 0$ and $0 \leq n \leq 1$, the Torelli group of $\Sigma_{g,n}$, denoted $\mathcal{I}(\Sigma_{g,n})$, is the normal subgroup of the mapping class group $\mathcal{M}(\Sigma_{g,n})$ of $\Sigma_{g,n}$ that acts trivially on $H_1(\Sigma_{g,n};\mathbb{Z})$. We have the following exact sequence:

$$1 \to \mathcal{I}(\Sigma_{g,n}) \to \mathcal{M}(\Sigma_{g,n}) \to \mathrm{Sp}(2g,\mathbb{Z}) \to 1.$$

In this exact sequence $\mathrm{Sp}(2g,\mathbb{Z})$ is well understood, being a matrix group. We therefore see that understanding the mapping class group boils down to understanding $\mathcal{I}(\Sigma_{g,n})$. Moreover, the Torelli group arises in research areas such as algebraic geometry and 3-manifold theory. For instance, integral homology 3-spheres are obtained by using the Torelli group.

Inductive arguments on subsurfaces are implicit in most proofs involving mapping class groups [18, 19, 12]. In such arguments a certain functoriality property of the mapping class group is absolutely essential [26]. Proofs in the mapping class group theory are typically structured as follows: the theorem is proven on a subsurface such as a pant or a 4-holed sphere. Functoriality is then used to extend the result to a larger surface.

In [26], Putman defined the subsurface Torelli groups in order to use inductive arguments in the Torelli group. An embedding of a subsurface $\Sigma_{g,n}$ into a larger surface $\Sigma_{g'}$ gives a partition $\mathcal{P}$ of the boundary components of $\Sigma_{g,n}$ recording which of the boundary components of $\Sigma_{g,n}$ become homologous in $\Sigma_{g'}$ [5]. Putman [26] defined the subsurface Torelli group $\mathcal{I}(\Sigma_{g,n},\mathcal{P})$ by restricting $\mathcal{I}(\Sigma_{g'})$ to $\Sigma_{g,n}$. Different embeddings of $\Sigma_{g,n}$ give different Torelli groups. See Figure 3.1 in Chapter 3. This is because different embeddings induce different maps from the homology of $\Sigma_{g,n}$ into the homology of $\Sigma_{g'}$. In order to capture this information, Putman defined the notion of a partitioned surface $(\Sigma_{g,n}, \mathcal{P})$, where $\mathcal{P}$ is the partition of the boundary components of $\Sigma_{g,n}$.

The resulting subsurface Torelli groups, $\mathcal{I}(\Sigma_{g,n}, \mathcal{P})$, defined in [26] restore functoriality and are therefore of central importance to the study of the Torelli group. For instance, they were used to give the first complete, verifiable proof that the Torelli group of surfaces with genus at least 2 is generated by bounding pair maps and Dehn twists around separating simple closed curves.[1] The subsurface Torelli groups were also used to obtain an efficient generating set for the Torelli group in [27], Theorem A and Theorem B.

The Johnson homomorphism $\tau : \mathcal{I}(\Sigma_{g,1}) \to \bigwedge^3 H_1(\Sigma_{g,1};\mathbb{Z})$ is a surjective homomorphism. The Johnson homomorphism determines the abelianization of $\mathcal{I}(\Sigma_{g,1})$ mod torsion in the following sense:

Theorem 1.0.1 ([16], Theorem 3). *For $g \geq 3$, $H_1(\mathcal{I}(\Sigma_{g,1});\mathbb{Z}) \cong W \oplus \bigwedge^3 H_1(\Sigma_{g,1};\mathbb{Z})$, where W consists of 2-torsion and $\bigwedge^3 H_1(\Sigma_{g,1};\mathbb{Z})$ is the image of the Johnson homomorphism.*

As no finite presentation for the Torelli group is known, the finiteness information inherent in the abelianization of the Torelli group is an important tool. For example, Putman showed in [27], Theorem A, that $\mathcal{I}(\Sigma_{g,1})$ has a generating set growing cubically with respect to genus. Theorem 1.0.1 shows that $H_1(\mathcal{I}(\Sigma_{g,1});\mathbb{Z})$ has rank cubic in the genus. Since $H_1(\mathcal{I}(\Sigma_{g,1});\mathbb{Z})$ is the quotient of $\mathcal{I}(\Sigma_{g,1})$ by its commutator subgroup $[\mathcal{I}(\Sigma_{g,1}), \mathcal{I}(\Sigma_{g,1})]$, the number of elements in a generating set of $\mathcal{I}(\Sigma_{g,1})$ is greater than or equal to the number of elements in a generating set of $H_1(\mathcal{I}(\Sigma_{g,1});\mathbb{Z})$. Therefore, the number of elements in a generating set for $\mathcal{I}(\Sigma_{g,1})$ must grow at least cubically in g.

Analogues of the Johnson homomorphism can be found in different parts of mathematics, such as 3- manifold topology (e.g. [9], Section 2.3), the geometry of the

[1] This fact is usually attributed to Birman and Powell [1, 25]. However, the Birman- Powell proof was based on unpublished material, and generally considered too unwieldy to be checked.

moduli space of curves (e.g. [11], Section 3 and 4), and number theory (e.g. [20]).

The Chillingworth class is the tensor contraction of the image of the Johnson homomorphism. More details can be found in Subsection 2.2.2. The Chillingworth homomorphism $t : \mathcal{I}(\Sigma_{g,1}) \to H_1(\Sigma_{g,1}; \mathbb{Z})$ is a homomorphism sending each $f \in \mathcal{I}(\Sigma_{g,1})$ to the Chillingworth class of f.

In [22], the Johnson homomorphism was extended by Morita to give a crossed homomorphism on the entire mapping class group. Thus, the Chillingworth homomorphism can be extended to $\mathcal{M}(\Sigma_{g,1})$ by composing the extended Johnson homomorphism with the tensor contraction.

Morita [21] also proved the following isomorphisms:

$$H^1(\mathcal{M}(\Sigma_{g,1}); H^1(\Sigma_g; \mathbb{Z})) \cong H^1(\mathcal{M}(\Sigma_{g,*}); H^1(\Sigma_g; \mathbb{Z})) \cong \mathbb{Z}.$$

Here $\Sigma_{g,*}$ is obtained by attaching a disc to $\partial\Sigma_{g,1}$ with a fixed point $*$, where each element of the mapping class group $\mathcal{M}(\Sigma_{g,*})$ is assumed to fix $*$. Since the cohomology group $H^1(\mathcal{M}(\Sigma_{g,1}); H_1(\Sigma_{g,1}; \mathbb{Z}))$ is infinite cyclic, any two crossed homomorphisms from $\mathcal{M}(\Sigma_{g,1})$ to $H_1(\Sigma_{g,1}; \mathbb{Z})$ differ by a multiplicative constant and addition of a coboundary. Therefore, any crossed homomorphism $\mathcal{M}(\Sigma_{g,1}) \to H_1(\Sigma_{g,1}; \mathbb{Z})$ restricts to a constant multiple of the Chillingworth homomorphism on $\mathcal{I}(\Sigma_{g,1})$. Earle [7] first constructed a crossed homomorphism from $\mathcal{M}(\Sigma_{g,*})$ to $H_1(\Sigma_g; (1/(2g-2))\mathbb{Z})$. By multiplying this crossed homomorphism by $(2g-2)$ we get a crossed homomorphism $\mathcal{M}(\Sigma_{g,*}) \to H_1(\Sigma_g; \mathbb{Z})$. Hence we attribute the definition of the Chillingworth class to Earle.

In this thesis, we construct a combinatorial description of the Chillingworth class of the subsurface Torelli groups via winding numbers in the projective tangent bundle. Given the definition of Putman's subsurface Torelli groups, the difficulty in finding a combinatorial description via winding numbers is to make sense of the winding number of an arc with end points on the boundary of the subsurface. By defining a difference cocycle on the projective tangent bundle of the surface we are able to make sense of the winding number of the difference of two arcs.

The rest of this thesis is structured as follows:

In Chapter 2, basic definitions and theorems related to the Torelli group, the Johnson homomorphism and the Chillingworth class are given. We have provided a survey of different constructions given by Trapp in [29] and Furuta (see [23]) for obtaining crossed homomorphisms on the mapping class group using winding numbers. Both of these constructions have been shown to be equivalent by using difference cocycles.

Chapter 3 presents subsurface Torelli groups defined by Putman in [26]. The partitioned Johnson homomorphism constructed by Church [5] is discussed.

In Chapter 4, we construct a well-defined map $\widetilde{e}_X : \mathcal{I}(\Sigma_{g,n}, \mathcal{P}) \to H_1^{\mathcal{P}}(\Sigma_{g,n}, \mathbb{Z})$ using the projective tangent bundle of $\Sigma_{g,n}$. Here, X is a nonvanishing vector field on $\Sigma_{g,n}$ and $H_1^{\mathcal{P}}(\Sigma_{g,n}, \mathbb{Z})$ denotes the homology group defined by Putman [26]. We show that $\widetilde{e}_X$ is a homomorphism. We define a symplectic basis for the homology group $H_1^{\mathcal{P}}(\Sigma_{g,n}, \mathbb{Z})$ and call the dual of $\widetilde{e}_X(f)$ *the Chillingworth class of* f. One reason for calling this dual the Chillingworth class, is that it is shown to factor through the partitioned Johnson homomorphism. Therefore, we obtain a combinatorial description of the Chillingworth class of the subsurface Torelli groups using the projective tangent bundle of $\Sigma_{g,n}$.

We use the Torelli category $\mathcal{T}\mathrm{Surf}$ defined by Church [5], which is the refinement of the category TSur defined by Putman [26]. The Torelli group is a functor from $\mathcal{T}\mathrm{Surf}$ to the category of groups and homomorphisms [26]. For a morphism $i : (\Sigma_{g,n}, \mathcal{P}) \to (\Sigma_{g',n'}, \mathcal{P}')$ of $\mathcal{T}\mathrm{Surf}$ and a nonvanishing vector field X on $\Sigma_{g',n'}$, we prove the following:

Theorem 1.0.2. *There exists a homomorphism i'_* such that the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{I}(\Sigma_{g,n}, \mathcal{P}) & \xrightarrow{\quad i_* \quad} & \mathcal{I}(\Sigma_{g',n'}, \mathcal{P}') \\ \downarrow{\scriptstyle \widetilde{e}_Y} & & \downarrow{\scriptstyle \widetilde{e}_X} \\ \mathrm{Hom}(H_1^{\mathcal{P}}(\Sigma_{g,n}; \mathbb{Z}), \mathbb{Z}) & \xrightarrow[i'_*]{} & \mathrm{Hom}(H_1^{\mathcal{P}'}(\Sigma_{g',n'}; \mathbb{Z}), \mathbb{Z}) \end{array} \tag{1.1}$$

Here Y is the restriction of X to $\Sigma_{g,n}$.

We also prove that $\widetilde{e}_Y$ is unique in the sense that it is the only nontrivial homomorphism such that diagram (1.1) commutes. We also get a commutative diagram for the Chillingworth homomorphism $t_{(\Sigma_{g,n},\mathcal{P})} : \mathcal{I}(\Sigma_{g,n}, \mathcal{P}) \to H_1^{\mathcal{P}}(\Sigma_{g,n}; \mathbb{Z})$.

CHAPTER 2

BACKGROUND

In the first section of this chapter, we review the Torelli group. In the second section, we give definitions of the Johnson homomorphism, which is an important tool in the study of the Torelli group. Moreover, the Chillingworth class, the contraction of the Johnson homomorphism, and some of its properties are examined.

2.1 The Torelli Group

We start this section by giving basic definitions.

Let $\Sigma_{g,n}$ be a compact connected oriented smooth surface of genus g with n boundary components. If $n = 0$, we denote the surface as Σ_g. When the genus and the number of boundary components are not important, we will use Σ to denote the surface. Let $\mathrm{Diff}^+(\Sigma_{g,n}, \partial\Sigma_{g,n})$ be the group of orientation-preserving diffeomorphisms of $\Sigma_{g,n}$ onto itself which fix the boundary components of $\Sigma_{g,n}$ pointwise. The *mapping class group of* $\Sigma_{g,n}$ is the group of all isotopy classes of elements of $\mathrm{Diff}^+(\Sigma_{g,n}, \partial\Sigma_{g,n})$ where isotopies fix the boundary pointwise. Let $\mathcal{M}(\Sigma_{g,n})$ denote the mapping class group of $\Sigma_{g,n}$.

Throughout this thesis, we will be working with representatives of mapping classes that fix a neighborhood of the boundary pointwise. We will use the notation $f \circ h$ or fh to denote the composition of maps, where h is assumed to be applied first.

Definition 2.1.1. *A simple closed curve on* $\Sigma_{g,n}$ *is an embedding* $S^1 \to \Sigma_{g,n}$. *An arc on* $\Sigma_{g,n}$ *is an embedding* $\alpha : [0, 1] \to \Sigma_{g,n}$ *such that* $\alpha^{-1}(\partial\Sigma_{g,n}) = \{0, 1\}$.

When we say a closed curve or an arc, we will generally refer to their images.

We will denote a curve and its isotopy class by the same notation. A diffeomorphism and its isotopy class will also be denoted by the same symbol.

When we cut a surface along a simple closed curve γ on the surface, if we obtain more than one connected components, then we call γ a *separating curve*. Otherwise, γ is called a *nonseparating curve*.

Let γ be a simple closed curve on $\Sigma_{g,n}$. A tubular neighborhood of γ is homeomorphic to an annulus $S^1 \times [0,1]$. Let Ψ denote a diffeomorphism from the tubular neighborhood of γ to $S^1 \times [0,1]$. Represent elements of S^1 by complex numbers of norm 1. Define a diffeomorphism $\Phi : S^1 \times [0,1] \to S^1 \times [0,1]$ mapping (z,t) to $(e^{-2\pi i t}z, t)$. Notice that Φ is the identity on the boundary components. The composition $\Psi^{-1} \circ \Phi \circ \Psi$ gives a self diffeomorphism on the chosen tubular neighborhood of γ. Extending $\Psi^{-1} \circ \Phi \circ \Psi$ to the entire surface by the identity map in the complement we get a self diffeomorphism of $\Sigma_{g,n}$. This extended diffeomorphism T_γ is called a *Dehn twist around* γ.

We will now define the symplectic representation and the Torelli group.

There is an action of the mapping class group on the first homology group of the surface defined as follows:

Let ϕ be a diffeomorphism of the surface $\Sigma_{g,1}$ whose restriction to the boundary is the identity. It is well known that ϕ induces an automorphism $\phi_* : H_1(\Sigma_{g,1};\mathbb{Z}) \to H_1(\Sigma_{g,1};\mathbb{Z})$. If ϕ is isotopic to ψ, then we get $\phi_* = \psi_*$. Therefore, we have a well-defined homomorphism

$$\rho : \mathcal{M}(\Sigma_{g,1}) \to \text{Aut}(H_1(\Sigma_{g,1};\mathbb{Z})).$$

Since $H_1(\Sigma_{g,1};\mathbb{Z}) \cong H_1(\Sigma_g;\mathbb{Z}) \cong \mathbb{Z}^{2g}$, an isomorphism between $\text{Aut}(H_1(\Sigma_{g,1};\mathbb{Z}))$ and $\text{GL}(2g,\mathbb{Z})$ is obtained by choosing a basis for homology. Choose a symplectic basis $\{a_1, a_2, \ldots, a_g, b_1, b_2, \ldots, b_g\}$ of $H_1(\Sigma_{g,1};\mathbb{Z})$, i.e. a basis with the property that

$$a_i \cdot a_j = b_i \cdot b_j = 0,$$

$$a_i \cdot b_j = \delta_{ij},$$

for all $1 \leq i,j \leq g$, where δ_{ij} is the Kronecker delta and $a_i \cdot b_j$ denotes the algebraic intersection number of the homology classes a_i with b_j. The mapping class group

$\mathcal{M}(\Sigma_{g,1})$ preserves the orientation of $\Sigma_{g,1}$ and the intersection pairing. Hence we have a representation

$$\rho : \mathcal{M}(\Sigma_{g,1}) \to \mathrm{Sp}(2g, \mathbb{Z}).$$

The homomorphism $\rho : \mathcal{M}(\Sigma_{g,1}) \to \mathrm{Sp}(2g, \mathbb{Z})$ is a surjective homomorphism (see [8], Section 6.3.2). It is called the symplectic representation of $\mathcal{M}(\Sigma_{g,1})$. Applying the same argument we also get the surjective homomorphism

$$\rho : \mathcal{M}(\Sigma_g) \to \mathrm{Sp}(2g, \mathbb{Z}).$$

The subgroup of $\mathcal{M}(\Sigma_g)$ acting trivially on $H_1(\Sigma_g; \mathbb{Z})$ is a normal subgroup of $\mathcal{M}(\Sigma_g)$ and is called the *Torelli group*. In other words, the Torelli group is the kernel of the symplectic representation $\rho : \mathcal{M}(\Sigma_g) \to \mathrm{Sp}(2g, \mathbb{Z})$. It will be denoted by the symbol $\mathcal{I}(\Sigma_g)$. The notation $\mathcal{I}(\Sigma_{g,1})$ will be used for the kernel of the symplectic representation $\rho : \mathcal{M}(\Sigma_{g,1}) \to \mathrm{Sp}(2g, \mathbb{Z})$. The following exact sequence is obtained:

$$1 \to \mathcal{I}(\Sigma_{g,1}) \to \mathcal{M}(\Sigma_{g,1}) \to \mathrm{Sp}(2g, \mathbb{Z}) \to 1.$$

Birman and Powell [1, 25] proved that $\mathcal{I}(\Sigma_{g,1})$ is generated by Dehn twists about separating simple closed curves and bounding pair maps. A bounding pair (α, β) is a pair of disjoint, homologous, nonseparating, simple closed curves. A bounding pair map is given by $T_\alpha T_\beta^{-1}$ for a bounding pair (α, β). The genus of a bounding pair is the genus of the subsurface of $\Sigma_{g,1}$ not containing the boundary of $\Sigma_{g,1}$. In [17], Theorem 2, Johnson showed that for $g \geq 3$, $\mathcal{I}(\Sigma_{g,1})$ is generated by bounding pair maps of genus 1.

2.2 The Johnson Homomorphism and The Chillingworth Class

In this section, we give equivalent definitions of the Johnson homomorphism. We also give the combinatorial description of the Chillingworth class for surfaces with one boundary component and for closed surfaces.

2.2.1 The Johnson Homomorphism

There are various definitions of the Johnson homomorphism. In this subsection, we will review some of them. For further information, see [14, 15, 8].

The Johnson homomorphism was first defined by using the action of $\mathcal{I}(\Sigma_{g,1})$ on the universal two step nilpotent quotient of the free group $\pi := \pi_1(\Sigma_{g,1}, *)$ where $* \in \partial\Sigma_{g,1}$ [14].

Define elements of the lower central series of π as follows:

$$\pi_1 := \pi, \qquad \pi_2 := [\pi, \pi], \qquad \pi_i := [\pi, \pi_{i-1}].$$

Consider the following central short exact sequence

$$1 \to \pi_2/\pi_3 \to \pi/\pi_3 \to \pi/\pi_2 \to 1.$$

Take two elements a and b of π/π_2, noting that π/π_2 can be identified with $H_1(\Sigma_{g,1};\mathbb{Z})$. If $\tilde{a}, \tilde{b} \in \pi/\pi_3$ are lifts of a and b, we obtain an isomorphism between $\bigwedge^2 H_1(\Sigma_{g,1};\mathbb{Z})$ and π_2/π_3 by sending $a \wedge b$ to $[\tilde{a}, \tilde{b}]$.

The Johnson homomorphism $\tau : \mathcal{I}(\Sigma_{g,1}) \to \mathrm{Hom}(H_1(\Sigma_{g,1};\mathbb{Z}), \pi_2/\pi_3)$ is the homomorphism defined as $\tau(f)(a) = f(\tilde{a})\tilde{a}^{-1}$. By the previous paragraph we can identify $\mathrm{Hom}(H_1(\Sigma_{g,1};\mathbb{Z}), \pi_2/\pi_3)$ with $\mathrm{Hom}(H_1(\Sigma_{g,1};\mathbb{Z}), \bigwedge^2 H_1(\Sigma_{g,1};\mathbb{Z}))$. There is a canonical isomorphism of $\mathrm{Hom}(H_1(\Sigma_{g,1};\mathbb{Z}), \bigwedge^2 H_1(\Sigma_{g,1};\mathbb{Z}))$ with $H_1^*(\Sigma_{g,1};\mathbb{Z}) \otimes \bigwedge^2 H_1(\Sigma_{g,1};\mathbb{Z})$. It is also known that $H_1^*(\Sigma_{g,1};\mathbb{Z})$ is isomorphic to $H_1(\Sigma_{g,1};\mathbb{Z})$ by the algebraic intersection pairing. Consequently, the Johnson homomorphism is of the form $\tau : \mathcal{I}(\Sigma_{g,1}) \to H_1(\Sigma_{g,1};\mathbb{Z}) \otimes \bigwedge^2 H_1(\Sigma_{g,1};\mathbb{Z})$. Moreover, Johnson showed that the image of the Johnson homomorphism is exactly $\bigwedge^3 H_1(\Sigma_{g,1};\mathbb{Z})$. In this way, a homomorphism

$$\tau : \mathcal{I}(\Sigma_{g,1}) \to \bigwedge^3 H_1(\Sigma_{g,1};\mathbb{Z})$$

is obtained [14].

Johnson also defined the Johnson homomorphism on closed surfaces [14].

Let D denote a closed disc in Σ_g and $\Sigma_{g,1}$ denote $\Sigma_g \setminus D^\circ$, where D° is the interior of D. Any diffeomorphism of $\Sigma_{g,1}$ can be extended by the identity diffeomorphism on D. Therefore, a surjective homomorphism $p_D : \mathcal{M}(\Sigma_{g,1}) \to \mathcal{M}(\Sigma_g)$ is obtained. Consider the following exact sequence

$$1 \to \text{Ker}\, p_D \to \mathcal{I}(\Sigma_{g,1}) \to \mathcal{I}(\Sigma_g) \to 1.$$

Since any two liftings f' and f'' of $f \in \mathcal{I}(\Sigma_g)$ differ by a multiple $k \in \text{Ker}\, p_D$, $\tau(f') = \tau(f'') + \tau(k)$. Therefore, the Johnson homomorphism on a closed surface is a surjective homomorphism

$$\tau_1 : \mathcal{I}(\Sigma_g) \to \bigwedge^3 H_1(\Sigma_g;\mathbb{Z})/\tau(\text{Ker}\, p_D).$$

Moreover, $\bigwedge^3 H_1(\Sigma_g;\mathbb{Z})/\tau(\text{Ker}\, p_D)$ and $\tau_1(f)$ are independent of the choice of D. It follows that τ_1 is a well- defined homomorphism.

Let $\{a_1, b_1, a_2, b_2, \ldots, a_g, b_g\}$ be a symplectic basis for $H_1(\Sigma_g;\mathbb{Z})$. Consider the map $u : H_1(\Sigma_g;\mathbb{Z}) \to \bigwedge^3 H_1(\Sigma_g;\mathbb{Z})$ taking x to $(\sum_{i=1}^{g} a_i \wedge b_i) \wedge x$ which is injective when $g \geq 2$. Note that $\sum_{i=1}^{g} a_i \wedge b_i$ does not depend on the chosen symplectic basis [14].

Lemma 2.2.1 ([14], Lemma 7A)**.** $\tau(\text{Ker}\, p_D) = \text{Im}\, u$*, where* Im u *denotes the image of* u*.*

Notice that u is an isomorphism onto its image. Identifying Im u with $H_1(\Sigma_g;\mathbb{Z})$, we have the Johnson homomorphism in the form

$$\tau_1 : \mathcal{I}(\Sigma_g) \to \bigwedge^3 H_1(\Sigma_g;\mathbb{Z})/H_1(\Sigma_g;\mathbb{Z}).$$

Another definition of the Johnson homomorphism was given by using mapping tori [15]. Let $f \in \mathcal{I}(\Sigma_{g,1})$. Build the mapping torus

$$M_f = \frac{\Sigma_{g,1} \times [0,1]}{(x,1) \sim (f(x),0)}.$$

Since f is an element of the Torelli group, for an oriented simple closed curve γ in $\Sigma_{g,1}$, $f([\gamma])$ is homologous to $[\gamma]$. Therefore, there exists an immersed surface S in $\Sigma_{g,1} \times \{0\}$ whose boundary is $\gamma - f(\gamma)$. The union $\gamma \times [0,1] \cup S$ is a closed surface Σ_γ in M_f.

Let $x \wedge y \wedge z \in \bigwedge^3 H_1(\Sigma_{g,1};\mathbb{Z})$ and let us represent each homology class x, y, z by an oriented multicurve in $\Sigma_{g,1}$. Let $\Sigma_x, \Sigma_y, \Sigma_z$ be the closed oriented surfaces obtained as in the previous paragraph. Then a homomorphism $\tau_2 : \mathcal{I}(\Sigma_{g,1}) \to \mathrm{Hom}(\bigwedge^3 H_1(\Sigma_{g,1};\mathbb{Z}),\mathbb{Z})$ that agrees with τ is defined to be $\tau_2(f)(x \wedge y \wedge z) = \Sigma_x \cdot \Sigma_y \cdot \Sigma_z$, where $\Sigma_x \cdot \Sigma_y \cdot \Sigma_z$ denotes the algebraic intersection number.

In [15], Johnson gave a third definition of the Johnson homomorphism via the Jacobi variety of $\Sigma_{g,*}$. Recall that $\Sigma_{g,*}$ is obtained by attaching a disc to $\partial\Sigma_{g,1}$ with a fixed point $*$ and each element of the mapping class group $\mathcal{M}(\Sigma_{g,*})$ fixes $*$. For further information, see [11, 6].

In [24], a combinatorial description of the Johnson homomorphism on $\Sigma_{g,*}$ is given by using the action of the mapping class group on the decorated Teichmüller space.

2.2.2 The Chillingworth Class

In this subsection, we will give basic definitions and constructions. This section also provides a survey of the different ways in which it is possible to use difference cocycle to obtain crossed homomorphisms on the mapping class group.

Let us choose a Riemannian metric on $\Sigma_{g,n}$ with which we define a norm on $T_x\Sigma_{g,n}$, the tangent space to $\Sigma_{g,n}$ at $x \in \Sigma_{g,n}$, for each $x \in \Sigma_{g,n}$. The *unit tangent bundle* $UT(\Sigma_{g,n})$ is a fiber bundle that consists of vectors of unit length in the tangent bundle $T\Sigma_{g,n}$.

Winding Number: Now we will give the definition of the winding number given by Chillingworth from [3]. Informally, given a nonvanishing vector field X, the winding number of a smooth closed oriented curve γ on a surface is defined as the number of rotations its tangent vector makes with respect to X as γ is traversed once in the positive direction.

Recall that, to start off with, we are assuming the surface has nonempty boundary so that a nonvanishing vector field X on $\Sigma_{g,n}$ exists. By choosing an appropriate parametrisation for a smooth closed curve, it can be assumed without loss of generality that the curve has a nonvanishing tangent vector at each point of the curve.

Let $\mathfrak{s} : S^1 \to \Sigma_{g,n}$ be a smooth closed oriented curve with $\mathfrak{s}(S^1) = \gamma$ and $pr : UT(\Sigma_{g,n}) \to \Sigma_{g,n}$ be the natural projection map sending each unit vector $v \in T_x\Sigma_{g,n}$

to x. We can define a pullback of the unit tangent bundle by $\mathfrak{s}$. The total space of the pullback bundle is defined as follows:

$$\{(z, v) \in S^1 \times UT(\Sigma_{g,n}) : \mathfrak{s}(z) = pr(v)\}$$

If $pr^{\mathfrak{s}} : \mathbb{T}^2 \to S^1$ denotes the projection map onto the first factor, there is a map F which is an isomorphism on each fiber such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{T}^2 & \xrightarrow{F} & UT(\Sigma_{g,n}) \\ {\scriptstyle pr^{\mathfrak{s}}}\downarrow & & \downarrow{\scriptstyle pr} \\ S^1 & \xrightarrow[\mathfrak{s}]{} & \Sigma_{g,n} \end{array} \tag{2.1}$$

Here $\mathbb{T}^2$ denotes a torus, $S^1 \times S^1$.

Given a vector field X, we can construct a section $\widetilde{X}^{\mathfrak{s}} : S^1 \to \mathbb{T}^2$ such that $F \circ \widetilde{X}^{\mathfrak{s}} = \widetilde{X} \circ \mathfrak{s}$ where $\widetilde{X}(x) = X(x)/\|X(x)\|$ for $x \in \Sigma_{g,n}$. This section is defined so that $\widetilde{X}^{\mathfrak{s}}(z) = (z, \widetilde{X}(\mathfrak{s}(z)))$ for every $z \in S^1$. As one can notice that $\widetilde{X}^{\mathfrak{s}}$ represents an element of $\pi_1(\mathbb{T}^2)$. Since the fundamental group of the torus is abelian, we do not write the base point in the expression of the fundamental group of $\mathbb{T}^2$.

By using the tangent map $d\mathfrak{s} : TS^1 \to T\Sigma_{g,n}$, we get a map $d_0\mathfrak{s} : S^1 \to UT(\Sigma_{g,n})$ defined by

$$d_0\mathfrak{s}(z) = d\mathfrak{s}((z, 1))/\|d\mathfrak{s}((z, 1))\|_{\mathfrak{s}(z)}.$$

The latter map pulls back to a unique section $Y^{\mathfrak{s}} : S^1 \to \mathbb{T}^2$ such that $F \circ Y^{\mathfrak{s}} = d_0\mathfrak{s}$. The projection map $pr^{\mathfrak{s}}$ induces the homomorphism $pr^{\mathfrak{s}}_* : \pi_1(\mathbb{T}^2) \to \pi_1(S^1)$. If E_0 denotes the fiber, S^1, and $i^{\mathfrak{s}} : E_0 \to \mathbb{T}^2$ is the inclusion map, we have the following exact sequence

$$0 \to \pi_1(E_0) \to \pi_1(\mathbb{T}^2) \to \pi_1(S^1).$$

Since $pr^{\mathfrak{s}}_*(X^{\mathfrak{s}}) = pr^{\mathfrak{s}}_*(Y^{\mathfrak{s}})$, we get

$$i^{\mathfrak{s}}_*(w^{\mathfrak{s}}) = Y^{\mathfrak{s}}(X^{\mathfrak{s}})^{-1}$$

for some unique $w^{\mathfrak{s}} \in \pi_1(E_0)$. A choice of orientation of $T_x\Sigma_{g,n}$ for $x \in \Sigma_{g,n}$ induces an orientation of E_0. Therefore, we can identify $w^{\mathfrak{s}}$ with an integer. The *winding number* $w_X(\gamma)$ *of* γ *with respect to* X is the integer $w^{\mathfrak{s}}$.

Lemma 2.2.2 ([3], Lemma 2.6)**.** *Let X be a nonvanishing vector field on $\Sigma_{g,n}$. If γ is a nullhomotopic, smooth simple closed curve, then $w_X(\gamma) = \pm 1$ depending on the orientation of γ.*

Theorem 2.2.3 ([3], Theorem 5.3)**.** *Let X be a nonvanishing vector field on $\Sigma_{g,n}$. Let γ_1 and γ_2 be smooth homotopic closed curves which are not null homotopic. Assume that both of them are in minimal position. Then they satisfy*

$$w_X(\gamma_1) = w_X(\gamma_2).$$

Let x be a point on $\Sigma_{g,n}$. We can define a function $\overline{w_X} : \pi_1(\Sigma_{g,n}, x) \to \mathbb{Z}$ such that for any nontrivial element $\beta \in \pi_1(\Sigma_{g,n}, x)$, $\overline{w_X}(\beta)$ is defined to be $w_X(\gamma)$ where γ is a smooth closed oriented curve in minimal position which is homotopic to β. See [3], Definition 2.8.

By Lemma 2.2.2, it is easily seen that $\overline{w_X} : \pi_1(\Sigma_{g,n}, x) \to \mathbb{Z}$ is not a homomorphism, as illustrated by the following example from [3]. Let $\Sigma_{1,1}$ be a punctured torus and X be a parallel vector field on $\Sigma_{1,1}$ with respect to a Euclidean metric, as shown in Figure 2.1. Let x be a point on $\Sigma_{1,1}$. Choose generators of $\pi_1(\Sigma_{1,1}, x)$ such that the tangent to a is everywhere parallel to X, and the tangent to b is everywhere perpendicular to X. Then $w_X(a) = w_X(b) = 0$. The commutator of a and b is freely homotopic to the boundary of $\Sigma_{1,1}$. However, in Figure 2.1 one can see that $w_X(aba^{-1}b^{-1}) = +1 \neq w_X(a) + w_X(b) - w_X(a) - w_X(b)$.

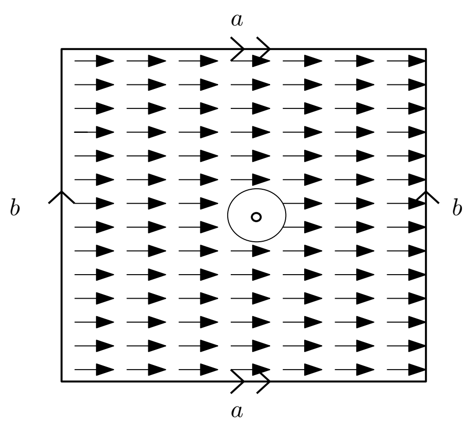


Figure 2.1: A punctured torus and a parallel vector field on it.

Lemma 2.2.4 ([3], Lemma 5.7)**.** *Let $\partial_1, \partial_2, \ldots, \partial_n$ be boundary components of $\Sigma_{g,n}$ with the orientations induced from $\Sigma_{g,n}$. If X is a nonvanishing vector field on $\Sigma_{g,n}$,* *then*

$$\sum_{i=1}^{n} w_X(\partial_i) = \pm(n + 2g - 2).$$

The sign depends on the orientation of $\Sigma_{g,n}$.

Difference Cocycle: We will define the difference cocyle introduced by Chillingworth in [3]. In fact, this definition is based on the obstruction theory exposed in [28]. In Part III of [28], the difference cochain is explained in detail.

For two nonvanishing vector fields X_1 and X_2 and for a Riemannian metric on $\Sigma_{g,n}$, one can define sections $\widetilde{X}_1$ and $\widetilde{X}_2$ of $UT(\Sigma_{g,n})$ by $\widetilde{X}_1(x) := X_1(x)/\|X_1(x)\|$ and $\widetilde{X}_2(x) := X_2(x)/\|X_2(x)\|$ for $x \in \Sigma_{g,n}$. Let $\gamma = \mathfrak{s}(S^1)$ be a closed oriented curve. Consider the diagram (2.1). The compositions $\widetilde{X}_1\mathfrak{s}$ and $\widetilde{X}_2\mathfrak{s}$ pull back to unique sections $\widetilde{X}_1^{\mathfrak{s}}$ and $\widetilde{X}_2^{\mathfrak{s}}$ from S^1 to $\mathbb{T}^2$, respectively. We have $i_*^{\mathfrak{s}}(u^{\mathfrak{s}}) = \widetilde{X}_1^{\mathfrak{s}}(\widetilde{X}_2^{\mathfrak{s}})^{-1}$ for some $u^{\mathfrak{s}} \in \pi_1(E_0) \cong \mathbb{Z}$. We can identify $u^{\mathfrak{s}}$ with an integer.

The following definition is given by Chillingworth in [3], Section 4.

Definition 2.2.5. *Let* $v_1, v_2, \ldots, v_{2g+n-1}$ *be smooth simple closed curves on* $\Sigma_{g,n}$, *based at* x, *whose homotopy classes generate* $\pi_1(\Sigma_{g,n}, x)$. *Then* $\{[v_1], [v_2], \ldots, [v_{2g+n-1}]\}$ *is a basis of* $H_1(\Sigma_{g,n}; \mathbb{Z})$. *A difference cocycle is a homomorphism*

$$d(X_1, X_2) : H_1(\Sigma_{g,n}; \mathbb{Z}) \to \mathbb{Z}$$

sending each basis element $[v_i]$ *to the corresponding number* $u^{\mathfrak{s}_i}$, *where* $v_i = \mathfrak{s}_i(S^1)$.

Let $\gamma = \mathfrak{s}(S^1)$ be any closed oriented curve. Since γ is freely homotopic to a product of v_i or their inverses, from the definition of $u^{\mathfrak{s}}$ we have

$$u^{\mathfrak{s}} = d(X_1, X_2)[\gamma].$$

From the definitions of difference cocyle and winding number of a smooth closed oriented curve γ, we get the following lemma:

Lemma 2.2.6 ([3], Lemma 4.1). $d(X_1, X_2)[\gamma] = w_{X_2}(\gamma) - w_{X_1}(\gamma)$.

It may not be clear that $d(X_1, X_2)$ is well-defined. That is, one may not see that the image of $[\gamma]$ does not change for different choice of representatives. First, consider the

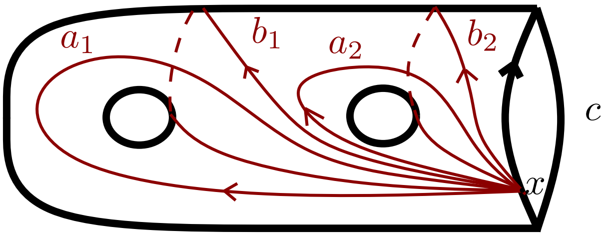


Figure 2.2: Orientable genus-2 surface with one boundary.

following example. Then we will explain well-definedness of the difference cocycle in the general case.

Example 2.2.7. *Consider a cellular decomposition of $\Sigma_{2,1}$. Take a 0-cell x on the boundary of $\Sigma_{2,1}$. Choose 1-cells a_1, b_1, a_2, b_2, c as shown in Figure 2.2. There is a single 2-cell $\Sigma_{2,1}\setminus\{a_1,b_1,a_2,b_2,c\}$ which will be denoted by τ. Let $\langle\tau\rangle$ and $\langle x\rangle$ be the free abelian groups with bases $\{\tau\}$ and $\{x\}$, respectively. Let $\langle a_1,b_1,a_2,b_2,c\rangle$ be the free abelian group with basis $\{a_1,b_1,a_2,b_2,c\}$. We have the cellular chain complex*

$$0 \to \langle\tau\rangle \overset{\partial_2}{\to} \langle a_1,b_1,a_2,b_2,c\rangle \overset{0}{\to} \langle x\rangle \to 0.$$

The image of ∂_2 is $\partial_2(\tau)=a_1+b_1-a_1-b_1+a_2+b_2-a_2-b_2+c=c$. Here ∂_2 denotes the boundary map.

For two nonvanishing vector fields X_1 and X_2 on $\Sigma_{2,1}$, we have a map $d(X_1,X_2)$ assigning an integer to each generating 1-cycle. This map is defined on the free abelian group generated by $\{a_1,b_1,a_2,b_2,c\}$ and has image in $\mathbb{Z}$. We want to show that it induces a map from $H_1(\Sigma_{2,1};\mathbb{Z})$ to $\mathbb{Z}$. Since the boundary of the two cell is c, we need to show that $d(X_1,X_2)(c)=0$.

By Lemma 2.2.4, we have $w_{X_1}(c)=w_{X_2}(c)=-3$. By Lemma 2.2.6, $d(X_1,X_2)(c)=w_{X_2}(c)-w_{X_1}(c)=0$ is obtained. Moreover, again by Lemma 2.2.4 and Lemma 2.2.6 it can easily be seen that for any separating curve γ we have $d(X_1,X_2)(\gamma)=0$. Hence we have a homomorphism

$$d(X_1,X_2):H_1(\Sigma_{2,1};\mathbb{Z})\to\mathbb{Z}.$$

The map is defined analogously for $\Sigma_{g,n}$ when $g\neq 2$ and $n\geq 1$. We need to show that $d(X_1,X_2)(c_1+c_2+\ldots+c_n)=0$, where $\{c_i\}$ are 1-cells around the boundary components. By Lemma 2.2.4, we have

$$\sum_{i=1}^{n} w_{X_1}(c_i) = \sum_{i=1}^{n} w_{X_2}(c_i).$$

By Lemma 2.2.6, we conclude that

$$\sum_{i=1}^{n} d(X_1, X_2)(c_i) = \sum_{i=1}^{n} w_{X_2}(c_i) - \sum_{i=1}^{n} w_{X_1}(c_i) = 0.$$

Since $d(X_1, X_2)$ is defined on the free abelian group generated by 1-cycles and has image in $\mathbb{Z}$, it is a homomorphism. Therefore,

$$\sum_{i=1}^{n} d(X_1, X_2)(c_i) = d(X_1, X_2)(c_1 + \ldots + c_n) = 0,$$

as desired. Thus, a cell complex structure on $\Sigma_{g,n}$ gives an analogous homomorphism $d(X_1, X_2) : H_1(\Sigma_{g,n}; \mathbb{Z}) \to \mathbb{Z}$.

The Chillingworth Class: Difference cocyles will enable us to define the Chillingworth class introduced first by Earle in [7]. See also [4]. We will give the Johnson's definition from [14]. In [14], Johnson defined the following homomorphism

$$e : \mathcal{I}(\Sigma_{g,1}) \to H^1(\Sigma_{g,1}; \mathbb{Z})$$

such that $e(f)([\gamma]) = w_X(f\gamma) - w_X(\gamma)$. To show that $e(f)$ is really a cohomology class, the difference cocycle will be used. Observe that $w_{fX}(f\gamma) = w_X(\gamma)$. Therefore, we have the equality $w_X(f\gamma) = w_{f^{-1}X}(\gamma)$. We obtain that

$$e(f)[\gamma] = w_X(f\gamma) - w_X(\gamma) = w_{f^{-1}X}(\gamma) - w_X(\gamma) = d(X, f^{-1}X)[\gamma].$$

We conclude that $e(f)$ is a difference cocycle.

Lemma 2.2.8 ([14], Lemma $5A$)**.** *The cohomology class $e(f)$ does not depend on the chosen vector field X.*

Lemma 2.2.9 ([14], Lemma $5B$)**.** *The map e is a homomorphism.*

For $f \in \mathcal{I}(\Sigma_{g,1})$, in Section 5 of [14], Johnson dualized the class $e(f)$ to a homology class $t(f)$ defined as follows: $[\gamma] \cdot t(f) = e(f)[\gamma]$. The homology class $t(f)$ is called the *Chillingworth class* of f.

Another combinatorial description of the Chillingworth class can be given by using curve complexes. In [13], Irmer proved that the Chillingworth class can be defined to be a signed stable length function. However, we will not explain this definition here.

Let $C : \bigwedge^3 H_1(\Sigma_{g,1};\mathbb{Z}) \to H_1(\Sigma_{g,1};\mathbb{Z})$ be the tensor contraction map given by

$$C(x \wedge y \wedge z) = 2[(x \cdot y)z + (y \cdot z)x + (z \cdot x)y],$$

where $\cdot$ denotes the intersection pairing of homology classes.

For example, $C(a_1 \wedge b_1 \wedge b_2) = 2b_2$ where $\{a_1, b_1, a_2, b_2, \ldots, a_g, b_g\}$ is a symplectic basis of $H_1(\Sigma_{g,1};\mathbb{Z})$.

Theorem 2.2.10 ([14], Theorem 2)**.** *For $f \in \mathcal{I}(\Sigma_{g,1})$, $t(f) = C(\tau(f))$, where τ is the Johnson homomorphism. Equivalently, the following diagram is commutative:*

$$\begin{array}{ccc} \mathcal{I}(\Sigma_{g,1}) & \xrightarrow{\quad\tau\quad} & \bigwedge^3 H_1(\Sigma_{g,1};\mathbb{Z}) \\ & {\scriptstyle t}\searrow \quad \swarrow{\scriptstyle C} & \\ & H_1(\Sigma_{g,1};\mathbb{Z}) & \end{array}$$

Corollary 2.2.11 ([14], Corollary 1)**.** *Let $f = T_\beta T_\delta^{-1}$ with (β, δ) a pair of disjoint, oriented, simple closed curves bounding a subsurface of $\Sigma_{g,1}$ of genus k. Then $t(f) = 2k[\beta]$.*

Now we want to define the first cohomology group $H^1(\mathcal{M}(\Sigma_{g,1}); H^1(\Sigma_{g,1};\mathbb{Z}))$ of $\mathcal{M}(\Sigma_{g,1})$ with coefficients in $H^1(\Sigma_{g,1};\mathbb{Z})$ [2, 21].

There is an action of $\mathcal{M}(\Sigma_{g,1})$ on $H_1(\Sigma_{g,1};\mathbb{Z})$ via the symplectic representation ρ as in Section 2.1. If we identify $H^1(\Sigma_{g,1};\mathbb{Z})$ with $Hom(H_1(\Sigma_{g,1};\mathbb{Z}), \mathbb{Z})$, the action of $\mathcal{M}(\Sigma_{g,1})$ on $H^1(\Sigma_{g,1};\mathbb{Z})$ is defined to be $\phi_1 u(x) = u((\phi_1)_*^{-1}(x)) = u(\rho(\phi_1^{-1})(x))$, where $\phi_1 \in \mathcal{M}(\Sigma_{g,1})$, $u \in H^1(\Sigma_{g,1};\mathbb{Z})$ and $x \in H_1(\Sigma_{g,1};\mathbb{Z})$.

Let $Z^1(\mathcal{M}(\Sigma_{g,1}); H^1(\Sigma_{g,1};\mathbb{Z}))$ denote the set of all crossed homomorphisms $d : \mathcal{M}(\Sigma_{g,1}) \to H^1(\Sigma_{g,1};\mathbb{Z})$. A *crossed homomorphism* d is a function $d : \mathcal{M}(\Sigma_{g,1}) \to H^1(\Sigma_{g,1};\mathbb{Z})$ such that $d(\phi_1\phi_2) = d(\phi_1) + \phi_1 d(\phi_2)$ for any $\phi_1, \phi_2 \in \mathcal{M}(\Sigma_{g,1})$.

Let m be a fixed element of $H^1(\Sigma_{g,1};\mathbb{Z})$. Define a function $d_m : \mathcal{M}(\Sigma_{g,1}) \to H^1(\Sigma_{g,1};\mathbb{Z})$ by $d_m(\phi_1) = \phi_1 m - m$. Such a function d_m is called a *principal crossed* *homomorphism.* Let $B^1(\mathcal{M}(\Sigma_{g,1}); H^1(\Sigma_{g,1};\mathbb{Z}))$ denote the set of all such principal crossed homomorphisms.

The first cohomology group of $\mathcal{M}(\Sigma_{g,1})$ with coefficients in $H^1(\Sigma_{g,1};\mathbb{Z})$ is defined as $Z^1(\mathcal{M}(\Sigma_{g,1}); H^1(\Sigma_{g,1};\mathbb{Z}))/B^1(\mathcal{M}(\Sigma_{g,1}); H^1(\Sigma_{g,1};\mathbb{Z}))$.

Similarly, we can define $H^1(\mathcal{M}(\Sigma_{g,1}); H_1(\Sigma_{g,1};\mathbb{Z}))$. Let $Z^1(\mathcal{M}(\Sigma_{g,1}); H_1(\Sigma_{g,1};\mathbb{Z}))$ be the set of all crossed homomorphisms from $\mathcal{M}(\Sigma_{g,1})$ to $H_1(\Sigma_{g,1};\mathbb{Z})$. For a fixed $m' \in H_1(\Sigma_{g,1};\mathbb{Z})$, a principal crossed homomorphism $d_{m'}$ is defined as $d_{m'}(\phi_1) = (\phi_1)_* m' - m'$. Let $B^1(\mathcal{M}(\Sigma_{g,1}); H_1(\Sigma_{g,1};\mathbb{Z}))$ be the set of all such principal crossed homomorphisms. Then

$$H^1(\mathcal{M}(\Sigma_{g,1}); H_1(\Sigma_{g,1};\mathbb{Z})) := Z^1(\mathcal{M}(\Sigma_{g,1}); H_1(\Sigma_{g,1};\mathbb{Z}))/B^1(\mathcal{M}(\Sigma_{g,1}); H_1(\Sigma_{g,1};\mathbb{Z})).$$

We can obtain crossed homomorphisms via difference cocycles. Two crossed homomorphisms defined by Trapp and Furuta will be defined and then it will be shown that these definitions are equivalent by realizing that they are difference cocycles. Of course, this is not a new result. Morita and Trapp were aware of this.

In [29], Trapp defined a crossed homomorphism

$$e_X : \mathcal{M}(\Sigma_{g,1}) \to H^1(\Sigma_{g,1};\mathbb{Z})$$

as follows:

$$e_X(f)[\gamma] = w_X(f\gamma) - w_X(\gamma)$$

for any $f \in \mathcal{M}(\Sigma_{g,1})$, $[\gamma] \in H_1(\Sigma_{g,1};\mathbb{Z})$. Here X is a nonvanishing vector field on $\Sigma_{g,1}$. Trapp showed that e_X is a crossed homomorphism such that

$$e_X(fh) = e_X(f)\rho(h) + e_X(h)$$

for any $f, h \in \mathcal{M}(\Sigma_{g,1})$, where $\rho : \mathcal{M}(\Sigma_{g,1}) \to \mathrm{Sp}(2g;\mathbb{Z})$ is the symplectic representation.

We now outline the construction of a crossed homomorphism from [23], Section 4.

Recall that there is a Riemannian metric on $\Sigma_{g,1}$. Let $f \in \mathcal{M}(\Sigma_{g,1})$ and X be a nonvanishing vector field on $\Sigma_{g,1}$. Consider the vector field fX. Note that fX is nonvanishing since X is nonvanishing. Let S^1 denote the set of angles mod 2π.

Furuta defined a mapping

$$\Theta_f : \Sigma_{g,1} \to S^1$$

as follows: Let the map $\Theta_f(p) := \angle(X_p, (fX)_p)$ measure the angle mod 2π from X_p to $(fX)_p$ with respect to the fixed orientation on $\Sigma_{g,1}$. Consider the cohomology class $[\Theta_f] := \Theta_f^*([S^1]) \in H^1(\Sigma_{g,1};\mathbb{Z})$, where $[S^1]$ is the generator of $H^1(S^1;\mathbb{Z})$. Taking the Poincaré dual of $[\Theta_f]$, we get $k_X(f) \in H_1(\Sigma_{g,1};\mathbb{Z})$. Namely, Furuta defined a mapping $k_X : \mathcal{M}(\Sigma_{g,1}) \to H_1(\Sigma_{g,1};\mathbb{Z})$ which is well-defined depending only on the choice of the nonvanishing vector field X.

Proposition 2.2.12 ([23], Proposition 4.1)**.** *The map $k_X : \mathcal{M}(\Sigma_{g,1}) \to H_1(\Sigma_{g,1};\mathbb{Z})$ is a crossed homomorphism. Its cohomology class $[k_X]$ does not depend on the nonvanishing vector field X and is a generator of $H^1(\mathcal{M}(\Sigma_{g,1}); H_1(\Sigma_{g,1};\mathbb{Z}))$.*

Now our aim is to relate these two crossed homomorphisms introduced by Furuta and Trapp.

Proposition 2.2.13. *The crossed homomorphism defined by Trapp is related to Furuta's definition by $[\Theta_{f^{-1}}][\gamma] = e_X(f)[\gamma]$ for any $f \in \mathcal{M}(\Sigma_{g,1})$ and any $[\gamma] \in H_1(\Sigma_{g,1};\mathbb{Z})$.*

Proof. Let X be a nonvanishing vector field on $\Sigma_{g,1}$. By the construction of the difference cocycle, if $\widetilde{X}$ and $\widetilde{f^{-1}X}$ rotate m and n-times, respectively, around the fiber on the unit tangent bundle $UT(\Sigma_{g,1})$ restricted to γ, $\widetilde{X}^{\mathfrak{s}}$ and $\widetilde{f^{-1}X}^{\mathfrak{s}}$ are homotopic to maps sending θ to $(\theta, m\theta)$ and θ to $(\theta, n\theta)$, respectively. Here S^1 denotes the set of angles mod 2π. By composing with the projection map on the second component $pr_2 : \mathbb{T}^2 \to S^1$, we get $pr_2 \circ \widetilde{X}^{\mathfrak{s}} : S^1 \to S^1$ sending θ to $m\theta$ and $pr_2 \circ \widetilde{f^{-1}X}^{\mathfrak{s}} : S^1 \to S^1$ sending θ to $n\theta$. We can consider the image of the difference cocyle $d(X, f^{-1}X)[\gamma]$ as the degree of the map $S^1 \to S^1$ sending θ to $(m-n)\theta$.

Now consider the map $\Theta_{f^{-1}} : \Sigma_{g,1} \to S^1$ such that $\Theta_{f^{-1}}(p) = \angle(X)_p, (f^{-1}X)_p)$. Observe that $d(X, f^{-1}X)[\gamma] = deg(\Theta_{f^{-1}}|_\gamma)$. Indeed, the induced map $(\Theta_{f^{-1}})_* : H_1(\Sigma_{g,1};\mathbb{Z}) \to H_1(S^1;\mathbb{Z})$ will give us a cohomology class, which is $d(X, f^{-1}X)$, after identifying $H_1(S^1;\mathbb{Z})$ with $\mathbb{Z}$.

Consequently, we have

$$[\Theta_{f^{-1}}][\gamma] = (\Theta_{f^{-1}})_*([\gamma]) = deg(\Theta_{f^{-1}}|_\gamma) = d(X, f^{-1}X)[\gamma] = e_X(f)[\gamma]$$

for any smooth representative γ. □

Dualize $e_X(f)$ to a homology class $t_X(f)$, i.e. $t_X(f)$ is defined by $[\gamma] \cdot t_X(f) = e_X(f)[\gamma]$. Then $t_X : \mathcal{M}(\Sigma_{g,1}) \to H_1(\Sigma_{g,1}; \mathbb{Z})$ depends on the choice of the non-vanishing vector field X. By using the same method used in the proof of Proposition 2.2.12 we can show that t_X is a crossed homomorphism and the cohomology class $[t_X] \in H^1(\mathcal{M}(\Sigma_{g,1}); H_1(\Sigma_{g,1}; \mathbb{Z}))$ is independent of the vector field X.

The Chillingworth Class for Closed Surfaces: When a surface of genus at least 2 is closed, any vector field on the surface has singularities. This makes it necessary to modify the definition of the Chillingworth class for closed surfaces.

First, assume that X is a vector field on $\mathbb{R}^2$ and it has an isolated singularity at the origin. Consider a circle around 0 so that there is no other singularity inside this circle. Define the index of X at 0, denoted by $\text{ind}_0(X)$, to be the degree of the map

$$u : S^1 \to S^1 \qquad u(z) = \frac{X(z)}{\|X(z)\|}.$$

Namely, the index of a singularity of a vector field is the number of revolutions counted with sign made by the vector field when S^1 is traversed once in a counterclockwise direction. If the vector field makes one counterclockwise turn along S^1, then the index of the singularity is $+1$. If the vector field makes a clockwise revolution, the singularity of the vector field has index -1.

Now, let X be a vector field on the closed oriented surface Σ_g and let X have an isolated singularity at v. For every point $y \in \Sigma_g$, there is a neighbourhood U of y in Σ_g diffeomorphic to an open disc $D \subset \mathbb{R}^2$. Let $\phi : D \to \Sigma_g$ be a local parametrization sending the origin of $\mathbb{R}^2$ to v. Notice that for each $x \in D$, there is an isomorphism $d\phi_x$ from $\mathbb{R}^2$ to the tangent space of Σ_g at $\phi(x)$. If we pull back X on Σ_g by ϕ, we get a vector field on D. Note that $d\phi_x$ is an isomorphism of $\mathbb{R}^2$ with $T_{\phi(x)}\Sigma_g$ for each $x \in D$. The pullback of X is given by

$$\phi^* X(x) = d\phi_x^{-1} X(\phi(x)).$$

Notice that if X has a singularity at v, then $\phi^* X$ has a singularity at 0. Define the index of X at v as $\text{ind}_v(X) = \text{ind}_0(\phi^* X)$. This definition is independent of the choice local parametrization. For more information, see [10].

Theorem 2.2.14 (Poincaré)**.** *Let X be a smooth vector field on a smooth closed oriented surface Σ with only finitely many singularities $p_1, p_2, \ldots, p_n$. Then the sum of the indices of X equals the Euler characteristic of the surface:*

$$\sum_{i=1}^{n} \text{ind}_{p_i}(X) = \chi(\Sigma).$$

Poincaré proved this theorem in 1885. Then Hopf generalized this theorem for higher dimensional manifolds in 1927. In the general case, it is called the Poincaré- Hopf Theorem.

By using this theorem we can conclude that if the genus of a closed surface Σ_g is greater than 1, it will not admit any continuous nonvanishing vector field.

Chillingworth defined the winding number of homotopy classes of curves on surfaces. To make this work, it is necessary to assume that all the singularities are concentrated at one point v with index $2 - 2g$.

Theorem 2.2.15 ([3], Theorem 6.1)**.** *Let X be a vector field with only one singularity v. Let γ_1 and γ_2 be two smooth, homotopic closed curves not passing through v which are not null homotopic. Assume that both of them are in minimal position. Then they satisfy*

$$w_X(\gamma_1) = w_X(\gamma_2) \mod (2g-2).$$

We are now ready to give the definition of the Chillingworth class on closed surfaces Σ_g (c.f. [14], Section 6). By following the same method in the definition of the Chillingworth class of $f' \in \mathcal{I}(\Sigma_{g,1})$, we obtain a well-defined class $t(f)$ in $2H_1(\Sigma_g; \mathbb{Z})$ mod $(2g-2)$ for $f \in \mathcal{I}(\Sigma_g)$.

The following theorem shows that $t(f) = C(\tau(f))$ for any $f \in \mathcal{I}(\Sigma_g)$.

Theorem 2.2.16 ([14], Theorem 3)**.** *For $g \geq 2$, we have a commutative diagram*

$$
\begin{array}{ccccccccc}
 & & & & \mathcal{I}(\Sigma_g) & & & & \\
 & & & \overset{\tau}{\swarrow} & & \overset{\frac{1}{2}t}{\searrow} & & & \\
0 \longrightarrow K & \longrightarrow & \bigwedge^3 H_1(\Sigma_g;\mathbb{Z})/H_1(\Sigma_g;\mathbb{Z}) & & \xrightarrow[\frac{1}{2}C]{} & & H_1(\Sigma_g;\mathbb{Z}) \,\mathrm{mod}\, (g-1) & \longrightarrow & 0
\end{array}
$$

where $C : \bigwedge^3 H_1(\Sigma_g;\mathbb{Z}) \to H_1(\Sigma_g;\mathbb{Z})$ *is the tensor contraction map and* $K =$ Ker C. *The row is exact and the Johnson homomorphism* τ *is surjective.*

CHAPTER 3

SUBSURFACE TORELLI GROUPS

In this chapter, we introduce the subsurface Torelli groups and the partitioned Johnson homomorphism. This chapter is required background for the next chapter.

3.1 Subsurface Torelli Groups

We start this section by giving basic definitions from the category theory. We then define the subsurface Torelli groups and give properties of them. This section is based on the work of Putman in [26].

Definition 3.1.1. *A category $\mathcal{C}$ consists of the following data:*

- *A collection* $\mathrm{Obj}(\mathcal{C})$ *of objects* $A, B, C, \dots$
- *For every pair of objects* $A, B \in \mathrm{Obj}(\mathcal{C})$*, a collection* $\mathrm{Morp}_{\mathcal{C}}(\mathrm{A}, \mathrm{B})$ *of morphisms* $f, h, k, \dots$

  Each morphism specifies domain and codomain objects. The notation $f : A \to B$ *denotes a morphism with domain* A *and codomain* B*.*
- *Each* $A \in \mathrm{Obj}(\mathcal{C})$ *has an element* id_A *of* $\mathrm{Morp}_{\mathcal{C}}(\mathrm{A}, \mathrm{A})$*, called the identity morphism on* A*.*
- *Given morphisms* $f : A \to B$ *and* $h : B \to C$*, there is a composition* $h \circ f : A \to C$*.*

These data must satisfy the following properties:

- *If* $f \in \mathrm{Morp}_{\mathcal{C}}(\mathrm{A},\mathrm{B})$*,* $h \in \mathrm{Morp}_{\mathcal{C}}(\mathrm{B},\mathrm{C})$ *and* $k \in \mathrm{Morp}_{\mathcal{C}}(\mathrm{C},\mathrm{D})$*, then* $k \circ (h \circ f) = (k \circ h) \circ f$*.*

- *If* $f \in \mathrm{Morp}_{\mathcal{C}}(\mathrm{A},\mathrm{B})$*, then* $id_B \circ f = f = f \circ id_A$*.*

Definition 3.1.2. *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be two categories. A functor* F *from* $\mathcal{C}$ *to* $\mathcal{D}$*, denoted* $F : \mathcal{C} \to \mathcal{D}$*, is a morphism sending each object* $A \in \mathcal{C}$ *to an object* $F(A)$ *of* $\mathcal{D}$ *and each morphism* $f \in \mathrm{Morp}_{\mathcal{C}}(\mathrm{A},\mathrm{B})$ *to a morphism in* $F(f) \in \mathrm{Morp}_{\mathcal{D}}(\mathrm{F(A)},\mathrm{F(B)})$ *such that*

- $F(h \circ f) = F(h) \circ F(f)$ *for all* $f \in \mathrm{Morp}_{\mathcal{C}}(\mathrm{A},\mathrm{B})$ *and* $h \in \mathrm{Morp}_{\mathcal{C}}(\mathrm{B},\mathrm{C})$*,*

- $F(id_A) = id_{F(A)}$ *for all* $A \in \mathrm{Obj}(\mathcal{C})$*.*

Definition 3.1.3. *Let* F *and* G *be two functors from* $\mathcal{C}$ *to* $\mathcal{D}$*. A natural transformation is an assignment to each* $A \in \mathcal{C}$ *a morphism* $\eta_A : F(A) \to G(A)$ *in* $\mathcal{D}$ *such that for every morphism* $f : A \to B$ *in* $\mathcal{C}$*, the following is commutative in* $\mathcal{D}$*.*

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(B) \\ {\scriptstyle \eta_A}\downarrow & & \downarrow{\scriptstyle \eta_B} \\ G(A) & \xrightarrow[G(f)]{} & G(B) \end{array} \tag{3.1}$$

Suppose that a compact connected oriented surface Σ with boundary is embedded in a compact connected oriented surface Σ'. Let $i : \Sigma \hookrightarrow \Sigma'$ be an embedding and identify Σ with $i(\Sigma)$. The inclusion map i induces a map $i_* : \mathcal{M}(\Sigma) \to \mathcal{M}(\Sigma')$ between mapping class groups, defined by extending diffeomorphism $\Sigma \to \Sigma$ to $\Sigma' \to \Sigma'$ by the identity on $\Sigma' \setminus \Sigma$. In other words, let $\mathcal{C}$ be the category such that objects are surfaces and morphisms are embeddings. Let $\mathcal{D}$ be a category whose objects are groups and whose morphisms are homomorphisms. The mapping class group, $\mathcal{M}$, is a functor from $\mathcal{C}$ to $\mathcal{D}$.

Putman defined in [26] the subsurface Torelli groups such that the Torelli group is a functor. In order to get functoriality he introduced the notion of partitioned surfaces. The following example shows the necessity of the partitioning of boundary components of the subsurface.

Example 3.1.4. *Assume that we have two embeddings of $\Sigma_{1,5}$ into closed oriented surfaces of genus 4, Σ_4 and Σ'_4 as in Figure 3.1. Let $i : \Sigma_{1,5} \hookrightarrow \Sigma_4$ be the embedding on the left-hand side and $j : \Sigma_{1,5} \hookrightarrow \Sigma'_4$ be the embedding on the right-hand side. Assume that we have a subgroup $\mathcal{I}(\Sigma_{1,5})$ satisfying $i_*(\mathcal{I}(\Sigma_{1,5})) \subset \mathcal{I}(\Sigma_4)$, $i_*^{-1}(\mathcal{I}(\Sigma_4)) = \mathcal{I}(\Sigma_{1,5})$ and $j_*(\mathcal{I}(\Sigma_{1,5})) \subset \mathcal{I}(\Sigma'_4)$, $j_*^{-1}(\mathcal{I}(\Sigma'_4)) = \mathcal{I}(\Sigma_{1,5})$.*

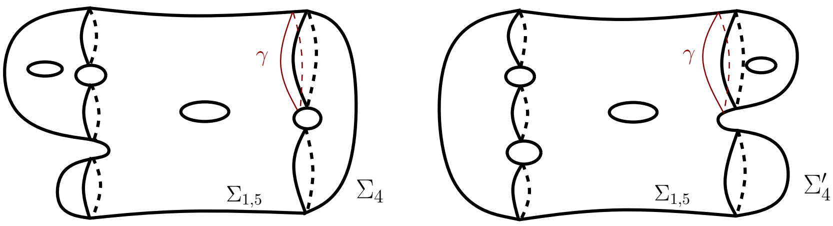


Figure 3.1: Two different embeddings of $\Sigma_{1,5}$ into Σ_4 and Σ'_4.

On the right-hand side in Figure 3.1, T_γ is an element of $\mathcal{I}(\Sigma'_4)$. From the equality $j_*^{-1}(\mathcal{I}(\Sigma'_4)) = \mathcal{I}(\Sigma_{1,5})$, we get that T_γ is an element of $\mathcal{I}(\Sigma_{1,5})$. By $i_*(\mathcal{I}(\Sigma_{1,5})) \subset \mathcal{I}(\Sigma_4)$, we obtain that T_γ is an element of $\mathcal{I}(\Sigma_4)$. But, since γ is not a separating curve on Σ_4, T_γ is not an element of $\mathcal{I}(\Sigma_4)$. We get a contradiction. Thus, there is not such a group. We conclude that $i_*^{-1}(\mathcal{I}(\Sigma_4))$ and $j_*^{-1}(\mathcal{I}(\Sigma'_4))$ are different groups.

Therefore, it is necessary to define the notion of a partitioned surface. A partitioned surface is the pair $(\Sigma, \mathcal{P})$ consisting of a surface Σ and a partition $\mathcal{P}$ of the boundary components of Σ. Each element P_k of $\mathcal{P}$ is called a block.

For a given embedding $i : \Sigma \hookrightarrow \Sigma_g$, let the connected components of $\Sigma_g \setminus \Sigma^\circ$ be $\{S_0, S_1, \ldots, S_m\}$ and let P_k denote the set of boundary components of S_k for each $k \in \{0, \ldots, m\}$. Here, Σ° denotes the interior of Σ. Consider the partition

$$\mathcal{P} = \{P_0, P_1, \ldots, P_m\}.$$

Then $i : \Sigma \hookrightarrow \Sigma_g$ is called a capping of $(\Sigma, \mathcal{P})$ (c.f. [26]).

Definition 3.1.5 ([26])**.** *For a partitioned surface $(\Sigma, \mathcal{P})$, the subsurface Torelli group $\mathcal{I}(\Sigma, \mathcal{P})$ is defined as the subgroup $i_*^{-1}(\mathcal{I}(\Sigma_g))$ of $\mathcal{M}(\Sigma)$ for any capping $i : \Sigma \hookrightarrow \Sigma_g$.*

In [26], Section 3, a special homology group $H_1^{\mathcal{P}}(\Sigma; \mathbb{Z})$ is defined on a partitioned surface $(\Sigma, \mathcal{P})$ such that $\mathcal{I}(\Sigma, \mathcal{P})$ is the kernel of $\mathcal{M}(\Sigma) \to \mathrm{Aut}(\mathrm{H}_1^{\mathcal{P}}(\Sigma; \mathbb{Z}))$.

Consider a partition

$$\mathcal{P} = \{\{\partial_1^1, \ldots, \partial_{k_1}^1\}, \ldots, \{\partial_1^m, \ldots, \partial_{k_m}^m\}\}.$$

Suppose the boundary components ∂_i^j are oriented so that $\sum_{i,j}[\partial_i^j] = 0$ in $H_1(\Sigma;\mathbb{Z})$. Define the homology group

$$\overline{H}_1^{\mathcal{P}}(\Sigma;\mathbb{Z}) := H_1(\Sigma;\mathbb{Z})/\partial H_1^{\mathcal{P}}(\Sigma;\mathbb{Z}),$$

where

$$\partial H_1^{\mathcal{P}}(\Sigma;\mathbb{Z}) = \big\langle([\partial_1^1] + \ldots + [\partial_{k_1}^1]), \ldots, ([\partial_1^m] + \ldots + [\partial_{k_m}^m])\big\rangle \subset H_1(\Sigma;\mathbb{Z}).$$

Definition 3.1.6 ([26], Section 3.1)**.** *Let $(\Sigma, \mathcal{P})$ be a partitioned surface, and let $\mathcal{Q}$ denote a set containing one point from each boundary component of Σ. The homology group $H_1^{\mathcal{P}}(\Sigma;\mathbb{Z})$ is defined to be the image of the following subgroup of $H_1(\Sigma, \mathcal{Q};\mathbb{Z})$ in $H_1(\Sigma, \mathcal{Q};\mathbb{Z})/\partial H_1^{\mathcal{P}}(\Sigma;\mathbb{Z})$:*

$$\langle\{[h] \in H_1(\Sigma, \mathcal{Q};\mathbb{Z}) | \text{ either } h \text{ is a simple closed curve or } h \text{ is an arc } a \text{ connecting two boundary curves in the same block of } \mathcal{P} \text{ and with } \partial a \in \mathcal{Q}\}\rangle$$

One can easily see that $\mathcal{M}(\Sigma)$ acts on $H_1^{\mathcal{P}}(\Sigma;\mathbb{Z})$.

Theorem 3.1.7 ([26], Theorem 3.3)**.** *The subsurface Torelli group $\mathcal{I}(\Sigma, \mathcal{P})$ is the subgroup of $\mathcal{M}(\Sigma)$ that acts trivially on $H_1^{\mathcal{P}}(\Sigma;\mathbb{Z})$.*

A $\mathcal{P}$-separating curve on a partitioned surface $(\Sigma, \mathcal{P})$ is a simple closed curve γ with $[\gamma] = 0$ in $H_1^{\mathcal{P}}(\Sigma;\mathbb{Z})$. A $\mathcal{P}$-separating curve is a separating curve on Σ_g for any capping $\Sigma \hookrightarrow \Sigma_g$. A twist about $\mathcal{P}$-bounding pair is defined to be $T_{\gamma_1}T_{\gamma_2}^{-1}$, where γ_1 and γ_2 are disjoint, nonisotopic simple closed curves and $[\gamma_1] = [\gamma_2]$ in $H_1^{\mathcal{P}}(\Sigma;\mathbb{Z})$. Similarly, a $\mathcal{P}$-bounding pair is also a bounding pair on Σ_g for any capping $\Sigma \hookrightarrow \Sigma_g$.

Theorem 3.1.8 ([26], Theorem 1.3)**.** *For $g \geq 1$, $\mathcal{I}(\Sigma_{g,n}, \mathcal{P})$ is generated by twists about $\mathcal{P}$-separating curves and twists about $\mathcal{P}$-bounding pairs.*

A category TSur was defined in [26]. The objects of TSur are partitioned surfaces $(\Sigma, \mathcal{P})$ and the morphisms from $(\Sigma_{g_1,n_1}, \mathcal{P}_1)$ to $(\Sigma_{g_2,n_2}, \mathcal{P}_2)$ are exactly those embeddings $i : \Sigma_{g_1,n_1} \hookrightarrow \Sigma_{g_2,n_2}$ which induce morphisms $i_* : \mathcal{I}(\Sigma_{g_1,n_1}, \mathcal{P}_1) \to \mathcal{I}(\Sigma_{g_2,n_2}, \mathcal{P}_2)$. The embeddings satisfy the following condition: for any $\mathcal{P}_1$-separating curve γ, the curve $i(\gamma)$ must be a $\mathcal{P}_2$-separating curve. In this thesis, we will use the refinement of this category defined by Church in [5]. See Definition 3.2.2 in the next section.

Theorem 3.1.9 ([26], Theorem 3.6)**.** *Let* $(\Sigma_{g_1,n_1}, \mathcal{P}_1)$ *and* $(\Sigma_{g_2,n_2}, \mathcal{P}_2)$ *denote partitioned surfaces. If there is an embedding* $i : \Sigma_{g_1,n_1} \hookrightarrow \Sigma_{g_2,n_2}$*, then*

$$i_*(\mathcal{I}(\Sigma_{g_1,n_1}, \mathcal{P}_1)) \subset \mathcal{I}(\Sigma_{g_2,n_2}, \mathcal{P}_2)$$

if and only if i *is a morphism in the category* TSur.

The next theorem shows that partitioning the boundary components is a minimum requirement for obtaining functoriality.

Theorem 3.1.10 ([26], Theorem 3.8)**.** *Let* $i : \Sigma_{g_1,n_1} \hookrightarrow \Sigma_{g_2,n_2}$ *be an embedding. If* $\mathcal{P}_2$ *is a partition of the boundary components of* Σ_{g_2,n_2}*, then there is some partition* $\mathcal{P}_1$ *of the boundary components of* Σ_{g_1,n_1} *so that* $\mathcal{I}(\Sigma_{g_1,n_1}, \mathcal{P}_1) = i_*^{-1}(\mathcal{I}(\Sigma_{g_2,n_2}, \mathcal{P}_2))$.

3.2 The Partitioned Johnson Homomorphism

In this section, we define the partitioned Johnson homomorphism given by Church in [5].

A *totally separated surface* is a surface with a partition of the boundary components such that each element of the partition contains only one boundary component.

Remark 3.2.1. *Given a partitioned surface, a minimal totally separated surface containing* Σ *can be constructed as follows: Let* Σ *be given with a partition* $\mathcal{P}$*. For each* $P \in \mathcal{P}$ *with* $|P| = n$*, we attach a sphere with* $n+1$ *boundary components to the* n *boundary components in* P *of* Σ *to obtain* $\widehat{\Sigma}$ *with a partition* $\widehat{\mathcal{P}}$*. Each element of the partition* $\widehat{\mathcal{P}}$ *contains only one boundary component. If it is needed to fix basepoints* $*$ *on* $\partial\Sigma$ *and* $\widehat{*}$ *on* $\partial\widehat{\Sigma}$*, the assumption is also made so that basepoints* $\widehat{*}$ *and* $*$ *are on the boundary of the same* $n+1$ *holed sphere attached to* Σ*. The resulting surface is an example of what will be called a totally separated surface. See Figure 3.2 as an example.*

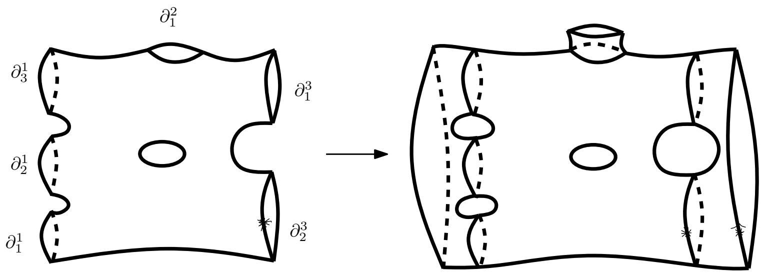


Figure 3.2: A partitioned surface $(\Sigma_{1,6}, \{\{\partial_1^1, \partial_2^1, \partial_3^1\}, \{\partial_1^2\}, \{\partial_1^3, \partial_2^3\}\})$ and a totally separated surface containing it.

Notation: Given a partitioned surface $(\Sigma, \mathcal{P})$, the partitioned surface $(\widehat{\Sigma}, \widehat{\mathcal{P}})$ will denote a minimal totally separated surface containing Σ.

Note that $H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma}; \mathbb{Z})$ is isomorphic to $H_1^P(\Sigma; \mathbb{Z})$ and it can be considered as the first homology group of the closed surface $\overline{\Sigma}$ obtained by attaching a disc to each boundary component of $\widehat{\Sigma}$.

Definition 3.2.2 ([5], Section 2.3)**.** *The Torelli category $\mathcal{T}\mathrm{Surf}$ is the category defined as in the previous section with some additional conditions. Objects of this category are partitioned surfaces $(\Sigma, \mathcal{P}, *)$. A morphism from $(\Sigma_1, \mathcal{P}_1, *_1)$ to $(\Sigma_2, \mathcal{P}_2, *_2)$ is an embedding $i : \Sigma_1 \hookrightarrow \Sigma_2$ satisfying the following conditions:*

- *i takes $\mathcal{P}_1$-separating and $\mathcal{P}_1$-nonseparating curves to $\mathcal{P}_2$-separating and $\mathcal{P}_2$-nonseparating curves, respectively.*
- *$*_1$ and $*_2$ can be connected by an arc in $\Sigma_2 \setminus \Sigma_1^\circ$.*
- *i extends to an embedding $\widehat{i} : \widehat{\Sigma}_1 \hookrightarrow \widehat{\Sigma}_2$.*

The embedding illustrated in Figure 3.3 is not a morphism in $\mathcal{T}\mathrm{Surf}$. Because there is no embedding from $\widehat{\Sigma}_{2,5}$ to $\widehat{\Sigma}_{6,6}$.

Note that if we have a morphism $i : (\Sigma_1, \mathcal{P}_1, *_1) \to (\Sigma_2, \mathcal{P}_2, *_2)$, then there is an embedding $\widehat{\Sigma}_1 \hookrightarrow \Sigma_2$.

If we are not dealing with the fundamental group of the partitioned surface $(\Sigma, \mathcal{P}, *)$, we will not mention $*$ and denote the partitioned surface simply by $(\Sigma, \mathcal{P})$.

By means of the third condition for a morphism $i : (\Sigma_1, \mathcal{P}_1) \to (\Sigma_2, \mathcal{P}_2)$, one

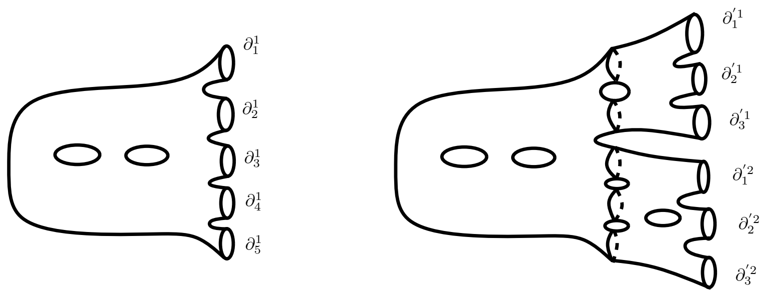


Figure 3.3: An embedding $(\Sigma_{2,5}, \{P_1\})$ into $(\Sigma_{6,6}, \{P_1', P_2'\})$, where $P_1 = \{\partial_1^1, \partial_2^1, \partial_3^1, \partial_4^1, \partial_5^1\}$, $P_1' = \{\partial_1'^1, \partial_2'^1, \partial_3'^1\}$ and $P_2' = \{\partial_1'^2, \partial_2'^2, \partial_3'^2\}$.

can define the induced map $H_1(\widehat{\Sigma}_1;\mathbb{Z}) \to H_1(\widehat{\Sigma}_2;\mathbb{Z})$ canonically. We get a map $H_1^{\widehat{\mathcal{P}}_1}(\widehat{\Sigma}_1;\mathbb{Z}) \to H_1^{\widehat{\mathcal{P}}_2}(\widehat{\Sigma}_2;\mathbb{Z})$ by using the first condition in the definition.

In order to define the Johnson homomorphism on a partitioned surface, Church constructed the lower central series of the fundamental group on a partitioned surface $(\Sigma, \mathcal{P}, *)$. Take $\pi := \pi_1(\Sigma, *)$ and define $T(\Sigma, \mathcal{P})$ to be the kernel of the composition $\pi_1(\Sigma, *) \to H_1(\Sigma;\mathbb{Z}) \to H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma};\mathbb{Z})$.

Lemma 3.2.3 ([5], Lemma 3.1)**.** *Let $(\Sigma, \mathcal{P}, *)$ be a partitioned surface and let $\mathcal{P} = \{P_0, P_1, \dots, P_k\}$. Consider a $\mathcal{P}$- separating curve γ_i which separates all boundary components in the block P_i from all boundary components in the block P_j for $i \neq j$ when the surface is cut along it. Let $\zeta_i \in \pi$ be a representative of γ_i. Then $T(\Sigma, \mathcal{P})$ is generated by $[\pi, \pi]$ together with $\zeta_1, \dots, \zeta_k$ (c.f. Figure 3.4).*

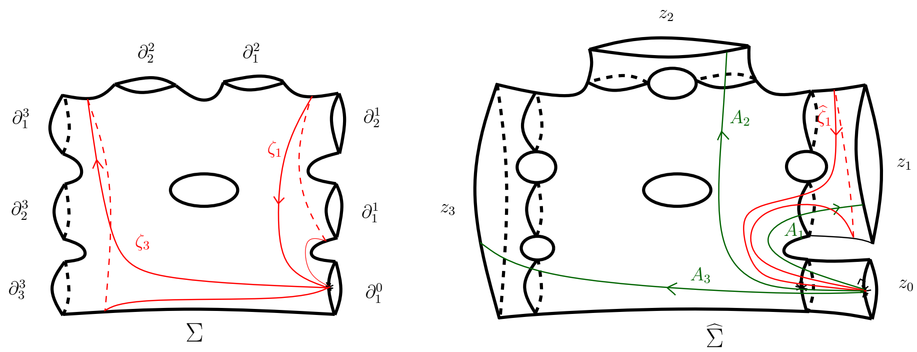


Figure 3.4: A partitioned surface $(\Sigma_{1,8}, \{\{\partial_1^0\}, \{\partial_1^1, \partial_2^1\}, \{\partial_1^2, \partial_2^2\}, \{\partial_1^3, \partial_2^3, \partial_3^3\}\})$ and a totally separated surface containing it.

Consider the central series defined by

$$\pi_1^T = \pi, \qquad \pi_2^T = T(\Sigma, \mathcal{P}), \qquad \pi_j^T = \langle [\pi_1^T, \pi_{j-1}^T], [\pi_2^T, \pi_{j-2}^T] \rangle \text{ for } j \geq 3.$$

Let $\widehat{\pi}$ denote $\pi_1(\widehat{\Sigma}, \widehat{*})$. Since $T(\widehat{\Sigma}, \mathcal{P})$ is the kernel of the composition $\widehat{\pi} \to H_1(\widehat{\Sigma}; \mathbb{Z}) \to H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma}; \mathbb{Z})$ and the composition is surjective, we obtain that $\widehat{\pi}/T(\widehat{\Sigma}, \mathcal{P}) \cong H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma}; \mathbb{Z})$.

To define the partitioned Johnson homomorphism, start with the short exact sequence

$$1 \to T(\widehat{\Sigma}, \mathcal{P})/[T(\widehat{\Sigma}, \mathcal{P}), \widehat{\pi}] \to \widehat{\pi}/[T(\widehat{\Sigma}, \mathcal{P}), \widehat{\pi}] \to \widehat{\pi}/T(\widehat{\Sigma}, \mathcal{P}) \to 1,$$

or renaming the terms in this short exact sequence, we get

$$1 \to N(\Sigma, \mathcal{P}) \to E(\Sigma, \mathcal{P}) \to H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma}; \mathbb{Z}) \to 1.$$

Considering this exact sequence, the partitioned Johnson homomorphism is defined similarly to the original definition of the Johnson homomorphism after guaranteeing that the Torelli group $\mathcal{I}(\Sigma, \mathcal{P})$ acts trivially on $N(\Sigma, \mathcal{P})$ and $H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma}; \mathbb{Z})$.

The Torelli group $\mathcal{I}(\Sigma, \mathcal{P})$ acts trivially on $N(\Sigma, \mathcal{P})$ and $H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma}; \mathbb{Z})$ (c.f. [5]).

Definition 3.2.4 ([5], Chapter 5)**.** *The action of $\mathcal{I}(\Sigma, \mathcal{P})$ on $E(\Sigma, \mathcal{P}) = \widehat{\pi}/[T(\widehat{\Sigma}, \mathcal{P}), \widehat{\pi}]$ gives a homomorphism*

$$\tau_{(\Sigma,\mathcal{P})} : \mathcal{I}(\Sigma, \mathcal{P}) \to \mathrm{Hom}(H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma}; \mathbb{Z}), N(\Sigma, \mathcal{P}))$$

which will be called the partitioned Johnson homomorphism. The partitioned Johnson homomorphism is defined as follows:

$$\tau_{(\Sigma,\mathcal{P})}(\varphi)(x) = \varphi(\tilde{x})\tilde{x}^{-1},$$

where $\tilde{x} \in \widehat{\pi}$ is a lift of $x \in H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma}; \mathbb{Z})$.

Note that well-definedness of $\tau_{(\Sigma,\mathcal{P})}$ comes from the definition of the Johnson homomorphism. More explicitly, if $\tilde{x}'$ is another lifting of x, we have $\tilde{x}' = \tilde{x}y'$ for some $y' \in N(\Sigma, \mathcal{P})$. Let $\varphi \in \mathcal{I}(\Sigma, \mathcal{P})$. Then $\varphi(x) = x$ and $\varphi(\tilde{x}) = y\tilde{x}$ for some $y \in N(\Sigma, \mathcal{P})$. Since $\mathcal{I}(\Sigma, \mathcal{P})$ acts trivially on $N(\Sigma, \mathcal{P})$, we obtain that $\varphi(\tilde{x}') = \varphi(\tilde{x}y') = \varphi(\tilde{x})\varphi(y') = \varphi(\tilde{x})y'$. Therefore, we get $\varphi(\tilde{x}')\tilde{x}'^{-1} = \varphi(\tilde{x})y'(\tilde{x}y')^{-1} = \varphi(\tilde{x})\tilde{x}^{-1} = y$. As one can notice, we use the same notation φ for the induced maps on $\widehat{\pi}$, $E(\Sigma, \mathcal{P})$ and $H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma}; \mathbb{Z})$. By using the fact that $N(\Sigma, \mathcal{P})$ is central, it can be shown that $\tau_{(\Sigma,\mathcal{P})}(\varphi)$ is a homomorphism. By using the fact that $\mathcal{I}(\Sigma, \mathcal{P})$ acts trivially on $N(\Sigma, \mathcal{P})$ and $H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma}; \mathbb{Z})$, we get that $\tau_{(\Sigma,\mathcal{P})}$ is a homomorphism.

Given a surface $(\Sigma, \mathcal{P}, *)$ with $\mathcal{P} = \{P_0, P_1, \dots, P_k\}$, let z_i be the boundary component of $\widehat{\Sigma}$ corresponding to $P_i \in \mathcal{P}$ for each $1 \leq i \leq k$. The image of the generator $\widehat{\zeta_i}$ of $T(\widehat{\Sigma}, \mathcal{P})$ in $N(\Sigma, \mathcal{P})$ is the generator z_i.

Let A_i be an arc from the basepoint $\widehat{*}$ to z_i, (see Figure 3.4 as an example). For each P_i, we obtain a map $N(\Sigma, \mathcal{P}) \to \mathbb{Z}$ by intersecting elements with arcs A_i. In particular, if $\tilde{x} \in \widehat{\pi}$ represents the class $x \in N(\Sigma, \mathcal{P})$, let $x \cdot A_i$ denote the algebraic intersection number of $\tilde{x}$ with A_i. This number is independent of the choice of A_i [5].

Definition 3.2.5 ([5], Definition 5.5)**.** *The homomorphism*

$$\delta_i : \mathrm{Hom}(H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma}; \mathbb{Z}), N(\Sigma, \mathcal{P})) \to H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma}; \mathbb{Z})$$

is defined to be

$$f(x) \cdot A_i = x \cdot \delta_i(f).$$

The image of $\tau_{(\Sigma,\mathcal{P})}$: Unlike the classical Johnson homomorphism $N(\Sigma, \mathcal{P})$ is not isomorphic to $\bigwedge^2 H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma}; \mathbb{Z})$. We have the isomorphism $N(\Sigma, \mathcal{P}) \cong \bigwedge^2 H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma}; \mathbb{Z}) \oplus \mathbb{Z}^k$. Here, $\bigwedge^2 H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma}; \mathbb{Z})$ is the image of $[\widehat{\pi}, \widehat{\pi}]$ and the $\mathbb{Z}^k$ factor is spanned by $z_1, \dots, z_k$. Note that the intersection $y \mapsto y \cdot A_i$ vanishes on $\bigwedge^2 H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma}; \mathbb{Z})$ and satisfies $z_j \cdot A_i = \delta_{ij}$.

The projection $N(\Sigma, \mathcal{P}) \twoheadrightarrow \bigwedge^2 H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma}; \mathbb{Z})$ induces a projection

$$\begin{aligned}\mathrm{Hom}(H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma}; \mathbb{Z}), N(\Sigma, \mathcal{P})) \twoheadrightarrow & \mathrm{Hom}(H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma}; \mathbb{Z}), \bigwedge^2 H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma}; \mathbb{Z})) \\ \cong & \bigwedge^2 H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma}; \mathbb{Z}) \otimes H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma}; \mathbb{Z}). \qquad (3.2)\end{aligned}$$

Let $D(\Sigma, \mathcal{P}) \leq H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma}; \mathbb{Z})$ be the subspace spanned by the homology classes of the boundary components of Σ and let $D_i \leq D(\Sigma, \mathcal{P})$ be the subspace spanned by the homology classes of the boundary components in $P_i \in \mathcal{P}$. Note that P_0 is the block that contains the boundary component $\partial_{i_0}^0$ such that the basepoint $* \in \partial_{i_0}^0$. Finally, let $D(\Sigma, \mathcal{P})^{\perp}$ denote the subspace of $H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma}; \mathbb{Z})$ spanned by $H_1(\Sigma; \mathbb{Z})$.

Definition 3.2.6 ([5], Definition 5.8)**.** *The subspace* $W_{(\Sigma,\mathcal{P})} \leq \mathrm{Hom}(H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma}; \mathbb{Z}), N(\Sigma, \mathcal{P}))$ *consists of homomorphisms* $f : H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma}; \mathbb{Z}) \to N(\Sigma, \mathcal{P})$ *satisfying the following conditions:*

(I) the image in $\bigwedge^2 H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma};\mathbb{Z}) \otimes H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma};\mathbb{Z})$ *of* f *under the projection* (3.2) *is contained in the subspace* $\bigwedge^3 H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma};\mathbb{Z}) \leq \bigwedge^2 H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma};\mathbb{Z}) \otimes H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma};\mathbb{Z})$.

(II) for $i \geq 1$, $\delta_i(f) \in D(\Sigma,\mathcal{P})^{\perp}$ *and for any* $a \in D_i$, $f(a) = \delta_i(f) \wedge a$.

(III) for any $a \in D_0$, $f(a) = 0$.

Theorem 3.2.7 ([5], Theorem 5.9)**.** $W_{(\Sigma,\mathcal{P})} = \text{Im}\ \tau_{(\Sigma,\mathcal{P})}$.

Recall that as Church stated in [5], Definition 5.12, $W_{(\Sigma,\mathcal{P})}$ can be considered as a subspace of $\bigwedge^3 H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma};\mathbb{Z}) \oplus (\mathbb{Z}^k \otimes H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma};\mathbb{Z}))$ where $k = |\widehat{\mathcal{P}}| - 1$. Basis elements of $W_{(\Sigma,\mathcal{P})}$ will be shown to be $a \wedge b \wedge c$ for the $\bigwedge^3 H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma};\mathbb{Z})$ component and as $z_i \wedge x$ for $(\mathbb{Z}^k \otimes H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma};\mathbb{Z}))$.

Definition 3.2.8 ([5], Definition 5.13)**.** *Let* $(\Sigma,\mathcal{P})$ *and* $(\Sigma',\mathcal{P}')$ *be partitioned surfaces and let* z_j *be the boundary component of* $\widehat{\Sigma}$ *corresponding to* $P_i \in \mathcal{P}$ *for each* $1 \leq i \leq k$. *If* Σ' *is obtained from* Σ *by attaching a disk to* z_j, p_i *is defined as follows:*

$$\begin{aligned} p_i : W_{(\Sigma,\mathcal{P})} &\to W_{(\Sigma',\mathcal{P}')} \\ a \wedge b \wedge c &\mapsto a \wedge b \wedge c \\ z_j \wedge x &\mapsto 0 \\ z_k \wedge x &\mapsto z_k \wedge x \quad \text{for } k \neq j \end{aligned}$$

Now consider a morphism $i : (\Sigma,\mathcal{P}) \to (\Sigma',\mathcal{P}')$ *satisfying the condition that for each connected component* U *of* $\Sigma' \setminus \Sigma^{\circ}$, *the set of boundary components of* U *is not contained in the set of boundary components of* Σ. *Decompose* $\widehat{\Sigma}' \setminus \widehat{\Sigma}^{\circ}$ *into subsurfaces* $\{U_j\}$, *where* $U_j \cap \widehat{\Sigma} = z_j$. *The group* $H_1^{\mathcal{P}_j}(\widehat{U}_j;\mathbb{Z})$ *is identified with a subspace of* $H_1^{\mathcal{P}'}(\widehat{\Sigma}';\mathbb{Z})$. *Let* $w_{U_j} \in \bigwedge^2 H_1^{\mathcal{P}_j}(\widehat{U}_j;\mathbb{Z})$ *be the intersection form on* $H_1^{\mathcal{P}_j}(\widehat{U}_j;\mathbb{Z})$, *and let* $z_j^1,\ldots,z_j^l$ *be the boundary components of* U_j *such that* $z_j^k \cap \widehat{\Sigma} = \emptyset$ *for* $1 \leq k \leq l$. *We define* p_i *as follows.*

$$\begin{aligned} p_i : W_{(\Sigma,\mathcal{P})} &\to W_{(\Sigma',\mathcal{P}')} \\ a \wedge b \wedge c &\mapsto a \wedge b \wedge c \\ z_j \wedge x &\mapsto (w_{U_j} + z_j^1 + \ldots + z_j^l) \wedge x \end{aligned}$$

Note that in $N(U_j,\mathcal{P}_j)$ *we have* $z_j = w_{U_j} + z_j^1 + \ldots + z_j^l$.

Theorem 3.2.9 ([5], Theorem 5.14)**.** *For any morphism* $i : (\Sigma, \mathcal{P}) \to (\Sigma', \mathcal{P}')$ *of* $\mathcal{T}\mathrm{Surf}$ *and morphism* $i_* : \mathcal{I}(\Sigma, \mathcal{P}) \to \mathcal{I}(\Sigma', \mathcal{P}')$*,* $p_i : W_{(\Sigma,\mathcal{P})} \to W_{(\Sigma',\mathcal{P}')}$ *satisfies* $\tau_{(\Sigma',\mathcal{P}')} i_*(\varphi) = p_i \tau_{(\Sigma,\mathcal{P})}(\varphi)$ *for all* $\varphi \in \mathcal{I}(\Sigma, \mathcal{P})$*.*

Proposition 3.2.10 ([5], Proposition 6.1)**.** *If* $(\Sigma, \mathcal{P})$ *is a partitioned surface and* γ *is a* $\mathcal{P}$*-separating curve, then we have* $\tau_{(\Sigma,\mathcal{P})}(T_\gamma) = 0$*.*

Proposition 3.2.11 ([5], Proposition 6.3)**.** *Let* $(\Sigma, \mathcal{P})$ *be a partitioned surface. Let* $T_\alpha T_\beta^{-1}$ *be a twist about* $\mathcal{P}$*-bounding pair and* ζ *be a* $\mathcal{P}$*-separating curve that forms the boundary of a pair of pants with* $\alpha \cup \beta$*. Assign an orientation to each curve so that the pair of pants lies on the left side of* ζ *and the right side of* α *and* β *(or vice versa). If the homology class of* α *is* $[\alpha]$ *and* z *denotes the class of* ζ *in* $N(\Sigma, \mathcal{P})$*, then we get* $\tau_{(\Sigma,\mathcal{P})}(T_\alpha T_\beta^{-1}) = z \wedge [\alpha]$ *in* $W_{(\Sigma,\mathcal{P})}$*.*

CHAPTER 4

GENERALIZED CHILLINGWORTH CLASSES

In this chapter, we establish and prove our main results. We construct a well-defined map $\widetilde{e}_X$ by means of the projective tangent bundle. We prove that $\widetilde{e}_X$ and the homomorphism obtained by taking the dual of $\widetilde{e}_X(f)$ for any $f \in \mathcal{I}(\Sigma, \mathcal{P})$, which is the Chillingworth homomorphism from the subsurface Torelli groups to $H_1^{\mathcal{P}}(\Sigma; \mathbb{Z})$, satisfy the naturality property. Moreover, we show that $\widetilde{e}_X$ is the unique homomorphism satisfying naturality. Finally, we give the relation between the Chillingworth classes of the subsurface Torelli groups and the partitioned Johnson homomorphism.

In this chapter, if $(\Sigma_1, \mathcal{P}_1)$ and $(\Sigma_2, \mathcal{P}_2)$ are partitioned surfaces, then by an embedding $i : (\Sigma_1, \mathcal{P}_1) \hookrightarrow (\Sigma_2, \mathcal{P}_2)$ of partitioned surfaces, we mean a morphism $i : (\Sigma_1, \mathcal{P}) \to (\Sigma_2, \mathcal{P})$ of $\mathcal{T}\mathrm{Surf}$ given in Definition 3.2.2.

4.1 Winding Number In The Projective Tangent Bundle

This section starts with the definition of the projective tangent bundle and aims to find a well-defined map $\widetilde{e}_X : \mathcal{I}(\Sigma, \mathcal{P}) \to \mathrm{Hom}(H_1^{\mathcal{P}}(\Sigma; \mathbb{Z}), \mathbb{Z})$ with the naturality property.

Let Σ be a smooth compact connected oriented surface with nonempty boundary. Let us choose a Riemannian metric on Σ. Let $UT(\Sigma)$ be the unit tangent bundle of Σ. Since Σ has nonempty boundary, there are nonvanishing vector fields on Σ. Choice of two nonvanishing vector fields which are orthogonal to each other gives a parallelization of Σ. The unit tangent bundle $UT(\Sigma)$ is therefore homeomorphic to $\Sigma \times S^1$.

By using this unit tangent bundle, the projective tangent bundle $PT(\Sigma)$ is constructed as follows: By identifying antipodal points in each fiber S^1, we obtain a fiber bundle whose fiber is $\mathbb{RP}^1$, which is homeomorphic to S^1. The projective tangent bundle $PT(\Sigma)$ is also homeomorphic to the product $\Sigma \times S^1$ since Σ has nonempty boundary.

Let $\{[\alpha_i]\}_{i\in I} \cup \{[x_j], [y_j]\}_{j\in J}$ be a basis for $H_1^{\mathcal{P}}(\Sigma; \mathbb{Z})$. Here, I and J are finite index sets and each α_i is an arc, and each x_j, y_j is a simple closed curve. We assume that all representatives are smooth and are in minimal position. If $(\Sigma, \mathcal{P})$ is a totally separated surface, then I is the empty set. If Σ is a sphere with boundary components, J is the empty set.

In this thesis, we always take representatives of mapping classes fixing points in a regular neighborhood of each boundary component. Therefore, $f(\alpha_i)$ and α_i have the same tangent vectors on a small neighborhood of the boundary components. We denote by $f(\alpha_i) * \alpha_i^{-1}$ the closed curve obtained by first traversing the arc $f(\alpha_i)$ then α_i with the reversed orientation. The resulting closed curve has two nondifferentiable points on the boundary of the subsurface. Since $f(\alpha_i)$ and α_i have the same tangent vectors at the end points, in the projective tangent bundle we can calculate the winding number of closed oriented curves having two such nondifferentiable points on the boundary. When we concatenate arcs to obtain a closed curve, we will assume that the tangent spaces of the arcs at the end points coincide.

Winding Number: The winding number in the projective tangent bundle is defined in analogy to the winding number in the tangent bundle. Changing the unit tangent bundle in the diagram (2.1) with the projective tangent bundle $PT(\Sigma)$ and applying the same argument in the construction of the winding number, we obtain the construction of the winding number in the projective tangent bundle. Winding number in the projective tangent bundle will be defined for smooth closed oriented curves or for closed oriented curves constructed by concatenating a pair of smooth arcs as just described.

Denote the winding number in the projective tangent bundle of a closed oriented curve γ with respect to a nonvanishing vector field X by $\widetilde{w}_X(\gamma)$. Since S^1 is a double cover of $\mathbb{RP}^1$, for a smooth closed oriented curve γ we have $w_X(\gamma) = \frac{\widetilde{w}_X(\gamma)}{2}$.

Construction of $\widetilde{e}_X$: Let X be a nonvanishing vector field on a partitioned surface

$(\Sigma, \mathcal{P})$ and f be an element of the subsurface Torelli group. Choose a set of simple closed curves representing a basis of $H_1(\Sigma; \mathbb{Z})$. Assigning an integer to each basis element determines a homomorphism from $H_1(\Sigma; \mathbb{Z})$ to $\mathbb{Z}$. This integer is chosen to be the total number of times that X rotates relative to $f^{-1}X$ as we traverse the basis element. This homomorphism, denoted by $d(X, f^{-1}X)$, is defined in Definition 2.2.5. By Lemma 2.2.6, we have

$$d(X, f^{-1}X)[\gamma] = w_X(f\gamma) - w_X(\gamma),$$

for any smooth closed oriented curve γ. In the projective tangent bundle we get

$$d(X, f^{-1}X)[\gamma] = \frac{\widetilde{w}_X(f\gamma) - \widetilde{w}_X(\gamma)}{2}.$$

Since f fixes every boundary component of Σ, $d(X, f^{-1}X)[\partial] = 0$ for any boundary component ∂. Therefore, $d(X, f^{-1}X)$ induces a homomorphism $\overline{d}(X, f^{-1}X) : \overline{H}_1^{\mathcal{P}}(\Sigma; \mathbb{Z}) \to \mathbb{Z}$ defined by

$$\overline{d}(X, f^{-1}X)[\gamma] = \frac{\widetilde{w}_X(f\gamma) - \widetilde{w}_X(\gamma)}{2}.$$

Now our aim is to get a well-defined map

$$\widetilde{d}(X, f^{-1}X) : H_1^{\mathcal{P}}(\Sigma; \mathbb{Z}) \to \mathbb{Z}$$

mapping an element $[\alpha]$ of $H_1^{\mathcal{P}}(\Sigma; \mathbb{Z})$ to the half of the number of times that X rotates relative to $f^{-1}X$ in the projective tangent bundle as we traverse α.

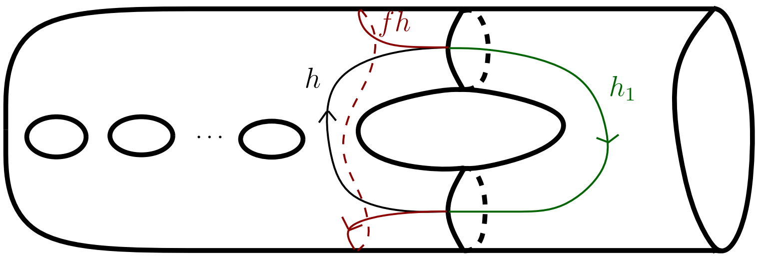


Figure 4.1: Extension of a subsurface with two boundary components to a surface with one boundary component by attaching a sphere with 3 holes.

For a closed oriented curve γ, we define

$$\widetilde{d}(X, f^{-1}X)[\gamma] = \overline{d}(X, f^{-1}X)[\gamma].$$

Now, let h be a smooth oriented arc whose endpoints are on the boundary components of Σ contained in the same element of $\mathcal{P}$ and let $f \in \mathcal{I}(\Sigma, \mathcal{P})$. Since f fixes all points of a regular neighborhood of the boundary components, h and $f(h)$ have the same tangent spaces at the end points and $f(h) * h^{-1}$ is a closed oriented curve with two cusps. We define

$$\widetilde{d}(X, f^{-1}X)[h] := \frac{\widetilde{w}_X(f(h) * h^{-1})}{2}.$$

For each $P \in \mathcal{P}$ with $|P| = n$, let us attach a sphere with $n+1$ boundary components to the n boundary components in P of Σ to obtain $\widehat{\Sigma}$ with a partition $\widehat{\mathcal{P}}$ as in Remark 3.2.1. Thus, $(\widehat{\Sigma}, \widehat{\mathcal{P}})$ is totally separated. Extend X to the obtained larger surface $\widehat{\Sigma}$ so that it is again a nonvanishing vector field on $\widehat{\Sigma}$. For simplicity, the extension will also be denoted by X. Let h_1 be a smooth oriented arc in the complement of Σ whose end points are ∂h (c.f. Figure 4.1). Let $\gamma := h * h_1$ denote the smooth closed oriented curve obtained by concatenating h and h_1. Notice that we choose a consistent orientation for h_1 to get a closed oriented curve γ. We parametrize γ such that its initial and terminal points are on one of the boundary components of the subsurface Σ. Then $f\gamma$ is isotopic to $f(h) * h_1$.

Remark 4.1.1. *The winding number in the tangent bundle of the concatenation of smooth closed oriented curves is not equal to the sum of the winding numbers of each smooth closed oriented curve if tangent vectors of the curves at the concatenation point are not parallel. That is, if α and β are smooth closed oriented curves with a common basepoint, in general we have $w_X(\alpha * \beta) \neq w_X(\alpha) + w_X(\beta)$. However, the winding number in the projective tangent bundle of the concatenation of smooth closed oriented curves is equal to the sum of the winding numbers of each smooth closed oriented curve if the tangent spaces of the curves at the end points are the same. Therefore, we obtain the following equalities:*

$$\begin{aligned} \frac{\widetilde{w}_X(f\gamma) - \widetilde{w}_X(\gamma)}{2} &= \frac{\widetilde{w}_X(f\gamma * \gamma^{-1})}{2} \\ &= \frac{\widetilde{w}_X(f(h) * h_1 * (h * h_1)^{-1})}{2} \\ &= \frac{\widetilde{w}_X(f(h) * h^{-1})}{2}. \end{aligned}$$

One can easily observe that the obtained equality

$$\frac{\widetilde{w}_X(f\gamma) - \widetilde{w}_X(\gamma)}{2} = \frac{\widetilde{w}_X(f(h) * h^{-1})}{2}$$

does not depend on the choice of the arc representative h_1 on $\widehat{\Sigma} \setminus \Sigma^\circ$.

Lemma 4.1.2. *Let h be a smooth oriented arc representing a homology class $[h]$ in $H_1^{\mathcal{P}}(\Sigma;\mathbb{Z})$. Then the number $\frac{\widetilde{w}_X(f(h)*h^{-1})}{2}$ is independent of the choice of the representative of $[h]$.*

Proof. Let $[h]=[h']$ be in $H_1^{\mathcal{P}}(\Sigma;\mathbb{Z})$. Then we have $[h*h'^{-1}]=0$ in $H_1^{\mathcal{P}}(\Sigma;\mathbb{Z})$. Since the embedding $(\Sigma,\mathcal{P})\hookrightarrow(\widehat{\Sigma},\widehat{\mathcal{P}})$ of partitioned surfaces takes $\mathcal{P}$-separating curves to $\widehat{\mathcal{P}}$-separating curves by the first condition of Definition 3.2.2, we get $[h*h'^{-1}]=0$ in $H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma};\mathbb{Z})$. We have $[\gamma]=[\gamma']$ by using the following equalities:

$$[h*h'^{-1}]=[h*h_1*h_1^{-1}*h'^{-1}]=[h*h_1]-[h'*h_1]=0,$$

where $[\gamma']=[h'*h_1]$.

Since we have

$$\frac{\widetilde{w}_X(f\gamma)-\widetilde{w}_X(\gamma)}{2}=\frac{\widetilde{w}_X(f\gamma')-\widetilde{w}_X(\gamma')}{2},$$

for any smooth homologous simple closed curves γ and γ' in $H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma};\mathbb{Z})$, we get

$$\frac{\widetilde{w}_X(f(h)*h^{-1})}{2}=\frac{\widetilde{w}_X(f(h')*h'^{-1})}{2}.$$

□

Lemma 4.1.3. *The map $\widetilde{d}(X,f^{-1}X):H_1^{\mathcal{P}}(\Sigma,\mathbb{Z})\to\mathbb{Z}$ is a homomorphism.*

Proof. For smooth closed oriented curves γ_1 and γ_2 by the definition of $d(X,f^{-1}X)$, we have

$$\widetilde{d}(X,f^{-1}X)[\gamma_1*\gamma_2]=\widetilde{d}(X,f^{-1}X)[\gamma_1]+\widetilde{d}(X,f^{-1}X)[\gamma_2].$$

Let h_1 and h_2 be smooth oriented arcs whose endpoints are on the boundary components of Σ contained in the same element of $\mathcal{P}$ and let us assume that initial point of h_2 is the same as the terminal point of h_1. Let $[h]$ denote the sum of two homology classes $[h_1]$ and $[h_2]$. We obtain the following equalities:

$$\begin{aligned}
\widetilde{d}(X, f^{-1}X)[h_1] + \widetilde{d}(X, f^{-1}X)[h_2] &= \frac{\widetilde{w}_X(f(h_1) * h_1^{-1})}{2} + \frac{\widetilde{w}_X(f(h_2) * h_2^{-1})}{2} \\
&= \frac{\widetilde{w}_X(h_1^{-1} * f(h_1))}{2} + \frac{\widetilde{w}_X(f(h_2) * h_2^{-1})}{2} \\
&= \frac{\widetilde{w}_X(h_1^{-1} * f(h_1) * f(h_2) * h_2^{-1})}{2} \\
&= \frac{\widetilde{w}_X(f(h_1) * f(h_2) * h_2^{-1} * h_1^{-1})}{2} \\
&= \frac{\widetilde{w}_X(f(h_1 * h_2) * (h_1 * h_2)^{-1})}{2} \\
&= \widetilde{d}(X, f^{-1}X)[h_1 * h_2] \\
&= \widetilde{d}(X, f^{-1}X)[h].
\end{aligned}$$

Now let γ be a smooth oriented arc whose homology class $[\gamma]$ is the sum of a homology class $[h']$ whose representatives are arcs and a homology class $[\alpha]$ with closed curve representatives. As in the previous paragraph of Remark 4.1.1, we can obtain a smooth closed oriented curve α' by concatenating h' with a smooth oriented arc in the complement of Σ. Hence, we have

$$\begin{aligned}
\widetilde{d}(X, f^{-1}X)[h'] + \widetilde{d}(X, f^{-1}X)[\alpha] &= \frac{\widetilde{w}_X(f(h') * h'^{-1})}{2} + \frac{\widetilde{w}_X(f\alpha) - \widetilde{w}_X(\alpha)}{2} \\
&= \frac{\widetilde{w}_X(f\alpha') - \widetilde{w}_X(\alpha')}{2} + \frac{\widetilde{w}_X(f\alpha) - \widetilde{w}_X(\alpha)}{2} \\
&= \widetilde{d}(X, f^{-1}X)[\alpha' * \alpha] \\
&= \widetilde{d}(X, f^{-1}X)[h' * \alpha] \quad \text{(by Remark 4.1.1)} \\
&= \widetilde{d}(X, f^{-1}X)[\gamma].
\end{aligned}$$

□

Definition 4.1.4. *The map* $\widetilde{e}_X : \mathcal{I}(\Sigma, \mathcal{P}) \to \mathrm{Hom}(H_1^{\mathcal{P}}(\Sigma; \mathbb{Z}), \mathbb{Z})$ *is defined to be* $\widetilde{e}(f) := \widetilde{d}(X, f^{-1}X)$. *More explicitly, it is defined as follows:*

If $[\gamma]$ *has a smooth closed curve representative* γ,

$$\widetilde{e}_X(f)[\gamma] := \frac{\widetilde{w}_X(f\gamma) - \widetilde{w}_X(\gamma)}{2}.$$

If h *is a smooth oriented arc representing a homology class* $[h]$ *in* $H_1^{\mathcal{P}}(\Sigma; \mathbb{Z})$,

$$\widetilde{e}_X(f)[h] := \frac{\widetilde{w}_X(f(h) * h^{-1})}{2}.$$

Lemma 4.1.5. *The map* $\widetilde{e}_X : \mathcal{I}(\Sigma, \mathcal{P}) \to \mathrm{Hom}(H_1^{\mathcal{P}}(\Sigma; \mathbb{Z}), \mathbb{Z})$ *is a homomorphism.*

Proof. It is easy to see that $\widetilde{e}_X(fg)[\gamma] = \widetilde{e}_X(f)[\gamma] + \widetilde{e}_X(g)[\gamma]$ for a smooth closed oriented curve γ by Lemma 2.2.9.

For a smooth oriented arc α_i,

$$\begin{aligned} \widetilde{e}_X(fg)[\alpha_i] &= \frac{\widetilde{w}_X(fg(\alpha_i) * \alpha_i^{-1})}{2} \\ &= \frac{\widetilde{w}_X(f(g\alpha_i) * g(\alpha_i^{-1}) * g(\alpha_i) * \alpha_i^{-1})}{2} \\ &= \frac{\widetilde{w}_X(f(g\alpha_i) * g(\alpha_i^{-1}))}{2} + \frac{\widetilde{w}_X(g(\alpha_i) * \alpha_i^{-1})}{2} \\ &= \widetilde{e}_X(f)[g(\alpha_i)] + \widetilde{e}_X(g)[\alpha_i]. \end{aligned}$$

Since $g \in \mathcal{I}(\Sigma, \mathcal{P})$, $g(\alpha_i)$ and α_i represent the same element of $H_1^{\mathcal{P}}(\Sigma; \mathbb{Z})$. Hence we get

$$\widetilde{e}_X(fg) = \widetilde{e}_X(f) + \widetilde{e}_X(g).$$

□

The homomorphism $\widetilde{e}_X$ depends on the choice of the nonvanishing vector field X. For instance, consider surfaces S_1 and S_2 containing S shown as in Figure 4.2.

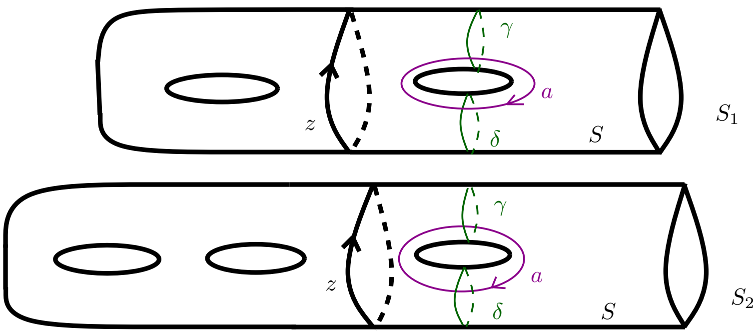


Figure 4.2: Different extensions of S with respect to the same partition.

Choose nonvanishing vector fields X_1 and X_2 on surfaces S_1 and S_2, respectively. Consider the restrictions of the vector fields X_1 and X_2 to S. Winding numbers of z with respect to $X_1|S$ and $X_2|S$ are different by Lemma 2.2.4. Indeed, we have $w_{X_1|S}(z) = -1$ and $w_{X_2|S}(z) = -3$. By using again Lemma 2.2.4, we can see that $\widetilde{e}_{X_1|S}(T_\gamma T_\delta^{-1})[a]$ and $\widetilde{e}_{X_2|S}(T_\gamma T_\delta^{-1})[a]$ are different numbers. These numbers are 2 and 4, respectively.

4.2 Symplectic Basis for $H_1^{\mathcal{P}}(\Sigma;\mathbb{Z})$

In this section, we introduce a symplectic basis for $H_1^{\mathcal{P}}(\Sigma;\mathbb{Z})$.

Let $(\Sigma,\mathcal{P})$ be a partitioned surface of genus g with the partition

$$\mathcal{P}=\{\{\partial_1^1,\dots,\partial_{k_1}^1\},\dots,\{\partial_1^m,\dots,\partial_{k_m}^m\}\}.$$

Let $\mathcal{Q}$ be a subset of the boundary $\partial\Sigma$ containing exactly one point from each boundary component.

Let us choose a set of simple closed curves $\{x_1,y_1,x_2,y_2,\dots,x_g,y_g\}$ on Σ satisfying

- $x_i\cap x_j=\emptyset,\quad x_i\cap y_j=\emptyset,\quad y_i\cap y_j=\emptyset$ for $i\neq j$,
- x_i intersects y_i transversely at one point, and
- under the filling map
$$H_1(\Sigma;\mathbb{Z})\to H_1(\overline{\Sigma};\mathbb{Z})$$
$\{[x_i],[y_i]\mid i=1,\dots,g\}$ maps to a symplectic basis of $H_1(\overline{\Sigma};\mathbb{Z})$. Here, $\overline{\Sigma}$ denotes the closed surface obtained by gluing a disc along each boundary component.

For each $l=1,2,\dots,m$, choose oriented arcs h_j^l connecting $\partial_j^l\cap\mathcal{Q}$ to $\partial_{j+1}^l\cap\mathcal{Q}$ for $j=1,2,\dots,k_l-1$ such that

- h_j^l are disjoint from x_i,y_i,
- h_j^l are pairwise disjoint except perhaps at endpoints,
- each h_j^l is oriented so that the algebraic intersection number of the homology classes $[h_j^l]$ and $[\partial_1^l+\cdots+\partial_j^l]$ is 1, where the orientations of the boundary components are induced from the orientation of the surface.

The union of the sets

- $\{[x_1],[y_1],\dots,[x_g],[y_g]\}$,

- $\{[h^1_1], [h^1_2], \ldots, [h^1_{k_1-1}], [h^2_1], \ldots, [h^2_{k_2-1}], \ldots, [h^m_1], \ldots, [h^m_{k_m-1}]\}$,
- $\{ [\partial^1_1], [\partial^1_1 + \partial^1_2], \ldots, [\partial^1_1 + \cdots + \partial^1_{k_1-1}], [\partial^2_1], \ldots, [\partial^2_1 + \cdots + \partial^2_{k_2-1}], \ldots, [\partial^m_1], \ldots, [\partial^m_1 + \cdots + \partial^m_{k_m-1}]\}$

is a basis $\mathscr{B}$ of $H_1^{\mathcal{P}}(\Sigma; \mathbb{Z})$.

In this basis, $\{x_i, y_i\}$ are closed curves, the $\{h^l_{i_l}\}$s are arcs, and $\{\partial^l_{i_l}\}$s are boundary curves as shown in Figure 4.3.

This basis $\mathscr{B}$ has the following properties:

- $\widehat{i}([x_i], [x_j]) = \widehat{i}([y_i], [y_j]) = 0, \qquad \widehat{i}([x_i], [y_j]) = \delta_{ij}, \quad$ for all $1 \leq i, j \leq g$,
- $\widehat{i}([h^l_i], [\partial^l_1 + \cdots + \partial^l_j]) = \delta_{ij}, \quad$ for all $1 \leq i, j \leq k_l - 1, 1 \leq l \leq m$,
- $\widehat{i}([h^l_j], [x_i]) = \widehat{i}([h^l_j], [y_i]) = 0,$ for all $1 \leq i \leq g, 1 \leq j \leq k_l - 1, 1 \leq l \leq m$,
- $\widehat{i}([\partial^l_1 + \cdots + \partial^l_j], [x_i]) = \widehat{i}([\partial^l_1 + \cdots + \partial^l_j], [y_i]) = 0,$ for all $1 \leq i \leq g, 1 \leq j \leq k_l - 1, 1 \leq l \leq m$.

Here, δ_{ij} denotes the Kronecker delta and $\widehat{i}(\cdot, \cdot)$ denotes the algebraic intersection number. Note that although the endpoints of the representatives of homology basis elements $[h^l_j]$ coincide with $[h^l_{j+1}]$ on $\partial\Sigma$, we define the algebraic intersection of arcs $\widehat{i}([h^l_j], [h^l_{j+1}])$ to be 0 for all $1 \leq j \leq k_l - 1, 1 \leq l \leq m$.

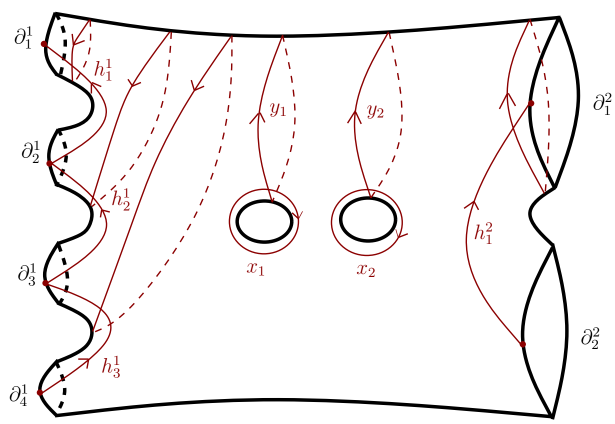


Figure 4.3: An example illustrating homology basis elements of $H_1^{\mathcal{P}}(\Sigma_{2,6}; \mathbb{Z})$, where $\mathcal{P} = \{\{\partial^1_1, \partial^1_2, \partial^1_3, \partial^1_4\}, \{\partial^2_1, \partial^2_2\}\}$.

We now define the dual of a homology class of $H_1^{\mathcal{P}}(\Sigma;\mathbb{Z})$ by using this intersection form. Note that the intersection form $\widehat{i}$ is nondegenerate. Therefore the map

$$D : H_1^{\mathcal{P}}(\Sigma;\mathbb{Z}) \to \mathrm{Hom}(H_1^{\mathcal{P}}(\Sigma;\mathbb{Z}),\mathbb{Z})$$

sending $[x] \in H_1^{\mathcal{P}}(\Sigma;\mathbb{Z})$ to $\widehat{i}(\cdot,[x])$ is an isomorphism.

4.3 Naturality and Uniqueness of $\widetilde{e}_X$

In this section, we show that $\widetilde{e}_X$ is natural and that it is the unique homomorphism from $\mathcal{I}(\Sigma,\mathcal{P})$ to $\mathrm{Hom}(H_1^{\mathcal{P}}(\Sigma;\mathbb{Z}),\mathbb{Z})$ satisfying the naturality.

Remark 4.3.1. *Suppose that $(\Sigma,\mathcal{P})$ is a totally separated surface with boundary components $z_1, z_2, \ldots, z_n$, so that $\mathcal{P} = \{\{z_1\},\ldots,\{z_n\}\}$ and that Σ' is a partitioned surface with a partition $\mathcal{P}'$ such that there is an embedding $(\Sigma,\mathcal{P}) \hookrightarrow (\Sigma',\mathcal{P}')$ of partitioned surfaces. For $1 \le j \le n$, let V_j be a connected component of $\Sigma' \setminus \Sigma^\circ$ containing z_j as a boundary component and let $\mathcal{P}_j$ be the partition of the boundary of V_j consisting of $\{z_j\}$ and a subset of $\mathcal{P}'$. Hence, if $P \in \mathcal{P}_j$ and $P \ne \{z_j\}$ then $P \in \mathcal{P}'$. If $i \ne j$, then $V_i \cap V_j = \emptyset$ and hence $H_1^{\mathcal{P}_i}(V_i;\mathbb{Z}) \cap H_1^{\mathcal{P}_j}(V_j;\mathbb{Z}) = \{0\}$. Since $V_j \cap \Sigma = z_j$, we have $H_1^{\mathcal{P}_j}(V_j;\mathbb{Z}) \cap H_1^{\mathcal{P}}(\Sigma;\mathbb{Z}) = \{[z_j]\} = \{0\}$. By identifying $H_1^{\mathcal{P}_j}(V_j;\mathbb{Z})$ and $H_1^{\mathcal{P}}(\Sigma;\mathbb{Z})$ with their images in $H_1^{\mathcal{P}'}(\Sigma';\mathbb{Z})$, we can write*

$$H_1^{\mathcal{P}'}(\Sigma';\mathbb{Z}) = H_1^{\mathcal{P}}(\Sigma;\mathbb{Z}) \oplus H_1^{\mathcal{P}_1}(V_1;\mathbb{Z}) \oplus \cdots \oplus H_1^{\mathcal{P}_n}(V_n;\mathbb{Z}).$$

If Σ is totally separated with the partition $\mathcal{P}$ and if $i : (\Sigma,\mathcal{P}) \hookrightarrow (\Sigma',\mathcal{P}')$ is an embedding of partitioned surfaces, then there is a natural projection

$$r_* : H_1^{\mathcal{P}'}(\Sigma';\mathbb{Z}) \to H_1^{\mathcal{P}}(\Sigma;\mathbb{Z})$$

which gives a natural homomorphism

$$r^* : \mathrm{Hom}(H_1^{\mathcal{P}}(\Sigma;\mathbb{Z}),\mathbb{Z}) \to \mathrm{Hom}(H_1^{\mathcal{P}'}(\Sigma';\mathbb{Z}),\mathbb{Z}).$$

Note also that if Σ is not totally separated, $H_1^{\mathcal{P}_j}(V_j;\mathbb{Z}) \cap H_1^{\mathcal{P}}(\Sigma;\mathbb{Z})$ has a nontrivial element for some j. Therefore, in this case $H_1^{\mathcal{P}'}(\Sigma';\mathbb{Z})$ is not isomorphic to $H_1^{\mathcal{P}}(\Sigma;\mathbb{Z}) \oplus H_1^{\mathcal{P}_1}(V_1;\mathbb{Z}) \oplus \cdots \oplus H_1^{\mathcal{P}_n}(V_n;\mathbb{Z})$.

Proposition 4.3.2. *Let Σ be a totally separated surface with the partition $\mathcal{P}$ and let $i : (\Sigma, \mathcal{P}) \hookrightarrow (\Sigma', \mathcal{P}')$ be an embedding of partitioned surfaces. Let X be a nonvanishing vector field on Σ' and let Y denote the restriction of X to Σ. Then the homomorphism $\widetilde{e}_Y$ is natural in the sense that the diagram*

$$\begin{array}{ccc} \mathcal{I}(\Sigma, \mathcal{P}) & \xrightarrow{i_*} & \mathcal{I}(\Sigma', \mathcal{P}') \\ \downarrow{\scriptstyle \widetilde{e}_Y} & & \downarrow{\scriptstyle \widetilde{e}_X} \\ \mathrm{Hom}(H_1^{\mathcal{P}}(\Sigma;\mathbb{Z}),\mathbb{Z}) & \xrightarrow[r^*]{} & \mathrm{Hom}(H_1^{\mathcal{P}'}(\Sigma';\mathbb{Z}),\mathbb{Z}) \end{array} \tag{4.1}$$

commutes.

Proof. Let $f \in \mathcal{I}(\Sigma, \mathcal{P})$, and let $i_*(f) = \widetilde{f}$. Thus (the class of) the diffeomorphism $\widetilde{f}$ is equal to f on Σ and is the identity on the complement $\Sigma' \setminus \Sigma$. We show that $r^*(\widetilde{e}_Y(f)) = \widetilde{e}_X(\widetilde{f})$.

Let γ be a smooth oriented simple closed curve in Σ representing a basis element of $H_1^{\mathcal{P}}(\Sigma;\mathbb{Z})$. Then, we have

$$r^*(\widetilde{e}_Y(f))[\gamma] = \widetilde{e}_Y(f)(r_*[\gamma]) = \widetilde{e}_Y(f)[\gamma] = \frac{\widetilde{w}_Y(f\gamma) - \widetilde{w}_Y(\gamma)}{2}$$

and

$$\widetilde{e}_X(\widetilde{f})[\gamma] = \frac{\widetilde{w}_X(\widetilde{f}\gamma) - \widetilde{w}_X(\gamma)}{2} = \frac{\widetilde{w}_X(f\gamma) - \widetilde{w}_X(\gamma)}{2}.$$

Since Y is the restriction of X to Σ, we have $r^*(\widetilde{e}_Y(f))[\gamma] = \widetilde{e}_X(\widetilde{f})[\gamma]$.

Now let γ' be a smooth closed oriented curve or smooth oriented arc in some V_j representing a homology basis element in $H_1^{\mathcal{P}_j}(V_j;\mathbb{Z})$. In this case, $r^*(\widetilde{e}_Y(f))[\gamma'] = \widetilde{e}_Y(f)(r_*([\gamma'])) = 0$ because $r_*([\gamma']) = 0$. Since $f(\gamma') = \gamma'$, we have

$$\widetilde{e}_X(\widetilde{f})[\gamma'] = \frac{\widetilde{w}_X(\widetilde{f}\gamma') - \widetilde{w}_X(\gamma')}{2} = \frac{\widetilde{w}_X(\gamma') - \widetilde{w}_X(\gamma')}{2} = 0.$$

Since $H_1^{\mathcal{P}'}(\Sigma';\mathbb{Z})$ is the direct sum of $H_1^{\mathcal{P}}(\Sigma;\mathbb{Z})$ and $H_1^{\mathcal{P}_j}(V_j;\mathbb{Z})$, it follows that $r^*(\widetilde{e}_Y(f)) = \widetilde{e}_X(\widetilde{f})$ for every f in $\mathcal{I}(\Sigma, \mathcal{P})$, and hence $r^*\widetilde{e}_Y = \widetilde{e}_X i_*$. □

Suppose now that Σ is any surface with a partition $\mathcal{P} = \{P_1, P_2, \ldots, P_n\}$, $|P_l| = n_l$, and that $\widehat{\Sigma}$ is the totally separated surface, with the partition $\widehat{\mathcal{P}}$, obtained by gluing a sphere S_l with $n_l + 1$ holes along the boundary components in P_l, i.e. the minimal totally separated surface containing Σ (c.f. Remark 3.2.1). For an $l = 1, 2, \dots, n$, suppose that $P_l = \{\partial_1^l, \partial_2^l, \dots, \partial_{n_l}^l\}$. For each $j = 1, 2, \dots, n_l - 1$, choose smooth arcs k_j^l on the complement $\widehat{\Sigma} \setminus \Sigma^\circ$ connecting $\mathcal{Q} \cap \partial_j^l$ to $\mathcal{Q} \cap \partial_{j+1}^l$. Here, k_j^l are pairwise disjoint except perhaps at endpoints. Let us orient each k_j^l so that concatenation $h_j^l * k_j^l$ is a smooth closed oriented curve in $\widehat{\Sigma}$, where $[h_j^l]$ is an element of the basis $\mathscr{B}$ defined in Section 4.2. Let $\mathcal{P}_l = \{P_l, \{z_l\}\}$ be the partition of the boundary of S^l, where z_l is the boundary component of $\widehat{\Sigma}$. Then $K_l = \{[k_j^l]\}$ is a set of basis elements with arc representatives of $H_1^{\mathcal{P}_l}(S_l; \mathbb{Z})$. Let $\mathscr{K}$ denote the union $K_1 \cup K_2 \cup \cdots \cup K_n$.

Let us fix the symplectic basis $\mathscr{B}$ of $H_1^{\mathcal{P}}(\Sigma; \mathbb{Z})$ defined as in Section 4.2.

We then have an isomorphism

$$\psi_{\mathscr{K}} : H_1^{\mathcal{P}}(\Sigma; \mathbb{Z}) \to H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma}; \mathbb{Z})$$

mapping the basis elements with closed curve representatives to itself and $[h_j^l]$ to $[h_j^l * k_j^l]$.

By using $\psi_{\mathscr{K}}$, we get the isomorphism

$$\psi_{\mathscr{K}}^* : \mathrm{Hom}(H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma}; \mathbb{Z}), \mathbb{Z}) \to \mathrm{Hom}(H_1^{\mathcal{P}}(\Sigma; \mathbb{Z}), \mathbb{Z})$$

defined to be $\psi_{\mathscr{K}}^*(\chi) = \chi \circ \psi_{\mathscr{K}}$ for any $\chi \in \mathrm{Hom}(H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma}; \mathbb{Z}), \mathbb{Z})$.

Proposition 4.3.3. *Let $(\Sigma, \mathcal{P})$ be a partitioned surface and let $(\widehat{\Sigma}, \widehat{\mathcal{P}})$ be the minimal totally separated surface containing Σ. Let $i : (\Sigma, \mathcal{P}) \hookrightarrow (\widehat{\Sigma}, \widehat{\mathcal{P}})$ be the inclusion so that it is an embedding of partitioned surfaces. Let X be a nonvanishing vector field on $\widehat{\Sigma}$ and let Y denote the restriction of X to Σ. Then the homomorphism $\widetilde{e}_Y$ is natural in the sense that the diagram*

$$\begin{array}{ccc} \mathcal{I}(\Sigma, \mathcal{P}) & \xrightarrow{i_*} & \mathcal{I}(\widehat{\Sigma}, \widehat{\mathcal{P}}) \\ \downarrow \widetilde{e}_Y & & \downarrow \widetilde{e}_X \\ \mathrm{Hom}(H_1^{\mathcal{P}}(\Sigma; \mathbb{Z}), \mathbb{Z}) & \xleftarrow{\psi_{\mathscr{K}}^*} & \mathrm{Hom}(H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma}; \mathbb{Z}), \mathbb{Z}) \end{array} \tag{4.2}$$

commutes.

Proof. Let $f \in \mathcal{I}(\Sigma, \mathcal{P})$, and let $i_*(f) = \widetilde{f}$. Thus $\widetilde{f}$ is equal to f on Σ and is the identity on the complement $\widehat{\Sigma} \setminus \Sigma$. We show that $\widetilde{e}_Y(f) = \psi_{\mathscr{K}}^* \widetilde{e}_X(\widetilde{f})$.

For any homology basis element $[\gamma] \in H_1^{\mathcal{P}}(\Sigma; \mathbb{Z})$ with a smooth closed oriented curve representative γ, we have

$$\widetilde{e}_Y(f)[\gamma] = \frac{\widetilde{w}_Y(f\gamma) - \widetilde{w}_Y(\gamma)}{2}$$

and

$$\begin{aligned}
\psi_{\mathscr{K}}^* \widetilde{e}_X(\widetilde{f})([\gamma]) &= \widetilde{e}_X(\widetilde{f})(\psi_{\mathscr{K}}[\gamma]) \\
&= \widetilde{e}_X(\widetilde{f})[\gamma] \\
&= \frac{\widetilde{w}_X(\widetilde{f}\gamma) - \widetilde{w}_X(\gamma)}{2} \\
&= \frac{\widetilde{w}_X(f\gamma) - \widetilde{w}_X(\gamma)}{2}.
\end{aligned}$$

Since $X = Y$ on Σ, we get the desired equality.

For any homology basis element $[h_j^l] \in H_1^{\mathcal{P}}(\Sigma; \mathbb{Z})$ with a smooth oriented arc representative h_j^l, we have

$$\widetilde{e}_Y(f)[h_j^l] = \frac{w_Y(f(h_j^l) * (h_j^l)^{-1})}{2}$$

and

$$\begin{aligned}
\psi_{\mathscr{K}}^* \widetilde{e}_X(\widetilde{f})[h_j^l] &= \widetilde{e}_X(\widetilde{f})(\psi_{\mathscr{K}}[h_j^l]) \\
&= \widetilde{e}_X(\widetilde{f})[h_j^l * k_j^l] \\
&= \frac{\widetilde{w}_X(\widetilde{f}(h_j^l * k_j^l)) - \widetilde{w}_X(h_j^l * k_j^l)}{2}.
\end{aligned}$$

Since we are working in the projective tangent bundle and we assume that representatives of mapping classes fix a regular neighborhood of the boundary components, we get

$$\begin{aligned}
\psi_{\mathscr{K}}^* \widetilde{e}_X(\widetilde{f})[h_j^l] &= \frac{\widetilde{w}_X(\widetilde{f}(h_j^l * k_j^l) * (h_j^l * k_j^l)^{-1})}{2} \\
&= \frac{\widetilde{w}_X(f(h_j^l) * k_j^l * (k_j^l)^{-1} * (h_j^l)^{-1})}{2} \\
&= \frac{\widetilde{w}_X(f(h_j^l) * (h_j^l)^{-1})}{2} \\
&= \frac{\widetilde{w}_Y(f(h_j^l) * (h_j^l)^{-1})}{2}.
\end{aligned}$$

Therefore, we obtain the equality $\widetilde{e}_Y = \psi_{\mathscr{K}}^* \widetilde{e}_X i_*$. This finishes the proof. $\square$

Note that commutativity of diagram (4.2) does not depend on the choice of basis $\{[k_j^l]\} \in H_1^{\mathcal{P}_l}(S_l;\mathbb{Z})$.

Proposition 4.3.2 and Proposition 4.3.3 imply the following theorem.

Theorem 4.3.4. *Let $(\Sigma,\mathcal{P})$ and $(\Sigma',\mathcal{P}')$ be partitioned surfaces and $i : (\Sigma,\mathcal{P}) \hookrightarrow (\Sigma',\mathcal{P}')$ be an embedding of partitioned surfaces. Let X be a nonvanishing vector field on Σ' and let Y denote the restriction of X to Σ. Then there exists a homomorphism i'_* such that the homomorphism $\widetilde{e}_Y$ is natural in the sense that the diagram*

$$\begin{array}{ccc} \mathcal{I}(\Sigma,\mathcal{P}) & \xrightarrow{\quad i_* \quad} & \mathcal{I}(\Sigma',\mathcal{P}') \\ {\scriptstyle \widetilde{e}_Y}\downarrow & & \downarrow{\scriptstyle \widetilde{e}_X} \\ \mathrm{Hom}(H_1^{\mathcal{P}}(\Sigma;\mathbb{Z}),\mathbb{Z}) & \xrightarrow[i'_*]{} & \mathrm{Hom}(H_1^{\mathcal{P}'}(\Sigma';\mathbb{Z}),\mathbb{Z}) \end{array} \tag{4.3}$$

commutes.

Proof. Let Σ be a surface with a partition $\mathcal{P} = \{P_1, P_2, \ldots, P_n\}$, $|P_l| = n_l$. For an $l = 1, 2, \ldots, n$, suppose that $P_l = \{\partial_1^l, \partial_2^l, \ldots, \partial_{n_l}^l\}$. For each $j = 1, 2, \ldots, n_l - 1$, choose smooth oriented simple arcs k_j^l on the complement $\Sigma' \setminus \Sigma^\circ$ connecting $\mathcal{Q} \cap \partial_j^l$ to $\mathcal{Q} \cap \partial_{j+1}^l$. Here, k_j^l are pairwise disjoint except perhaps at endpoints. We consider a closed tubular neighbourhood of the union $\partial_1^l \cup \partial_2^l \cup \cdots \cup \partial_{n_l}^l \cup k_1^l \cup \cdots k_{n_l-1}^l$. This tubular neighbourhood is homeomorphic to a sphere S_l with $n_l + 1$ holes. Let us consider now the minimal totally separated surface $(\widehat{\Sigma}, \widehat{\mathcal{P}})$ containing Σ and all S_l as a subsurface.

Let us fix bases $\mathscr{B}$ and $\mathscr{K}$ as in Proposition 4.3.3.

Consider the composition of the embedding $\widehat{j} : (\Sigma,\mathcal{P}) \hookrightarrow (\widehat{\Sigma}, \widehat{\mathcal{P}})$ of partitioned surfaces with the embedding $j' : (\widehat{\Sigma}, \widehat{\mathcal{P}}) \hookrightarrow (\Sigma',\mathcal{P}')$ of partitioned surfaces. Let $\widehat{Y}$ denote the restriction of X to $\widehat{\Sigma}$. After showing that both diagrams in (4.4) are commutative, our proof will be complete.

$$\begin{array}{ccccc} \mathcal{I}(\Sigma,\mathcal{P}) & \xrightarrow{\quad \widehat{j}_* \quad} & \mathcal{I}(\widehat{\Sigma},\widehat{\mathcal{P}}) & \xrightarrow{\quad j'_* \quad} & \mathcal{I}(\Sigma',\mathcal{P}') \\ {\scriptstyle \widetilde{e}_Y}\downarrow & & \downarrow{\scriptstyle \widetilde{e}_{\widehat{Y}}} & & \downarrow{\scriptstyle \widetilde{e}_X} \\ \mathrm{Hom}(H_1^{\mathcal{P}}(\Sigma;\mathbb{Z}),\mathbb{Z}) & \xrightarrow[(\psi^*_{\mathscr{K}})^{-1}]{} & \mathrm{Hom}(H_1^{\widehat{P}}(\widehat{\Sigma};\mathbb{Z}),\mathbb{Z}) & \xrightarrow[r^*]{} & \mathrm{Hom}(H_1^{\mathcal{P}'}(\Sigma';\mathbb{Z}),\mathbb{Z}) \end{array} \tag{4.4}$$

Proposition 4.3.2 implies the commutativity of the right-hand side in diagram (4.4). Proposition 4.3.3 gives the commutativity of the left-hand side in diagram (4.4).

□

Remark 4.3.5. *Theorem 4.3.4 remains true for any capping $i : (\Sigma, \mathcal{P}) \hookrightarrow \Sigma_g$ under the condition that the chosen vector field X on Σ_g has only one singularity in the complement of $\widehat{\Sigma}$.*

Proposition 4.3.6. *The homomorphism $\widetilde{e}_Y$ is unique in the sense that it is the only nontrivial homomorphism from $\mathcal{I}(\Sigma, \mathcal{P})$ to $\mathrm{Hom}(H_1^{\mathcal{P}}(\Sigma; \mathbb{Z}), \mathbb{Z})$ such that diagram (4.3) commutes.*

Proof. Let $\widehat{\Sigma}$ be a totally separated surface obtained from Σ as in Remark 3.2.1. Since an embedding $i : (\Sigma, \mathcal{P}) \hookrightarrow (\Sigma', \mathcal{P}')$ of partitioned surfaces can be considered to be the composition of the two embeddings $(\Sigma, \mathcal{P}) \hookrightarrow (\widehat{\Sigma}, \widehat{\mathcal{P}}) \hookrightarrow (\Sigma', \mathcal{P}')$ of partitioned surfaces, we will consider the diagram (4.4).

First, we show the uniqueness of $\widetilde{e}_{\widehat{Y}}$ such that the right side of diagram (4.4) is commutative. The second step will be to show the uniqueness of $\widetilde{e}_Y$ such that the left side of diagram (4.4) is commutative. This will finish our proof.

Now let us consider the embedding $(\widehat{\Sigma}, \widehat{\mathcal{P}}) \hookrightarrow (\Sigma', \mathcal{P}')$ of partitioned surfaces. We have $r^* \circ \widetilde{e}_{\widehat{Y}} = \widetilde{e}_X \circ j'_*$. Let us assume that there is another homomorphism $G : \mathcal{I}(\widehat{\Sigma}, \widehat{\mathcal{P}}) \to \mathrm{Hom}(H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma}; \mathbb{Z}), \mathbb{Z})$ satisfying the naturality condition, $r^* \circ G = \widetilde{e}_X \circ j'_*$. Our aim is to show that $\widetilde{e}_{\widehat{Y}} = G$, hence proving the proposition for this case. Since both G and $\widetilde{e}_{\widehat{Y}}$ satisfy the naturality condition, we get $r^* \circ \widetilde{e}_{\widehat{Y}} = r^* \circ G$. Since r_* is onto, r^* is injective, which implies that $\widetilde{e}_{\widehat{Y}} = G$.

Now consider the embedding $(\Sigma, \mathcal{P}) \hookrightarrow (\widehat{\Sigma}, \widehat{P})$ of partitioned surfaces for the second part of the proof. We need to show that $\widetilde{e}_Y : \mathcal{I}(\Sigma, \mathcal{P}) \to \mathrm{Hom}(H_1^{\mathcal{P}}(\Sigma; \mathbb{Z}), \mathbb{Z})$ is the unique homomorphism satisfying the naturality property. Let $F : \mathcal{I}(\Sigma, \mathcal{P}) \to \mathrm{Hom}(H_1^{\mathcal{P}}(\Sigma; \mathbb{Z}), \mathbb{Z})$ be another homomorphism such that $\widetilde{e}_{\widehat{Y}} \circ \widehat{j}_* = (\psi^*_{\mathscr{K}})^{-1} \circ F$. Recall that $(\psi^*_{\mathscr{K}})^{-1} : \mathrm{Hom}(H_1^{\mathcal{P}}(\Sigma; \mathbb{Z}), \mathbb{Z}) \to \mathrm{Hom}(H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma}; \mathbb{Z}), \mathbb{Z})$ is defined such that $(\psi^*_{\mathscr{K}})^{-1}(\chi) = \chi \circ \psi^{-1}_{\mathscr{K}}$ for any $\chi \in \mathrm{Hom}(H_1^{\mathcal{P}}(\Sigma; \mathbb{Z}), \mathbb{Z})$. Observe that $(\psi^*_{\mathscr{K}})^{-1}$ is an isomorphism because $\psi_{\mathscr{K}}$ is an isomorphism. Hence by composing both sides of

$(\psi^*_{\mathscr{K}})^{-1} \circ \widetilde{e}_Y = (\psi^*_{\mathscr{K}})^{-1} \circ F$ with $\psi^*_{\mathscr{K}}$, we get the equality $\widetilde{e}_Y = F$.

This finishes the proof. □

Corollary 4.3.7. *For an embedding $i : (\Sigma, \mathcal{P}) \hookrightarrow \Sigma_{g,1}$ of partitioned surfaces, $\widetilde{e}_Y$ depends only on the embedding. That is, $\widetilde{e}_Y$ is independent of the vector field Y obtained by restricting the vector field X on $\Sigma_{g,1}$ to Σ.*

Proof. Choose a nonvanishing vector field X_1 different from X on $\Sigma_{g,1}$. Let Y_1 be the restricted vector field of X_1 to Σ.

We aim to show that for any $f \in \mathcal{I}(\Sigma, \mathcal{P})$, $\widetilde{e}_Y(f) = \widetilde{e}_{Y_1}(f)$. By Theorem 4.3.4, we have $\widetilde{e}_X(i_*(f)) = i'_*(\widetilde{e}_Y(f))$ and $\widetilde{e}_{X_1}(i_*(f)) = i'_*(\widetilde{e}_{Y_1}(f))$. By Lemma 2.2.8, $e_X(\widetilde{f})$ defined by $e_X(\widetilde{f})[\gamma] = w_X(\widetilde{f}\gamma) - w_X(\gamma)$ is independent of the choice of X. Since by the definition of $\widetilde{e}_X$ we have $e_X(\widetilde{f}) = \widetilde{e}_X(\widetilde{f})$ for any $\widetilde{f} \in \mathcal{I}(\Sigma_{g,1})$, $\widetilde{e}_X(\widetilde{f})$ is also independent of the choice of X. Therefore, we obtain $\widetilde{e}_X(\widetilde{f}) = \widetilde{e}_{X_1}(\widetilde{f})$.

Now we have two homomorphisms $\widetilde{e}_Y$ and $\widetilde{e}_{Y_1}$ from $\mathcal{I}(\Sigma, \mathcal{P})$ to $\mathrm{Hom}(H_1^{\mathcal{P}}(\Sigma; \mathbb{Z}), \mathbb{Z})$ such that diagram (4.3) commutes. By Proposition 4.3.6, we conclude that $\widetilde{e}_Y(f) = \widetilde{e}_{Y_1}(f)$. □

4.4 Naturality of the Chillingworth Homomorphism

In this section, we show that the Chillingworth homomorphism is natural. We find a relation between the Chillingworth classes of the subsurface Torelli groups and the partitioned Johnson homomorphism. Finally, we give an example to see this relation.

For an element $f \in \mathcal{I}(\Sigma, \mathcal{P})$, let us define the dual of $\widetilde{e}_Y(f)$ which we call the Chillingworth class of f. The algebraic intersection form for $H_1^{\mathcal{P}}(\Sigma; \mathbb{Z})$ gives $t_{(\Sigma,\mathcal{P})}(f)$ defined by:

$$\widehat{i}([\gamma], t_{(\Sigma,\mathcal{P})}(f)) = \widetilde{e}_Y(f)[\gamma].$$

Therefore, we get the Chillingworth homomorphism:

$$t_{(\Sigma,\mathcal{P})} : \mathcal{I}(\Sigma, \mathcal{P}) \to H_1^{\mathcal{P}}(\Sigma; \mathbb{Z}).$$

Let $(\Sigma, \mathcal{P}) \hookrightarrow (\Sigma', \mathcal{P}')$ be an embedding of partitioned surfaces. Fix a symplectic basis $\mathscr{B}$ of $H_1^{\mathcal{P}}(\Sigma; \mathbb{Z})$ defined in Section 4.2. Recall that $H_1^{\mathcal{P}'}(\Sigma'; \mathbb{Z})$ is isomorphic to $H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma}; \mathbb{Z}) \oplus H_1^{\mathcal{P}_1}(V_1; \mathbb{Z}) \oplus H_1^{\mathcal{P}_2}(V_2; \mathbb{Z}) \oplus \cdots \oplus H_1^{\mathcal{P}_n}(V_n; \mathbb{Z})$ as in Remark 4.3.1.

As in the previous section, take a nonvanishing vector field X on Σ'. Restrict X to the subsurface Σ and call the restriction Y.

Lemma 4.4.1. *Let $s_* : H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma}; \mathbb{Z}) \to H_1^{\mathcal{P}'}(\Sigma'; \mathbb{Z})$ be the inclusion map and D be the isomorphism defined in Section 4.2. Then the following diagram commutes:*

$$\begin{CD}
\mathcal{I}(\Sigma, \mathcal{P}) @>{i_*}>> \mathcal{I}(\Sigma', \mathcal{P}') \\
@V{\widetilde{e}_Y}VV @VV{\widetilde{e}_X}V \\
\mathrm{Hom}(H_1^{\mathcal{P}}(\Sigma; \mathbb{Z}), \mathbb{Z}) @>{i'_*}>> \mathrm{Hom}(H_1^{\mathcal{P}'}(\Sigma'; \mathbb{Z}), \mathbb{Z}) \\
@V{D^{-1}}VV @VV{D^{-1}}V \\
H_1^{\mathcal{P}}(\Sigma; \mathbb{Z}) @>{s_* \circ \psi_{\mathscr{K}}}>> H_1^{\mathcal{P}'}(\Sigma'; \mathbb{Z})
\end{CD} \tag{4.5}$$

Proof. We showed in Theorem 4.3.4 that the upper square in diagram (4.5) commutes. Hence our aim is to show that the lower square also commutes. Here, the image of $i'_* : \mathrm{Hom}(H_1^{\mathcal{P}}(\Sigma; \mathbb{Z}), \mathbb{Z}) \to \mathrm{Hom}(H_1^{\mathcal{P}'}(\Sigma'; \mathbb{Z}), \mathbb{Z})$ is defined as the composition of $(\psi^*_{\mathscr{K}})^{-1}$ and r^* defined as before. Let $\chi \in \mathrm{Hom}(H_1^{\mathcal{P}}(\Sigma; \mathbb{Z}), \mathbb{Z})$. Then $i'_*(\chi)[\gamma] = \chi(\psi^{-1}_{\mathscr{K}} r_*)[\gamma]$ for any $[\gamma] \in H_1^{\mathcal{P}'}(\Sigma'; \mathbb{Z})$. Recall that r_* is the projection of $H_1^{\mathcal{P}'}(\Sigma'; \mathbb{Z})$ on $H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma}; \mathbb{Z})$. Commutativity of the lower square is proven by showing commutativity of diagram (4.6).

$$\begin{CD}
\mathrm{Hom}(H_1^{\mathcal{P}}(\Sigma; \mathbb{Z}), \mathbb{Z}) @>{(\psi^*_{\mathscr{K}})^{-1}}>> \mathrm{Hom}(H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma}; \mathbb{Z}), \mathbb{Z}) @>{r^*}>> \mathrm{Hom}(H_1^{\mathcal{P}'}(\Sigma'; \mathbb{Z}), \mathbb{Z}) \\
@A{D}AA @A{D}AA @A{D}AA \\
H_1^{\mathcal{P}}(\Sigma; \mathbb{Z}) @>{\psi_{\mathscr{K}}}>> H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma}; \mathbb{Z}) @>{s_*}>> H_1^{\mathcal{P}'}(\Sigma'; \mathbb{Z})
\end{CD} \tag{4.6}$$

We will analyze 2 cases for the square on the left of diagram (4.6).

Clearly, $\psi^{-1}_{\mathscr{K}} : H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma}; \mathbb{Z}) \to H_1^{\mathcal{P}}(\Sigma; \mathbb{Z})$ preserves the algebraic intersection form, i.e. for any $a, b \in H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma}; \mathbb{Z})$ we have $\widehat{i}(a, b) = \widehat{i}(\psi^{-1}_{\mathscr{K}}(a), \psi^{-1}_{\mathscr{K}}(b))$.

<u>Case 1</u>: For any homology class $[x]$ of $H_1^{\mathcal{P}}(\Sigma; \mathbb{Z})$ with a closed curve representative, we have

$$(\psi^*_{\mathscr{K}})^{-1}(D([x]))[\gamma] = D([x])(\psi^{-1}_{\mathscr{K}}([\gamma])) = \widehat{i}(\psi^{-1}_{\mathscr{K}}[\gamma], [x]) \tag{4.7}$$

and

$$D(\psi_{\mathscr{K}}([x]))[\gamma] = D([x])([\gamma]) = \widehat{i}([\gamma],[x]). \tag{4.8}$$

For simplicity, $[x]$ denotes both an element in $H_1^{\mathcal{P}}(\Sigma;\mathbb{Z})$ and its image under the isomorphism $\psi_{\mathscr{K}}$. In (4.7), $[x]$ is an element of $H_1^{\mathcal{P}}(\Sigma;\mathbb{Z})$, whereas in (4.8), $[x]$ is an element of $H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma};\mathbb{Z})$. Hence, we can consider $[x]$ in (4.7) to be $\psi_{\mathscr{K}}^{-1}([x])$. The following gives the commutativity for $[x]$:

$$\begin{aligned}(\psi_{\mathscr{K}}^*)^{-1}(D([x]))[\gamma] &= \widehat{i}(\psi_{\mathscr{K}}^{-1}[\gamma],[x])\\ &= \widehat{i}(\psi_{\mathscr{K}}^{-1}[\gamma],\psi_{\mathscr{K}}^{-1}([x]))\\ &= \widehat{i}([\gamma],[x])\\ &= D(\psi_{\mathscr{K}}([x]))[\gamma]\end{aligned}$$

for all $[\gamma]$ in $H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma};\mathbb{Z})$.

Case 2: For the basis elements $[h_j^l]$ of $H_1^{\mathcal{P}}(\Sigma;\mathbb{Z})$, we will show that left-hand side of diagram (4.6) commutes. We have

$$(\psi_{\mathscr{K}}^*)^{-1}(D([h_j^l]))[\gamma] = D([h_j^l])(\psi_{\mathscr{K}}^{-1}([\gamma])) = \widehat{i}(\psi_{\mathscr{K}}^{-1}[\gamma],[h_j^l]) \tag{4.9}$$

and

$$D(\psi_{\mathscr{K}}([h_j^l]))[\gamma] = D([h_j^l * k_j^l])([\gamma]) = \widehat{i}([\gamma],[h_j^l * k_j^l]). \tag{4.10}$$

As in Case 1, the last term in (4.9) denotes the algebraic intersection number in $H_1^{\mathcal{P}}(\Sigma;\mathbb{Z})$ and the last term in (4.10) denotes the algebraic intersection number in $H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma};\mathbb{Z})$.

By the same reasoning as in the previous case,

$$\widehat{i}(\psi_{\mathscr{K}}^{-1}[\gamma],[h_j^l]) = \widehat{i}(\psi_{\mathscr{K}}^{-1}[\gamma],\psi_{\mathscr{K}}^{-1}([h_j^l * k_j^l])) = \widehat{i}([\gamma],[h_j^l * k_j^l])$$

for all $[\gamma]$ in $H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma};\mathbb{Z})$.

Finally, for the left-hand side of diagram (4.6) we have $(\psi_{\mathscr{K}}^*)^{-1}\circ D = \psi_{\mathscr{K}}\circ D$. This shows commutativity for the left-hand side of diagram (4.6).

Now our aim is to show that the square in the right-hand side of diagram (4.6) commutes. Recall that $s_* : H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma};\mathbb{Z}) \to H_1^{\mathcal{P}'}(\Sigma';\mathbb{Z})$ is the inclusion map.

Let $[x]$ be an element of $H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma};\mathbb{Z})$. For any homology basis element $[\gamma] \in H_1^{\mathcal{P}'}(\Sigma';\mathbb{Z})$ the lemma follows:

$$r^*(D([x]))[\gamma] = D([x])(r_*([\gamma])) = \widehat{i}(r_*([\gamma]), [x])$$

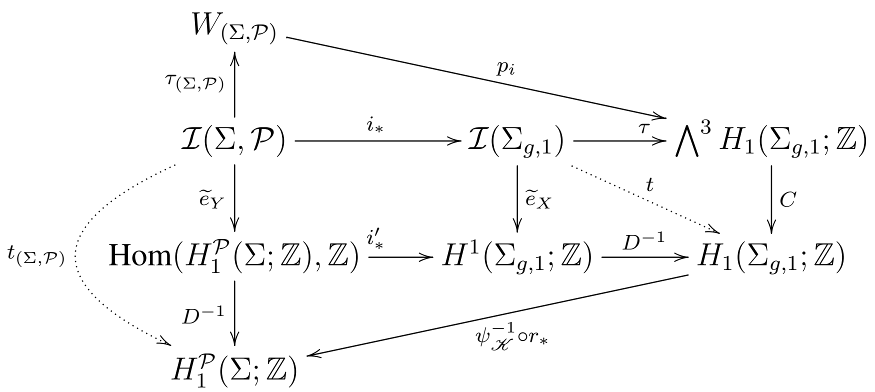
$$D(s_*([x]))[\gamma] = D([x])([\gamma]) = \widehat{i}([\gamma], [x]).$$

If a representative of $[\gamma]$ is contained in the complement of $\widehat{\Sigma}$, $r_*([\gamma]) = 0$. Hence $r^*(D([x]))[\gamma] = 0$. Since a representative of $[x]$ is contained in $\widehat{\Sigma}$, $D(s_*([x]))[\gamma] = 0$ is obtained.

If a representative of $[\gamma]$ is contained in $\widehat{\Sigma}$, $r_*([\gamma]) = [\gamma]$ and so $r^*(D([x]))[\gamma] = D(s_*([x]))[\gamma]$ as desired.

Consequently, we have proven that diagram (4.6) commutes. Since dual maps D are isomorphisms, we obtain that diagram (4.5) is also commutative. We conclude that $s_* \circ \psi_{\mathscr{K}} \circ t_{(\Sigma,\mathcal{P})} = t_{(\Sigma',\mathcal{P}')} \circ i_*$ by diagram (4.5). □

Corollary 4.4.2. *The following diagram is commutative and hence we get the following equality:* $t_{(\Sigma,\mathcal{P})} = \psi_{\mathscr{K}}^{-1} \circ r_* \circ C \circ p_i \circ \tau_{(\Sigma,\mathcal{P})}$, *where* C *is the contraction map. Here,* $\tau_{(\Sigma,\mathcal{P})}$ *is the partitioned Johnson homomorphism defined in Definition 3.2.4,* p_i *is the map defined in Definition 3.2.8.*



(4.11)

Proof. We need to confirm that each triangle and square is commutative. The partitioned Johnson homomorphism is natural (c.f. Theorem 3.2.9). Hence, the upper triangle is commutative. We showed in Theorem 4.3.4 that the left square in the middle part is commutative. The commutativity of the right square in the middle follows from Theorem 2.2.10 and the definition of the Chillingworth class. Finally, the commutativity of the lower triangle follows from Lemma 4.4.1. □

We now provide an example to verify the equality $t_{(\Sigma,\mathcal{P})} = \psi_{\mathscr{K}}^{-1} \circ r_* \circ C \circ p_i \circ \tau_{(\Sigma,\mathcal{P})}$ in Corollary 4.4.2. We assume that there is an embedding $\Sigma \hookrightarrow \Sigma_{g,1}$.

Example 4.4.3. *Let $(\Sigma, \mathcal{P})$ be a partitioned surface of genus g. Choose a basis for the fundamental group of the partitioned surface Σ as in Figure 4.4. Notice that*

$$[\alpha_1, \beta_1] \dots [\alpha_g, \beta_g]\zeta_n \dots \zeta_0 = 1.$$

Note that if $[\alpha_i]$ and $[\beta_i]$ denote the homology classes of α_i and β_i in $H_1^{\mathcal{P}}(\Sigma;\mathbb{Z})$, respectively, the image of the partitioned Johnson homomorphism can be written as

$$\tau_{(\Sigma,\mathcal{P})}(f) = \sum_{i=1}^{g} \tau_{(\Sigma,\mathcal{P})}(f)([\alpha_i]) \otimes [\beta_i] - \tau_{(\Sigma,\mathcal{P})}(f)([\beta_i]) \otimes [\alpha_i],$$

for any $f \in \mathcal{I}(\Sigma, \mathcal{P})$ (c.f. [5, 14]).

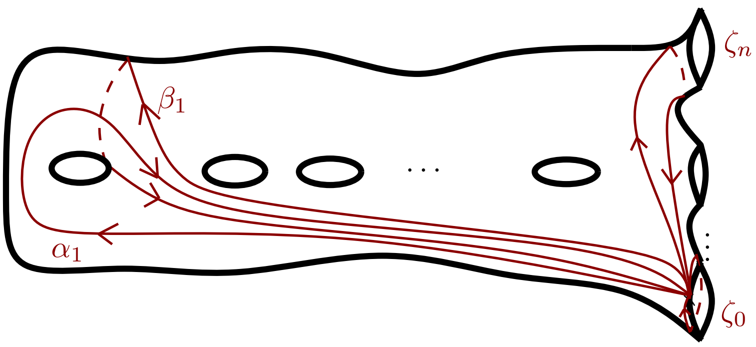


Figure 4.4: An example illustrating basis elements of $\pi_1(\Sigma, *)$.

Now, for simplicity let $(\Sigma, \mathcal{P})$ denote $(\Sigma_{1,2}, \{\{z_0\}, \{z\}\})$ and $i : (\Sigma, \mathcal{P}) \hookrightarrow \Sigma_{g,1}$ be an embedding of partitioned surfaces. Assume that in the complement $\Sigma_{g,1} \setminus \Sigma^\circ$, z bounds a genus k subsurface which does not contain the boundary of $\Sigma_{g,1}$ as a boundary component. Let X be a nonvanishing vector field on $\Sigma_{g,1}$ and Y be the restriction of X to Σ.

We first compute the right-hand side of the equation $t_{(\Sigma,\mathcal{P})} = \psi_{\mathscr{K}}^{-1} \circ r_* \circ C \circ p_i \circ \tau_{(\Sigma,\mathcal{P})}$ in the case that $T_\gamma T_\delta^{-1}$ is a twist about $\mathcal{P}$-bounding pair as in Figure 4.5.

Notice that $T_\gamma T_\delta^{-1}(\alpha) = \alpha\zeta$, so $T_\gamma T_\delta^{-1}(\alpha)\alpha^{-1} = [\alpha, \zeta]\zeta$.

We have $T_\gamma T_\delta^{-1}(\beta) = \zeta^{-1}\beta\zeta$, so $T_\gamma T_\delta^{-1}(\beta)\beta^{-1} = [\zeta^{-1}, \beta]$.

Let $[\alpha], [\beta]$ be the homology classes of α and β in $H_1^{\mathcal{P}}(\Sigma;\mathbb{Z})$, and let z denote the class of ζ in $N(\Sigma, \mathcal{P})$. Therefore, we have

$$\tau_{(\Sigma,\mathcal{P})}(T_\gamma T_\delta^{-1}) = z \otimes [\beta].$$

Since z bounds a genus k subsurface in $\Sigma_{g,1}$, we have $z = \sum_{i=1}^{k}([\alpha_i] \wedge [\beta_i])$ in $\bigwedge^2 H_1(\Sigma_{k,1};\mathbb{Z})$ where $\{[\alpha_1],[\beta_1],\ldots,[\alpha_k],[\beta_k]\}$ is a symplectic basis of $H_1(\Sigma_{k,1};\mathbb{Z})$. Therefore, we have

$$p_i \circ \tau_{(\Sigma,\mathcal{P})}(T_\gamma T_\delta^{-1}) = (\sum_{i=1}^{k}([\alpha_i] \wedge [\beta_i])) \otimes [\beta].$$

After composing with the tensor contraction map C, we have

$$C \circ p_i \circ \tau_{(\Sigma,\mathcal{P})}(T_\gamma T_\delta^{-1}) = 2k[\beta].$$

Finally,

$$\psi_{\mathscr{K}}^{-1} \circ r_* \circ C \circ p_i \circ \tau_{(\Sigma,\mathcal{P})}(T_\gamma T_\delta^{-1}) = 2k[\beta].$$

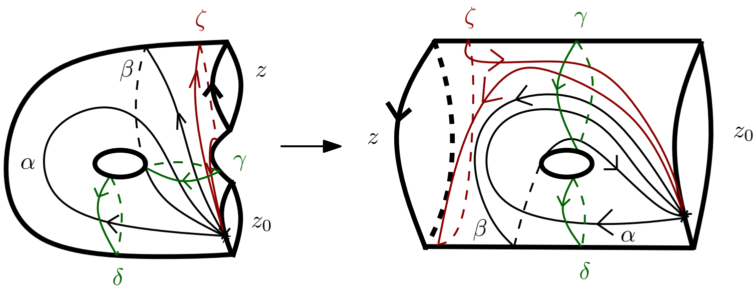


Figure 4.5: A basis for the fundamental group of $\Sigma_{1,2}$ and bounding pair (γ,δ) embedded in $\Sigma_{1,2}$.

We now calculate the left-hand side of the equation $t_{(\Sigma,\mathcal{P})} = \psi_{\mathscr{K}}^{-1} \circ r_* \circ C \circ p \circ \tau_{(\Sigma,\mathcal{P})}$. We need to show that $D^{-1}(\widetilde{e}_Y(T_\gamma T_\delta^{-1})) = 2k[\beta]$. As seen in Figure 4.5, $T_\gamma T_\delta^{-1}$ is the identity on every homology basis element except $[\alpha]$. Recall that Y is the restriction of the nonvanishing vector field X on $\Sigma_{g,1}$ to $\Sigma_{1,2}$. We may assume that Y is orthogonal to α by using Corollary 4.3.7. Therefore, $\widetilde{w}_Y(\alpha) = 0$. Since we have

$$\widetilde{e}_Y(T_\gamma T_\delta^{-1})[\alpha] = \frac{\widetilde{w}_Y(T_\gamma T_\delta^{-1}(\alpha)) - \widetilde{w}_Y(\alpha)}{2},$$

we need to find $\widetilde{w}_Y(T_\gamma T_\delta^{-1}(\alpha))$ to compute $\widetilde{e}_Y(T_\gamma T_\delta^{-1})[\alpha]$. We will use the argument in Lemma 2.2.4. If one wants to calculate $\widetilde{w}_Y(T_\gamma T_\delta^{-1}(\alpha))$ without knowledge of genus in the connected component of $\Sigma_{g,1} \setminus \Sigma_{1,2}^{\circ}$ not containing the boundary component, $\widetilde{w}_Y(z)$ needs to be known. If we did not fix any embedding and a vector field on $\Sigma_{g,1}$, $\widetilde{w}_Y(z)$ could take on any even integer value.

As a corollary of Lemma 2.2.4, Chillingworth states that $w_X(z) = \pm(2k-1)$ where the sign of $w_X(z)$ depends on the orientation of z. In our example, it can be shown that $w_X(z) = 2k-1$ and so $\widetilde{w}_Y(z) = 2(2k-1)$. Let us cut $\Sigma_{1,2}$ along α and $T_\gamma T_\delta^{-1}(\alpha)$. Then consider the pair of pants with one of the boundary components z. Glue discs D_1, D_2 and D_3 along α, $T_\gamma T_\delta^{-1}(\alpha)$ and z, respectively. Extend the nonvanishing vector field Y to the resulting sphere such that in each D_i, $i = 1,2,3$, there will be at most one singularity. By the Poincaré-Hopf Theorem, the sum of the indices of the extended vector field is 2. Since Y is orthogonal to α, in D_1 the extended vector field has a singularity of index 1. By a diffeomorphism, we can consider the gluing discs as unit discs in $\mathbb{R}^2$. There is therefore a notion of constant vector field on the discs. Since $w_X(z) = 2k-1$, the index of the singularity on D_3 is calculated as follows:

Let X' denote a constant vector field on the discs. Recall the definition of $d(Y, X')$ form Subsection 2.2.2.

By Lemma 2.2.6, we have

$$w_{X'}(z) - d(Y, X')[z] = w_Y(z).$$

Since the boundary of D_3 is clockwise oriented, $-d(Y, X')[z]$ corresponds to the index of the singularity in D_3. We find that the singularity v_3 in D_3 has index $2k$. Therefore, index of the singularity v_2 in D_2 needs to be $\mathrm{ind}_{v_2}(Y) = 1-2k$. Then by using the formula

$$w_{X'}(T_\gamma T_\delta^{-1}(\alpha)) - d(Y, X')[T_\gamma T_\delta^{-1}(\alpha)] = w_Y(T_\gamma T_\delta^{-1}(\alpha)),$$

we get $w_Y(T_\gamma T_\delta^{-1}(\alpha)) = 2k$. Here since the boundary of D_2 has the orientation in the counterclockwise direction, $d(Y, X')[T_\gamma T_\delta^{-1}(\alpha)] = \mathrm{ind}_{v_2}(Y) = 1-2k$. As a conclusion, we get $\widetilde{e}_Y(T_\gamma T_\delta^{-1})[\alpha] = 2k$ and so,

$$D^{-1}(\widetilde{e}_Y(T_\gamma T_\delta^{-1})) = t_{(\Sigma,\mathcal{P})}(T_\gamma T_\delta^{-1}) = 2k[\beta].$$

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