MINIMAL EXTENSION OF EINSTEIN'S GRAVITY AT THE QUARTIC ORDER

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submitted by **ESİN KENAR** in partial fulfillment of the requirements for the degree of **Doctor of Philosophy in Physics Department, Middle East Technical University** by,

Prof. Dr. Halil Kalıpçılar
Dean, Graduate School of **Natural and Applied Sciences** ________________

Prof. Dr. Altuğ Özpineci
Head of Department, **Physics** ________________

Prof. Dr. Bayram Tekin
Supervisor, **Physics Dept., METU** ________________

Examining Committee Members:

Prof. Dr. Atalay Karasu
Physics Dept., METU ________________

Prof. Dr. Bayram Tekin
Physics Dept., METU ________________

Assoc. Prof. Dr. İsmet Yurduşen
Mathematics Dept., HU ________________

Assoc. Prof. Dr. Tahsin Çağrı Şişman
Astronautical Engineering Dept., UTAA ________________

Assist. Prof. Dr. Özgür Kelekçi
Electrical-Electronics Engineering Dept.,UTAA ________________

Date: ________________

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Name, Last Name: ESİN KENAR

Signature :

ABSTRACT

MINIMAL EXTENSION OF EINSTEIN'S GRAVITY AT THE QUARTIC ORDER

Kenar, Esin
Ph.D., Department of Physics
Supervisor : Prof. Dr. Bayram Tekin

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We study an extension of Einstein general relativity theory at the quartic order in the curvature. The extended theory has a unique vacuum and a single massless spin-2 excitation about this vacuum, just like general relativity, hence it is called a minimal extension. The extended theory can also be obtained from a particular form of Born-Infeld gravity. We show that the Schwarzschild and Kerr black holes are not exact solutions and the Kretschmann scalar obeys a non-linear wave equation, suggesting that black hole singularities might be avoided.

Keywords: Modified Gravity, Born-Infeld Gravity, Quantum Gravity, Schwarzschild Singularity, Maximally Symmetric Vacuum, Black Hole Solutions, Massless graviton.

ÖZ

EINSTEIN KÜTLEÇEKİMİNİN DÖRDÜNCÜ DERECEDEN MİNİMAL GENİŞLETİLMESİ

Kenar, Esin
Doktora, Fizik Bölümü
Tez Yöneticisi : Prof. Dr. Bayram Tekin

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Einstein genel görelilik teorisinin eğrilik bakımından dördüncü dereceye kadar genişletilmesini çalışmaktayız. Genişletilmiş teori, genel görelilikte olduğu gibi, tek vakuma ve bu vakum etrafında tek bir kütlesiz spin-2 eksitasyona sahiptir ve dolayısıyla minimal genişletme olarak adlandırılır. Genişletilmiş teori ayrıca Born-Infeld kütleçekiminin özel bir formundan da elde edilebilir. Schwarzschild ve Kerr kara deliklerinin kesin çözüm olmadıklarını ve Kretschmann skalerinin doğrusal olmayan dalga denklemini sağladığını gösterdik ve bu sonuç kara delik tekilliklerinin önlenebileceğini öngörmektedir.

Anahtar Kelimeler: Modifiye Kütleçekim, Born-Infeld Kütleçekim, Kuantum Kütleçekim, Schwarzschild Tekilliği, Maksimal olarak Simetrik Vakum, Kara Delik Çözümleri, Kütlesiz Graviton.

to my beloved son

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BI	Born-Infeld
dS	de Sitter
AdS	Anti-de Sitter
SR	Special relativity
GR	General relativity

CHAPTER 1

INTRODUCTION

Einstein's special relativity (SR) theory describes the universe by unifying the space and time then the resulting structure is called as "spacetime". But this spacetime does not have gravity in it and hence it is a flat manifold. Unlike SR, general relativity (GR) theory includes gravitation. According to the GR any energy momentum distribution curves the spacetime and then particles move in this curved geometry which means that these two interpretations are just two perspectives that directly show us the curvature-gravitation relation [1]. This duality relation is expressed by Einstein's equation

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{1.1}$$

Here G is the Newton's gravitation constant and $T_{\mu\nu}$ is the energy momentum tensor of matter fields. We have taken the speed of light to be $c = 1$. The metric is denoted by $g_{\mu\nu}$, a symmetric (0,2) tensor field. $R_{\mu\nu}$ is the Ricci tensor and R is the Ricci scalar. They are notationally shown as $R^{\mu}{}_{\alpha\mu\beta} = R_{\alpha\beta}$, $R^{\mu}{}_{\mu} = R$ and derived from contractions of the Riemann tensor which reads

$$R^{\mu}{}_{\nu\rho\sigma} = \partial_\rho \Gamma^{\mu}_{\sigma\nu} - \partial_\sigma \Gamma^{\mu}_{\rho\nu} + \Gamma^{\mu}_{\rho\lambda}\Gamma^{\lambda}_{\sigma\nu} - \Gamma^{\mu}_{\sigma\lambda}\Gamma^{\lambda}_{\rho\nu}. \tag{1.2}$$

$\Gamma^{\mu}_{\nu\sigma}$ is the Christoffel connection which is given explicitly for a metric-compatible connection as

$$\Gamma^{\mu}_{\nu\sigma} = \frac{1}{2}g^{\mu\rho}(\partial_\nu g_{\sigma\rho} + \partial_\sigma g_{\rho\nu} - \partial_\rho g_{\nu\sigma}), \tag{1.3}$$

and it is symmetric $\Gamma^{\mu}_{\nu\sigma} = \Gamma^{\mu}_{\sigma\nu}$.

The left hand side of the Eq.(1.1) is exactly the Einstein tensor presented in Eq.(A.23) which is

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} \tag{1.4}$$

[1] We must of course note that even without any energy-momentum distribution, due to the non-linearity of gravity, spacetime can be curved.

with vanishing covariant divergence $\nabla^\mu G_{\mu\nu} = 0$. This conservation also implies the conservation of energy-momentum $\nabla^\mu T_{\mu\nu} = 0$.
Since Einstein's equation is a non-linear second order differential equation, it is really hard to solve directly. The Schwarzschild solution is one of the most important exact solutions assuming spherically symmetry. In fact spherical symmetry also leads to a static spacetime. So in GR, the Schwarzschild metric is the unique spherically symmetric static metric. It can be considered as the vacuum $T_{\mu\nu} = 0$ solution or it can be considered as the region outside a spherically symmetric matter distribution.
The Schwarzschild metric written in spherical coordinates is

$$ds^2 = -(1 - \frac{2GM}{r})dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{1.5}$$

where M can be considered as the mass of the source [1]. Besides, if there is a non-zero cosmological constant Λ_0, the Einstein's equation and the Schwarzschild solution are modified as [2]

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda_0 g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{1.6}$$

$$ds^2 = -(1 - \frac{2GM}{r} - \frac{\Lambda_0 r^2}{3})dt^2 + \left(1 - \frac{2GM}{r} - \frac{\Lambda_0 r^2}{3}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \tag{1.7}$$

Turning back to the Eq.(1.5) , $r = 0$ and $r = 2GM$ seem to be singular points. However the terms that contain these points are coordinate dependent and change after a coordinate transformation. Then it is clear that this testing argument is not appropriate for finding the singular points. We should analyse the coordinate independent scalars like R ,$R^{\mu\nu}R_{\mu\nu}$, $R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$ *etc*. For this Schwarzschild metric, calculations indicate that the point $r = 0$ is a real singularity and there is no other singular point [1]. This can be seen from the Kretschmann scalar $R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = \frac{48M^2G^2}{r^6}$ which diverges at $r = 0$. This invariant divergence method works well for the Schwarzschild metric. But unfortunately we can not generalize this approach to all singular geometries. Actually the issue of singularity has been studied for years and called as *singularity problem* in general relativity [8], [9],[26], [27]. As seen in the Schwarzschild metric example, the curvature invariant blows at the singularity point which means that we can not work at this point. Strictly speaking, $r = 0$ does not seem to be the part of the smooth spacetime manifold. But the metric tensor $g_{\mu\nu}$ is defined for all points and we expect to study the whole geometry of the manifold which is surely governed by

$g_{\mu\nu}$. Then some physicists consider the singularity as a boundary but not part of the spacetime. This perspective gives us a manifold with holes (by removing the singular points) resulting another problem about differentiation (due to the neighbourhood issue). After years of study, incompleteness of geodesics was accepted as a singularity signal. S.W. Hawking defines the singularity as [4]: *" A spacetime is singular if it is timelike or null geodesically incomplete, but can not be embedded in a larger spacetime "* . These nonspacelike incomplete geodesics could be divided into two: past-directed and future-directed. Past directed incomplete geodesics, like the Big Bang singularity, can be illustrated up to a point (singularity) in the past while it is geodesically complete in the future. The other type, incomplete geodesics, which are future directed indicate the black holes. The world lines could be drawn up to an end point (black hole singularity). These type of singularities have two main properties which are the singularity point and the horizon. For a singularity evolution period (collapsing of a star), if the singularity is formed before the horizon then we have a naked singularity allowing data transportation between the singularity and an observer outside the singularity. However if the horizon is formed initially, then the final geometry is a black hole with no information flow outside the event horizon. However, trusting the *cosmic censorship conjecture*, we do not expect to see any signal of a naked singularity.

Singularity theorems tells that the general theory of relativity admits singularities for some cases [23]. The singularities are considered to be the shortcomings of general relativity and expected to be overcome with a quantum theory extension. Actually singularity is not the only problem in GR. Beside the fact that the outcomes of GR theory fits well to the experimental results at intermediate scales (solar system etc), there remains some other problems unsolved like the current accelerated expansion of the universe, the rotation speeds of spiral galaxies *etc*. To solve these problems we need to modify Einstein's gravity perhaps even replace with a quantum gravity theory [24]. There are many research avenues along this direction. One such avenue is the Born-Infeld (BI) type gravity that has a unitary massless spin-2 excitation and a unique viable vacuum similar to Einstein's gravity [10], [11], [12].

The BI action is defined as

$$\mathcal{I} = \frac{1}{2\kappa_0\gamma}\int d^4x \left[\sqrt{-\det(g_{\mu\nu}+4\gamma A_{\mu\nu})} - (4\gamma\Lambda_0+1)\sqrt{-g}\right], \tag{1.8}$$

where the $A_{\mu\nu}$ tensor is given as

$$\begin{aligned} A_{\mu\nu} &= R_{\mu\nu} + cS_{\mu\nu} \\ &+ 4\gamma\left(aC_{\mu\rho\nu\sigma}R^{\rho\sigma} + \frac{c+1}{4}R_{\mu\rho}R^{\rho}_{\nu} + \left(\frac{c\,(c+2)}{2} - 2 - b\right)S_{\mu\rho}S^{\rho}_{\nu}\right) \\ &+ \gamma g_{\mu\nu}\left(\frac{9}{8}C_{\rho\sigma\lambda\gamma}C^{\rho\sigma\lambda\gamma} - \frac{c}{4}R_{\rho\sigma}R^{\rho\sigma} + bS_{\rho\sigma}S^{\rho\sigma}\right). \end{aligned} \tag{1.9}$$

Here g is the determinant of the metric tensor $g_{\mu\nu}$. a, b and c are dimensionless parameters. γ is the BI parameter and $\kappa_0 = 8\pi G$. $S_{\mu\nu}$ is the traceless Ricci tensor

$$S_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R, \tag{1.10}$$

$C_{\mu\nu\rho\sigma}$ is the Weyl tensor defined as

$$\begin{aligned} C_{\mu\nu\rho\sigma} &= R_{\mu\nu\rho\sigma} - \frac{1}{(n-2)}\left(R_{\mu\rho}g_{\nu\sigma} + R_{\nu\sigma}g_{\mu\rho} - R_{\mu\sigma}g_{\nu\rho} - R_{\nu\rho}g_{\mu\sigma}\right) \\ &+ \frac{1}{(n-1)(n-2)}R\left(g_{\mu\rho}g_{\nu\sigma} - g_{\nu\rho}g_{\mu\sigma}\right) \end{aligned} \tag{1.11}$$

for n dimensional spacetime. In four dimensions the square of the Weyl tensor reads

$$C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} \equiv R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2. \tag{1.12}$$

The BI theory Eq.(1.8) with Eq.(1.9) was constructed as a theory that extends Einstein's gravity while keeping its important features intact. These are: the uniqueness of the vacuum, the existence of a single massless graviton about this vacuum. Moreover, the theory reproduces Einstein's gravity at the lowest order in the curvature expansion. Further details can be found in the thesis [29] devoted to a detailed study on this theory.

Throughout this thesis we will mostly use geometrical units and take the signature as (-,+,+,+). In some sections we use new notations which are indicated in relevant parts. In general all necessary calculations are explicitly shown and placed in the chapters or the appendix sections. Basically the outline of the thesis is as follows: In Chapter 2 we give some background information on relativity. Actually some other chapters also include such calculations especially the appendices. In Chapter 3 we firstly give the field equations of the general extended $f(R^{\mu\nu}_{\alpha\beta})$ action that is formed by the Riemann tensor and its contractions. Then we write our special action where we fix $a = 0$, $b = -5/2$ and $c = -1$ and calculate the field equations and the trace. Secondly we show the detailed derivation of the Riemann tensor in the maximally symmetric spacetime then we calculate the vacuum equation in our theory. We show that we have a unique viable vacuum and a massless spin-2 excitation using the linearization method. We also calculate the effective Newton's constant. In Chapter 3 we do the basic calculations of the Schwarzschild black holes. In Chapter 4 we give the Ricci flat solutions and black hole search of the BI theory. We show that Gauss-Bonnet invariant (which is the Kretschmann scalar in this section) satisfies a wavelike equation and so the Schwarzschild solution is not included in BI theory. We also argue that other types of black hole solutions are allowed to study deeply. In this chapter, we also discuss the approximate spherically symmetric solutions.

This thesis includes the detailed calculations of a collaborative study [30].

While writing this thesis we used the computer programmes: LaTeX for typing, Mathematica for computing, Desmos and Paint for drawing figures.

CHAPTER 2

BACKGROUND ON CURVED SPACETIME AND GENERAL RELATIVITY

Assuming a spherically symmetric metric for a compact object we can find an exact solution of Einstein's field equations. As a starting point we introduce the spherically symmetric Minkowskian metric which is

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2, \tag{2.1}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. In order to generalize this simple metric we introduce coefficients (C_1, C_2, C_3 and C_4) preserving the spherical symmetry. Then we have [3]

$$ds^2 = -C_1(r,t)dt^2 + C_2(r,t)dr^2 + C_3(r,t)drdt + C_4(r,t)r^2 d\Omega^2. \tag{2.2}$$

If each of the metric components does not depend on time we get

$$ds^2 = -C_1(r)dt^2 + C_2(r)dr^2 + C_3(r)drdt + C_4(r)r^2 d\Omega^2, \tag{2.3}$$

which is called a *stationary* metric. After redefining the time coordinate [25], this metric can be expressed as

$$ds^2 = -C_1(r)dt^2 + C_2(r)dr^2 + C_4(r)r^2 d\Omega^2. \tag{2.4}$$

These metrics are named as *static* metrics. Although we only assumed a spherically symmetric and stationary metric, we additionally obtained the staticity characteristic finally, with the added assumption that $t \to -t$ is a symmetry of the spacetime. Hence a static metric is a stationary metric with a time reflection symmetry. In any gravity theory, due to their high symmetry, these are the metrics one studies first to understand the properties of the given theory.

2.1 Schwarzschild coordinates[1]

As the most important example of a static metric, let us study the Schwarzschild solution of Einstein's gravity.

The Schwarzschild solution in spherical coordinates is described by the metric

$$ds^2 = -(1 - \frac{2GM}{r})dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \tag{2.5}$$

Here t and r are not the ordinary coordinates that we are familiar in flat spacetime. r is the area coordinate that ensures the area of a sphere with radius r at a fixed time as $4\pi r^2$. We can see the distinction with simple calculations below.

Let us start with the radius analysis. The distance between the spheres with different radii can be calculated by taking $dt = d\theta = d\phi = 0$;

$$\triangle r = \int_{r_1}^{r_2} ds = \int_{r_1}^{r_2} \left(1 - \frac{2GM}{r}\right)^{-1/2} dr. \tag{2.6}$$

Carrying out the integral we find

$$\begin{aligned} \triangle r &= \left(r\sqrt{1-2GM/r} + 2GM \ln\left(\sqrt{r-2GM} + \sqrt{r}\right)\right)\Big|_{r_2}^{r_1} \\ &= r_2\sqrt{1-2GM/r_2} - r_1\sqrt{1-2GM/r_1} \\ &+ 2GM \ln\left(\sqrt{r_2 - 2GM} + \sqrt{r_2}\right) \\ &- 2GM \ln\left(\sqrt{r_1 - 2GM} + \sqrt{r_1}\right). \end{aligned} \tag{2.7}$$

As seen above, the distance $\triangle r$ is different from the distance measured in flat spacetime $(\triangle r)_{flat} = r_2 - r_1$ $(\triangle r > (\triangle r)_{flat})$. Of course this is due to the spacetime being curved.

Now let us study the time intervals using the Schwarzschild metric. Taking $dr = d\theta = d\phi = 0$ we get

$$\triangle t = \left(1 - \frac{2GM}{r}\right)^{1/2} dt. \tag{2.8}$$

Time differences are measured to be $(\triangle t)_{flat} = t_2 - t_1$ in a flat spacetime. Since $r > 2GM$, we conclude that $\triangle t < (\triangle t)_{flat}$. The measurement differences $(\triangle t - (\triangle t)_{flat})$ and $(\triangle r - (\triangle r)_{flat})$ decrease when we move away from the source as expected due to the asymptotical flatness. So in some sense (t, r, θ, ϕ) coordinates of the Schwarzschild metric are just labelling the points of spacetime. But they match with

[1] In this section we generally follow the book [16].

the Minkowski metric and the usual notion at asymptotic infinity. The notion of asymptotic flatness can be defined more rigorously, but in this thesis we will not need that.

2.2 Geometric Unit System, World Lines and Lightcones[2]

When we consider the events of a particle through some period of time and add these points we get its *world line*. Light has a constant velocity; $c = 3.10^{10}$ cm/sec. We can use this number as a scaling. For example for a distance of d cm, we do a scaling such that $d' = \frac{d\ cm}{c\ cm/sec} = \frac{d}{3.10^{10}} sec$. Then the unit of length is second and the velocities are unitless. This scaling is called the geometric unit system. $c = 1$ in geometric units. As a good example, let us consider a flash. We light the flash standing at the origin at $t = 0$ and take pictures at some time t. In a 3 dimensional space, the photons coming out of the flash form a sphere with radius t

$$r^2 = x^2 + y^2 + z^2 = t^2. \tag{2.9}$$

For simplicity let us take $z = 0$. Then we have a circle with radius t governed by $x^2 + y^2 = t^2$. Now if we include the time coordinate and plot the (x, y, t)- spacetime, then we have a cone (Figure 2.1). If we take $y = 0$ and plot in (x, t)- spacetime we get Figure 2.2 and this trajectory is called as future directed *lightcone*. This is the path of a photon emerging from the origin in a 2 dimensional spacetime in geometrical units. When we continue to draw the lines for the region $x < 0$ we obtain the past directed lightcone together with the future directed one (Figure 2.3). The final form of the lightcone is illustrated in Figure 2.4.

The events lying inside the past lightcone can affect the event at the origin; the other regions can not affect this event. This is due to the fact that the light has a fixed and so restricted velocity. The world lines outside the lightcone have a slope greater than 1, then these world lines are spacelike whereas the inside region corresponds to timelike particles. As stated before the cone itself is the world line of the photon, which is also called null or lightlike trajectory.

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[2] In this section we generally follow the books [15], [16] .

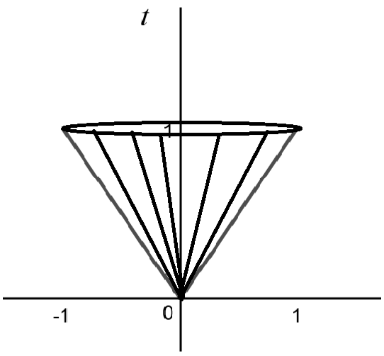

Figure 2.1: The light coming out of a flash (at the origin 0) travels in (x, y, t)- spacetime forming a cone.

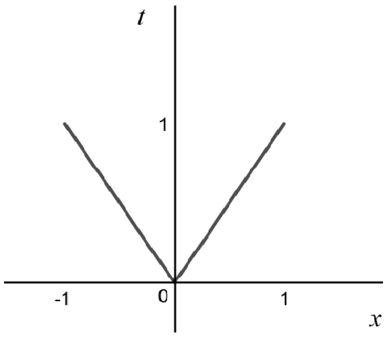

Figure 2.2: The light coming out of a flash (at the origin 0) travels in (x, t)- spacetime forming a future directed lightcone.

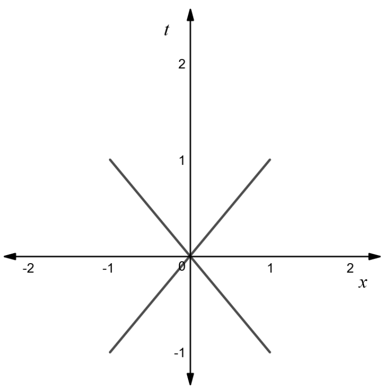

Figure 2.3: Future directed lightcone can be drawn continuously to include the past.

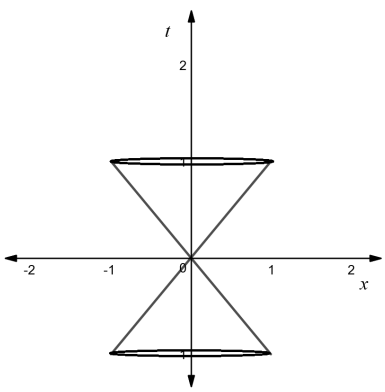

Figure 2.4: The lightcone of the photons .

2.3 The Cosmological constant

2.3.1 Energy-Momentum Tensor[3]

For a particle moving with velocity v with mass m, its energy can be expressed as

$$E = \frac{mc^2}{\sqrt{1 - v^2/c^2}}, \tag{2.10}$$

where $v^2 = v_x^2 + v_y^2 + v_z^2$. Let us consider the noninteracting particles (dust) in a unit volume. Assuming that there are n particles in this unit volume (and taking $c = 1$), we can write the energy density as

$$T^{00} = \frac{nm}{\sqrt{1 - v^2}}. \tag{2.11}$$

Now we calculate the energy flux density. It is the flow of energy through unit area per unit time t. We choose the coordinate x for example for a flow of distance l:

$$\text{Energy flux density} = \frac{nm}{\sqrt{1 - v^2}} \frac{(area)l}{(area)t} = \frac{nm}{\sqrt{1 - v^2}} v_x, \tag{2.12}$$

here v_x is the velocity in x direction. For any direction i we simply write the energy flux density as

$$T^{0i} = T^{i0} = \frac{nm}{\sqrt{1 - v^2}} v^i. \tag{2.13}$$

This is also defined to be the density of momentum. And the momentum flux can be written as

$$T^{ij} = T^{ji} = \frac{nm}{\sqrt{1 - v^2}} v^i v^j. \tag{2.14}$$

This is the flux of momentum in the i-direction flowing through j-direction. Then we have a T-matrix that has 16 components formed by $T^{00}, T^{0i}, T^{i0}, T^{ij}, T^{ji}$ and T^{ii}. We will simply show that this matrix is a tensor. $T^{\mu\nu}$ can be expressed as

$$T^{\mu\nu} = n_0 m u^\mu u^\nu, \tag{2.15}$$

here n_0 is the proper particle density, $n_0 = n\sqrt{1 - v^2}$ and u^μ is the four-velocity , $u^\mu = \left(\frac{1}{\sqrt{1-v^2}}, \frac{v_x}{\sqrt{1-v^2}}, \frac{v_y}{\sqrt{1-v^2}}, \frac{v_z}{\sqrt{1-v^2}} \right)$. n_0 is a scalar [4] and $u^\mu u^\nu$ is a tensor [5]. Then our

[3] In this section we generally summarise the Ohanian's discussions[2].
[4] n_0 is measured in the rest frame of particles.
[5] u^μ is a vector and here we have a multiplication of vectors which yields a tensor.

$T^{\mu\nu}$ matrix is a tensor with rank 2. The $T^{\mu\nu}$ tensor is called as the energy-momentum (or stress) tensor and is symmetric, $T^{\mu\nu} = T^{\nu\mu}$. We can rewrite Eq.(2.15) such that

$$T^{\mu\nu} = \rho_0 u^\mu u^\nu \tag{2.16}$$

where $\rho_0 = n_0 m$ is the proper mass density. $T^{\mu\nu}$ obeys the conservation law such that

$$\nabla_\mu T^{\mu\nu} = 0. \tag{2.17}$$

This last relation can be assumed to be correct or as $T_{\mu\nu}$ couples to Einstein's gravity, it follows from the theory.

2.4 Perfect Fluids[6]

In the rest frame of the fluid the velocity is zero, $\vec{u} = 0$. Still particles may interact with each other and have thermal energies. Surely we can read the energy-momentum tensor to observe these kinds of properties for more realistic fluids. We list the physical meanings of the components of the $T^{\mu\nu}$ tensor in zero-momentum frame :
T^{00}: total energy density,
T^{0i}: energy flux due to heat conduction,
T^{i0}: momentum density due to heat conduction,
T^{ij}: momentum flux, specifically T^{ii} denotes the isotropic pressure and T^{ij} is the viscous stress.
Energy-momentum tensor for a perfect fluid in the rest frame is written as

$$T^{\mu\nu} = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{bmatrix}.$$

In the rest frame we have $u^\mu = (1, 0, 0, 0)$. Then we can write the $T^{\mu\nu}$ tensor using u^μ;

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + p\eta^{\mu\nu}. \tag{2.18}$$

[6] In this section we generally follow the reference book [14].

For a general form we simply change $\eta^{\mu\nu}$ by $g^{\mu\nu}$ to arrive at the energy-momentum tensor of a perfect fluid:

$$T^{\mu\nu} = (\rho + p)u^{\mu}u^{\nu} + pg^{\mu\nu}. \tag{2.19}$$

2.5 Cosmological Field Equations[7]

In the Newtonian theory, the gravity field equation is expressed by the Poisson's equation:

$$\vec{\nabla}^2\Phi = 4\pi G\rho, \tag{2.20}$$

where Φ is the gravitational potential caused by the gravitational matter density ρ. In Einstein's theory, Poisson's equation can be generalised as follows. On the right hand side (RHS) of Eq.(2.20) matter density can be replaced by the full energy-momentum tensor. In the weak field approximation, for noninteracting particles with mass density ρ, we have

$$g_{00} = -(1 + 2\Phi). \tag{2.21}$$

Using the above equation and $T_{00} = \rho$, Eq.(2.20) becomes

$$\vec{\nabla}^2 g_{00} = -8\pi G T_{00}. \tag{2.22}$$

Now on the left hand side of this equation, we have the second derivatives of the metric. For a covariant equation we surely need a tensor with additional properties:

1. It must be symmetric rank-2 tensor since on the RHS we have $T^{\mu\nu}$.
2. It must be conserved ($\nabla_{\mu}T^{\mu\nu} = 0$) .
3. It must consist of derivatives of the metric up to the second order.

The Einstein tensor $G_{\mu\nu}$ (derived in Appendix A.1) satisfies these expectations. Then the Einstein field equations are written as

$$G_{\mu\nu} = \kappa T_{\mu\nu} \tag{2.23}$$

[7] In this section we generally follow the reference book [14].

with $\kappa = 8\pi G$. When $T_{\mu\nu} = 0$, we have the vacuum field equations

$$R_{\mu\nu} = 0, \tag{2.24}$$

which are called as Ricci-flat metrics.

Suppose we add a new term to the Einstein tensor: a constant Λ_0 multiplied by the metric tensor $g_{\mu\nu}$. Then we get

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda_0 g_{\mu\nu}. \tag{2.25}$$

Λ_0 is called as the bare cosmological constant and the new field equations are cosmological Einstein equations written as:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda_0 g_{\mu\nu} = \kappa T_{\mu\nu}. \tag{2.26}$$

We still have $\nabla^\mu G_{\mu\nu} = 0$ since $\nabla^\mu g_{\mu\nu} = 0$.

Now let us do a simple search for the physical meaning of the cosmological constant. In the weak field limit of the cosmological Einstein equations we have

$$\vec{\nabla}^2\Phi = 4\pi G\rho - \Lambda_0, \tag{2.27}$$

which yields

$$\vec{g} = -\vec{\nabla}\Phi = -\frac{GM}{r^2}\hat{r} + \frac{\Lambda_0 r}{3}\hat{r}. \tag{2.28}$$

Then in addition to the usual attractive field $\left(-\frac{GM}{r^2}\right)$ we obtain a positive term $\left(\frac{\Lambda_0 r}{3}\right)$ meaning a repulsive field, for $\Lambda_0 > 0$.

Lorentz invariant vacuum dictates that [1], [25]

$$p_{vac} = -\rho_{vac} \tag{2.29}$$

for the vacuum. Observe that otherwise $T^{\mu\nu}$ has a $u^\mu u^\nu$ part and the existence of a u^μ vector dictates a choice of a Lorentz frame and hence non-invariance of the vacuum which we do not want or observe. Then the energy momentum tensor is

$$T^{\mu\nu}_{vac} = p_{vac}\, g^{\mu\nu} = \rho_{vac}\, g^{\mu\nu}. \tag{2.30}$$

Then we need to add this $T^{\mu\nu}_{vac}$ to $T^{\mu\nu}$ of matter ($T^{\mu\nu}_M$):

$$T^{\mu\nu} = T^{\mu\nu}_{vac} + T^{\mu\nu}_M. \tag{2.31}$$

We rewrite the field equations:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa\left((T_{\mu\nu})_M - \rho_{vac}\, g_{\mu\nu}\right). \tag{2.32}$$

Then the bare cosmological constant Λ_0 could be related to vacuum energy density as

$$\rho_{vac} = \frac{\Lambda_0}{\kappa}. \tag{2.33}$$

Besides, effective cosmological constant (Λ_{eff}) [5] [6][25] is defined to be

$$\Lambda_{eff} = \Lambda_0 + \Lambda_{other} \tag{2.34}$$

where Λ_{other} denotes other additional effects contributed by for example scalar fields or zero-point energies of quantum fields. Theoretically we estimate effective cosmological constant adding all contributions. When we devide this estimated value $(\Lambda_{theo.})$ to the observed effective cosmological constant $(\Lambda_{obs.})$ we find that [5]

$$\frac{\Lambda_{theo.}}{\Lambda_{obs.}} \approx 10^{120}. \tag{2.35}$$

This unfortunate misfit is called the cosmological constant problem.
Here we finalise the introductory background information on Einstein's general relativity theory and with the next chapter we will proceed with the study of modified gravity starting with the field equations of the quartic theory.

CHAPTER 3

VACUUM AND SPECTRUM OF THE QUARTIC GRAVITY THEORY

Aiming a modified gravity theory the procedure is as follows: We add the functions of the Riemann tensor with its contractions to the Einstein-Hilbert action (with a cosmological constant). Respecting the minimality condition we do not consider the derivatives of the Riemann tensor. Then our action consists of the Riemann tensor and its contractions and any powers of these. In order to simplify the following calculations we prefer to study with a (2,2) Riemann tensor (${R^{\sigma\rho}}_{\mu\nu}$) instead of a standard (1,3) Riemann tensor (${R^{\sigma}}_{\rho\mu\nu}$) . Besides, we notationally write (2,2) Riemann tensor as $R^{\sigma\rho}_{\mu\nu}$ just for practical simplicity. Then the generic theory we are interested in has the action

$$\mathcal{I} = \frac{1}{\kappa_0}\int d^4x\sqrt{-g}f(R^{\mu\nu}_{\alpha\beta}), \tag{3.1}$$

where f is a smooth function of its argument of course, what is tacitly assumed here is that we have a diffeomorphism invariant theory.

We will use the variation method to find the field equations for this action. The usual contraction rules hold for our preferred form of the Riemann tensor $R^{\sigma\rho}_{\mu\nu}$. When we contract the first and third indices of the Riemann tensor we get the Ricci tensor. This can be seen with a simple calculation: Starting with the following

$$R^{\mu\nu}_{\alpha\beta} = g^{\nu\sigma}{R^{\mu}}_{\sigma\alpha\beta} \tag{3.2}$$

and multiplying both sides with δ^{α}_{μ} we get

$$R^{\mu\nu}_{\mu\beta} = g^{\nu\sigma}R_{\sigma\beta}, \tag{3.3}$$

$$R^{\mu\nu}_{\mu\beta} = R^{\nu}_{\beta}, \tag{3.4}$$

so there is no need for the metric tensor for contraction. And for the Ricci scalar we have $R^{\nu}_{\nu} = R$.

Considering $\mathcal{I}$ as a functional of the metric tensor and $R^{\mu\nu}_{\alpha\beta}$ and taking the variation of Eq.(3.1) we get

$$\begin{aligned}\delta\mathcal{I} &= \frac{1}{\kappa_0}\int d^4x\left[(\delta\sqrt{-g})f+\sqrt{-g}\delta f(R^{\mu\nu}_{\alpha\beta})\right]\\ &= \frac{1}{\kappa_0}\int d^4x\left[(\delta\sqrt{-g})f+\sqrt{-g}\frac{\partial f}{\partial R^{\mu\nu}_{\rho\sigma}}\delta R^{\mu\nu}_{\rho\sigma}\right],\end{aligned} \tag{3.5}$$

where we used the chain rule in the second line.
Using the identity $\delta\sqrt{-g}=-\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}$ and $\delta R^{\mu\nu}_{\rho\sigma}$ given in Eq.(A.30) we get

$$\begin{aligned}\delta\mathcal{I} &= \frac{1}{\kappa_0}\int d^4x\left[-\frac{1}{2}g_{\mu\nu}\sqrt{-g}f\left(R^{\alpha\beta}_{\rho\sigma}\right)\delta g^{\mu\nu}\right]\\ &+ \frac{1}{2\kappa_0}\int d^4x\,\sqrt{-g}\frac{\partial f}{\partial R^{\mu\nu}_{\rho\sigma}}\Big(g_{\alpha\rho}\nabla_\sigma\nabla^\nu-g_{\alpha\sigma}\nabla_\rho\nabla^\nu\Big)\delta g^{\mu\alpha}\\ &+ \frac{1}{2\kappa_0}\int d^4x\,\sqrt{-g}\frac{\partial f}{\partial R^{\mu\nu}_{\rho\sigma}}\Big(g_{\alpha\sigma}\nabla_\rho\nabla^\mu-g_{\alpha\rho}\nabla_\sigma\nabla^\mu\Big)\delta g^{\alpha\nu}\\ &- \frac{1}{2\kappa_0}\int d^4x\,\sqrt{-g}\frac{\partial f}{\partial R^{\mu\nu}_{\rho\sigma}}\Big(R_{\rho\sigma}{}^{\nu}{}_{\alpha}\delta g^{\mu\alpha}-R_{\rho\sigma}{}^{\mu}{}_{\alpha}\delta g^{\alpha\nu}\Big).\end{aligned} \tag{3.6}$$

Now we can manipulate the following term

$$\begin{aligned}\int d^4x\,\sqrt{-g}\frac{\partial f}{\partial R^{\mu\nu}_{\rho\sigma}}g_{\alpha\rho}\nabla_\sigma\nabla^\nu\delta g^{\mu\alpha} &= \int d^4x\,\sqrt{-g}\Big[\nabla_\sigma\left(\frac{\partial f}{\partial R^{\mu\nu}_{\rho\sigma}}g_{\alpha\rho}\nabla^\nu\delta g^{\mu\alpha}\right)\\ &- \nabla_\sigma\left(\frac{\partial f}{\partial R^{\mu\nu}_{\rho\sigma}}g_{\alpha\rho}\right)\nabla^\nu\delta g^{\mu\alpha}\Big]\\ &= -\int d^4x\,\sqrt{-g}\nabla_\sigma\left(\frac{\partial f}{\partial R^{\mu\nu}_{\rho\sigma}}g_{\alpha\rho}\right)\nabla^\nu\delta g^{\mu\alpha}\\ &= -\int d^4x\,\sqrt{-g}\Big[\nabla^\nu\Big[\nabla_\sigma\left(\frac{\partial f}{\partial R^{\mu\nu}_{\rho\sigma}}g_{\alpha\rho}\right)\delta g^{\mu\alpha}\Big]\\ &- \nabla^\nu\nabla_\sigma\left(\frac{\partial f}{\partial R^{\mu\nu}_{\rho\sigma}}g_{\alpha\rho}\right)\delta g^{\mu\alpha}\Big]\\ &= \int d^4x\,\sqrt{-g}\nabla^\nu\nabla_\sigma\left(\frac{\partial f}{\partial R^{\mu\nu}_{\rho\sigma}}g_{\alpha\rho}\right)\delta g^{\mu\alpha}\Big],\end{aligned} \tag{3.7}$$

where we used the Gauss' theorem. After doing some other necessary manipulations we arrive at the field equations as

$$\begin{aligned}&\frac{1}{2}\Big(g_{\nu\rho}\nabla^\lambda\nabla_\sigma-g_{\nu\sigma}\nabla^\lambda\nabla_\rho\Big)\frac{\partial f}{\partial R^{\mu\lambda}_{\rho\sigma}}-\frac{1}{2}\Big(g_{\mu\rho}\nabla^\lambda\nabla_\sigma-g_{\mu\sigma}\nabla^\lambda\nabla_\rho\Big)\frac{\partial f}{\partial R^{\lambda\nu}_{\rho\sigma}}\\ &\qquad -\frac{1}{2}\left(\frac{\partial f}{\partial R^{\mu\lambda}_{\rho\sigma}}R_{\rho\sigma}{}^{\lambda}{}_{\nu}-\frac{\partial f}{\partial R^{\lambda\nu}_{\rho\sigma}}R_{\rho\sigma}{}^{\lambda}{}_{\mu}\right)-\frac{1}{2}g_{\mu\nu}f\left(R^{\alpha\beta}_{\rho\sigma}\right)=0.\end{aligned} \tag{3.8}$$

These are the field equations for a most general $f(R^{\mu\nu}_{\alpha\beta})$ action Eq.(3.1). But we will be interested in a subclass of these theories, which are the Born-Infeld theories given by the action Eq.(1.8) and Eq.(1.9). This action contains 3 dimensionless parameters a, b and c to be determined by the theoretical arguments or experiments. Now we fix these parameters as $c = -1$, $a = 0$, $b = -\frac{5}{2}$. The motivation about this fixing is that with this special action we obtain a much simpler theory compared to the generic BI gravity. Understanding the properties of this theory will help us understand the generic BI gravity.

The reduced $A_{\mu\nu}$ tensor becomes

$$A_{\mu\nu} = R_{\mu\nu} - S_{\mu\nu} + \gamma g_{\mu\nu}\left(\frac{9}{8}C_{\rho\sigma\lambda\gamma}C^{\rho\sigma\lambda\gamma} + \frac{1}{4}R_{\rho\sigma}R^{\rho\sigma} - \frac{5}{2}S_{\rho\sigma}S^{\rho\sigma}\right). \tag{3.9}$$

When we substitute $S_{\mu\nu}$ Eq.(1.10) and $C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$ Eq.(1.12) into $A_{\mu\nu}$, we find

$$A_{\mu\nu} = \frac{1}{4}g_{\mu\nu}R + \gamma g_{\mu\nu}\left(\frac{9}{8}R_{\rho\sigma\lambda\gamma}R^{\rho\sigma\lambda\gamma} - 2R_{\rho\sigma}R^{\rho\sigma} + \frac{3}{8}R^2 - \frac{5}{2}S_{\rho\sigma}S^{\rho\sigma}\right). \tag{3.10}$$

Calculating the square of the traceless Ricci tensor as

$$\begin{aligned} S_{\rho\sigma}S^{\rho\sigma} &= (R_{\rho\sigma} - \frac{1}{4}g_{\rho\sigma}R)(R^{\rho\sigma} - \frac{1}{4}g^{\rho\sigma}R) \\ &= R_{\rho\sigma}R^{\rho\sigma} - \frac{1}{4}R^2, \end{aligned} \tag{3.11}$$

and substituting into $A_{\mu\nu}$ tensor Eq.(3.10) we have

$$A_{\mu\nu} = \frac{1}{4}g_{\mu\nu}R + \gamma g_{\mu\nu}\left[\frac{9}{8}\left(R_{\rho\sigma\lambda\gamma}R^{\rho\sigma\lambda\gamma} - 4R_{\rho\sigma}R^{\rho\sigma} + R^2\right) - \frac{1}{8}R^2\right], \tag{3.12}$$

which can be simply written as

$$A_{\mu\nu} = \frac{1}{4}g_{\mu\nu}R + \gamma g_{\mu\nu}\left(\frac{9}{8}\mathcal{G} - \frac{1}{8}R^2\right). \tag{3.13}$$

Here $\mathcal{G}$ is the Gauss-Bonnet invariant which is given as

$$\mathcal{G} \equiv R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}. \tag{3.14}$$

Finding the $A_{\mu\nu}$ tensor for a special case, now we substitute into the action Eq.(1.8) and get

$$\begin{aligned} \mathcal{I} = \frac{1}{2\kappa_0\gamma}\int d^4x\Bigg\{&\sqrt{-\det\left[g_{\mu\nu}\left(1 + \gamma R + 4\gamma^2\left(\frac{9}{8}\mathcal{G} - \frac{1}{8}R^2\right)\right)\right]} \\ &- (4\gamma\Lambda_0 + 1)\sqrt{-g}\Bigg\}. \end{aligned} \tag{3.15}$$

Using $\det(g_{\mu\nu}\, a) = a^4 g$ we have

$$\mathcal{I} = \frac{1}{2\kappa_0\gamma}\int d^4x\Bigg\{\sqrt{-g\left(1+\gamma R+4\gamma^2\left(\frac{9}{8}\mathcal{G}-\frac{1}{8}R^2\right)\right)^4} \\ -(4\gamma\Lambda_0+1)\sqrt{-g}\Bigg\}. \tag{3.16}$$

Simplifying the above action and defining $\lambda_0 = \gamma\Lambda_0$ we get the quartic theory

$$\mathcal{I} = \frac{1}{2\kappa_0\gamma}\int d^4x\sqrt{-g}\left\{\left[1+\gamma R+\frac{9}{2}\gamma^2\left(\mathcal{G}-\frac{1}{9}R^2\right)\right]^2-(4\lambda_0+1)\right\}. \tag{3.17}$$

We can recast this more explicitly as

$$\mathcal{I} = \frac{1}{\kappa_0}\int d^4x\sqrt{-g}\Bigg\{R-\frac{1}{2}\gamma^2R^3+\frac{1}{8}\gamma^3R^4+\frac{9}{2}\gamma\mathcal{G}-\frac{9}{2}\gamma^2R\mathcal{G} \\ -\frac{9}{4}\gamma^3\mathcal{G}R^2+\frac{81}{8}\gamma^3\mathcal{G}^2-\frac{2}{\gamma}\lambda_0\Bigg\}. \tag{3.18}$$

Here a summary of the above calculations could be given as follows: We started with a somehow long and determinantal action. Then we took the dimensionless parameters as $a = 0$, $b = -\frac{5}{2}$, $c = -1$. And after some manipulations we now have a purely polynomial and so simpler action as a function of R and Gauss-Bonnet invariant $\mathcal{G}$. Next we can write this action Eq.(3.18) in a notationally simpler form such that

$$\mathcal{I} = \frac{1}{2\kappa_0}\int d^4x\sqrt{-g}\mathcal{F}(R,\mathcal{G})\,, \tag{3.19}$$

and

$$2\gamma\mathcal{F} \equiv \left(1+\gamma R-\frac{1}{2}\gamma^2(R^2-9\mathcal{G})\right)^2-4\lambda_0-1. \tag{3.20}$$

Here we use the notation $\mathcal{F} \equiv \mathcal{F}(R,\mathcal{G})$ for simplicity.

Now let us calculate the terms in Eq.(3.8) separately to make the calculations clear and find the field equations for our special action Eq.(3.19). We need to calculate the partial derivative of $\mathcal{F}$ with respect to $R^{\mu\lambda}_{\rho\sigma}$ which is

$$\frac{\partial\mathcal{F}}{\partial R^{\mu\lambda}_{\rho\sigma}} = \frac{\partial\mathcal{F}}{\partial\mathcal{G}}\frac{\partial\mathcal{G}}{\partial R^{\mu\lambda}_{\rho\sigma}}+\frac{\partial\mathcal{F}}{\partial R}\frac{\partial R}{\partial R^{\mu\lambda}_{\rho\sigma}}. \tag{3.21}$$

Let us now find the derivative of $\mathcal{G}$ with respect to $R^{\mu\lambda}_{\rho\sigma}$:

$$
\begin{aligned}
\frac{\partial \mathcal{G}}{\partial R^{\mu\lambda}_{\rho\sigma}} &= 2R^{cd}_{ab}\frac{\partial R^{ab}_{cd}}{\partial R^{\mu\lambda}_{\rho\sigma}} - 8R^{d}_{b}\delta^{c}_{a}\frac{\partial R^{ab}_{cd}}{\partial R^{\mu\lambda}_{\rho\sigma}} + 2R\frac{\partial R}{\partial R^{\mu\lambda}_{\rho\sigma}} \\
&= 2R^{cd}_{ab}\left[\frac{1}{4}\delta^{a}_{\mu}\delta^{b}_{\lambda}\left(\delta^{\rho}_{c}\delta^{\sigma}_{d} - \delta^{\sigma}_{c}\delta^{\rho}_{d}\right) - \frac{1}{4}\delta^{a}_{\lambda}\delta^{b}_{\mu}\left(\delta^{\rho}_{c}\delta^{\sigma}_{d} - \delta^{\sigma}_{c}\delta^{\rho}_{d}\right)\right] \\
&- 8R^{d}_{b}\left[\frac{1}{4}\delta^{c}_{\mu}\delta^{b}_{\lambda}\left(\delta^{\rho}_{c}\delta^{\sigma}_{d} - \delta^{\sigma}_{c}\delta^{\rho}_{d}\right) - \frac{1}{4}\delta^{c}_{\lambda}\delta^{b}_{\mu}\left(\delta^{\rho}_{c}\delta^{\sigma}_{d} - \delta^{\sigma}_{c}\delta^{\rho}_{d}\right)\right] \\
&+ R\left(\delta^{\rho}_{\mu}\delta^{\sigma}_{\lambda} - \delta^{\sigma}_{\mu}\delta^{\rho}_{\lambda}\right) \\
&= 2R^{\rho\sigma}_{\mu\lambda} - 2\delta^{\rho}_{\mu}R^{\sigma}_{\lambda} + 2\delta^{\sigma}_{\mu}R^{\rho}_{\lambda} + 2\delta^{\rho}_{\lambda}R^{\sigma}_{\mu} - 2\delta^{\sigma}_{\lambda}R^{\rho}_{\mu} \\
&+ R\left(\delta^{\rho}_{\mu}\delta^{\sigma}_{\lambda} - \delta^{\sigma}_{\mu}\delta^{\rho}_{\lambda}\right) \qquad (3.22)
\end{aligned}
$$

where we used

$$
\frac{\partial R^{ab}_{cd}}{\partial R^{\mu\lambda}_{\rho\sigma}} = \frac{1}{4}\delta^{a}_{\mu}\delta^{b}_{\lambda}\left(\delta^{\rho}_{c}\delta^{\sigma}_{d} - \delta^{\sigma}_{c}\delta^{\rho}_{d}\right) - \frac{1}{4}\delta^{a}_{\lambda}\delta^{b}_{\mu}\left(\delta^{\rho}_{c}\delta^{\sigma}_{d} - \delta^{\sigma}_{c}\delta^{\rho}_{d}\right), \qquad (3.23)
$$

and

$$
R = \frac{1}{2}R^{ab}_{cd}\left(\delta^{c}_{a}\delta^{d}_{b} - \delta^{d}_{a}\delta^{c}_{b}\right), \qquad (3.24)
$$

with

$$
\frac{\partial R}{\partial R^{\mu\lambda}_{\rho\sigma}} = \frac{1}{2}\left(\delta^{\rho}_{\mu}\delta^{\sigma}_{\lambda} - \delta^{\sigma}_{\mu}\delta^{\rho}_{\lambda}\right). \qquad (3.25)
$$

Then we can continue to calculate the partial derivative of $\mathcal{F}$ with respect to $R^{\mu\lambda}_{\rho\sigma}$ which is

$$
\begin{aligned}
\frac{\partial \mathcal{F}}{\partial R^{\mu\lambda}_{\rho\sigma}} =& \frac{\partial \mathcal{F}}{\partial \mathcal{G}}\left\{2R^{\rho\sigma}_{\mu\lambda} - 2\delta^{\rho}_{\mu}R^{\sigma}_{\lambda} + 2\delta^{\sigma}_{\mu}R^{\rho}_{\lambda} + 2\delta^{\rho}_{\lambda}R^{\sigma}_{\mu} - 2\delta^{\sigma}_{\lambda}R^{\rho}_{\mu} + R\left(\delta^{\rho}_{\mu}\delta^{\sigma}_{\lambda} - \delta^{\sigma}_{\mu}\delta^{\rho}_{\lambda}\right)\right\} \\
&+ \frac{\partial \mathcal{F}}{\partial R}\frac{1}{2}(\delta^{\rho}_{\mu}\delta^{\sigma}_{\lambda} - \delta^{\rho}_{\lambda}\delta^{\sigma}_{\mu}). \qquad (3.26)
\end{aligned}
$$

Now let us introduce the notation $\frac{\partial \mathcal{F}}{\partial \mathcal{G}} \equiv \mathcal{F}_{\mathcal{G}}$ and $\frac{\partial \mathcal{F}}{\partial R} \equiv \mathcal{F}_{R}$ for simplicity. Then we have

$$
\begin{aligned}
\frac{\partial \mathcal{F}}{\partial R^{\mu\lambda}_{\rho\sigma}} &= \left(\mathcal{F}_{\mathcal{G}}R + \frac{1}{2}\mathcal{F}_{R}\right)(\delta^{\rho}_{\mu}\delta^{\sigma}_{\lambda} - \delta^{\rho}_{\lambda}\delta^{\sigma}_{\mu}) \\
&+ 2\mathcal{F}_{\mathcal{G}}\Big(R^{\rho\sigma}_{\mu\lambda} - \delta^{\rho}_{\mu}R^{\sigma}_{\lambda} + \delta^{\sigma}_{\mu}R^{\rho}_{\lambda} + \delta^{\rho}_{\lambda}R^{\sigma}_{\mu} - \delta^{\sigma}_{\lambda}R^{\rho}_{\mu}\Big). \qquad (3.27)
\end{aligned}
$$

Let us calculate the other necessary term which is

$$
\begin{aligned}
g_{\nu\rho}\nabla^{\lambda}\nabla_{\sigma}\left(\frac{\partial \mathcal{F}}{\partial R^{\mu\lambda}_{\rho\sigma}}\right) &= g_{\nu\rho}\nabla^{\lambda}\nabla_{\sigma}\left(\mathcal{F}_{\mathcal{G}}R + \frac{1}{2}\mathcal{F}_{R}\right)\left(\delta^{\rho}_{\mu}\delta^{\sigma}_{\lambda} - \delta^{\rho}_{\lambda}\delta^{\sigma}_{\mu}\right) \\
+\ & 2g_{\nu\rho}\nabla^{\lambda}\nabla_{\sigma}\left[\mathcal{F}_{\mathcal{G}}\left(R^{\rho\sigma}_{\mu\lambda} - \delta^{\rho}_{\mu}R^{\sigma}_{\lambda} + \delta^{\sigma}_{\mu}R^{\rho}_{\lambda} + \delta^{\rho}_{\lambda}R^{\sigma}_{\mu} - \delta^{\sigma}_{\lambda}R^{\rho}_{\mu}\right)\right]. \qquad (3.28)
\end{aligned}
$$

To continue, firstly we calculate the first part of the previous equation which is

$$
\begin{aligned}
g_{\nu\rho}\nabla^{\lambda}\nabla_{\sigma}&\left(\mathcal{F}_{\mathcal{G}}R+\frac{1}{2}\mathcal{F}_{R}\right)\left(\delta^{\rho}_{\mu}\delta^{\sigma}_{\lambda}-\delta^{\rho}_{\lambda}\delta^{\sigma}_{\mu}\right)\\
&=\left(g_{\nu\mu}\nabla^{\lambda}\nabla_{\lambda}-g_{\nu\lambda}\nabla^{\lambda}\nabla_{\mu}\right)\left(\mathcal{F}_{\mathcal{G}}R+\frac{1}{2}\mathcal{F}_{R}\right)\\
&=g_{\mu\nu}\Box\left(R\mathcal{F}_{\mathcal{G}}+\frac{1}{2}\mathcal{F}_{R}\right)-\nabla_{\nu}\nabla_{\mu}\left(R\mathcal{F}_{\mathcal{G}}+\frac{1}{2}\mathcal{F}_{R}\right),
\end{aligned}
\tag{3.29}
$$

and then calculating the second part we get

$$
\begin{aligned}
&2g_{\nu\rho}\nabla^{\lambda}\nabla_{\sigma}\left[\mathcal{F}_{\mathcal{G}}\left(R^{\rho\sigma}_{\mu\lambda}-\delta^{\rho}_{\mu}R^{\sigma}_{\lambda}+\delta^{\sigma}_{\mu}R^{\rho}_{\lambda}+\delta^{\rho}_{\lambda}R^{\sigma}_{\mu}-\delta^{\sigma}_{\lambda}R^{\rho}_{\mu}\right)\right]\\
&\quad=2\nabla^{\lambda}\nabla_{\sigma}[\mathcal{F}_{\mathcal{G}}(R_{\nu}{}^{\sigma}{}_{\mu\lambda}-g_{\mu\nu}R^{\sigma}_{\lambda}+R_{\nu\lambda}\delta^{\sigma}_{\mu}+R^{\sigma}_{\mu}g\nu_{\lambda}-R_{\mu\nu}\delta^{\sigma}_{\lambda})].
\end{aligned}
\tag{3.30}
$$

Summing up these two equations we have

$$
\begin{aligned}
g_{\nu\rho}\nabla^{\lambda}\nabla_{\sigma}\left(\frac{\partial\mathcal{F}}{\partial R^{\mu\lambda}_{\rho\sigma}}\right) &= g_{\mu\nu}\Box\left(R\mathcal{F}_{\mathcal{G}}+\frac{1}{2}\mathcal{F}_{R}\right)-\nabla_{\nu}\nabla_{\mu}\left(R\mathcal{F}_{\mathcal{G}}+\frac{1}{2}\mathcal{F}_{R}\right)\\
&+ 2\nabla^{\lambda}\nabla_{\sigma}\left[\mathcal{F}_{\mathcal{G}}\left(R_{\nu}{}^{\sigma}{}_{\mu\lambda}-g_{\mu\nu}R^{\sigma}_{\lambda}+R_{\nu\lambda}\delta^{\sigma}_{\mu}+R^{\sigma}_{\mu}g\nu_{\lambda}-R_{\mu\nu}\delta^{\sigma}_{\lambda}\right)\right]\\
&= (g_{\mu\nu}\Box-\nabla_{\nu}\nabla_{\mu})\left(R\mathcal{F}_{\mathcal{G}}+\frac{1}{2}\mathcal{F}_{R}\right)\\
&+ 2\left(R_{\nu}{}^{\sigma}{}_{\mu\lambda}-g_{\mu\nu}R^{\sigma}_{\lambda}+R_{\nu\lambda}\delta^{\sigma}_{\mu}+R^{\sigma}_{\mu}g\nu_{\lambda}-R_{\mu\nu}\delta^{\sigma}_{\lambda}\right)\nabla^{\lambda}\nabla_{\sigma}\mathcal{F}_{\mathcal{G}}\\
&+ 2\mathcal{F}_{\mathcal{G}}\nabla^{\lambda}[\nabla_{\sigma}R_{\nu}{}^{\sigma}{}_{\mu\lambda}-\nabla_{\sigma}(g_{\mu\nu}R^{\sigma}_{\lambda})+\nabla_{\sigma}(R_{\nu\lambda}\delta^{\sigma}_{\mu})+\nabla_{\sigma}(R^{\sigma}_{\mu}g\nu_{\lambda})\\
&\quad-\nabla_{\sigma}(R_{\mu\nu}\delta^{\sigma}_{\lambda})].
\end{aligned}
\tag{3.31}
$$

Now to continue the calculation we can simply write

$$\nabla_{\sigma}R_{\nu}{}^{\sigma}{}_{\mu\lambda}=\nabla^{\sigma}R_{\nu\sigma\mu\lambda}=-\nabla_{\sigma}R^{\sigma}{}_{\nu\mu\lambda} \tag{3.32}$$

and using the Bianchi identities

$$\nabla_{\sigma}R^{\alpha}{}_{\beta\alpha\nu}+\nabla_{\alpha}R^{\alpha}{}_{\beta\nu\sigma}+\nabla_{\nu}R^{\alpha}{}_{\beta\sigma\alpha}=0, \tag{3.33}$$

$$\nabla_{\sigma}R_{\beta\nu}+\nabla_{\alpha}R^{\alpha}{}_{\beta\nu\sigma}-\nabla_{\nu}R_{\beta\sigma}=0, \tag{3.34}$$

we get

$$\nabla_{\sigma}R^{\sigma}{}_{\nu\mu\lambda}=\nabla_{\mu}R_{\nu\lambda}-\nabla_{\lambda}R_{\nu\mu}. \tag{3.35}$$

Hence we have

$$\nabla_{\sigma}R_{\nu}{}^{\sigma}{}_{\mu\lambda}=\nabla_{\lambda}R_{\nu\mu}-\nabla_{\mu}R_{\nu\lambda}. \tag{3.36}$$

Then we conclude that

$$
\begin{aligned}
\nabla_{\sigma}R_{\nu}{}^{\sigma}{}_{\mu\lambda}-\nabla_{\sigma}(g_{\mu\nu}R^{\sigma}_{\lambda})+\nabla_{\sigma}(R_{\nu\lambda}\delta^{\sigma}_{\mu})&+\nabla_{\sigma}(R^{\sigma}_{\mu}g\nu_{\lambda})-\nabla_{\sigma}(R_{\mu\nu}\delta^{\sigma}_{\lambda})\\
&=\frac{1}{2}g_{\nu\lambda}\nabla_{\mu}R-\frac{1}{2}g_{\mu\nu}\nabla_{\lambda}R
\end{aligned}
\tag{3.37}
$$

where we used $\nabla^\mu R_{\nu\mu} = \frac{1}{2}\nabla_\nu R$.
Substituting Eq.(3.37) into Eq.(3.31) we get

$$
\begin{aligned}
g_{\nu\rho}\nabla^\lambda\nabla_\sigma\left(\frac{\partial\mathcal{F}}{\partial R^{\mu\lambda}_{\rho\sigma}}\right) &= \left(g_{\mu\nu}\Box - \nabla_\nu\nabla_\mu\right)\left(R\mathcal{F}_\mathcal{G} + \frac{1}{2}\mathcal{F}_R\right) \\
&+ 2\left(R_\nu{}^\sigma{}_{\mu\lambda} - g_{\mu\nu}R^\sigma_\lambda + R_{\nu\lambda}\delta^\sigma_\mu + R^\sigma_\mu g_{\nu\lambda} - R_{\mu\nu}\delta^\sigma_\lambda\right)\nabla^\lambda\nabla_\sigma\mathcal{F}_\mathcal{G} \\
&+ 2\mathcal{F}_\mathcal{G}\nabla^\lambda\left(\frac{1}{2}g_{\nu\lambda}\nabla_\mu R - \frac{1}{2}g_{\mu\nu}\nabla_\lambda R\right) \\
&= \left(g_{\mu\nu}\Box - \nabla_\nu\nabla_\mu\right)\left(R\mathcal{F}_\mathcal{G} + \frac{1}{2}\mathcal{F}_R\right) + 2R_\nu{}^\sigma{}_{\mu\lambda}\nabla^\lambda\nabla_\sigma\mathcal{F}_\mathcal{G} \\
&- 2g_{\mu\nu}R^\sigma_\lambda\nabla^\lambda\nabla_\sigma\mathcal{F}_\mathcal{G} + 2R_{\nu\lambda}\nabla^\lambda\nabla_\mu\mathcal{F}_\mathcal{G} + 2R^\sigma_\mu\nabla_\nu\nabla_\sigma\mathcal{F}_\mathcal{G} \\
&- 2R_{\mu\nu}\Box\mathcal{F}_\mathcal{G} + \mathcal{F}_\mathcal{G}\nabla_\nu\nabla_\mu R - \mathcal{F}_\mathcal{G}g_{\mu\nu}\Box R \\
&= R\left(g_{\mu\nu}\Box - \nabla_\nu\nabla_\mu\right)\mathcal{F}_\mathcal{G} + \frac{1}{2}\left(g_{\mu\nu}\Box - \nabla_\nu\nabla_\mu\right)\mathcal{F}_R \\
&+ 2R_\nu{}^\sigma{}_{\mu\lambda}\nabla^\lambda\nabla_\sigma\mathcal{F}_\mathcal{G} - 2g_{\mu\nu}R^\sigma_\lambda\nabla^\lambda\nabla_\sigma\mathcal{F}_\mathcal{G} + 2R_{\nu\lambda}\nabla^\lambda\nabla_\mu\mathcal{F}_\mathcal{G} \\
&+ 2R_{\sigma\mu}\nabla_\nu\nabla^\sigma\mathcal{F}_\mathcal{G} - 2R_{\mu\nu}\Box\mathcal{F}_\mathcal{G}. \qquad (3.38)
\end{aligned}
$$

Hence we can write

$$
\begin{aligned}
\frac{1}{2}\left(g_{\nu\rho}\nabla^\lambda\nabla_\sigma - g_{\nu\sigma}\nabla^\lambda\nabla_\rho\right)\frac{\partial f}{\partial R^{\mu\lambda}_{\rho\sigma}} =& R\left(g_{\mu\nu}\Box - \nabla_\nu\nabla_\mu\right)\mathcal{F}_\mathcal{G} + \frac{1}{2}\left(g_{\mu\nu}\Box - \nabla_\nu\nabla_\mu\right)\mathcal{F}_R \\
&+2R_\nu{}^\sigma{}_{\mu\lambda}\nabla^\lambda\nabla_\sigma\mathcal{F}_\mathcal{G} - 2g_{\mu\nu}R^\sigma_\lambda\nabla^\lambda\nabla_\sigma\mathcal{F}_\mathcal{G} \\
&+2R_{\nu\sigma}\nabla^\sigma\nabla_\mu\mathcal{F}_\mathcal{G} + 2R_{\sigma\mu}\nabla_\nu\nabla^\sigma\mathcal{F}_\mathcal{G} - 2R_{\mu\nu}\Box\mathcal{F}_\mathcal{G}
\end{aligned} \qquad (3.39)
$$

and the similar term is simply

$$
\begin{aligned}
\frac{1}{2}\left(g_{\mu\rho}\nabla^\lambda\nabla_\sigma - g_{\mu\sigma}\nabla^\lambda\nabla_\rho\right)\frac{\partial f}{\partial R^{\lambda\nu}_{\rho\sigma}} =& - R\left(g_{\mu\nu}\Box - \nabla_\nu\nabla_\mu\right)\mathcal{F}_\mathcal{G} \\
&-\frac{1}{2}\left(g_{\mu\nu}\Box - \nabla_\nu\nabla_\mu\right)\mathcal{F}_R \\
&-2R_\mu{}^\sigma{}_{\nu\lambda}\nabla^\lambda\nabla_\sigma\mathcal{F}_\mathcal{G} + 2g_{\mu\nu}R^\sigma_\lambda\nabla^\lambda\nabla_\sigma\mathcal{F}_\mathcal{G} \\
&-2R_{\mu\sigma}\nabla^\sigma\nabla_\nu\mathcal{F}_\mathcal{G} - 2R_{\sigma\nu}\nabla_\mu\nabla^\sigma\mathcal{F}_\mathcal{G} + 2R_{\mu\nu}\Box\mathcal{F}_\mathcal{G}.
\end{aligned} \qquad (3.40)
$$

We also need to calculate the term

$$
\begin{aligned}
\frac{\partial f}{\partial R^{\mu\lambda}_{\rho\sigma}} R_{\rho\sigma}{}^{\lambda}{}_{\nu} &= \Big(\mathcal{F}_{\mathcal{G}} R + \frac{1}{2}\mathcal{F}_R\Big)\left(\delta^{\rho}_{\mu}\delta^{\sigma}_{\lambda} - \delta^{\rho}_{\lambda}\delta^{\sigma}_{\mu}\right) R_{\rho\sigma}{}^{\lambda}{}_{\nu} \\
&+ 2\mathcal{F}_{\mathcal{G}}\left(R^{\rho\sigma}_{\mu\lambda} - \delta^{\rho}_{\mu} R^{\sigma}_{\lambda} + \delta^{\sigma}_{\mu} R^{\rho}_{\lambda} + \delta^{\rho}_{\lambda} R^{\sigma}_{\mu} - \delta^{\sigma}_{\lambda} R^{\rho}_{\mu}\right) R_{\rho\sigma}{}^{\lambda}{}_{\nu} \\
&= \Big(\mathcal{F}_{\mathcal{G}} R + \frac{1}{2}\mathcal{F}_R\Big)\Big(R_{\mu\lambda}{}^{\lambda}{}_{\nu} - R_{\lambda\mu}{}^{\lambda}{}_{\nu}\Big) \\
&+ 2\mathcal{F}_{\mathcal{G}}\Big(R^{\rho\sigma}_{\mu\lambda} R_{\rho\sigma}{}^{\lambda}{}_{\nu} - R^{\sigma}_{\lambda} R_{\mu\sigma}{}^{\lambda}{}_{\nu} + R^{\rho}_{\lambda} R_{\rho\mu}{}^{\lambda}{}_{\nu} + R^{\sigma}_{\mu} R_{\lambda\sigma}{}^{\lambda}{}_{\nu} - R^{\rho}_{\mu} R_{\rho\lambda}{}^{\lambda}{}_{\nu}\Big) \\
&= \Big(\mathcal{F}_{\mathcal{G}} R + \frac{1}{2}\mathcal{F}_R\Big)\Big(R_{\mu\lambda}{}^{\lambda}{}_{\nu} - R_{\lambda\mu}{}^{\lambda}{}_{\nu}\Big) \\
&+ 2\mathcal{F}_{\mathcal{G}}\Big(R^{\rho\sigma}_{\mu\lambda} R_{\rho\sigma}{}^{\lambda}{}_{\nu} + 2R^{\rho}_{\lambda} R_{\rho\mu}{}^{\lambda}{}_{\nu} + 2R^{\rho}_{\mu} R_{\lambda\rho}{}^{\lambda}{}_{\nu}\Big) \\
&= -2R_{\mu\nu}\Big(\mathcal{F}_{\mathcal{G}} R + \frac{1}{2}\mathcal{F}_R\Big) \\
&+ 2\mathcal{F}_{\mathcal{G}}\Big(R^{\rho\sigma}_{\mu\lambda} R_{\rho\sigma}{}^{\lambda}{}_{\nu} + 2R_{\rho\mu}{}^{\lambda}{}_{\nu} R^{\rho}_{\lambda} + 2R^{\sigma}_{\mu} R_{\sigma\nu}\Big). \qquad (3.41)
\end{aligned}
$$

Similarly we compute

$$
\frac{\partial f}{\partial R^{\lambda\nu}_{\rho\sigma}} R_{\rho\sigma}{}^{\lambda}{}_{\mu} = 2R_{\mu\nu}\Big(\mathcal{F}_{\mathcal{G}} R + \frac{1}{2}\mathcal{F}_R\Big) - 2\mathcal{F}_{\mathcal{G}}\Big(R^{\rho\sigma}_{\nu\lambda} R_{\rho\sigma}{}^{\lambda}{}_{\mu} + 2R_{\rho\nu}{}^{\lambda}{}_{\mu} R^{\rho}_{\lambda} + 2R^{\sigma}_{\nu} R_{\sigma\mu}\Big). \qquad (3.42)
$$

So the field equations with a source follow as

$$
\begin{aligned}
&2R(g_{\mu\nu}\Box - \nabla_{\mu}\nabla_{\nu})\mathcal{F}_{\mathcal{G}} + (g_{\mu\nu}\Box - \nabla_{\mu}\nabla_{\nu})\mathcal{F}_R + 2\Big(R_{\mu}{}^{\sigma}{}_{\nu\lambda} + R_{\nu}{}^{\sigma}{}_{\mu\lambda}\Big)\nabla^{\lambda}\nabla_{\sigma}\mathcal{F}_{\mathcal{G}} \\
&- 4g_{\mu\nu} R^{\sigma}_{\lambda}\nabla^{\lambda}\nabla_{\sigma}\mathcal{F}_{\mathcal{G}} + 4\Big(R_{\mu\sigma}\nabla^{\sigma}\nabla_{\nu} + R_{\nu\sigma}\nabla^{\sigma}\nabla_{\mu}\Big)\mathcal{F}_{\mathcal{G}} - 4R_{\mu\nu}\Box\mathcal{F}_{\mathcal{G}} \\
&+ 2R_{\mu\nu}\Big(\mathcal{F}_{\mathcal{G}} R + \frac{1}{2}\mathcal{F}_R\Big) - 2\mathcal{F}_{\mathcal{G}}\Big(2R_{\rho\mu}{}^{\lambda}{}_{\nu} R^{\rho}_{\lambda} + 2R^{\sigma}_{\mu} R_{\sigma\nu} + R^{\rho\sigma}_{\mu\lambda} R_{\rho\sigma}{}^{\lambda}{}_{\nu}\Big) \\
&- \frac{1}{2} g_{\mu\nu} f = \frac{\kappa}{2} T_{\mu\nu} \qquad (3.43)
\end{aligned}
$$

or we can recast them as

$$
\begin{aligned}
&\mathcal{F}_R\Big(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\Big) + \frac{1}{2} g_{\mu\nu}\Big(\mathcal{F}_R R - f\Big) \\
&- 2\mathcal{F}_{\mathcal{G}}\Big(- R_{\mu\nu} R + 2R_{\rho\mu}{}^{\lambda}{}_{\nu} R^{\rho}_{\lambda} + 2R^{\sigma}_{\mu} R_{\sigma\nu} + R^{\rho\sigma}_{\mu\lambda} R_{\rho\sigma}{}^{\lambda}{}_{\nu}\Big) \\
&+ (g_{\mu\nu}\Box - \nabla_{\mu}\nabla_{\nu})\mathcal{F}_R + 2R(g_{\mu\nu}\Box - \nabla_{\mu}\nabla_{\nu})\mathcal{F}_{\mathcal{G}} + 4\Big(R_{\mu\sigma}\nabla^{\sigma}\nabla_{\nu} \\
&+ R_{\nu\sigma}\nabla^{\sigma}\nabla_{\mu} - g_{\mu\nu} R^{\sigma}_{\lambda}\nabla^{\lambda}\nabla_{\sigma} - R_{\mu\nu}\Box\Big)\mathcal{F}_{\mathcal{G}} + 2\Big(R_{\mu}{}^{\sigma}{}_{\nu\lambda} + R_{\nu}{}^{\sigma}{}_{\mu\lambda}\Big)\nabla^{\lambda}\nabla_{\sigma}\mathcal{F}_{\mathcal{G}} = \frac{\kappa_0}{2} T_{\mu\nu}. \qquad (3.44)
\end{aligned}
$$

Still, these field equations can be written in a simpler form using the Weyl tensor and some simple tricks.

The Weyl tensor in four dimensions is

$$
\begin{aligned}
C_{\mu\sigma\nu\lambda} &= R_{\mu\sigma\nu\lambda}+\frac{1}{2}\left(g_{\mu\lambda}R_{\nu\sigma}+g_{\sigma\nu}R_{\mu\lambda}-g_{\mu\nu}R_{\sigma\lambda}-g_{\sigma\lambda}R_{\nu\mu}\right)\\
&+ \frac{1}{6}\left(g_{\mu\nu}g_{\lambda\sigma}-g_{\mu\lambda}g_{\nu\sigma}\right)R. \qquad (3.45)
\end{aligned}
$$

Then we simply compute that

$$
\begin{aligned}
\left(2C_{\mu\sigma\nu\lambda}-R_{\mu\sigma\nu\lambda}\right)\nabla^\sigma\nabla^\lambda &= R_{\mu\sigma\nu\lambda}\nabla^\sigma\nabla^\lambda+g_{\mu\lambda}R_{\nu\sigma}\nabla^\sigma\nabla^\lambda+g_{\sigma\nu}R_{\mu\lambda}\nabla^\sigma\nabla^\lambda\\
&- g_{\mu\nu}R_{\sigma\lambda}\nabla^\sigma\nabla^\lambda-R_{\mu\nu}\Box\\
&+ \frac{1}{3}\left(g_{\mu\nu}R\Box-R\nabla_\nu\nabla_\mu\right). \qquad (3.46)
\end{aligned}
$$

For the next calculation let us start with the definition of the Gauss-Bonnet invariant:

$$
\mathcal{G}-R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho}+4R_{\mu\nu}R^{\mu\nu}-R^2=0. \qquad (3.47)
$$

Using

$$
R_{\alpha\beta}R^{\alpha\beta}=\frac{1}{2}R_{\alpha\beta}R^{\alpha\beta}+\frac{1}{2}R_{\alpha\beta}R^{\alpha\beta}=\frac{1}{2}(R^{\mu\rho}R^\nu_\rho-R^{\mu\rho\sigma\nu}R_{\rho\sigma})g_{\mu\nu} \qquad (3.48)
$$

and $\mathcal{G}=\frac{1}{4}g^{\mu\nu}\mathcal{G}g_{\mu\nu}$ and $R^2=RR^{\mu\nu}g_{\mu\nu}$ we can write

$$
\left(\frac{1}{4}g^{\mu\nu}\mathcal{G}-R^{\mu\alpha\sigma\rho}R^\nu_{\alpha\sigma\rho}+2\left(R^{\mu\rho}R^\nu_\rho-R^{\mu\rho\sigma\nu}R_{\rho\sigma}\right)-RR^{\mu\nu}\right)g_{\mu\nu}=0 \qquad (3.49)
$$

and we conclude that

$$
g^{\mu\nu}\mathcal{G}=4RR^{\mu\nu}+4R^{\mu\alpha\sigma\rho}R^\nu_{\alpha\sigma\rho}-8\left(R^{\mu\rho}R^\nu_\rho-R^{\mu\rho\sigma\nu}R_{\rho\sigma}\right). \qquad (3.50)
$$

Using the above equations we find the field equations as

$$
\begin{aligned}
&\mathcal{F}_R R_{\mu\nu}+\frac{1}{2}g_{\mu\nu}(\mathcal{G}\mathcal{F}_\mathcal{G}-\mathcal{F})+\Big(g_{\mu\nu}\Box-\nabla_\mu\nabla_\nu\Big)\mathcal{F}_R\\
&+4\left[\Big(2C_{\mu\sigma\nu\lambda}-R_{\mu\sigma\nu\lambda}\Big)\nabla^\sigma\nabla^\lambda+\frac{R}{6}\Big(g_{\mu\nu}\Box-\nabla_\mu\nabla_\nu\Big)\right]\mathcal{F}_\mathcal{G}=8\pi G_0T_{\mu\nu} \qquad (3.51)
\end{aligned}
$$

where

$$
\mathcal{F}_R=\frac{1}{2}(\gamma R-1)\left(\gamma R(\gamma R-2)-9\gamma^2\mathcal{G}-2\right), \qquad (3.52)
$$

$$
\mathcal{F}_\mathcal{G}=\frac{9}{4}\gamma(-\gamma^2R^2+9\gamma^2\mathcal{G}+2\gamma R+2) \qquad (3.53)
$$

for our special action Eq.(3.19). And for the trace of the field equations we simply compute

$$
R\mathcal{F}_R+2\mathcal{G}\mathcal{F}_\mathcal{G}-2\mathcal{F}+3\Box\mathcal{F}_R-4G_{\mu\nu}\nabla^\mu\nabla^\nu\mathcal{F}_\mathcal{G}=8\pi G_0T. \qquad (3.54)
$$

The trace equation, when computed in the vacuum state of a spacetime, give us the vacuum equation (namely the maximally symmetric solution) to study and interpret the vacuum state of the universe. In the following section we will study the maximally symmetric vacuum solution for our Born-Infeld universe .

3.1 Maximally Symmetric Vacuum Solution

In Einstein's theory, the vacuum solution is maximally symmetric i.e. the spacetime is homogeneous and isotropic in 4 dimensions (and so the curvature is constant at every point) [1]. In this section, we search for the maximally symmetric vacuum solutions of our theory. As an introduction we firstly give the brief information about the maximally symmetric spaces in general.

3.1.1 Maximally Symmetric Spaces[1]

Suppose we choose a coordinate system x^μ and after a coordinate transformation we get a new chart x'^μ. If under such a coordinate transformation the form of the metric does not change then the metric has symmetry which is called an isometry. We denote the transformed metric as $g'_{\mu\nu}(x'^\mu)$. Then the isometry can be expressed as

$$g'_{\mu\nu}(x'^\mu) = g_{\mu\nu}(x'^\mu) \tag{3.55}$$

for all x'^μ. Since the metric is a (0,2) tensor field, it transforms as

$$g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}(x) \tag{3.56}$$

at a point and the inverse transformation is

$$g_{\mu\nu}(x) = \frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\nu} g'_{\rho\sigma}(x'). \tag{3.57}$$

Using the isometry property $g'_{\rho\sigma}(x') = g_{\rho\sigma}(x')$ we have

$$g_{\mu\nu}(x) = \frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\nu} g_{\rho\sigma}(x'). \tag{3.58}$$

Let us consider an infinitesimal transformations for simplicity

$$x'^\mu = x^\mu + \epsilon\,\xi^\mu, \tag{3.59}$$

[1] In this section we generally follow the Weinberg's calculations[28].

where ϵ is a small quantity; $\epsilon << 1$ and ξ is a vector field . Taking the partial derivative of this transformation Eq.(3.59) we have,

$$\frac{\partial x'^{\mu}}{\partial x^{\rho}} = \frac{\partial x^{\mu}}{\partial x^{\rho}} + \epsilon \frac{\partial \xi^{\mu}}{\partial x^{\rho}}. \tag{3.60}$$

Then we substitute Eq.(3.60) into Eq.(3.58) and do the calculations to the first order;

$$\begin{aligned} g_{\mu\nu}(x) &= \left(\frac{\partial x^{\rho}}{\partial x^{\mu}} + \epsilon \frac{\partial \xi^{\rho}}{\partial x^{\mu}}\right)\left(\frac{\partial x^{\sigma}}{\partial x^{\nu}} + \epsilon \frac{\partial \xi^{\sigma}}{\partial x^{\nu}}\right) g_{\rho\sigma}(x') \\ &= \left(\frac{\partial x^{\rho}}{\partial x^{\mu}}\frac{\partial x^{\sigma}}{\partial x^{\nu}} + \epsilon \frac{\partial x^{\rho}}{\partial x^{\mu}}\frac{\partial \xi^{\sigma}}{\partial x^{\nu}} + \epsilon \frac{\partial x^{\sigma}}{\partial x^{\nu}}\frac{\partial \xi^{\rho}}{\partial x^{\mu}}\right) g_{\rho\sigma}(x'). \end{aligned} \tag{3.61}$$

Now we can do a Taylor series expansion (to the first order):

$$g_{\rho\sigma}(x') = g_{\rho\sigma}(x + \epsilon\,\xi) = g_{\rho\sigma}(x) + \epsilon\;\xi^{\alpha}\frac{g_{\rho\sigma}}{\partial x^{\alpha}} \tag{3.62}$$

and we insert Eq.(3.62) into Eq.(3.61)

$$g_{\mu\nu}(x) = \left(\frac{\partial x^{\rho}}{\partial x^{\mu}}\frac{\partial x^{\sigma}}{\partial x^{\nu}} + \epsilon \frac{\partial x^{\rho}}{\partial x^{\mu}}\frac{\partial \xi^{\sigma}}{\partial x^{\nu}} + \epsilon \frac{\partial x^{\sigma}}{\partial x^{\nu}}\frac{\partial \xi^{\rho}}{\partial x^{\mu}}\right)\left(g_{\rho\sigma}(x) + \epsilon\;\xi^{\alpha}\frac{g_{\rho\sigma}}{\partial x^{\alpha}}\right). \tag{3.63}$$

$$g_{\mu\nu}(x) = g_{\mu\nu}(x) + \epsilon g_{\mu\sigma}(x)\frac{\partial \xi^{\sigma}}{\partial x^{\nu}} + \epsilon g_{\rho\nu}(x)\frac{\partial \xi^{\rho}}{\partial x^{\mu}} + \epsilon \xi^{\alpha}\frac{\partial g_{\mu\nu}}{\partial x^{\alpha}}. \tag{3.64}$$

Then we have

$$g_{\mu\sigma}(x)\frac{\partial \xi^{\sigma}}{\partial x^{\nu}} + g_{\rho\nu}(x)\frac{\partial \xi^{\rho}}{\partial x^{\mu}} + \xi^{\alpha}\frac{\partial g_{\mu\nu}}{\partial x^{\alpha}} = 0. \tag{3.65}$$

Now we need to calculate the partial derivative of $\xi_{\mu} = g_{\sigma\mu}\xi^{\sigma}$;

$$\frac{\partial \xi_{\mu}}{\partial x^{\nu}} = \frac{\partial g_{\sigma\mu}}{\partial x^{\nu}}\xi^{\sigma} + g_{\sigma\mu}\frac{\partial \xi^{\sigma}}{\partial x^{\nu}}. \tag{3.66}$$

Then we have

$$g_{\sigma\mu}\frac{\partial \xi^{\sigma}}{\partial x^{\nu}} = \frac{\partial \xi_{\mu}}{\partial x^{\nu}} - \frac{\partial g_{\sigma\mu}}{\partial x^{\nu}}\xi^{\sigma} \tag{3.67}$$

and similarly

$$g_{\rho\nu}\frac{\partial \xi^{\rho}}{\partial x^{\mu}} = \frac{\partial \xi_{\nu}}{\partial x^{\mu}} - \frac{\partial g_{\rho\nu}}{\partial x^{\mu}}\xi^{\rho}. \tag{3.68}$$

Substituting above two equations into Eq.(3.65) we get

$$\frac{\partial \xi_{\mu}}{\partial x^{\nu}} + \frac{\partial \xi_{\nu}}{\partial x^{\mu}} + \xi^{\alpha}\left(\frac{\partial g_{\mu\nu}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\mu}}{\partial x^{\nu}} - \frac{\partial g_{\alpha\nu}}{\partial x^{\mu}}\right) = 0. \tag{3.69}$$

Now using the definition of the Christoffel symbol Eq.(A.27) we have

$$\frac{\partial \xi_{\mu}}{\partial x^{\nu}} + \frac{\partial \xi_{\nu}}{\partial x^{\mu}} - 2\Gamma^{\sigma}_{\nu\mu}\xi_{\sigma} = 0. \tag{3.70}$$

We can write this equation in a covariant form as

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0, \tag{3.71}$$

which is named as the "Killing equation". And ξ_μ is the Killing vector accepted by the metric $g_{\mu\nu}$ when there is an infinitesimal isometry.
Now we can continue to our calculation using the Eq.(A.39); rewriting for the Killing vector we have

$$\nabla_\rho \nabla_\nu \xi_\mu - \nabla_\nu \nabla_\rho \xi_\mu = \xi_\sigma R^\sigma{}_{\mu\nu\rho}. \tag{3.72}$$

We can also use the cyclic identity Eq.(A.9);

$$R^\sigma{}_{\mu\nu\rho} + R^\sigma{}_{\nu\rho\mu} + R^\sigma{}_{\rho\mu\nu} = 0. \tag{3.73}$$

Then we have

$$\nabla_\rho \nabla_\nu \xi_\mu - \nabla_\nu \nabla_\rho \xi_\mu + \nabla_\mu \nabla_\rho \xi_\nu - \nabla_\rho \nabla_\mu \xi_\nu + \nabla_\nu \nabla_\mu \xi_\rho - \nabla_\mu \nabla_\nu \xi_\rho = 0. \tag{3.74}$$

Now using Eq.(3.71), the above equation simplifies to

$$\nabla_\nu \nabla_\mu \xi_\rho + \nabla_\rho \nabla_\nu \xi_\mu + \nabla_\mu \nabla_\rho \xi_\nu = 0. \tag{3.75}$$

Then Eq.(3.72) turns out to be

$$-\nabla_\mu \nabla_\rho \xi_\nu - \nabla_\nu \nabla_\mu \xi_\rho - \nabla_\nu \nabla_\rho \xi_\mu = \xi_\sigma R^\sigma{}_{\mu\nu\rho}. \tag{3.76}$$

Again using the Killing equation Eq.(3.71) we find

$$\nabla_\mu \nabla_\rho \xi_\nu = -\xi_\sigma R^\sigma{}_{\mu\nu\rho}. \tag{3.77}$$

Up to here we studied the Killing vector at a point in the spacetime. Now let us give a short analysis on this equation. This analysis will enable us to study all the existing Killing vectors the metric has. Suppose we know the Killing vector ξ_μ and its covariant derivative $\nabla_\nu \xi_\mu$ at some point A in n dimensional spacetime. And assume that we know the metric; we can calculate the Christoffel connection and the Riemann tensor. Then we can simply find the partial derivative of the Killing vector using the definition of the covariant derivative $\left(\partial_\nu \xi_\mu = \nabla_\nu \xi_\mu + \Gamma^\alpha_{\nu\mu} \xi_\alpha\right)$. By the Eq.(3.77) we can specify the second covariant derivative of the ξ_μ such that it is proportional to ξ_σ:

$$\nabla_\mu \nabla_\rho \xi_\nu \propto \xi_\sigma. \tag{3.78}$$

We can continue to calculate the higher derivatives by taking derivatives of Eq.(3.77):

$$\nabla_\alpha \nabla_\mu \nabla_\rho \xi_\nu = -\nabla_\alpha (\xi_\sigma R^\sigma{}_{\mu\nu\rho}) \propto \nabla_\alpha \xi_\sigma. \tag{3.79}$$

And similarly

$$\nabla_\sigma \nabla_\alpha \nabla_\mu \nabla_\rho \xi_\nu \propto \nabla_\sigma \nabla_\alpha \xi_\sigma \propto \xi_\mu, \tag{3.80}$$

$$\nabla_\beta \nabla_\sigma \nabla_\alpha \nabla_\mu \nabla_\rho \xi_\nu \propto \nabla_\beta \xi_\mu. \tag{3.81}$$

Consequently, we can specify the higher derivatives of the ξ_μ in terms of ξ_μ and $\nabla_\nu \xi_\mu$.

Considering the Eq.(3.71) we simply observe that $\nabla_\nu \xi_\mu$ is an antisymmetric rank-2 tensor and by the antisymmetry property it has $n(n-1)/2$ linearly independent, non-zero components. Now we try to do a Taylor series expansion of a Killing vector ξ^*_μ in the neighbourhood of a fixed point A

$$\xi^*_\mu(x) = \xi^*_\mu(A+\zeta) = \xi^*_\mu(A) + \zeta^\nu \partial_\nu \xi^*_\mu(A) + ... \tag{3.82}$$

or

$$\xi^*_\mu(x) = X^\beta_\mu \xi^*_\beta(A) + Y^{\alpha\beta}_\mu \nabla_\alpha \xi^*_\beta(A). \tag{3.83}$$

Here ζ is a vector field. X^β_μ and $Y^{\alpha\beta}_\mu$ are expansion coefficients independent of $\xi^*_\beta(A)$ and $\nabla_\alpha \xi^*_\beta(A)$. Now we do a simple calculation: $\xi^*_\beta(A)$ has n components and besides there are $n(n-1)/2$ derivative terms. Then we find the total number of the terms constructing the Killing vector field as

$$n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}. \tag{3.84}$$

This is the maximum number of linearly independent Killing vectors that a spacetime of dimension n may possesses. When we add two or more Killing vectors with constant coefficients, the resultant vector is again a Killing vector. As a final point we conclude that if we know ξ_μ and $\nabla_\nu \xi_\mu$ at a point, we can calculate the complete form of the Killing vector field $\xi_\mu(x)$.

• *Homogeneity:* For a homogeneous spacetime, there is no special point. Then we can assign any existing Killing vector to any point we prefer.

•*Isotropy:* If at a fixed point, the Killing vector ξ_μ vanishes and if there is no restriction on $\nabla_\nu \xi_\mu$ except the antisymmetry property by the Killing equation, then it is called as an isotropic spacetime. It means there is no special direction.

• If the space is isotropic at any point, then it is also homogeneous.

• If the spacetime is both homogeneous and isotropic then it has the maximum number of Killing vectors or vice versa.

To investigate the Killing vectors further, we need to solve the Eq.(3.77). Instead, we can obtain a long but a simpler equation. We use the integrability (or compatibility) condition. For a (0,2) rank tensor the integrability condition is stated as Eq.(A.44):

$$[\nabla_\alpha, \nabla_\mu] X_{\rho\nu} = -R^\lambda{}_{\rho\alpha\mu} X_{\lambda\nu} - R^\lambda{}_{\nu\alpha\mu} X_{\rho\lambda}. \tag{3.85}$$

We rewrite this equation for $\nabla_\rho \xi_\nu$;

$$\nabla_\alpha \nabla_\mu \nabla_\rho \xi_\nu - \nabla_\mu \nabla_\alpha \nabla_\rho \xi_\nu = -R^\sigma{}_{\rho\alpha\mu} \nabla_\sigma \xi_\nu - R^\sigma{}_{\nu\alpha\mu} \nabla_\rho \xi_\sigma. \tag{3.86}$$

Now taking the derivative of Eq.(3.77) we also have

$$\nabla_\alpha \nabla_\mu \nabla_\rho \xi_\nu = -R^\sigma{}_{\mu\nu\rho} \nabla_\alpha \xi_\sigma - \xi_\sigma \nabla_\alpha R^\sigma{}_{\mu\nu\rho}. \tag{3.87}$$

Substituting this equation into Eq.(3.86) we get

$$R^\sigma{}_{\alpha\nu\rho} \nabla_\mu \xi_\sigma - R^\sigma{}_{\mu\nu\rho} \nabla_\alpha \xi_\sigma + R^\sigma{}_{\rho\alpha\mu} \nabla_\sigma \xi_\nu + R^\sigma{}_{\nu\alpha\mu} \nabla_\rho \xi_\sigma = (\nabla_\alpha R^\sigma{}_{\mu\nu\rho} - \nabla_\mu R^\sigma{}_{\alpha\nu\rho}) \xi_\sigma. \tag{3.88}$$

And inserting Eq.(3.71) into above equation we find

$$\left(R^\sigma{}_{\alpha\nu\rho} \delta^\beta_\mu + R^\sigma{}_{\nu\alpha\mu} \delta^\beta_\rho - R^\sigma{}_{\mu\nu\rho} \delta^\beta_\alpha - R^\sigma{}_{\rho\alpha\mu} \delta^\beta_\nu\right) \nabla_\beta \xi_\sigma = (\nabla_\alpha R^\sigma{}_{\mu\nu\rho} - \nabla_\mu R^\sigma{}_{\alpha\nu\rho}) \xi_\sigma. \tag{3.89}$$

This is the final equation we should solve to obtain the Killing vectors and to solve this we need to write down the equations by component analysis. Inserting the Riemann tensor components, Eq.(3.89) must hold. However, this is not always possible. If the metric has the maximum number of Killing vectors Eq.(3.84), then the spacetime is named to be maximally symmetric.

Now our aim is to construct the Riemann tensor using Eq.(3.89). We will choose a point to study. Considering the isotropy property that the Killing vector may vanish at a point the right hand side of Eq.(3.89) could be taken as zero. For the other side, we firstly anti-symmetrize the multiplier terms with respect to β and σ then equalize to zero:

$$R^\sigma{}_{\alpha\nu\rho} \delta^\beta_\mu + R^\sigma{}_{\nu\alpha\mu} \delta^\beta_\rho - R^\sigma{}_{\mu\nu\rho} \delta^\beta_\alpha - R^\sigma{}_{\rho\alpha\mu} \delta^\beta_\nu = R^\beta{}_{\alpha\nu\rho} \delta^\sigma_\mu + R^\beta{}_{\nu\alpha\mu} \delta^\sigma_\rho - R^\beta{}_{\mu\nu\rho} \delta^\sigma_\alpha - R^\beta{}_{\rho\alpha\mu} \delta^\sigma_\nu \tag{3.90}$$

Now we pass to the left hand side Eq.(3.89) which also becomes zero. By homogeneity condition we can choose any Killing vector at this point. Assigning a non-zero Killing vector we find

$$\nabla_\alpha R^\sigma{}_{\mu\nu\rho} = \nabla_\mu R^\sigma{}_{\alpha\nu\rho}. \tag{3.91}$$

We turn back to Eq.(3.90) and contract β and μ. For n dimensional spacetime we have,

$$(n-1)R_{\sigma\alpha\nu\rho} = R_{\rho\alpha}g_{\sigma\nu} - R_{\nu\alpha}g_{\sigma\rho} \tag{3.92}$$

where we used the property Eq.(A.9)

$$R^\sigma{}_{\alpha\nu\rho} + R^\sigma{}_{\nu\rho\alpha} + R^\sigma{}_{\rho\alpha\nu} = 0. \tag{3.93}$$

Since the Riemann tensor is antisymmetric with respect to σ and α in Eq.(3.92) we also anti-symmetrize the right hand side:

$$R_{\rho\alpha}g_{\sigma\nu} - R_{\nu\alpha}g_{\sigma\rho} = -R_{\rho\sigma}g_{\alpha\nu} + R_{\nu\sigma}g_{\alpha\rho}. \tag{3.94}$$

Contracting σ and ν we get

$$R_{\rho\alpha} = \frac{R}{n}g_{\alpha\rho}. \tag{3.95}$$

Finding the Ricci tensor, now we can substitute into Eq.(3.92);

$$R_{\sigma\alpha\nu\rho} = \frac{R}{n(n-1)}\left(g_{\rho\alpha}g_{\sigma\nu} - g_{\nu\alpha}g_{\sigma\rho}\right). \tag{3.96}$$

This is the Riemann tensor in a maximally symmetric n dimensional spacetime. Now in order to gain more information about the scalar curvature R, we calculate Eq.(A.22):

$$\nabla_\mu\left(R^{\mu\alpha} - \frac{1}{2}g^{\mu\alpha}R\right) = \nabla_\mu\left(\frac{R}{n}g^{\mu\alpha} - \frac{1}{2}g^{\mu\alpha}R\right) = 0. \tag{3.97}$$

Since R is a scalar we simply write

$$\left(\frac{1}{n} - \frac{1}{2}\right)\partial_\mu R = 0 \tag{3.98}$$

then we conclude that R is a constant. Now, to find R, let us calculate the field equations in 4 dimensions. Einstein tensor is computed to be

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\frac{R}{4}g_{\mu\nu}. \tag{3.99}$$

For a vacuum solution the Einstein tensor should vanish. Then we could fairly interpret this equation as a cosmological Einstein equation taking $\Lambda = R/4$ [23]. Finally, in a 4-dimensional maximally symmetric spacetime Riemann tensor can be written as [18]

$$R_{\mu\sigma\nu\rho} = \frac{\Lambda}{3}(g_{\mu\nu}g_{\sigma\rho} - g_{\mu\rho}g_{\sigma\nu}) \tag{3.100}$$

with its contractions $R_{\mu\nu} = \Lambda g_{\mu\nu}$ and $R = g^{\mu\nu}R_{\mu\nu} = 4\Lambda$. Here Λ is the effective cosmological constant. Then the Gauss-Bonnet invariant becomes

$$\mathcal{G} = R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho} - 4R_{\mu\nu}R^{\mu\nu} + R^2 = \frac{8}{3}\Lambda^2. \tag{3.101}$$

3.2 Maximally Symmetric Solutions

The Riemann tensor in a maximally symmetric 4-dimensional spacetime is written as Eq.(3.96)

$$R_{\sigma\alpha\nu\rho} = \frac{R}{12}\left(g_{\rho\alpha}g_{\sigma\nu} - g_{\nu\alpha}g_{\sigma\rho}\right). \tag{3.102}$$

Then we can list 3 different spaces according to the value of the R [22],[23]:

- If $R = 0$, the spacetime is Minkowskian ($R_{\sigma\alpha\nu\rho} = 0$, flat universe);
- If $R > 0$, we have a positive curvature and the spacetime is said to be *de Sitter* ;
- If $R < 0$, we get an *anti-de Sitter* spacetime implying a negative curvature.

Besides, these 3 spaces are the solutions of the conformally flat cosmological Einstein field equations [22]. Let us show this briefly. For a conformally flat metric we have $C_{\sigma\alpha\nu\rho} = 0$. Then the Riemann tensor becomes

$$R_{\mu\sigma\nu\lambda} = \frac{1}{2}\left(-g_{\mu\lambda}R_{\nu\sigma} - g_{\sigma\nu}R_{\mu\lambda} + g_{\mu\nu}R_{\sigma\lambda} + g_{\sigma\lambda}R_{\nu\mu}\right) + \frac{1}{6}\left(-g_{\mu\nu}g_{\lambda\sigma} + g_{\mu\lambda}g_{\nu\sigma}\right)R. \tag{3.103}$$

When we substitute $R_{\mu\nu} = \Lambda g_{\mu\nu} = \frac{1}{4}Rg_{\mu\nu}$ we get

$$\begin{aligned} R_{\mu\sigma\nu\lambda} &= \frac{R}{8}\left(-g_{\mu\lambda}g_{\nu\sigma} - g_{\sigma\nu}g_{\mu\lambda} + g_{\mu\nu}g_{\sigma\lambda} + g_{\sigma\lambda}g_{\nu\mu}\right) + \frac{R}{6}\left(-g_{\mu\nu}g_{\lambda\sigma} + g_{\mu\lambda}g_{\nu\sigma}\right) \\ &= \frac{R}{12}\left(g_{\mu\nu}g_{\sigma\lambda} - g_{\mu\lambda}g_{\nu\sigma}\right). \end{aligned} \tag{3.104}$$

With this expected Riemann tensor we finalise the proof. Then we continue with short introductions to de-Sitter and anti-de Sitter spacetimes before presenting the calculations of the maximally symmetric vacuum of the quartic theory.

3.2.1 de Sitter and anti-de Sitter Spacetimes

A simple example as a start: While studying a 2-sphere we use the idea of embedding to be able to visualize this surface. Let us remember the embedding tool [25]:
The 2-dimensional sphere is defined by

$$x_1^2 + x_2^2 + x_3^2 = r^2 \tag{3.105}$$

and it is embedded into 3-dimensional Euclidean space with the flat metric

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2. \tag{3.106}$$

Or, when we write the ambient (embedding) space in spherical coordinates we have

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \tag{3.107}$$

Fixing $r = R$, we get a hypersurface with a metric

$$ds^2 = R^2 d\theta^2 + R^2 \sin^2\theta d\phi^2 \tag{3.108}$$

which is the ordinary sphere. Of course the metric Eq.(3.108) is not valid everywhere on the two sphere as it is a curved surface.
We can use embedding in higher dimensions also.

3.2.1.1 de Sitter Spacetimes

For a de Sitter spacetime we need a 5-dimensional Euclidean metric as an ambient space

$$ds^2 = dx^2 + dy^2 + dz^2 + dw^2 + dv^2. \tag{3.109}$$

Embedded hypersurface is then defined by the equation

$$x^2 + y^2 + z^2 + w^2 + v^2 = a^2. \tag{3.110}$$

Here we do a coordinate transformation $v = it'$ [18], then we have

$$x^2 + y^2 + z^2 + w^2 - t'^2 = a^2. \tag{3.111}$$

Now the metric takes the form

$$ds^2 = dx^2 + dy^2 + dz^2 + dw^2 - dt'^2. \tag{3.112}$$

Now we do a transformation such that [22]

$$
\begin{aligned}
t' &= a\ \sinh\ \frac{t}{a}, \\
x &= a\ \cosh\ \frac{t}{a}\ \cos\chi, \\
y &= a\ \cosh\ \frac{t}{a}\ \sin\chi\ \cos\theta, \\
z &= a\ \cosh\ \frac{t}{a}\ \sin\chi\ \sin\theta\ \cos\phi, \\
w &= a\ \cosh\ \frac{t}{a}\ \sin\chi\ \sin\theta\ \sin\phi.
\end{aligned}
\tag{3.113}
$$

Then we get the metric

$$
ds^2 = -dt^2 + a^2 \cosh^2 \frac{t}{a} \left(d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)\right) \tag{3.114}
$$

which is named as *de-Sitter* (dS) metric.

3.2.1.2 Anti-de Sitter Spacetimes

This time the subspace has a constant negative curvature. We have

$$
-x^2 + y^2 + z^2 + w^2 - v^2 = -a^2 \tag{3.115}
$$

embedded in a space

$$
ds^2 = -dx^2 + dy^2 + dz^2 + dw^2 - dv^2. \tag{3.116}
$$

After using the transformation [22]

$$
\begin{aligned}
x &= a\ \cosh\ r'\ \sin\frac{t}{a}, \\
y &= a\ \sinh\ r'\ \cos\theta, \\
z &= a\ \sinh\ r'\ \sin\ \theta\ \cos\phi, \\
w &= a\ \sinh\ r\ \sin\theta\ \sin\phi, \\
v &= a\ \cosh\ r'\ \cos\frac{t}{a},
\end{aligned}
\tag{3.117}
$$

our metric turns out to be

$$
ds^2 = -\cosh^2 r' dt^2 + a^2\left(dr'^2 + \sinh^2 r'(d\theta^2 + \sin^2\theta d\phi^2)\right) \tag{3.118}
$$

and called as anti-de Sitter (AdS) metric. Here we can do one more transformation that $a\ \sinh\ r' = r$ to get a different version of AdS metric:

$$
ds^2 = -\left(1 + \frac{r^2}{a^2}\right) dt^2 + \left(1 + \frac{r^2}{a^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{3.119}
$$

3.2.2 Maximally Symmetric Vacuum of Quartic Gravity

After giving an introductory review of the maximally symmetric spaces in general, we turn to quartic gravity. Now finding the Riemann tensor in the maximally symmetric spacetime we can calculate the background values of the fields:

$$F_R = (1 - 4\lambda)(1 + 2\lambda)^2, \tag{3.120}$$

$$F_{\mathcal{G}} = \frac{9\gamma}{2}(1 + 2\lambda)^2, \tag{3.121}$$

$$F = \frac{1}{2\gamma}\left[(1 + 2\lambda)^4 - 4\lambda_0 - 1\right], \tag{3.122}$$

where we use a new notation $\lambda_0 = \gamma\Lambda_0$ and $\lambda = \gamma\Lambda$, with λ_0 and λ being dimensionless.

We can write the Eq.(3.54) for a vacuum solution as

$$R\,F_R + 2\mathcal{G}\,F_{\mathcal{G}} - 2F = 0 \tag{3.123}$$

since the derivative terms do not contribute. When we substitute the background fields we obtain the final equation:

$$4\lambda^4 + 4\lambda^3 - \lambda + \lambda_0 = 0. \tag{3.124}$$

We will try to solve this quartic equation and certainly take just the physically allowed ones. At this point this vacuum equation could be considered to have 4 possible roots being a quartic equation. The discriminant value plays an important role while solving an equation and is computed to be

$$\triangle = 16(1 + 4\lambda_0)^2(-11 + 64\lambda_0). \tag{3.125}$$

Now without solving the Eq.(3.124) we will find out that this equation has just one viable root using the discriminant analysis as mentioned in Appendix A.4.

3.2.3 Particle Spectrum of the Theory

In order to investigate the vacuum equation further we search for the excitations about the vacua using the linearization procedure.

We consider the metric $g_{\mu\nu}$ as a combination of a background metric $\overline{g}_{\mu\nu}$ and a perturbation $h_{\mu\nu}$ (linearized form of the metric $g_{\mu\nu}$) about this background;

$$g_{\mu\nu} = \overline{g}_{\mu\nu} + h_{\mu\nu}. \tag{3.126}$$

It is shown that $h_{\mu\nu}$ transforms like a component of a tensor field Eq.(C.6). Then we can consider in a different perspective that $h_{\mu\nu}$ *fields* are propagating around a background metric $\overline{g}_{\mu\nu}$ in vacuum [14]. With this approach we can investigate the excitations around the vacuum and study the particle spectrum of the theory.

We firstly linearize F_R Eq.(3.52) such that

$$\begin{aligned}
(\mathcal{F}_R)_L &= \frac{1}{2}(\gamma R_L)(\gamma^2\overline{R}^2 - 2\gamma\overline{R} - 9\gamma^2\overline{\mathcal{G}} - 2) \\
&+ \frac{1}{2}(\gamma\overline{R} - 1)(2\gamma^2\overline{R}R_L - 2\gamma R_L - 9\gamma^2\mathcal{G}_L) \\
&= -6\gamma\lambda(1+2\lambda)R_L.
\end{aligned} \tag{3.127}$$

Here the subscript L stands for the linearized form of the fields and overbar is placed for the background fields. $\mathcal{F}_{\mathcal{G}}$ and $\mathcal{F}$ fields under linearization are computed as

$$(\mathcal{F}_{\mathcal{G}})_L = \frac{9\gamma}{2}(1+2\lambda)R_L, \;\; \mathcal{F}_L = (1+2\lambda)^3 R_L, \tag{3.128}$$

and the linear form of the Gauss-Bonnet invariant is

$$\mathcal{G}_L = \frac{4}{3}\lambda R_L. \tag{3.129}$$

The explicit calculation of the Eq.(3.129) is shown in Appendix B.3.

Now we can linearize the trace equation to see if there is a spin-0 mode of excitation about the vacuum. Taking $T_{\mu\nu} = 0$, linearization of Eq.(3.54) takes the form below :

$$\begin{aligned}
&(\mathcal{F}_R)_L\overline{R} + \overline{\mathcal{F}}_R R_L + 2\mathcal{G}_L\overline{\mathcal{F}}_{\mathcal{G}} + 2\overline{\mathcal{G}}(\mathcal{F}_{\mathcal{G}})_L - 2\mathcal{F}_L + 3\overline{\Box}(\mathcal{F}_R)_L \\
&\qquad - 4\overline{R}_{\mu\nu}\overline{\nabla}^{\mu}\overline{\nabla}^{\nu}(\mathcal{F}_{\mathcal{G}})_L + 2\overline{R}\,\overline{\Box}(\mathcal{F}_{\mathcal{G}})_L = 0.
\end{aligned} \tag{3.130}$$

Inserting the linearized fields, all the $\Box R_L$ terms cancel each other, then we conclude that our theory does not have a spin-0 mode of graviton. And the remaining terms just reduce to

$$(1+2\lambda)^2(-1+4\lambda)R_L = 0. \tag{3.131}$$

Hence, the linearization of the trace equation produces a simple equation of R_L and we require to find out this gauge invariant term (invariance of R_L is discussed in

Appendix C.3). Then we choose $R_L = 0$ and $\lambda \neq -\frac{1}{2}$, $\lambda \neq \frac{1}{4}$.
Now we linearize the field equations Eq.(3.51) fixing $R_L = 0$. We again take $T_{\mu\nu} = 0$ since we study the vacuum case. Then we have

$$\begin{aligned}
&\overline{\mathcal{F}}_R(R_{\mu\nu})_L + (\mathcal{F}_R)_L\overline{R}_{\mu\nu} + \frac{1}{2}h_{\mu\nu}(\overline{\mathcal{G}}\,\overline{\mathcal{F}}_{\mathcal{G}} - \overline{\mathcal{F}}) \\
&+ \frac{1}{2}\overline{g}_{\mu\nu}\left(\mathcal{G}_L\overline{\mathcal{F}}_{\mathcal{G}} + \overline{\mathcal{G}}(\mathcal{F}_{\mathcal{G}})_L - \mathcal{F}_L\right) + \overline{g}_{\mu\nu}\overline{\Box}(\mathcal{F}_R)_L - \overline{\nabla}_\mu\overline{\nabla}_\nu(\mathcal{F}_R)_L \\
&+ 4\left[\left(2\overline{C}_{\mu\sigma\nu\lambda} - \overline{R}_{\mu\sigma\nu\lambda}\right)\overline{\nabla}^\sigma\overline{\nabla}^\lambda + \frac{\overline{R}}{6}\left(\overline{g}_{\mu\nu}\overline{\Box} - \overline{\nabla}_\mu\overline{\nabla}_\nu\right)\right](\mathcal{F}_{\mathcal{G}})_L = 0. \qquad (3.132)
\end{aligned}$$

The background valued Weyl tensor is zero i.e. $\overline{C}_{\mu\sigma\nu\lambda} = 0$ for the maximally symmetric background. Since we admit R_L to be zero we also have $(\mathcal{F}_R)_L = 0,\ (\mathcal{F}_{\mathcal{G}})_L = 0,\ \mathcal{F}_L = 0$, and $\mathcal{G}_L = 0$. The linearized field equations simplify to

$$\overline{\mathcal{F}}_R(R_{\mu\nu})_L + \frac{1}{2}h_{\mu\nu}(\overline{\mathcal{G}}\,\overline{\mathcal{F}}_{\mathcal{G}} - \overline{\mathcal{F}}) = 0. \qquad (3.133)$$

Substituting the background values we have

$$(1-4\lambda)(1+2\lambda)^2\left((R_{\mu\nu})_L - \Lambda h_{\mu\nu}\right) = 0. \qquad (3.134)$$

Then under the linearization process we recovered the linearized Einstein equation (with $R_L = 0$) which is $(R_{\mu\nu})_L - \Lambda h_{\mu\nu} = 0$ (studied in Appendix B.2). Then we can inherit from the Einstein's theory that our theory also has a single massless spin-2 excitation (also mentioned in Appendix B.4) and no other modes. If we couple the Eq.(3.134) to a energy-momentum tensor we find the effective Newton's constant:

$$G_{eff} = \frac{1}{(1-4\lambda)(1+2\lambda)^2}G_0. \qquad (3.135)$$

This equation confirms our previous conclusion inferred from Eq.(3.131) that $\lambda \neq -\frac{1}{2}$ and $\lambda \neq \frac{1}{4}$. And Newton's constant should be positive for attractive gravity which results in a additional restriction that $\lambda < \frac{1}{4}$. The relation of G_{eff}/G_0 vs. λ can be seen in Figure 3.1. One important observation is that the measured (effective) Newton's constant G_{eff} depends on the effective cosmological parameter λ and they are inversely proportional as $G_{eff} \approx -\frac{G_0}{16\lambda^3}$.

Now we can go back to the Eq.(3.124) and study whether we have a unique viable solution. The last restriction ($\lambda < \frac{1}{4}$) automatically implies that $\lambda_0 < \frac{11}{64}$ using Eq.(3.124) and the discriminant Eq.(3.125) takes a negative value which means that we have 2 real roots. However one of the real roots does not lie on the allowed region.

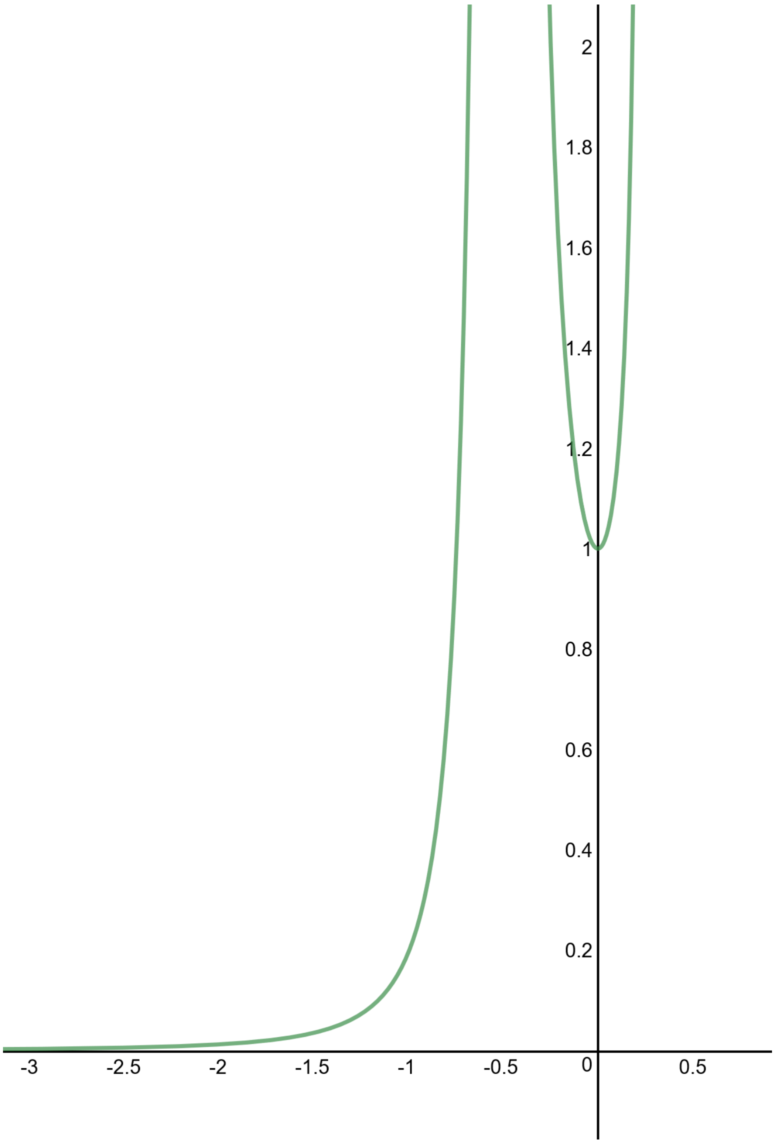

Figure 3.1: Graph of G_{eff}/G_0 *vs* λ plotted according to Eq.(3.135).

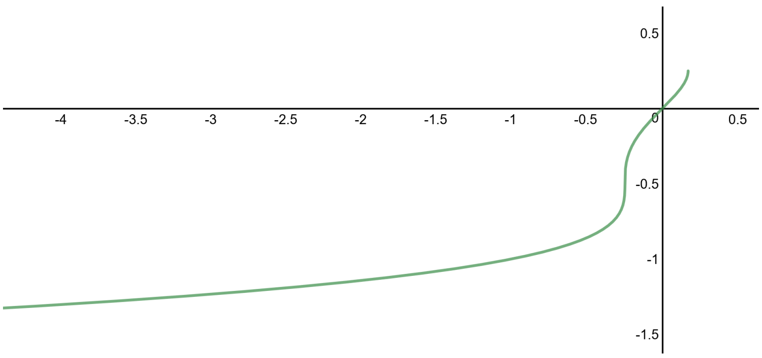

Figure 3.2: Graph of λ *vs* λ_0 plotted according to Eq.(3.124).

Then we conclude that our vacuum equation has just a single physically acceptable solution as desired. Figure 3.2 shows the graph of λ *vs* λ_0 according to Eq.(3.124). Since $\lambda \approx (-\lambda_0)^{1/4}$, as we give a large negative number for λ_0, the gap between λ and λ_0 increases dramatically and this is not an unexpected result for the modified curved geometries.

3.3 Truncation of the Theory

We remark again that having a viable unique vacuum is a crucial step for an extended gravity theory in order to survive as an ultimate quantum gravity theory that is required to be able to study the universe properly. Certainly, it is more challenging to keep this significant property even when we truncate the theory as we do below. Our theory is a quartic theory i.e. we have terms up to $\mathcal{O}(R^4)$ in our Lagrangian density. When we perform a truncation, for example, to the $\mathcal{O}(R^3)$ or $\mathcal{O}(R^2)$ then we can study our theory in lower energies also.
Let us start with the cubic truncation. Keeping the terms up to the $\mathcal{O}(R^3)$ we have

$$\mathcal{F} = R - 2\Lambda_0 + \frac{9}{2}\gamma\mathcal{G} + \frac{9}{2}\gamma^2\mathcal{G}R - \frac{1}{2}\gamma^2 R^3, \tag{3.136}$$

$$F_{\mathcal{G}} = \frac{9}{2}\gamma(1+\gamma R), \tag{3.137}$$

$$\mathcal{F}_R = 1 + \frac{9}{2}\gamma^2\mathcal{G} - \frac{3}{2}\gamma^2 R^2, \tag{3.138}$$

$$\overline{\mathcal{F}}_{\mathcal{G}} = \frac{9}{2}\gamma(1+4\lambda), \tag{3.139}$$

$$\overline{\mathcal{F}}_R = 1 - 12\lambda^2, \tag{3.140}$$

$$\overline{\mathcal{F}} = \frac{2}{\gamma}(2\lambda + 6\lambda^2 + 8\lambda^3 - \lambda_0). \tag{3.141}$$

We will restudy the vacuum equation for the truncated versions of the theory to see whether our theory still remains viable. When we substitute the relevant terms to the Eq.(3.123) we get the new vacuum equation

$$4\lambda^3 - \lambda + \lambda_0 = 0. \tag{3.142}$$

We again need to linearize the fields:

$$(\mathcal{F}_R)_L = -6\gamma\lambda R_L,\ (\mathcal{F}_{\mathcal{G}})_L = \frac{9}{2}\gamma^2 R_L,\ (\mathcal{F})_L = (1+6\lambda+12\lambda^2)R_L. \tag{3.143}$$

Substituting the fields into Eq.(3.130), again all the $\Box R_L$ terms vanish and we finally get

$$(-1 + 12\lambda^2)R_L = 0. \tag{3.144}$$

As mentioned before, $R_L = 0$, then we find that $\lambda \neq \pm\frac{1}{2\sqrt{3}}$.
And Eq.(3.133) becomes

$$(1 - 12\lambda^2)(R_{\mu\nu})_L + \frac{1}{\gamma}h_{\mu\nu}(\lambda_0 - 2\lambda + 16\lambda^3) = 0. \tag{3.145}$$

Using the Eq.(3.142) we get

$$(1 - 12\lambda^2)\left((R_{\mu\nu})_L - \Lambda h_{\mu\nu}\right) = 0. \tag{3.146}$$

Then, by cubic truncation, we obtain the linearized Einstein theory (with a cosmological constant) imposing an effective Newton's constant

$$G_{eff} = \frac{1}{1 - 12\lambda^2}G_0. \tag{3.147}$$

The requirement of positive Newton's constant gives a restriction that $-\frac{1}{2\sqrt{3}} < \lambda < \frac{1}{2\sqrt{3}}$. Using the prior finding that $\lambda < \frac{1}{4}$, we actually have

$$-\frac{1}{2\sqrt{3}} < \lambda < \frac{1}{4}. \tag{3.148}$$

Then we compute that $-\frac{1}{3\sqrt{3}} < \lambda_0 < \frac{3}{16}$ using the Eq.(3.142). Again considering the prior result that $\lambda_0 < \frac{11}{64}$ we get

$$-\frac{1}{3\sqrt{3}} < \lambda_0 < \frac{11}{64}. \tag{3.149}$$

And the discriminant value is computed to be $\Delta = \frac{1}{16}(1 - 27\lambda_0^2)$. At cubic truncation we have a single solution in the allowed region.
At $\mathcal{O}(R^2)$ we have

$$\mathcal{F} = R - 2\Lambda_0 + \frac{9}{2}\gamma\mathcal{G} \tag{3.150}$$

which is the Lagrangian density for the Einstein-Gauss-Bonnet theory with a cosmological constant. And truncation to the $\mathcal{O}(R)$ gives the cosmological Einstein theory. By truncation we mean that we have studied the theory at $\mathcal{O}(R^2)$ and $\mathcal{O}(R^3)$ in addition to $\mathcal{O}(R^4)$.

CHAPTER 4

A BRIEF INTRODUCTION TO BLACK HOLES

An ordinary star balances its gravitational attraction with thermonuclear reactions by burning its source at the core of the star. After the burning steps, gases turn into denser metals. And finally when all the fuel is used up and transformed to iron we can list the alternatives for the fate of the star [13], [26], [32], [33] :

• If the mass of the star is under the Chandrasekhar limit (it is around $1.4M_\odot$ where $M_\odot$ is the mass of the Sun), it can balance its gravitational attraction by electron degeneracy pressure (due to the Pauli exclusion principle). Since it is an unlimited source of pressure, this star could continue its life in this mode. This kind of stars are named as *white dwarfs*.

• If the mass of the star is heavier than the Chandrasekhar limit, it can not balance its gravity and collapses. During this collapse, the neutron density increases and the star becomes a *neutron star*. Then the neutron degeneracy provides the stability.

• If the mass of the star is above $3M_\odot$, it continues to collapse and forms a *black hole*. It exerts such an extreme gravitation that anything (including photons) can not escape from this attraction once it passes through the black hole's *event horizon*.

4.1 Schwarzschild Black Holes[1]

Birkoff theorem states that the only spherically symmetric vacuum solution to Einstein's theory is the Schwarzschild metric:

$$ds^2 = -(1 - \frac{2GM}{r})dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \qquad (4.1)$$

---

[1] In this section we generally follow the calculations of [13] and also [14].

Although it is not assumed previously, we see that Schwarzschild metric is static i.e. it is independent of time. As discussed before, $r = 0$ is a real singularity and $r = r_S = 2GM$ is called as Schwarzschild radius. If we apply this solution to our Sun for example, we find that Schwarzschild radius of the Sun $\left(r_{S_\odot}\right)$ is smaller than its radius $(r_\odot)$:

$$r_{S_\odot} = 2G_\odot M_\odot < r_\odot \tag{4.2}$$

Since the Schwarzschild metric is found for a vacuum solution, we can not use it for this kind of objects but we can approximately use it as an exterior solution in the static spherically symmetric case. We will in this section assume that we are studying a compact object that is applicable to the Schwarzschild metric.

Let us consider the Schwarzschild metric while passing from the $r > r_S$ region through the $r < r_S$ region. Firstly, for interior regions the sign of the g_{tt} and g_{rr} components change:

g_{tt}: its sign is (-) for $r > r_S$; leads to a timelike coordinate whereas it is (+) for $r < r_S$ which is spacelike.

g_{rr}: its sign is (+) for $r > r_S$; leads to a spacelike coordinate whereas it turns to a timelike coordinate for $r < r_S$.

This odd behaviour of coordinates is a sign of improper coordinate chart. Anyway, let us study the geodesic equations to probe deeply. For simplicity we choose radially moving light rays. Then we have $d\theta = 0$, $d\phi = 0$, $ds^2 = 0$ and the metric becomes

$$0 = -(1 - \frac{2GM}{r})dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 \tag{4.3}$$

which yields

$$\frac{dt}{dr} = \mp \frac{1}{\left(1 - \frac{2GM}{r}\right)}. \tag{4.4}$$

For an outgoing photon as t increases r must increase since r is measured from the origin of the source. Then (+) sign refers to the outgoing ray and (-) sign refers to the incoming one.

Now let us start solving the Eq.(4.4) for an outgoing photon;

$$\int_0^t dt = \int_0^r \frac{1}{\left(1 - \frac{2GM}{r}\right)} dr = \int_0^r \left(\frac{2GM}{r - 2GM} + 1\right) dr, \tag{4.5}$$

$$\int_0^t dt = 2GM \int_0^r \frac{1}{r - 2GM} dr + \int_0^r dr. \tag{4.6}$$

Then we simply calculate that

$$t_1 = r_1 + 2GM \ln \left| \frac{r_1}{2GM} - 1 \right| + const. \tag{4.7}$$

for the outgoing photon (t_1, r_1). For the incoming photon (t_2, r_2) we find

$$t_2 = -r_2 - 2GM \ln \left| \frac{r_2}{2GM} - 1 \right| + const. \tag{4.8}$$

We can list some of the outcomes of these equations:

•There is a singularity at $r = 2GM$ as seen in the metric equation.

• If $t \to -t$, incoming photons are replaced by outcoming ones and vice versa.

•For the outgoing ray;

If $r < 2GM$: when r increases, t decreases!

If $r > 2GM$: when r increases, t increases also.

•For the incoming ray;

If $r < 2GM$: when r decreases, t also decreases!

If $r > 2GM$: when r decreases, t increases.

The incoming and outgoing rays are plotted in Figure 4.1; the apparently singular point $(r = 2GM)$ can be recognized easily (dashed line). We also see the odd behaviour of the rays coming from a point $r > r_S$; these rays need infinite time to cross this singular point. The incoming rays inside the Schwarzschild radius move until they reach to the real singular point $r = 0$. An outgoing ray at $r < r_S$ can move to the singularity $(r = 2GM)$ but it takes infinite time to cross this point; this means that it is actually trapped inside an unseen surface with radius $r = r_S$.

Peculiar results might be due to the inappropriate coordinates. Then let us choose a different chart for a better understanding. We continue to study radial null rays and introduce a new transformation:

$$\bar{t} = t \mp 2GM \ln \left(\frac{r}{2GM} - 1 \right), \tag{4.9}$$

and

$$d\bar{t} = dt \mp \frac{2GM}{r - 2GM} dr. \tag{4.10}$$

We firstly consider the (+) sign and calculate the new metric

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) d\bar{t}^2 + \frac{4GM}{r} dr d\bar{t} + \left(1 + \frac{2GM}{r}\right) dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2, \tag{4.11}$$

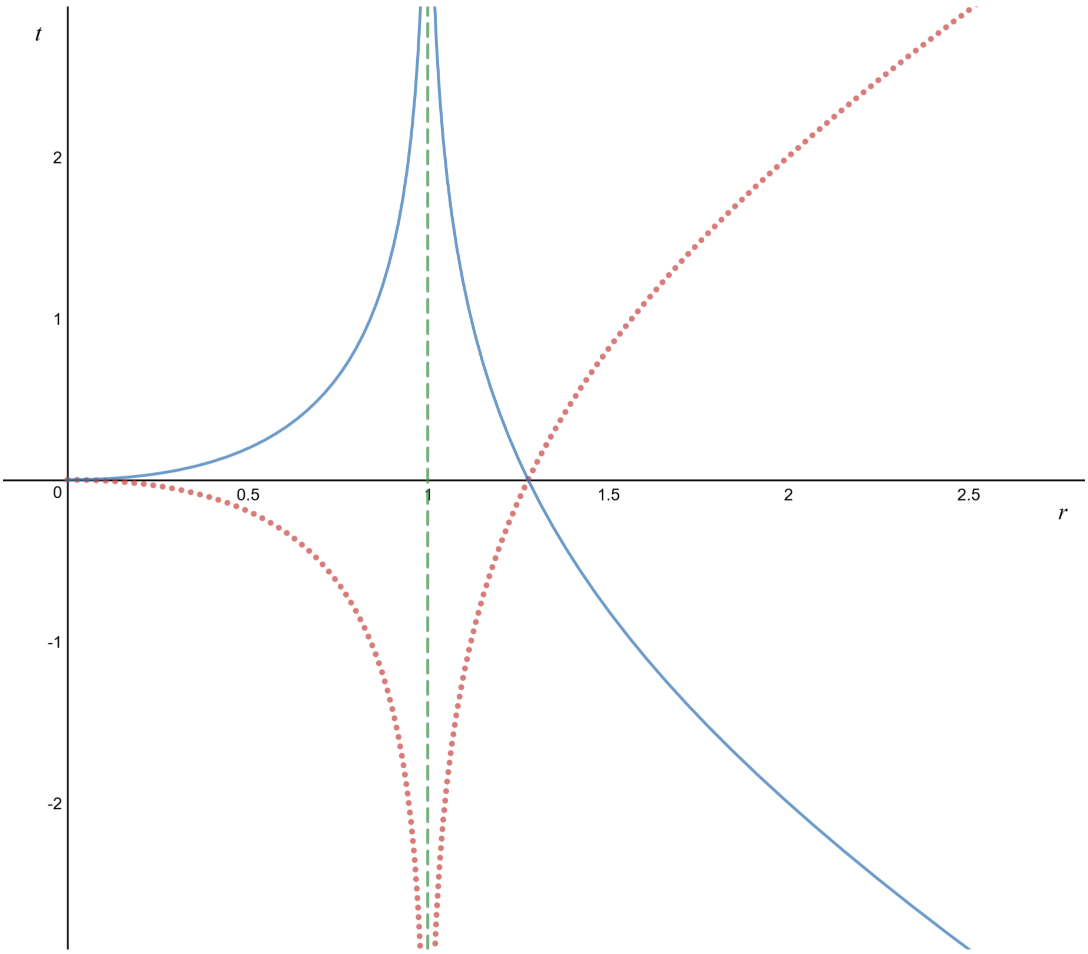

Figure 4.1: Incoming (blue solid lines) and outgoing (red dotted lines) rays in Schwarzschild coordinates. The dashed vertical line indicates the event horizon.

written in terms of the *advanced Eddington-Finkelstein* coordinates $(\bar{t}, r, \theta, \phi)$. Then the singularity problem at $r = 2GM$ seems to be solved. We can study the metric for the region $0 < r < \infty$. Now let us study radially moving null rays for this metric. Taking $ds^2 = d\theta = d\phi = 0$ we obtain

$$\left(\frac{r - 2GM}{r}\right)\frac{d\bar{t}^2}{dr^2} - \frac{4MG}{r}\frac{d\bar{t}}{dr} - \left(\frac{r + 2GM}{r}\right) = 0. \tag{4.12}$$

Solving this equation by ordinary methods we find two solutions. For the incoming photons we have

$$\frac{d\bar{t}}{dr} = -1 \tag{4.13}$$

which yields

$$\bar{t} = -r + const. \tag{4.14}$$

The second solution corresponds to outgoing rays

$$\frac{d\bar{t}}{dr} = \frac{r + 2GM}{r - 2GM}, \tag{4.15}$$

and when we solve this equation we obtain

$$\bar{t} = r + 4MG \ln\left|\frac{r}{2GM} - 1\right| + const. \tag{4.16}$$

Then we see that the incoming photons can cross through the surface $r = 2GM$ (Figure:4.2). Outgoing rays at $r < 2GM$ region never pass through this surface. Outgoing rays at $r > 2GM$ region can travel to infinity. We conclude that $r = 2GM$ serves like a one-way membrane. This boundary is called as the *event horizon* which is a defining property for black holes.

Now we turn back to Eq.(4.10) and study the remaining alternative which is

$$d\bar{t} = dt - \frac{2GM}{r - 2GM}dr \tag{4.17}$$

then the metric becomes

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)d\bar{t}^2 - \frac{4GM}{r}drd\bar{t} + \left(1 + \frac{2GM}{r}\right)dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2. \tag{4.18}$$

These coordinates are named as the *retarded Eddington-Finkelstein* coordinates. Next, we search for radially moving null rays again:

$$\left(\frac{r - 2GM}{r}\right)\frac{d\bar{t}^2}{dr^2} + \frac{4MG}{r}\frac{d\bar{t}}{dr} - \left(\frac{r + 2GM}{r}\right) = 0. \tag{4.19}$$

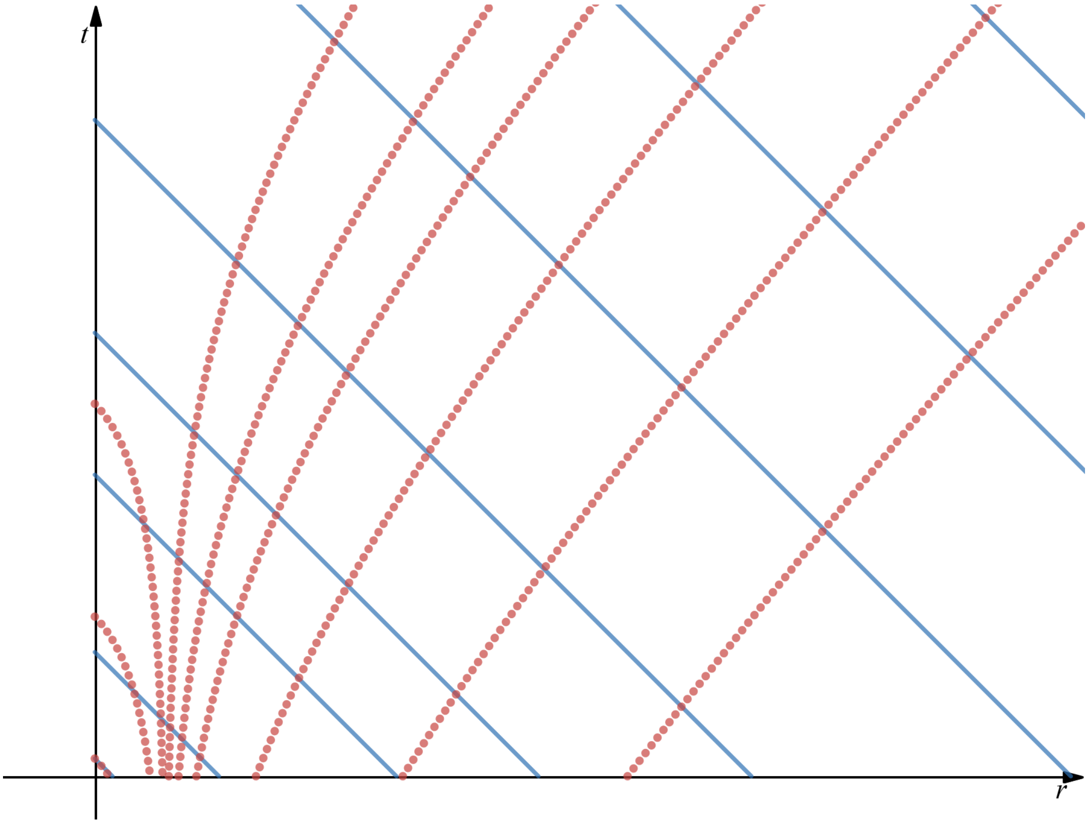

Figure 4.2: Incoming (blue solid lines) and outgoing (red dotted lines) rays in Eddington-Finkelstein coordinates indicating a black hole.

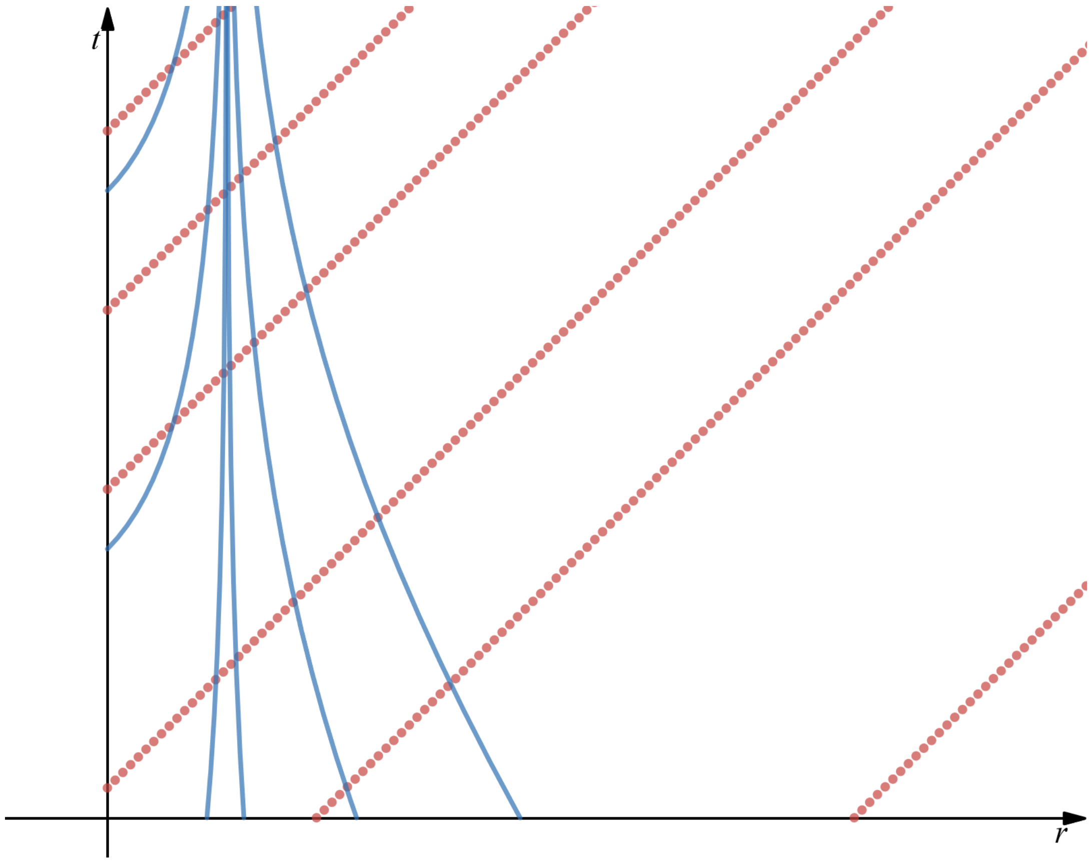

Figure 4.3: Incoming (blue solid lines) and outgoing (red dotted lines) rays in retarded Eddington-Finkelstein coordinates indicating a white hole.

The first solution implies the outgoing rays that $d\bar{t}/dr = 1$, so passing through the singularity these rays could travel to infinity (Figure:4.3). The second solution gives the incoming photons that

$$\frac{d\bar{t}}{dr} = \frac{r + 2GM}{-r + 2GM}. \tag{4.20}$$

Integration yields

$$\bar{t} = -r - 4MG \ln \left| \frac{-r}{2GM} + 1 \right| + const. \tag{4.21}$$

This time we see that incoming photons at region $r > 2GM$ do not pass through $r = 2GM$ surface. These objects are called as *white holes*.

These are the basic calculations on general black holes in Einstein's general relativity theory. Now we will study the black holes within the Ricci flatness assumption in our modified quartic theory and present an approximate spherically symmetric solution.

CHAPTER 5

RICCI FLAT, BLACK HOLE AND APPROXIMATE SPHERICALLY SYMMETRIC SOLUTIONS OF THE QUARTIC THEORY

5.1 Ricci Flat Solutions

In this section we continue to study the solutions of Einstein's gravity to search whether they also solve the quartic theory. For the vacuum case ($T_{\mu\nu} = 0$) and with $\Lambda_0 = 0$, Einstein's theory reduces to finding the Ricci flat ($R_{\mu\nu} = 0$) metrics. When we recalculate the fields taking into account these assumptions we have

$$\mathcal{F} = \frac{9\gamma\mathcal{G}}{2}\left(1 + \frac{9}{4}\gamma^2\mathcal{G}\right), \; \mathcal{F}_{\mathcal{G}} = \frac{9\gamma}{2}(1 + \frac{9}{2}\gamma^2\mathcal{G}), \; \mathcal{F}_R = 1 + \frac{9}{2}\gamma^2\mathcal{G}. \tag{5.1}$$

And the Gauss-Bonnet invariant turns out to be the Kretschmann scalar, for Ricci-flat metrics,

$$\mathcal{G} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}. \tag{5.2}$$

Since $G_{\mu\nu} = 0$ and $T_{\mu\nu} = 0$, trace of the field equations simplifies to

$$2\mathcal{G}\mathcal{F}_{\mathcal{G}} - 2\mathcal{F} + 3\Box\mathcal{F}_R = 0 \tag{5.3}$$

and substituting the fields we get a nonlinear wavelike equation for $\mathcal{G}$

$$\Box\mathcal{G} + \frac{3}{2}\mathcal{G}^2 = 0. \tag{5.4}$$

While solving an equation, we can always discard the undesired solutions and keep the physically good ones. With this reasoning we can eliminate the singular solutions of Eq.(5.4) even if there exist any. Then, we can conclude that our theory does not have a Kretschmann scalar-singularity. Recalling that this kind of singularity arises in Schwarzschild solution unavoidably, we see that our theory is free of a Schwarzschild singularity. Of course this is also true for the rotating (Kerr) solution which we have not discussed here.

Now this time let us find the traceless part of the field equations. We firstly write the field equations for Ricci flat metrics:

$$\frac{1}{2}g_{\mu\nu}(\mathcal{G}\mathcal{F}_{\mathcal{G}}-\mathcal{F})+\Big(g_{\mu\nu}\Box-\nabla_{\mu}\nabla_{\nu}\Big)\mathcal{F}_{R}+4\Big(2C_{\mu\sigma\nu\lambda}-R_{\mu\sigma\nu\lambda}\Big)\nabla^{\sigma}\nabla^{\lambda}\mathcal{F}_{\mathcal{G}}=0. \quad (5.5)$$

When we insert $\mathcal{F},\mathcal{F}_{\mathcal{R}}$ and $\mathcal{F}_{\mathcal{G}}$ we have

$$\frac{9}{8}g_{\mu\nu}\gamma\mathcal{G}^{2}+\left(g_{\mu\nu}\Box-\nabla_{\mu}\nabla_{\nu}\right)\mathcal{G}+18\gamma C_{\mu\sigma\nu\lambda}\nabla^{\sigma}\nabla^{\lambda}\mathcal{G}=0 \quad (5.6)$$

and then substituting the Eq.(5.4) we find the traceless part of the field equations in Ricci flat spacetime:

$$\left(\frac{1}{4}g_{\mu\nu}\Box-\nabla_{\mu}\nabla_{\nu}\right)\mathcal{G}+18\gamma C_{\mu\sigma\nu\lambda}\nabla^{\sigma}\nabla^{\lambda}\mathcal{G}=0. \quad (5.7)$$

As a final point, Ricci flat solutions survive in our theory; $\mathcal{G}$ being the Kretschmann scalar and satisfying Eq.(5.4) and Eq.(5.7).

5.2 Black Hole Solutions

The fact that our theory does not include a Schwarzschild singularity does not mean that our theory is free of all kinds of singularities. Still, we can search for a black hole type solution to investigate the theory better.

We consider a general static metric

$$ds^{2}=-N^{2}dt^{2}+h_{ab}dx^{a}dx^{b}. \quad (5.8)$$

Here h_{ab} is the 3-dimensional (spatial) part of the metric, so the indices a and b denotes the 3-dimensional spacetime. N and h_{ab} are the functions of spatial coordinates, since we assume staticity. For a black hole study we will use the method used in [19], [20], [21]. First of all we need a mathematical rewriting to continue. Considering a general coordinate dependent scalar $\psi=\psi(x_{a})$ we can do the simple calculation below

$$\begin{aligned}\Box\psi(x_{a}) &= g^{\mu\nu}\nabla_{\mu}\nabla_{\nu}\psi\\ &= g^{00}\nabla_{0}\nabla_{0}\psi+h^{ab}\nabla_{a}\nabla_{b}\psi\\ &= g^{00}\left(\partial_{0}\nabla_{0}\psi-\Gamma^{a}_{00}\nabla_{a}\psi\right)+h^{ab}\nabla_{a}\nabla_{b}\psi\\ &= -g^{00}\Gamma^{a}_{00}\nabla_{a}\psi+h^{ab}\nabla_{a}\nabla_{b}\psi \quad (5.9)\end{aligned}$$

since ψ is a time independent scalar i.e. $\nabla_0\psi = \partial_0\psi = 0$. Substituting $\Gamma^a_{00} = \frac{1}{2}h^{ab}\partial_b N^2 = h^{ab}N\partial_b N$ we find the result

$$\Box\psi(x_a) = \frac{1}{N^2}h^{ab}N\partial_b N\nabla_a\psi + h^{ab}\nabla_a\nabla_b\psi = \nabla^a\nabla_a\psi + \frac{1}{N}h^{ab}\partial_b N\nabla_a\psi. \quad (5.10)$$

We finally derived a new formulation of d'Alembertian operator for a general coordinate dependent (but time independent) scalar and we can use this new formulation to rewrite our wave function of $\mathcal{G}$. When we insert the above equation into Eq.(5.4) we get

$$\nabla^a\nabla_a\mathcal{G} + \frac{1}{N}\nabla^a N\nabla_a\mathcal{G} + \frac{3}{2}\gamma\mathcal{G}^2 = 0 \quad (5.11)$$

To continue we follow a procedure that we multiply Eq.(5.11) by $N\mathcal{G}$ and integrate over a 3-dimensional segment:

$$\int_S \sqrt{h}d^3x \left[N\mathcal{G}\nabla^a\nabla_a\mathcal{G} + \mathcal{G}\nabla^a N\nabla_a\mathcal{G} + \frac{3\gamma}{2}N\mathcal{G}^3\right] = 0. \quad (5.12)$$

After rewriting we have

$$\int_S \sqrt{h}d^3x \left[\nabla^a\left(N\mathcal{G}\nabla_a\mathcal{G}\right) - N\nabla^a\mathcal{G}\nabla_a\mathcal{G} + \frac{3\gamma}{2}N\mathcal{G}^3\right] = 0. \quad (5.13)$$

Now we need to interpret this equation carefully to understand whether our theory admits a black hole type solution or not. We take the integral from the horizon of the presupposed black hole to infinity (or a point very far away). If we decide on $\mathcal{G}$ to be zero we can not think of a black hole (then we have just a flat space since there is no gravitation any more). Back to the integral equation; the first term does not contribute due to the Gauss' theorem (N vanishes at the horizon by definition and $\mathcal{G}$ is taken to be zero at infinity by the asymptotical flatness). And since both N and $\mathcal{G}$ can be positive or negative, we can not decide that the remaining two integral terms are positive or negative definite. Then we allow $\mathcal{G}$ to survive as a Kretschmann scalar; it does not have to vanish. We can still consider black hole type solutions. Namely, our theory might have solutions with event horizons.

5.3 Approximate Spherically Symmetric Solutions

We assume a general spherically symmetric metric:

$$ds^2 = -g(r)^2 f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \quad (5.14)$$

where f and g are functions of r only. Now, using the computer program *Mathematica*, we will search for spherically symmetric solutions, for the case of zero cosmological constant.

Firstly, we observe that $f(r) = 1$ and $g(r) = 1$ is an exact solution (Minkowski metric). Namely, we have $\mathcal{E}_{\mu\nu} = 0$, here we label the field equations as $\mathcal{E}_{\mu\nu}$ for a simple notation ($\mathcal{E}_{\mu\nu} = 8\pi G_0 \, T_{\mu\nu}$) .

In order to study the Schwarzschild solution we fix $f(r) = 1 - \frac{2GM}{r}$ and $g(r) = 1$. Then we see that $\mathcal{E}_{\mu\nu}$ does not vanish as expected since we already found that Schwarzschild solution is not included in the quartic theory. For the Schwarzschild metric one obtains

$$\mathcal{E}_{tt} = -\frac{1296\gamma^2 G^2 m^2 \left(2Gm - r\right)\left(r^3 \left(11Gm - 5r\right) + 9\gamma Gm \left(67Gm - 32r\right)\right)}{r^{13}}, \tag{5.15}$$

$$\mathcal{E}_{rr} = \frac{1296\gamma^2 G^2 m^2 \left(r^3 \left(2r - 3Gm\right) + 9\gamma Gm \left(11Gm - 4r\right)\right)}{r^{11}\left(2GM - r\right)}, \tag{5.16}$$

$$\mathcal{E}_{\theta\theta} = \frac{1296\gamma^2 G^2 m^2 \left(2r^3 \left(3r - 7Gm\right) + 9\gamma Gm \left(41Gm - 18r\right)\right)}{r^{10}}, \tag{5.17}$$

$$\mathcal{E}_{\phi\phi} = \mathcal{E}_{\theta\theta} \sin^2\theta. \tag{5.18}$$

One can observe that for large r, one has

$\mathcal{E}_{tt} \approx \mathcal{O}\left(\frac{1}{r^8}\right),\ \mathcal{E}_{rr} \approx \mathcal{O}\left(\frac{1}{r^8}\right),\ \mathcal{E}_{\theta\theta} \approx \mathcal{O}\left(\frac{1}{r^6}\right), \mathcal{E}_{\phi\phi} \approx \mathcal{O}\left(\frac{1}{r^6}\right)$.

This says that even though the Schwarzschild metric is not an exact solution, it is an approximate solution up to $\mathcal{O}\left(\frac{1}{r^6}\right)$. The first corrections to the Schwarzschild metric come at $\mathcal{O}\left(\frac{1}{r^6}\right)$.

The corrected, approximate solution up to $\mathcal{O}\left(\frac{1}{r^8}\right)$ can be found as

$$f(r) = 1 - \frac{2GM}{r} - \frac{2592G^2m^2\gamma^2}{5r^6} + \frac{864G^3m^3\gamma^2}{r^7} + \mathcal{O}\left(\frac{1}{r^8}\right), \tag{5.19}$$

$$g(r) = 1 + \mathcal{O}\left(\frac{1}{r^8}\right). \tag{5.20}$$

Here we see that, at $\mathcal{O}\left(\frac{1}{r^{8+m}}\right)$ $(m = 0, 1, 2, ...)$, the metric does not satisfy the relation $g_{00}g^{rr} = -1$ due to the additional terms.

CHAPTER 6

CONCLUSIONS

To remedy the short-distance (or large field) regime problems of general relativity, there are many modified gravity models that extend general relativity. Most of these models are built on adding powers of the curvature tensor as $\alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu} + \gamma R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} + \mathcal{O}(R^3)$. Addition of these higher curvature terms do not affect the long-distance behaviour of the theory for small α, β, γ... parameters but yield an improved theory at short distances. Nevertheless, one can easily see that addition of these terms drastically change some salient features of Einstein's general relativity. For example, beside the massless graviton in general relativity, massive gravitons, massive scalar particles arise. Moreover, some of these particles, that contribute to gravity are ghosts or tachyons. This is quite disturbing both from the classical and quantum theory point of view. Another problem with these higher curvature theories is that, generically, in the presence of a cosmological constant or even in the absence of it, the maximally symmetric solution of the theory becomes degenerate. One can have Minkowski spacetime or de Sitter or anti-de Sitter spacetimes with different cosmological constants given by the parameters of the theory. In general, if one has R^n terms in the action one has up to n different vacua. These two problems prompted a recent research that led to a construction of higher curvature modifications of Einstein's gravity that has a unique vacuum and a single massless graviton as the only perturbative excitation about its vacuum. A class of theories in the Born-Infeld form was constructed that has this property. We studied a special form of the general action taking $a = 0$, $b = -5/2$ and $c = -1$, which is called the quartic gravity.

Using the linearization method we studied the particle spectrum of the theory. Linearized trace equation gives two restrictions that $\lambda = \gamma\Lambda \neq -\frac{1}{2}$ and $\lambda \neq \frac{1}{4}$ (with $R_L = 0$). Linearization procedure reveals that our theory admits linearized cosmo-

logical Einstein equations which ensure the single massless spin-2 excitation presence. Then we also calculated the effective Newton's constant. This study confirms the previous results ($\lambda \neq -\frac{1}{2}$ and $\lambda \neq \frac{1}{4}$) and gives a new restriction that $\lambda < \frac{1}{4}$. Then we also concluded that $\lambda_0 = \gamma\Lambda_0 < \frac{11}{64}$. And with the discriminant analysis we see that quartic gravity has single vacuum solution which is maximally symmetric.
Ricci flat and black hole solutions existing in Einstein's theory were studied for our quartic gravity. Ricci flatness condition gives an equation for Gauss-Bonnet invariant which is the Kretschmann scalar in our calculations. This equation is noteworthy to get rid of the Kretschmann scalar singularity which is inevitable in Einstein's theory leading to Schwarzschild or Kerr singularity. Our action does not have this kind of singularity but other black hole solutions can be searched. Using *Mathematica* we also showed that the Schwarzschild metric is not a solution in quartic gravity and we found an approximate solution for a spherically symmetric metric.
BI theory has been studied in the cosmological context: The inflation era of the BI universe was studied in [38]. This early stage of the universe was explained without introducing a field like an inflaton. The inflation is due to the ghostlike nature of the theory for the wrong vacuum. Another study is on the entropy of the universe [39]. Gibbons-Hawking entropy is reproduced and in this result G is replaced by K_{eff} leading a increased entropy in dS spacetimes.

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APPENDIX A

SOME BACKGROUND ON TENSORS

A.1 Bianchi Identity and the Einstein Tensor[1]

Covariant derivative is defined as

$$\nabla_\nu V_\mu = \partial_\nu V_\mu - \Gamma^\sigma_{\mu\nu} V_\sigma, \tag{A.1}$$

$$\nabla_\nu V^\mu = \partial_\nu V^\mu + \Gamma^\mu_{\sigma\nu} V^\sigma \tag{A.2}$$

for the covariant and contravariant components of a general vector V. A simple observation is that; if there is no curvature ($\Gamma^\sigma_{\mu\nu} = 0$) covariant derivative reduces to ordinary derivative.
The Riemann tensor is expressed as

$$R^\mu_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu_{\sigma\nu} - \partial_\sigma \Gamma^\mu_{\rho\nu} + \Gamma^\mu_{\rho\lambda}\Gamma^\lambda_{\sigma\nu} - \Gamma^\mu_{\sigma\lambda}\Gamma^\lambda_{\rho\nu} \tag{A.3}$$

and its covariant form is

$$\begin{aligned} R_{\alpha\beta\mu\nu} &= \frac{1}{2}\left(\partial_\nu\partial_\alpha g_{\beta\mu} - \partial_\nu\partial_\beta g_{\alpha\mu} + \partial_\mu\partial_\beta g_{\alpha\nu} - \partial_\mu\partial_\alpha g_{\beta\nu}\right) \\ &- g^{\sigma\rho}\left(\Gamma_{\sigma\alpha\mu}\Gamma_{\rho\beta\nu} - \Gamma_{\sigma\alpha\nu}\Gamma_{\rho\beta\mu}\right). \end{aligned} \tag{A.4}$$

Now suppose we study on a point A in geodesic coordinates. Then taking $\Gamma^\mu_{\alpha\beta} = 0$, Riemann tensor at this point becomes

$$R_{\alpha\beta\mu\nu} = \frac{1}{2}\left(\partial_\nu\partial_\alpha g_{\beta\mu} - \partial_\nu\partial_\beta g_{\alpha\mu} + \partial_\mu\partial_\beta g_{\alpha\nu} - \partial_\mu\partial_\alpha g_{\beta\nu}\right) \tag{A.5}$$

using Eq.(A.4). Now we can see the symmetries of the Riemann tensor

$$R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu}, \tag{A.6}$$

$$R_{\alpha\beta\mu\nu} = -R_{\alpha\beta\nu\mu}, \tag{A.7}$$

[1] In this section we generally follow the book [14].

$$R_{\alpha\beta\mu\nu} = R_{\mu\nu\alpha\beta}, \tag{A.8}$$

and the cyclic identity

$$R_{\alpha\beta\mu\nu} + R_{\alpha\mu\nu\beta} + R_{\alpha\nu\beta\mu} = 0. \tag{A.9}$$

Now let us take the covariant derivative of the Riemann tensor at this point A.

$$\begin{aligned}
\nabla_\sigma(R_{\mu\alpha\nu\beta}) &= \partial_\sigma(R_{\mu\alpha\nu\beta}) \\
&= \frac{1}{2}\partial_\sigma\left(\partial_\nu\partial_\alpha g_{\beta\mu} - \partial_\nu\partial_\beta g_{\alpha\mu} + \partial_\mu\partial_\beta g_{\alpha\nu} - \partial_\mu\partial_\alpha g_{\beta\nu}\right) \\
&= \partial_\sigma\partial_\nu\Gamma_{\mu\alpha\beta} - \partial_\sigma\partial_\beta\Gamma_{\mu\alpha\nu}.
\end{aligned} \tag{A.10}$$

Then it can be shown that

$$\nabla_\sigma R_{\mu\alpha\nu\beta} + \nabla_\nu R_{\mu\alpha\beta\sigma} + \nabla_\beta R_{\mu\alpha\sigma\nu} = 0. \tag{A.11}$$

This relation is named as Bianchi identity. Since we deal with tensorial equations, these results can be used at all other coordinates.

For a 4-rank tensor we generally have 6 contracted 2-rank tensors. However using the symmetries of the Riemann tensor we only get the *Ricci tensor*

$$R^\mu_{\alpha\mu\beta} = R_{\alpha\beta}. \tag{A.12}$$

In order to investigate the Ricci tensor let us do a ($\alpha - \mu$) contraction in (A.9):

$$R^\alpha_{\beta\alpha\nu} + R^\alpha_{\alpha\nu\beta} + R^\alpha_{\nu\beta\alpha} = 0. \tag{A.13}$$

$$R_{\beta\nu} + 0 - R^\alpha_{\nu\alpha\beta} = 0. \tag{A.14}$$

$$R_{\beta\nu} - R_{\nu\beta} = 0. \tag{A.15}$$

$$R_{\beta\nu} = R_{\nu\beta} \tag{A.16}$$

Then we see that Ricci tensor is symmetric. Now we could do one more contraction and derive the *Ricci scalar*

$$R^\mu_\mu = R. \tag{A.17}$$

Bianchi identity Eq.(A.11) can be written in a different form. We firstly contract β and μ;

$$\nabla_\sigma R^\beta_{\alpha\nu\beta} + \nabla_\nu R^\beta_{\alpha\beta\sigma} + \nabla_\beta R^\beta_{\alpha\sigma\nu} = 0. \tag{A.18}$$

$$-\nabla_\sigma R_{\alpha\nu} + \nabla_\nu R_{\alpha\sigma} + \nabla_\beta R^\beta_{\alpha\sigma\nu} = 0. \tag{A.19}$$

Now we perform a $(\alpha - \nu)$ contraction , then we have

$$-\nabla_\sigma R + \nabla_\nu R^\nu_\sigma + \nabla^\beta R_{\beta\sigma} = 0. \tag{A.20}$$

It can be written as

$$\nabla_\nu \left(R^\nu_\sigma - \frac{1}{2}\delta^\nu_\sigma R \right) = 0 \tag{A.21}$$

or

$$\nabla_\nu \left(R^{\nu\sigma} - \frac{1}{2} g^{\nu\sigma} R \right) = 0 \tag{A.22}$$

Then we find the Einstein tensor

$$G^{\nu\sigma} = \left(R^{\nu\sigma} - \frac{1}{2} g^{\nu\sigma} R \right) \tag{A.23}$$

with its conservation $\nabla_\nu G^{\nu\sigma} = 0$.

A.2 Variation of the Riemann tensor

Let us take the variation of the Riemann tensor Eq.(A.3)

$$\delta R^\mu_{\nu\rho\sigma} = \partial_\rho \delta\Gamma^\mu_{\sigma\nu} - \partial_\sigma \delta\Gamma^\mu_{\rho\nu} + \delta\Gamma^\mu_{\rho\lambda}\Gamma^\lambda_{\sigma\nu} + \Gamma^\mu_{\rho\lambda}\delta\Gamma^\lambda_{\sigma\nu} - \delta\Gamma^\mu_{\sigma\lambda}\Gamma^\lambda_{\nu\rho} - \Gamma^\mu_{\sigma\lambda}\delta\Gamma^\lambda_{\nu\rho}. \tag{A.24}$$

Using $\nabla_\rho \delta\Gamma^\mu_{\nu\sigma} = \partial_\rho \delta\Gamma^\mu_{\nu\sigma} + \Gamma^\mu_{\rho\beta}\delta\Gamma^\beta_{\nu\sigma} - \Gamma^\beta_{\rho\nu}\delta\Gamma^\mu_{\beta\sigma} - \Gamma^\beta_{\rho\sigma}\delta\Gamma^\mu_{\nu\beta}$, we get

$$\delta R^\mu{}_{\nu\rho\sigma} = \nabla_\rho \delta\Gamma^\mu_{\nu\sigma} - \nabla_\sigma \delta\Gamma^\mu_{\nu\rho}. \tag{A.25}$$

Note that

$$\begin{aligned} \delta R^{\mu\nu}_{\rho\sigma} &= \delta(g^{\nu\beta} R^\mu_{\beta\rho\sigma}) = \delta g^{\nu\beta} R^\mu_{\beta\rho\sigma} + g^{\nu\beta}\delta R^\mu_{\beta\rho\sigma} \\ &= \delta g^{\nu\beta} R^\mu_{\beta\rho\sigma} + g^{\nu\beta}(\nabla_\rho \delta\Gamma^\mu_{\beta\sigma} - \nabla_\sigma \delta\Gamma^\mu_{\beta\rho}) \end{aligned} \tag{A.26}$$

Now we need to calculate the variation of the Christoffel symbol. Starting from the usual formula

$$\Gamma^\mu_{\nu\sigma} = \frac{1}{2} g^{\mu\rho}(\partial_\nu g_{\sigma\rho} + \partial_\sigma g_{\rho\nu} - \partial_\rho g_{\nu\sigma}), \tag{A.27}$$

we find the variation as

$$\delta\Gamma^\mu_{\nu\sigma} = \frac{1}{2}\delta g^{\mu\rho}(\partial_\nu g_{\sigma\rho} + \partial_\sigma g_{\rho\nu} - \partial_\rho g_{\nu\sigma}) + \frac{1}{2} g^{\mu\rho}(\partial_\nu \delta g_{\sigma\rho} + \partial_\sigma \delta g_{\rho\nu} - \partial_\rho \delta g_{\nu\sigma}). \tag{A.28}$$

Using $\nabla_\nu \delta g_{\sigma\rho} = \partial_\nu \delta g_{\sigma\rho} - \Gamma^\gamma_{\nu\sigma}\delta g_{\gamma\rho} - \Gamma^\gamma_{\nu\rho}\delta g_{\sigma\gamma}$ and $\delta g_{\mu\nu} = -g_{\mu\sigma} g_{\nu\beta}\delta g^{\sigma\beta}$, we have

$$\delta\Gamma^\mu_{\nu\sigma} = \frac{1}{2} g^{\mu\rho}(\nabla_\nu g_{\sigma\rho} + \nabla_\sigma \delta g_{\rho\nu} - \nabla_\rho \delta g_{\nu\sigma}). \tag{A.29}$$

Using this result (A.29) we finally get

$$\begin{aligned}\delta R^{\mu\nu}_{\rho\sigma} =& \frac{1}{2}\left(g_{\alpha\rho}\nabla_\sigma\nabla^\nu - g_{\alpha\sigma}\nabla_\rho\nabla^\nu\right)\delta g^{\alpha\mu} + \frac{1}{2}\left(g_{\alpha\sigma}\nabla_\rho\nabla^\mu - g_{\alpha\rho}\nabla_\sigma\nabla^\mu\right)\delta g^{\alpha\nu} \\ &+ \frac{1}{2}R_{\rho\sigma}{}^{\mu}{}_{\alpha}\delta g^{\nu\alpha} - \frac{1}{2}R_{\rho\sigma}{}^{\nu}{}_{\alpha}\delta g^{\mu\alpha}.\end{aligned} \tag{A.30}$$

A.3 Integrability Condition [2]

Let us consider an arbitrary function F and its derivative in two different coordinate systems. The derivative of F in some coordinate chart is $\frac{\partial F}{\partial x^\mu}$ and it is $\frac{\partial F}{\partial x'^\mu}$ in some other coordinates. The corresponding transformation is written as

$$\frac{\partial F}{\partial x'^\mu} = \frac{\partial F}{\partial x^\nu}\frac{\partial x^\nu}{\partial x'^\mu} \tag{A.31}$$

or in terms of a covariant vector V;

$$V'_\mu = V_\nu \frac{\partial x^\nu}{\partial x'^\mu}. \tag{A.32}$$

Here V_μ is the derivative of the function F in x^μ-coordinates and V'_μ is the one in x'^μ-coordinates. And the inverse transformation can be stated as

$$V_\nu = V'_\mu \frac{\partial x'^\mu}{\partial x^\nu}. \tag{A.33}$$

In fact this covariant vector V is the gradient of the function F by definition. The covariant derivative of a function reduces to ordinary derivative and

$$\nabla_\mu\nabla_\nu F - \nabla_\nu\nabla_\mu F = \partial_\mu\partial_\nu F - \partial_\nu\partial_\mu F = 0. \tag{A.34}$$

Now we take the second covariant derivative of a (0,1) rank vector T_μ;

$$\nabla_\rho\nabla_\nu T_\mu = \nabla_\rho(\nabla_\nu T_\mu) = \partial_\rho(\nabla_\nu T_\mu) - \Gamma^\sigma_{\rho\nu}\nabla_\sigma T_\mu - \Gamma^\sigma_{\rho\mu}\nabla_\nu T_\sigma. \tag{A.35}$$

We substitute $\nabla_\nu T_\mu = \partial_\nu T_\mu - \Gamma^\sigma_{\nu\mu}T_\sigma$;

$$\begin{aligned}\nabla_\rho\nabla_\nu T_\mu =& \partial_\rho\partial_\nu T_\mu - T_\sigma\partial_\rho\Gamma^\sigma_{\nu\mu} - \Gamma^\sigma_{\nu\mu}\partial_\rho T_\sigma - \Gamma^\sigma_{\rho\nu}\partial_\sigma T_\mu + \Gamma^\sigma_{\rho\nu}\Gamma^\alpha_{\sigma\mu}T_\alpha \\ &- \Gamma^\sigma_{\rho\mu}\partial_\nu T_\sigma + \Gamma^\sigma_{\rho\mu}\Gamma^\alpha_{\nu\sigma}T_\alpha.\end{aligned} \tag{A.36}$$

Similarly

$$\begin{aligned}\nabla_\nu\nabla_\rho T_\mu =& \partial_\nu\partial_\rho T_\mu - T_\sigma\partial_\nu\Gamma^\sigma_{\rho\mu} - \Gamma^\sigma_{\rho\mu}\partial_\nu T_\sigma - \Gamma^\sigma_{\nu\rho}\partial_\sigma T_\mu + \Gamma^\sigma_{\nu\rho}\Gamma^\alpha_{\sigma\mu}T_\alpha \\ &- \Gamma^\sigma_{\nu\mu}\partial_\rho T_\sigma + \Gamma^\sigma_{\nu\mu}\Gamma^\alpha_{\rho\sigma}T_\alpha.\end{aligned} \tag{A.37}$$

[2] In this section we follow the book [17].

Then we calculate that

$$\nabla_\rho \nabla_\nu T_\mu - \nabla_\nu \nabla_\rho T_\mu = T_\sigma(\partial_\nu \Gamma^\sigma_{\rho\mu} - \partial_\rho \Gamma^\sigma_{\nu\mu}) + T_\alpha(\Gamma^\sigma_{\rho\mu}\Gamma^\alpha_{\nu\sigma} - \Gamma^\sigma_{\nu\mu}\Gamma^\alpha_{\rho\sigma}) \tag{A.38}$$

which can be written as [3]

$$\nabla_\rho \nabla_\nu T_\mu - \nabla_\nu \nabla_\rho T_\mu = T_\sigma R^\sigma{}_{\mu\nu\rho}. \tag{A.39}$$

Let us do the same calculation for a tensor $T_{\mu\nu}$. Its second covariant derivative is

$$\begin{aligned} \nabla_\rho \nabla_\sigma T_{\mu\nu} &= \nabla_\rho(\nabla_\sigma T_{\mu\nu}) \\ &= \partial_\rho(\nabla_\sigma T_{\mu\nu}) - \Gamma^\alpha_{\rho\sigma}\nabla_\alpha T_{\mu\nu} - \Gamma^\alpha_{\rho\mu}\nabla_\sigma T_{\alpha\nu} - \Gamma^\alpha_{\rho\nu}\nabla_\sigma T_{\mu\alpha}. \end{aligned} \tag{A.40}$$

Substituting $\nabla_\sigma T_{\mu\nu} = \partial_\sigma T_{\mu\nu} - \Gamma^\beta_{\sigma\mu} T_{\beta\nu} - \Gamma^\beta_{\sigma\nu} T_{\mu\beta}$, we have

$$\begin{aligned} \nabla_\rho \nabla_\sigma T_{\mu\nu} &= \partial_\rho \partial_\sigma T_{\mu\nu} - \left(\partial_\rho \Gamma^\beta_{\sigma\mu}\right) T_{\beta\nu} - \Gamma^\beta_{\sigma\mu}\partial_\rho T_{\beta\nu} - \left(\partial_\rho \Gamma^\beta_{\sigma\nu}\right) T_{\mu\beta} \\ &- \Gamma^\beta_{\sigma\nu}\partial_\rho T_{\mu\beta} - \Gamma^\alpha_{\rho\sigma}\left(\partial_\alpha T_{\mu\nu} - \Gamma^\beta_{\alpha\mu}T_{\beta\nu} - \Gamma^\beta_{\alpha\nu}T_{\mu\beta}\right) \\ &- \Gamma^\alpha_{\rho\mu}\left(\partial_\sigma T_{\alpha\nu} - \Gamma^\beta_{\sigma\alpha}T_{\beta\nu} - \Gamma^\beta_{\sigma\nu}T_{\alpha\beta}\right) \\ &- \Gamma^\alpha_{\rho\nu}\left(\partial_\sigma T_{\mu\alpha} - \Gamma^\beta_{\sigma\mu}T_{\beta\alpha} - \Gamma^\beta_{\sigma\alpha}T_{\mu\beta}\right). \end{aligned} \tag{A.41}$$

Similarly

$$\begin{aligned} \nabla_\sigma \nabla_\rho T_{\mu\nu} &= \partial_\sigma \partial_\rho T_{\mu\nu} - \left(\partial_\sigma \Gamma^\beta_{\rho\mu}\right) T_{\beta\nu} - \Gamma^\beta_{\rho\mu}\partial_\sigma T_{\beta\nu} - \left(\partial_\sigma \Gamma^\beta_{\rho\nu}\right) T_{\mu\beta} \\ &- \Gamma^\beta_{\rho\nu}\partial_\sigma T_{\mu\beta} - \Gamma^\alpha_{\sigma\rho}\left(\partial_\alpha T_{\mu\nu} - \Gamma^\beta_{\alpha\mu}T_{\beta\nu} - \Gamma^\beta_{\alpha\nu}T_{\mu\beta}\right) \\ &- \Gamma^\alpha_{\sigma\mu}\left(\partial_\rho T_{\alpha\nu} - \Gamma^\beta_{\rho\alpha}T_{\beta\nu} - \Gamma^\beta_{\rho\nu}T_{\alpha\beta}\right) \\ &- \Gamma^\alpha_{\sigma\nu}\left(\partial_\rho T_{\mu\alpha} - \Gamma^\beta_{\rho\mu}T_{\beta\alpha} - \Gamma^\beta_{\rho\alpha}T_{\mu\beta}\right). \end{aligned} \tag{A.42}$$

Then we get

$$\begin{aligned} \nabla_\rho \nabla_\sigma T_{\mu\nu} - \nabla_\sigma \nabla_\rho T_{\mu\nu} &= T_{\beta\nu}\left(\partial_\sigma \Gamma^\beta_{\rho\mu} - \partial_\rho \Gamma^\beta_{\sigma\mu} - \Gamma^\alpha_{\sigma\mu}\Gamma^\beta_{\rho\alpha} + \Gamma^\alpha_{\rho\mu}\Gamma^\beta_{\sigma\alpha}\right) \\ &+ T_{\mu\beta}\left(\partial_\sigma \Gamma^\beta_{\rho\nu} - \partial_\rho \Gamma^\beta_{\sigma\nu} - \Gamma^\alpha_{\sigma\nu}\Gamma^\beta_{\rho\alpha} + \Gamma^\alpha_{\rho\nu}\Gamma^\beta_{\sigma\alpha}\right). \end{aligned} \tag{A.43}$$

Using the definition of the Riemann tensor, we rewrite the above equation as

$$\nabla_\rho \nabla_\sigma T_{\mu\nu} - \nabla_\sigma \nabla_\rho T_{\mu\nu} = T_{\beta\nu} R^\beta_{\mu\sigma\rho} + T_{\mu\beta} R^\beta_{\nu\sigma\rho}. \tag{A.44}$$

[3] Taking double covariant derivative of a tensor preserves the tensorial property. Observe that the LHS of (A.39) is tensorial and on the RHS we have a multiplication of two independent quantities; $R^\sigma_{\mu\nu\rho}$ and a vector. Then by the quotient theorem $R^\sigma_{\mu\nu\rho}$ is a tensor [14].

We can generalize these findings for rank n tensor such that

$$\nabla_\rho \nabla_\sigma T_{\mu_1 \ldots \mu_n} - \nabla_\sigma \nabla_\rho T_{\mu_1 \ldots \mu_n} = \sum_{\alpha}^{1,\ldots,n} T_{\mu_1 \ldots \mu_{\alpha-1} \beta \mu_{\alpha+1} \ldots \mu_n} R^{\beta}_{\mu_\alpha \sigma \rho}. \tag{A.45}$$

This is the compatibility or the integrability condition. For consistency, a general rank-n tensor must satisfy this equation. The reasoning comes from the flat space. Recall that the integrability condition in a flat geometry is

$$\partial_\mu \partial_\nu V_\alpha - \partial_\nu \partial_\mu V_\alpha = 0. \tag{A.46}$$

In curved spaces we replace ordinary differentiation with covariant one. Then we find a condition related to Riemann tensor. For historical reasons this condition frequently known as *Ricci identity* [34].

A.4 Quartic Equations

A quartic equation is defined as

$$x^4 + ax^3 + bx^2 + cx + d = 0 \tag{A.47}$$

where a, b, c and d are coefficients. Quartic polynomials are solvable equations using some methods and L. Ferrari was the first to solve this kind of equations (in the 16th century). Besides, we could also get some information about the nature of the solutions via the discussion on the discriminant value.
For the quartic equations we calculate the discriminant using the below formula (Item 4 by Schroeppel in [36]).

$$\begin{aligned} \triangle = & - 27c^4 + 18abc^3 - 4a^3c^3 - 4b^3c^2 + a^2b^2c^2 \\ & + d\left(144bc^2 - 6a^2c^2 - 80ab^2 + 18a^3bc + 16b^4 - 4a^2b^3\right) \\ & + d^2\left(-192ac - 128b^2 + 144a^2b - 27a^4 - 256d\right). \end{aligned} \tag{A.48}$$

Then we can analyse the Eq.(A.47) such that [37] :
• If $\triangle > 0$, we have 4 real or 4 imaginary roots .
• If $\triangle < 0$, we have 2 real and 2 imaginary roots .
• If $\triangle = 0$, we have 2 or more equal roots.

APPENDIX B

SOME LINEARIZATION CALCULATIONS

B.1 Linearization of the Christoffel Connection

Christoffel connection (it is not a tensor) is defined as

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} (\partial_\alpha g_{\beta\nu} + \partial_\beta g_{\alpha\nu} - \partial_\nu g_{\alpha\beta}). \tag{B.1}$$

In the weak field approximation we can assume that $g_{\mu\nu} = \overline{g}_{\mu\nu} + h_{\mu\nu}$ and $g^{\mu\nu} = \overline{g}^{\mu\nu} - h^{\mu\nu}$. We directly substitute these assumptions into the Eq.(B.1):

$$\begin{aligned} \Gamma^{\mu}_{\alpha\beta} &= \frac{1}{2}(\overline{g}^{\mu\nu} - h^{\mu\nu}) \left[\partial_\alpha \left(\overline{g}_{\beta\nu} + h_{\beta\nu}\right) + \partial_\beta \left(\overline{g}_{\alpha\nu} + h_{\alpha\nu}\right) - \partial_\nu \left(\overline{g}_{\alpha\beta} + h_{\alpha\beta}\right)\right] \\ &= \frac{1}{2}\overline{g}^{\mu\nu} \left(\partial_\alpha \overline{g}_{\beta\nu} + \partial_\beta \overline{g}_{\alpha\nu} - \partial_\nu \overline{g}_{\alpha\beta}\right) + \frac{1}{2}\overline{g}^{\mu\nu} \left(\partial_\alpha h_{\beta\nu} + \partial_\beta h_{\alpha\nu} - \partial_\nu h_{\alpha\beta}\right) \\ &- \frac{1}{2} h^{\mu\nu} \left(\partial_\alpha \overline{g}_{\beta\nu} + \partial_\beta \overline{g}_{\alpha\nu} - \partial_\nu \overline{g}_{\alpha\beta}\right) \end{aligned} \tag{B.2}$$

to the first order in $h_{\mu\nu}$. We specify the Christoffel connection calculated in the background metric as

$$\overline{\Gamma}^{\mu}_{\alpha\beta} = \frac{1}{2}\overline{g}^{\mu\nu} \left(\partial_\alpha \overline{g}_{\beta\nu} + \partial_\beta \overline{g}_{\alpha\nu} - \partial_\nu \overline{g}_{\alpha\beta}\right). \tag{B.3}$$

Then we have

$$\Gamma^{\mu}_{\alpha\beta} = \overline{\Gamma}^{\mu}_{\alpha\beta} + \frac{1}{2}\overline{g}^{\mu\nu} \left(\partial_\alpha h_{\beta\nu} + \partial_\beta h_{\alpha\nu} - \partial_\nu h_{\alpha\beta}\right) - \frac{1}{2} h^{\mu\nu} \left(\partial_\alpha \overline{g}_{\beta\nu} + \partial_\beta \overline{g}_{\alpha\nu} - \partial_\nu \overline{g}_{\alpha\beta}\right). \tag{B.4}$$

Now using the metric compatibility $\left(\overline{\nabla}_\nu \overline{g}_{\alpha\beta} = 0\right)$ and the definition of the covariant derivative $\left(\overline{\nabla}_\alpha \overline{g}_{\beta\nu} = \partial_\alpha \overline{g}_{\beta\nu} - \overline{\Gamma}^{\sigma}_{\alpha\beta}\overline{g}_{\sigma\nu} - \overline{\Gamma}^{\sigma}_{\alpha\nu}\overline{g}_{\sigma\beta}\right)$ in background space, we can write down

$$\partial_\alpha \overline{g}_{\beta\nu} + \partial_\beta \overline{g}_{\alpha\nu} - \partial_\nu \overline{g}_{\alpha\beta} = 2\overline{\Gamma}^{\sigma}_{\alpha\beta}\, \overline{g}_{\sigma\nu} \tag{B.5}$$

and again from the definition $\overline{\nabla}_\alpha h_{\beta\nu} = \partial_\alpha h_{\beta\nu} - \overline{\Gamma}^{\sigma}_{\alpha\beta} h_{\sigma\nu} - \overline{\Gamma}^{\sigma}_{\alpha\nu} h_{\sigma\beta}$ we find

$$\partial_\alpha h_{\beta\nu} + \partial_\beta h_{\alpha\nu} - \partial_\nu h_{\alpha\beta} = \overline{\nabla}_\alpha h_{\beta\nu} + \overline{\nabla}_\beta h_{\alpha\nu} - \overline{\nabla}_\nu h_{\alpha\beta} + 2\overline{\Gamma}^{\sigma}_{\alpha\beta} h_{\sigma\nu}. \tag{B.6}$$

Using Eq.(B.5) and Eq.(B.6) we have

$$\Gamma^{\mu}_{\alpha\beta} = \overline{\Gamma}^{\mu}_{\alpha\beta} + \frac{1}{2}\overline{g}^{\mu\nu}\left(\overline{\nabla}_{\alpha}h_{\beta\nu} + \overline{\nabla}_{\beta}h_{\alpha\nu} - \overline{\nabla}_{\nu}h_{\alpha\beta}\right) + \overline{\Gamma}^{\sigma}_{\alpha\beta}\left(\overline{g}^{\mu\nu}h_{\sigma\nu} - \overline{g}_{\sigma\nu}h^{\mu\nu}\right). \quad (B.7)$$

Then we finally get the result

$$(\Gamma^{\mu}_{\alpha\beta})_{L} = \Gamma^{\mu}_{\alpha\beta} - \overline{\Gamma}^{\mu}_{\alpha\beta} = \frac{1}{2}\overline{g}^{\mu\nu}\left(\overline{\nabla}_{\alpha}h_{\beta\nu} + \overline{\nabla}_{\beta}h_{\alpha\nu} - \overline{\nabla}_{\nu}h_{\alpha\beta}\right). \quad (B.8)$$

B.2 Linearization of the source-free Einstein's Equation

Unlike the previous section (we used the original method there) , we will linearize the below equation using the somehow direct means. The procedure is as follows: We replace all the terms with their linearized version. If there is a multiplication of terms we linearize the first term multiplied with the background value of the second term and vice versa.
Cosmological Einstein equation without any source is:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 0. \quad (B.9)$$

Taking the linearization we have

$$(R_{\mu\nu})_{L} - \frac{1}{2}h_{\mu\nu}\overline{R} - \frac{1}{2}\overline{g}_{\mu\nu}R_{L} + \Lambda h_{\mu\nu} = 0. \quad (B.10)$$

In n dimensions (for this section we do the calculations in n dim.) we can write the background value of the Riemann tensor as

$$\overline{R}_{\mu\alpha\nu\beta} = \frac{2\Lambda}{(n-1)(n-2)}(\overline{g}_{\mu\nu}\overline{g}_{\alpha\beta} - \overline{g}_{\mu\beta}\overline{g}_{\alpha\nu}) \quad (B.11)$$

and by contraction we get $\overline{R}_{\mu\nu} = \frac{2\Lambda}{(n-2)}\overline{g}_{\mu\nu}$ and $\overline{R} = \frac{2n}{n-2}\Lambda$. Substituting $\overline{R}$ into Eq.(B.10) we have

$$(R_{\mu\nu})_{L} - \frac{1}{2}\overline{g}_{\mu\nu}R_{L} - \frac{2\Lambda}{n-2}h_{\mu\nu} = 0. \quad (B.12)$$

Now we need to calculate $(R_{\mu\nu})_{L}$ and R_{L}. Let us write the Riemann tensor again and linearize it . The Riemann tensor is

$$R^{\mu}_{\alpha\beta\nu} = \partial_{\beta}\Gamma^{\mu}_{\alpha\nu} - \partial_{\nu}\Gamma^{\mu}_{\alpha\beta} + \Gamma^{\sigma}_{\alpha\nu}\Gamma^{\mu}_{\sigma\beta} - \Gamma^{\sigma}_{\alpha\beta}\Gamma^{\mu}_{\sigma\nu} \quad (B.13)$$

and after the linearization we get

$$\begin{aligned}(R^{\mu}_{\alpha\beta\nu})_{L} &= \partial_{\beta}(\Gamma^{\mu}_{\alpha\nu})_{L} - \partial_{\nu}(\Gamma^{\mu}_{\alpha\beta})_{L} + (\Gamma^{\sigma}_{\alpha\nu})_{L}\,\overline{\Gamma}^{\mu}_{\sigma\beta} + \overline{\Gamma}^{\sigma}_{\alpha\nu}\,(\Gamma^{\mu}_{\sigma\beta})_{L} \\ &- (\Gamma^{\sigma}_{\alpha\beta})_{L}\,\overline{\Gamma}^{\mu}_{\sigma\nu} - \overline{\Gamma}^{\sigma}_{\alpha\beta}\,(\Gamma^{\mu}_{\sigma\nu})_{L}. \quad (B.14)\end{aligned}$$

Now consider the covariant derivative of the linearized Christoffel connection. Using

$$\overline{\nabla}_\beta(\Gamma^\mu_{\alpha\nu})_L = \partial_\beta(\Gamma^\mu_{\alpha\nu})_L + \overline{\Gamma}^\mu_{\beta\sigma}(\Gamma^\sigma_{\alpha\nu})_L - \overline{\Gamma}^\sigma_{\beta\alpha}(\Gamma^\mu_{\sigma\nu})_L - \overline{\Gamma}^\sigma_{\beta\nu}(\Gamma^\mu_{\alpha\sigma})_L \tag{B.15}$$

and

$$\overline{\nabla}_\nu(\Gamma^\mu_{\alpha\beta})_L = \partial_\nu(\Gamma^\mu_{\alpha\beta})_L + \overline{\Gamma}^\mu_{\nu\sigma}(\Gamma^\sigma_{\alpha\beta})_L - \overline{\Gamma}^\sigma_{\nu\alpha}(\Gamma^\mu_{\sigma\beta})_L - \overline{\Gamma}^\sigma_{\nu\beta}(\Gamma^\mu_{\alpha\sigma})_L \tag{B.16}$$

we find that

$$(R^\mu_{\alpha\beta\nu})_L = \overline{\nabla}_\beta(\Gamma^\mu_{\alpha\nu})_L - \overline{\nabla}_\nu(\Gamma^\mu_{\alpha\beta})_L. \tag{B.17}$$

And contraction gives the linearized Ricci tensor:

$$(R^\mu_{\alpha\mu\nu})_L = (R_{\alpha\nu})_L = \overline{\nabla}_\mu(\Gamma^\mu_{\alpha\nu})_L - \overline{\nabla}_\nu(\Gamma^\mu_{\alpha\mu})_L \tag{B.18}$$

or

$$(R_{\mu\nu})_L = \overline{\nabla}_\alpha(\Gamma^\alpha_{\mu\nu})_L - \overline{\nabla}_\nu(\Gamma^\alpha_{\mu\alpha})_L. \tag{B.19}$$

Now we can substitute the linearized Christoffel connection Eq.(B.8) into above equation;

$$\begin{aligned}(R_{\mu\nu})_L &= \overline{\nabla}_\alpha\left[\frac{1}{2}\overline{g}^{\alpha\sigma}\left(\overline{\nabla}_\mu h_{\nu\sigma} + \overline{\nabla}_\nu h_{\mu\sigma} - \overline{\nabla}_\sigma h_{\mu\nu}\right)\right] \\ &- \overline{\nabla}_\nu\left[\frac{1}{2}\overline{g}^{\alpha\sigma}\left(\overline{\nabla}_\mu h_{\alpha\sigma} + \overline{\nabla}_\alpha h_{\mu\sigma} - \overline{\nabla}_\sigma h_{\mu\alpha}\right)\right] \\ &= \frac{1}{2}\left(\overline{\nabla}^\sigma\overline{\nabla}_\mu h_{\nu\sigma} + \overline{\nabla}^\sigma\overline{\nabla}_\nu h_{\mu\sigma} - \overline{\Box} h_{\mu\nu} - \overline{\nabla}_\nu\overline{\nabla}_\mu h\right)\end{aligned} \tag{B.20}$$

where $h \equiv \overline{g}^{\mu\nu}h_{\mu\nu}$.

And the linearization of Ricci scalar is simply calculated as follows.

$$\begin{aligned}(R)_L &= (g^{\mu\nu}R_{\mu\nu})_L = (R_{\mu\nu})_L\,\overline{g}^{\mu\nu} - \overline{R}_{\mu\nu}h^{\mu\nu} \\ &= \frac{1}{2}\left(\overline{\nabla}^\sigma\overline{\nabla}_\mu h_{\nu\sigma} + \overline{\nabla}^\sigma\overline{\nabla}_\nu h_{\mu\sigma} - \overline{\Box} h_{\mu\nu} - \overline{\nabla}_\nu\overline{\nabla}_\mu h\right)\overline{g}^{\mu\nu} - \frac{2\Lambda}{(n-2)}\overline{g}_{\mu\nu}h^{\mu\nu} \\ &= \overline{\nabla}^\sigma\overline{\nabla}^\mu h_{\mu\sigma} - \overline{\Box} h - \frac{2\Lambda}{(n-2)}h.\end{aligned} \tag{B.21}$$

Now we complete the linearized form of the cosmological Einstein equation generally in n dimensions:

$$\begin{aligned}&\left(\overline{\nabla}^\sigma\overline{\nabla}_\mu h_{\nu\sigma} + \overline{\nabla}^\sigma\overline{\nabla}_\nu h_{\mu\sigma} - \overline{\Box} h_{\mu\nu} - \overline{\nabla}_\nu\overline{\nabla}_\mu h\right) \\ &-\overline{g}_{\mu\nu}\left(\overline{\nabla}^\sigma\overline{\nabla}^\alpha h_{\alpha\sigma} - \overline{\Box} h - \frac{4\Lambda}{(n-2)}h\right) - \frac{2\Lambda}{(n-2)}h_{\mu\nu} = 0.\end{aligned} \tag{B.22}$$

B.3 Linearization of the Gauss-Bonnet Invariant

In this section we perform the linearization for the Gauss-Bonnet term $\mathcal{G} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ in 4 dimensions. We again directly take the linearized form as follows.

$$\mathcal{G}_L = 2\overline{R}R_L - 4(R_{\mu\nu}R^{\mu\nu})_L + (R_{\mu\alpha\nu\beta}R^{\mu\alpha\nu\beta})_L. \tag{B.23}$$

We already know some of the terms from the previous sections. Let us calculate

$$\begin{aligned}
(R_{\mu\nu}R^{\mu\nu})_L &= (R_{\mu\nu}R_{\alpha\beta}g^{\alpha\mu}g^{\beta\nu})_L \\
&= (R_{\mu\nu})_L\overline{R}_{\alpha\beta}\,\overline{g}^{\alpha\mu}\overline{g}^{\beta\nu} + \overline{R}_{\mu\nu}(R_{\alpha\beta})_L\,\overline{g}^{\alpha\mu}\overline{g}^{\beta\nu} \\
&- \overline{R}_{\mu\nu}\overline{R}_{\alpha\beta}h^{\alpha\mu}\overline{g}^{\beta\nu} - \overline{R}_{\mu\nu}\overline{R}_{\alpha\beta}\,\overline{g}^{\alpha\mu}h^{\beta\nu} \\
&= \frac{\Lambda}{2}\left(\overline{\nabla}^{\sigma}\overline{\nabla}_{\mu}h_{\nu\sigma} + \overline{\nabla}^{\sigma}\overline{\nabla}_{\nu}h_{\mu\sigma} - \Box h_{\mu\nu} - \overline{\nabla}_{\nu}\overline{\nabla}_{\mu}h\right)\overline{g}_{\alpha\beta}\overline{g}^{\alpha\mu}\overline{g}^{\beta\nu} \\
&+ \frac{\Lambda}{2}\left(\overline{\nabla}^{\sigma}\overline{\nabla}_{\alpha}h_{\beta\sigma} + \overline{\nabla}^{\sigma}\overline{\nabla}_{\beta}h_{\mu\sigma} - \Box h_{\alpha\beta} - \overline{\nabla}_{\beta}\overline{\nabla}_{\alpha}h\right)\overline{g}_{\mu\nu}\overline{g}^{\alpha\mu}\overline{g}^{\beta\nu} \\
&- \Lambda^2\,\overline{g}_{\mu\nu}\overline{g}_{\alpha\beta}h^{\alpha\mu}\overline{g}^{\beta\nu} - \Lambda^2\overline{g}_{\mu\nu}\overline{g}_{\alpha\beta}\overline{g}^{\alpha\mu}h^{\beta\nu} \\
&= 2\Lambda(\overline{\nabla}^{\sigma}\overline{\nabla}^{\nu}h_{\nu\sigma} - \Box h - \Lambda h) \\
&= 2\Lambda R_L.
\end{aligned} \tag{B.24}$$

Now we need to calculate the linearized Riemann tensor:

$$\begin{aligned}
(R^{\mu}{}_{\alpha\beta\nu})_L &= \overline{\nabla}_{\beta}(\Gamma^{\mu}_{\alpha\nu})_L - \overline{\nabla}_{\nu}(\Gamma^{\mu}_{\alpha\beta})_L \\
&= \frac{1}{2}\overline{g}^{\mu\sigma}(\overline{\nabla}_{\beta}\overline{\nabla}_{\alpha}h_{\nu\sigma} + \overline{\nabla}_{\beta}\overline{\nabla}_{\nu}h_{\alpha\sigma} - \overline{\nabla}_{\beta}\overline{\nabla}_{\sigma}h_{\alpha\nu} - \overline{\nabla}_{\nu}\overline{\nabla}_{\alpha}h_{\beta\sigma} \\
&- \overline{\nabla}_{\nu}\overline{\nabla}_{\beta}h_{\alpha\sigma} + \overline{\nabla}_{\nu}\overline{\nabla}_{\sigma}h_{\alpha\beta})
\end{aligned} \tag{B.25}$$

where we used Eq.(B.8).

Then we continue as follows:

$$\begin{aligned}
(R_{\mu\alpha\nu\beta}R^{\mu\alpha\nu\beta})_L &= (R^{\mu}{}_{\alpha\beta\nu}R_{\mu}{}^{\alpha\beta\nu})_L = (R^{\mu}{}_{\alpha\beta\nu}R^{\sigma}{}_{\rho\gamma\eta}\,g_{\mu\sigma}g^{\alpha\rho}g^{\beta\gamma}g^{\nu\eta})_L \\
&= (R^{\mu}{}_{\alpha\beta\nu})_L\,\overline{R}^{\sigma}{}_{\rho\gamma\eta}\,\overline{g}_{\mu\sigma}\overline{g}^{\alpha\rho}\overline{g}^{\beta\gamma}\overline{g}^{\nu\eta} + \overline{R}^{\mu}{}_{\alpha\beta\nu}(R^{\sigma}{}_{\rho\gamma\eta})_L\,\overline{g}_{\mu\sigma}\overline{g}^{\alpha\rho}\overline{g}^{\beta\gamma}\overline{g}^{\nu\eta} \\
&= \overline{R}^{\mu}{}_{\alpha\beta\nu}\overline{R}^{\sigma}{}_{\rho\gamma\eta}h_{\mu\sigma}\overline{g}^{\alpha\rho}\overline{g}^{\beta\gamma}\overline{g}^{\nu\eta} - \overline{R}^{\mu}{}_{\alpha\beta\nu}\overline{R}^{\sigma}{}_{\rho\gamma\eta}\overline{g}_{\mu\sigma}h^{\alpha\rho}\overline{g}^{\beta\gamma}\overline{g}^{\nu\eta} \\
&= \overline{R}^{\mu}{}_{\alpha\beta\nu}\overline{R}^{\sigma}{}_{\rho\gamma\eta}\overline{g}_{\mu\sigma}\overline{g}^{\alpha\rho}h^{\beta\gamma}\overline{g}^{\nu\eta} - \overline{R}^{\mu}{}_{\alpha\beta\nu}\overline{R}^{\sigma}{}_{\rho\gamma\eta}\overline{g}_{\mu\sigma}\overline{g}^{\alpha\rho}\overline{g}^{\beta\gamma}h^{\nu\eta}.
\end{aligned} \tag{B.26}$$

Substituting all the terms, we find that $(R_{\mu\alpha\nu\beta}R^{\mu\alpha\nu\beta})_L = \frac{4}{3}\Lambda R_L$.

Then we finally find

$$\mathcal{G}_L = 8\Lambda R_L - 8\Lambda R_L + \frac{4}{3}\Lambda R_L = \frac{4}{3}\Lambda R_L. \tag{B.27}$$

B.4 Massless Graviton

In this section, let us show that Einstein's theory has a single massless graviton in (A)dS. The linearized Einstein tensor, after setting $R_L = 0$, is

$$(G_{\mu\nu})_L = (R_{\mu\nu})_L - \Lambda h_{\mu\nu}. \tag{B.28}$$

Observe that $R_L = 0$ comes from the linearization of the full trace equation $R = 4\Lambda$. Choosing $h = 0$, $\overline{\nabla}_\mu h^{\mu\nu} = 0$ (transverse traceless gauge), which is compatible with $R_L = 0$, we can simplify the linearized source free equation $((R_{\mu\nu})_L - \Lambda h_{\mu\nu} = 0)$ to a wave equation as follows.

We start calculating the linearized Ricci tensor. Taking $h = 0$, Eq.(B.20) becomes

$$(R_{\mu\nu})_L = \frac{1}{2}\left(\overline{\nabla}^\sigma\overline{\nabla}_\mu h_{\nu\sigma} + \overline{\nabla}^\sigma\overline{\nabla}_\nu h_{\mu\sigma} - \overline{\Box} h_{\mu\nu}\right). \tag{B.29}$$

We can rewrite the identity Eq.(A.44) for the background space as

$$\overline{\nabla}_\rho\overline{\nabla}_\sigma h_{\mu\nu} - \overline{\nabla}_\sigma\overline{\nabla}_\rho h_{\mu\nu} = h_{\beta\nu}\overline{R}^{\beta}_{\mu\sigma\rho} + h_{\mu\beta}\overline{R}^{\beta}_{\nu\sigma\rho}. \tag{B.30}$$

Using Eq.(3.100) we get

$$\overline{\nabla}_\rho\overline{\nabla}_\sigma h_{\mu\nu} - \overline{\nabla}_\sigma\overline{\nabla}_\rho h_{\mu\nu} = \frac{\Lambda}{3}(h_{\sigma\nu}\overline{g}_{\mu\rho} - h_{\rho\nu}\overline{g}_{\mu\sigma} + h_{\sigma\mu}\overline{g}_{\nu\rho} - h_{\rho\mu}\overline{g}_{\nu\sigma}). \tag{B.31}$$

Then we have

$$\overline{\nabla}^\sigma\overline{\nabla}_\mu h_{\nu\sigma} = \frac{4}{3}\Lambda h_{\mu\nu}, \tag{B.32}$$

where we used the transverse traceless gauge.

Then we find the linearized Ricci tensor as

$$(R_{\mu\nu})_L = \frac{4}{3}\Lambda h_{\mu\nu} - \frac{1}{2}\overline{\Box} h_{\mu\nu}, \tag{B.33}$$

and the linearized field equations are

$$\left(\overline{\Box} - \frac{2}{3}\Lambda\right) h_{\mu\nu} = 0, \tag{B.34}$$

subject to the conditions $h = 0$, $\overline{\nabla}_\mu h^{\mu\nu} = 0$. Although this equation stands as a massive wave equation, it can be converted to a massless wave equation in conformally flat backgrounds with a transformation $h_{\mu\nu} = \Omega H_{\mu\nu}$ as explained in [35]. This point can be understood with a scalar field example which has a simpler calculation given below.

We will evaluate $\Box\Phi$ where Φ is a scalar field. The metric $g_{\mu\nu}$ is a conformally flat metric that can be written as

$$g_{\mu\nu} = \Omega^2 \eta_{\mu\nu}, \tag{B.35}$$

with $\Omega = \left(1 - \frac{m^2 x^2}{4}\right)^{-1}$. Here, m is a parameter and $x^2 = x^\mu x_\mu$.

Then we can calculate that

$$\begin{aligned} \Box\Phi &= g^{\mu\nu}\nabla_\mu\nabla_\nu\Phi \\ &= \Omega^{-2}\eta^{\mu\nu}\left(\partial_\mu\partial_\nu\Phi - \Gamma^\alpha_{\mu\nu}\partial_\alpha\Phi\right). \end{aligned} \tag{B.36}$$

Now let us introduce a scaled scalar field ϕ as

$$\Phi = \Omega^w \phi = \Omega^{-1}\phi, \tag{B.37}$$

here w is named as the Weyl weight which is -1 for the scalar field. Then we have

$$\partial_\mu\Phi = \partial_\mu(\Omega^{-1}\phi) = \left(\partial_\mu\Omega^{-1}\right)\phi + \Omega^{-1}\partial_\mu\phi. \tag{B.38}$$

Substituting $\partial_\mu\Omega^{-1} = -\frac{m^2}{2}x_\mu$ we get

$$\partial_\mu\Phi = -\frac{m^2}{2}x_\mu\phi + \Omega^{-1}\partial_\mu\phi. \tag{B.39}$$

And we evaluate the second derivative of Φ such that

$$\partial_\nu\partial_\mu\Phi = -\frac{m^2}{2}\delta_{\mu\nu}\phi - \frac{m^2}{2}x_\mu\partial_\nu\phi - \frac{m^2}{2}x_\nu\partial_\mu\phi + \Omega^{-1}\partial_\nu\partial_\mu\phi. \tag{B.40}$$

Then the first term of Eq.(B.36) is

$$\Omega^{-2}\eta^{\mu\nu}\partial_\mu\partial_\nu\Phi = \Omega^{-2}\left(-2m^2\phi - m^2 x_\mu\partial^\mu\phi + \Omega^{-1}\partial_\mu\partial^\mu\phi\right). \tag{B.41}$$

Now we calculate the Christoffel symbol as

$$\begin{aligned} \Gamma^\alpha_{\mu\nu} &= \frac{1}{2}g^{\alpha\beta}(\partial_\mu g_{\nu\beta} + \partial_\nu g_{\mu\beta} - \partial_\beta g_{\mu\nu}) \\ &= \frac{1}{2}\Omega^{-2}\eta^{\alpha\beta}\left\{\partial_\mu\left(\Omega^2\eta_{\nu\beta}\right) + \partial_\nu\left(\Omega^2\eta_{\mu\beta}\right) - \partial_\beta\left(\Omega^2\eta_{\mu\nu}\right)\right\} \\ &= \frac{m^2}{2}\Omega\left(x_\mu\delta^\alpha_\nu + x_\nu\delta^\alpha_\mu - x^\alpha\eta_{\mu\nu}\right), \end{aligned} \tag{B.42}$$

and

$$\Omega^{-2}\eta^{\mu\nu}\Gamma^\alpha_{\mu\nu}\partial_\alpha\Phi = \frac{m^4}{2}\Omega^{-1}x^2\phi - m^2\Omega^{-2}x^\alpha\partial_\alpha\phi. \tag{B.43}$$

Finally we find that

$$\begin{aligned}
\Box\Phi &= \Omega^{-2}\left(-2m^2\phi - m^2 x_\mu\partial^\mu\phi + \Omega^{-1}\partial_\mu\partial^\mu\phi\right) - \frac{m^4}{2}\Omega^{-1}x^2\phi + m^2\Omega^{-2}x^\alpha\partial_\alpha\phi \\
&= \Omega^{-3}\partial_\mu\partial^\mu\phi - 2m^2\Omega^{-1}\phi \\
&= \Omega^{-3}\Box_0\phi - 2m^2\Omega^{-1}\phi
\end{aligned} \tag{B.44}$$

where $\Box_0 = \eta^{\mu\nu}\partial_\mu\partial_\nu$. We can rewrite this equation as

$$\left(\Box + 2m^2\right)\Phi = \Omega^{-3}\Box_0\phi. \tag{B.45}$$

Then we conclude that a massive-looking wave equation of Φ can be transformed to a massless wave equation of $\phi = \Omega^{-w}\Phi = \Omega\Phi$ for the conformally flat metrics. Here we see that the interaction between the field and the curved spacetime can be viewed as a mass. While studying the conformally flat metrics, introducing a suitable transformation taking into account the Weyl weight, as in Eq.(B.37), we can observe the masslessness of the field. A similar argument works for the case of the linearized wave equation Eq.(B.34). Following the arguments of [35], one can show that Eq.(B.34) reduces to

$$\Box_0\tilde{h}_{\mu\nu} = 0. \tag{B.46}$$

where $\tilde{h}_{\mu\nu}$ is the transformed field.

APPENDIX C

LINEARIZED GRAVITY

Suppose our metric can be decomposed as

$$g_{\alpha\beta} = \overline{g}_{\alpha\beta} + h_{\alpha\beta}, \quad |h_{\alpha\beta}| \ll 1. \tag{C.1}$$

We also note that $g^{\alpha\beta} = \overline{g}^{\alpha\beta} - h^{\alpha\beta}$ and $h = \overline{g}^{\alpha\beta} h_{\alpha\beta}$.

C.1 Global Lorentz coordinate transformations[1]

We consider a global Lorentz transformations from x^{μ} to $x^{\mu'}$ such that

$$x^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^{\nu}} x^{\nu} \equiv \Lambda^{\mu}{}_{\nu} x^{\nu} \tag{C.2}$$

where $\Lambda^{\mu}{}_{\nu}$ are the constant transformation matrix components with the property of

$$\overline{g}_{\mu'\nu'} = \Lambda_{\mu}{}^{\alpha} \Lambda_{\nu}{}^{\beta} \overline{g}_{\alpha\beta}. \tag{C.3}$$

And the metric components are transformed by the below equation

$$g_{\mu'\nu'} = \Lambda_{\mu}{}^{\alpha} \Lambda_{\nu}{}^{\beta} g_{\alpha\beta}. \tag{C.4}$$

Now we can substitute Eq.(C.1) into above equation;

$$g_{\mu'\nu'} = \Lambda_{\mu}{}^{\alpha} \Lambda_{\nu}{}^{\beta} (\overline{g}_{\alpha\beta} + h_{\alpha\beta}) = \overline{g}_{\mu'\nu'} + \Lambda_{\mu}{}^{\alpha} \Lambda_{\nu}{}^{\beta} h_{\alpha\beta}. \tag{C.5}$$

Inserting $g_{\mu'\nu'} = \overline{g}_{\mu'\nu'} + h_{\mu'\nu'}$ we get

$$h_{\mu'\nu'} = \Lambda_{\mu}{}^{\alpha} \Lambda_{\nu}{}^{\beta} h_{\alpha\beta}. \tag{C.6}$$

Then we see that $h_{\mu\nu}$ obeys the (0,2) rank tensor transformation under global coordinate transformations.

[1] In this section we used the book [14] as a reference.

C.2 Infinitesimal coordinate transformations[2]

Infinitesimal coordinate transformations can be expressed as

$$x^{\mu'} = x^{\mu} + \xi^{\mu}(x) \tag{C.7}$$

here ξ^{μ} is a vector field depending on the coordinates. Taking the derivation of both sides we have;

$$\frac{\partial x^{\mu'}}{\partial x^{\nu}} = \delta^{\mu}_{\nu} + \partial_{\nu}\xi^{\mu} \tag{C.8}$$

and the inverse transformation is

$$\frac{\partial x^{\mu}}{\partial x^{\nu'}} = \delta^{\mu}_{\nu} - \partial_{\nu}\xi^{\mu}. \tag{C.9}$$

Then the metric transformation can be calculated as below.

$$\begin{aligned} g_{\mu'\nu'}(x') &= \frac{\partial x^{\alpha}}{\partial x^{\mu'}}\frac{\partial x^{\beta}}{\partial x^{\nu'}} g_{\alpha\beta}(x) \\ &= \left(\delta^{\alpha}_{\mu} - \partial_{\mu}\xi^{\alpha}\right)\left(\delta^{\beta}_{\nu} - \partial_{\nu}\xi^{\beta}\right) g_{\alpha\beta}(x) \\ &= g_{\mu\nu}(x) - g_{\mu\beta}(x)\partial_{\nu}\xi^{\beta} - g_{\alpha\nu}(x)\partial_{\mu}\xi^{\alpha} + g_{\alpha\beta}(x)\partial_{\mu}\xi^{\alpha}\partial_{\nu}\xi^{\beta}. \end{aligned} \tag{C.10}$$

Assuming $|\partial_{\mu}\xi^{\alpha}| \ll 1$, we continue the calculation, to the first order, with;

$$g_{\mu'\nu'}(x') = g_{\mu\nu}(x) - g_{\mu\beta}(x)\partial_{\nu}\xi^{\beta} - g_{\alpha\nu}(x)\partial_{\mu}\xi^{\alpha}. \tag{C.11}$$

Now we do a Taylor expansion for $g_{\mu'\nu'}(x')$;

$$g_{\mu'\nu'}(x') = g_{\mu'\nu'}(x+\xi) \equiv g_{\mu'\nu'}(x) + \xi^{\sigma}\partial_{\sigma} g_{\mu'\nu'}(x). \tag{C.12}$$

Inserting this expression into Eq.(C.10) we have

$$g_{\mu'\nu'}(x) = g_{\mu\nu}(x) - g_{\mu\beta}(x)\partial_{\nu}\xi^{\beta} - g_{\alpha\nu}(x)\partial_{\mu}\xi^{\alpha} - \xi^{\sigma}\partial_{\sigma} g_{\mu'\nu'}(x). \tag{C.13}$$

Now using $g_{\mu\beta}(x)\partial_{\nu}\xi^{\beta} = \partial_{\nu}\xi_{\mu} - \xi^{\beta}\partial_{\nu} g_{\mu\beta}$ we get

$$\begin{aligned} g_{\mu'\nu'}(x) &= g_{\mu\nu}(x) - \partial_{\nu}\xi_{\mu} + \xi^{\alpha}\partial_{\nu} g_{\mu\alpha} - \partial_{\mu}\xi_{\nu} + \xi^{\alpha}\partial_{\mu} g_{\alpha\nu} - \xi^{\alpha}\partial_{\alpha} g_{\mu'\nu'} \\ &= g_{\mu\nu}(x) - \partial_{\nu}\xi_{\mu} - \partial_{\mu}\xi_{\nu} + 2\xi_{\beta}\Gamma^{\beta}_{\mu\nu} \\ &= g_{\mu\nu}(x) - \overline{\nabla}_{\mu}\xi_{\nu} - \overline{\nabla}_{\nu}\xi_{\mu} \end{aligned} \tag{C.14}$$

to the first order approximation [3].

[2] In this section we generally followed the books [14] and [31].

[3] In the first order approximation we simply take $\xi^{\alpha}\partial_{\alpha} g_{\mu'\nu'} = \xi^{\alpha}\partial_{\alpha} g_{\mu\nu}$ as inferred from Eq.(C.13).

C.3 Diffeomorphism invariance for $\mathrm{R_L}$

In this section we will search how the linearized Ricci scalar changes under the diffeomorphism $x^{\mu'} = x^\mu + \xi^\mu(x)$ which is mentioned in the previous section. Let us denote the change by δ. Then we can start as the following:

$$\delta R_L = \delta(g^{\mu\nu}R_{\mu\nu})_L = \delta\left(\overline{g}^{\mu\nu}(R_{\mu\nu})_L - h^{\mu\nu}\overline{R}_{\mu\nu}\right). \tag{C.15}$$

Substituting $\overline{R}_{\mu\nu} = \frac{2\Lambda}{(n-2)}\overline{g}_{\mu\nu}$ we obtain

$$\delta R_L = \overline{g}^{\mu\nu}\delta\left(R_{\mu\nu}\right)_L - \frac{2\Lambda}{(n-2)}\overline{g}_{\mu\nu}\delta h^{\mu\nu}. \tag{C.16}$$

Then we insert $(R_{\mu\nu})_L$ given in Eq.(B.20) and $\delta h^{\mu\nu}$ into above equation ;

$$\begin{aligned}
\delta R_L &= \frac{1}{2}\overline{g}^{\mu\nu}\delta\left(\overline{\nabla}^\sigma\overline{\nabla}_\mu h_{\nu\sigma} + \overline{\nabla}^\sigma\overline{\nabla}_\nu h_{\mu\sigma} - \Box h_{\mu\nu} - \overline{\nabla}_\nu\overline{\nabla}_\mu h\right) \\
&- \frac{2\Lambda}{(n-2)}\overline{g}_{\mu\nu}\left(\overline{\nabla}^\mu\xi^\nu + \overline{\nabla}^\nu\xi^\mu\right) \\
&= \delta\left(\overline{\nabla}^\sigma\overline{\nabla}^\mu h_{\sigma\mu} - \overline{\Box}h\right) - \frac{4\Lambda}{(n-2)}\overline{\nabla}^\mu\xi_\mu \\
&= \overline{\nabla}^\sigma\overline{\nabla}^\mu\delta h_{\sigma\mu} - \overline{\Box}\delta h - \frac{4\Lambda}{(n-2)}\overline{\nabla}^\mu\xi_\mu.
\end{aligned} \tag{C.17}$$

Now we substitute $\overline{\Box}\delta h = \overline{\Box}\delta(\overline{g}_{\mu\nu}h^{\mu\nu}) = \overline{g}_{\mu\nu}\overline{\Box}\delta h^{\mu\nu}$ and $\delta h^{\mu\nu}$;

$$\begin{aligned}
\delta R_L &= \overline{\nabla}^\sigma\overline{\nabla}^\mu\left(\overline{\nabla}_\sigma\xi_\mu + \overline{\nabla}_\mu\xi_\sigma\right) - \overline{g}_{\mu\nu}\overline{\Box}\left(\overline{\nabla}_\mu\xi_\nu + \overline{\nabla}_\nu\xi_\mu\right) - \frac{4\Lambda}{(n-2)}\overline{\nabla}^\mu\xi_\mu \\
&= \overline{\nabla}^\sigma\overline{\nabla}^\mu\overline{\nabla}_\sigma\xi_\mu + \overline{\nabla}^\sigma\overline{\Box}\xi_\sigma - 2\overline{\Box}\overline{\nabla}^\mu\xi_\mu - \frac{4\Lambda}{(n-2)}\overline{\nabla}^\mu\xi_\mu.
\end{aligned} \tag{C.18}$$

In order to find the first term we will use Eq.(A.39);

$$\left[\overline{\nabla}^\mu, \overline{\nabla}_\sigma\right]\xi_\mu = \overline{R}^\mu{}_{\sigma\mu\rho}\xi^\rho = \overline{R}_{\sigma\rho}\xi^\rho = \frac{2\Lambda}{(n-2)}\xi_\sigma, \tag{C.19}$$

then we calculate that

$$\begin{aligned}
\overline{\nabla}^\mu\overline{\nabla}_\sigma\xi_\mu &= \left[\overline{\nabla}^\mu, \overline{\nabla}_\sigma\right]\xi_\mu + \overline{\nabla}_\sigma\overline{\nabla}^\mu\xi_\mu \\
&= \frac{2\Lambda}{(n-2)}\xi_\sigma + \overline{\nabla}_\sigma\overline{\nabla}^\mu\xi_\mu.
\end{aligned} \tag{C.20}$$

Taking derivative we have

$$\overline{\nabla}^\sigma\overline{\nabla}^\mu\overline{\nabla}_\sigma\xi_\mu = \frac{2\Lambda}{(n-2)}\overline{\nabla}^\mu\xi_\mu + \overline{\Box}\,\overline{\nabla}^\mu\xi_\mu. \tag{C.21}$$

For the second term of Eq.(C.18) let us start with the below expression:

$$\left[\overline{\nabla}^{\sigma},\overline{\nabla}^{\mu}\right]\overline{\nabla}_{\mu}\xi_{\sigma}=\overline{\nabla}^{\sigma}\overline{\nabla}^{\mu}\overline{\nabla}_{\mu}\xi_{\sigma}-\overline{\nabla}^{\mu}\overline{\nabla}^{\sigma}\overline{\nabla}_{\mu}\xi_{\sigma}=\overline{\nabla}^{\sigma}\overline{\Box}\xi_{\sigma}-\overline{\nabla}^{\mu}\overline{\nabla}^{\sigma}\overline{\nabla}_{\mu}\xi_{\sigma}.$$

Then our second term is

$$\overline{\nabla}^{\sigma}\overline{\Box}\xi_{\sigma}=\left[\overline{\nabla}^{\sigma},\overline{\nabla}^{\mu}\right]\overline{\nabla}_{\mu}\xi_{\sigma}+\overline{\nabla}^{\mu}\overline{\nabla}^{\sigma}\overline{\nabla}_{\mu}\xi_{\sigma}. \tag{C.22}$$

We can continue to calculate using Eq.(3.86) ;

$$\begin{aligned}\left[\overline{\nabla}^{\sigma},\overline{\nabla}^{\mu}\right]\overline{\nabla}_{\mu}\xi_{\sigma} &= -\overline{R}^{\lambda}{}_{\mu}{}^{\sigma\mu}\overline{\nabla}_{\lambda}\xi_{\sigma}-\overline{R}^{\lambda}{}_{\sigma}{}^{\sigma\mu}\overline{\nabla}_{\mu}\xi_{\lambda}\\ &= -\overline{R}_{\lambda\sigma}\overline{\nabla}^{\lambda}\xi^{\sigma}+\overline{R}_{\lambda\mu}\overline{\nabla}^{\mu}\xi^{\lambda}\\ &= 0. \end{aligned}\tag{C.23}$$

And using Eq.(C.19) we can write

$$\left[\overline{\nabla}^{\sigma},\overline{\nabla}_{\mu}\right]\xi_{\sigma}=\overline{\nabla}^{\sigma}\overline{\nabla}_{\mu}\xi_{\sigma}-\overline{\nabla}_{\mu}\overline{\nabla}^{\sigma}\xi_{\sigma}=\frac{2\Lambda}{(n-2)}\xi_{\mu}. \tag{C.24}$$

Then we simply calculate that

$$\overline{\nabla}^{\mu}\overline{\nabla}^{\sigma}\overline{\nabla}_{\mu}\xi_{\sigma}=\frac{2\Lambda}{(n-2)}\overline{\nabla}^{\mu}\xi_{\mu}+\overline{\Box}\,\overline{\nabla}^{\mu}\xi_{\mu}. \tag{C.25}$$

So the second term becomes

$$\overline{\nabla}^{\sigma}\overline{\Box}\xi_{\sigma}=\frac{2\Lambda}{(n-2)}\overline{\nabla}^{\mu}\xi_{\mu}+\overline{\Box}\,\overline{\nabla}^{\mu}\xi_{\mu}. \tag{C.26}$$

We finally obtain

$$\begin{aligned}\delta R_L &= \frac{4\Lambda}{(n-2)}\overline{\nabla}^{\mu}\xi_{\mu}+2\overline{\Box}\,\overline{\nabla}^{\mu}\xi_{\mu}-2\overline{\Box}\,\overline{\nabla}^{\mu}\xi_{\mu}-\frac{4\Lambda}{(n-2)}\overline{\nabla}^{\mu}\xi_{\mu}\\ &= 0. \end{aligned}\tag{C.27}$$

Then we conclude that R_L is a diffeomorphism invariant under $x^{\mu'}=x^{\mu}+\xi^{\mu}(x)$.

CURRICULUM VITAE

PERSONAL INFORMATION

Surname, Name: Kenar, Esin
Nationality: Turkish (TC)
Date and Place of Birth: 1984, Zonguldak/Turkey
Contact: esin555@gmail.com

EDUCATION

Degree	Institution	Year of Graduation
M.S.	Physics, Middle East Technical University	2009
B.S.	Physics, Middle East Technical University	2007

PROFESSIONAL EXPERIENCE

Year	Place	Enrollment
2007-2017	Middle East Technical University	Research Assistant

PUBLICATION

- *"Minimal Extension of Einstein's Theory: The Quartic Gravity"*. Phys. Rev. D **93**, 084040 (2016). Atalay Karasu, Esin Kenar and Bayram Tekin.

MS. THESIS

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