THE INFLUENCE OF SOME EMBEDDING PROPERTIES OF SUBGROUPS ON THE STRUCTURE OF A FINITE GROUP

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

M.YASİR KIZMAZ

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY
IN
MATHEMATICS

AUGUST 2018

Approval of the thesis:

THE INFLUENCE OF SOME EMBEDDING PROPERTIES OF SUBGROUPS ON THE STRUCTURE OF A FINITE GROUP

submitted by **M.YASİR KIZMAZ** in partial fulfillment of the requirements for the degree of **Doctor of Philosophy in Mathematics Department, Middle East Technical University** by,

Prof. Dr. Prof. Dr. Halil Kalıpçılar
Dean, Graduate School of **Natural and Applied Sciences** ________________

Prof. Dr. Yıldıray Ozan
Head of Department, **Mathematics** ________________

Prof. Dr. Gülin Ercan
Supervisor, **Mathematics Department, METU** ________________

Examining Committee Members:

Prof. Dr. İsmail Şuayip Güloğlu
Mathematics Department, Doğuş University ________________

Prof. Dr. Gülin Ercan
Mathematics Department, METU ________________

Prof. Dr. Sait Halıcıoğlu
Mathematics Department, Ankara University ________________

Prof. Dr. Mustafa Hurşit Önsiper
Mathematics Department, METU ________________

Assoc. Prof. Dr. Ebru Solak
Mathematics Department, METU ________________

Date: ________________

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Name, Last Name: M.YASİR KIZMAZ

Signature :

ABSTRACT

THE INFLUENCE OF SOME EMBEDDING PROPERTIES OF SUBGROUPS ON THE STRUCTURE OF A FINITE GROUP

Kızmaz, M.Yasir
Ph.D., Department of Mathematics
Supervisor : Prof. Dr. Gülin Ercan

August 2018, 56 pages

In a finite group G, a subgroup H is called a TI-subgroup if H intersects trivially with distinct conjugates of itself. Suppose that H is a Hall π-subgroup of G which is also a TI-subgroup. A famous theorem of Frobenius states that G has a normal π-complement whenever H is self normalizing. In this case, H is called a Frobenius complement and G is said to be a Frobenius group. A first main result in this thesis is the following generalization of Frobenius' Theorem.

Theorem. *Let H be a TI-subgroup of G which is also a Hall subgroup of $N_G(H)$. Then H has a normal complement in $N_G(H)$ if and only if H has a normal complement in G. Moreover, if H is nonnormal in G and H has a normal complement in $N_G(H)$ then H is a Frobenius complement.*

In the above configuration, the group G need not be a Frobenius group, but the second part of the theorem guarantees the existence of a Frobenius group into which H can be embedded as a Frobenius complement.

Another contribution of this thesis is the following theorem, which extends a result of

Gow (see Theorem 1.0.1) to π-separable groups. This result shows that the structure of a π-separable group admitting a Hall π-subgroup which is also a TI-subgroup is very restricted.

Theorem. *Let H be a nonnormal TI-subgroup of the π-separable group G where π is the set of primes dividing the order of H. Further assume that H is a Hall subgroup of $N_G(H)$. Then the following hold:*

$a)$ *G has π-length 1 where $G = O_{\pi'}(G)N_G(H)$;*

$b)$ *there is an H-invariant section of G on which the action of H is Frobenius. This section can be chosen as a chief factor of G whenever $O_{\pi'}(G)$ is solvable;*

$c)$ *G is solvable if and only if $O_{\pi'}(G)$ is solvable and H does not involve a subgroup isomorphic to $SL(2,5)$.*

In the last chapter we focus on giving alternative proofs without character theory for the following two solvability theorems due to Isaacs ([5], Theorem 1 and Theorem 2). Our proofs depend on transfer theory and graph theory.

Theorem. *Let G be a finite group having a cyclic Sylow p-subgroup. Assume that every p'-subgroup of G is abelian. Then G is either p-nilpotent or p-closed.*

Theorem. *Let G be a finite group and let $p \neq 2$ and q be primes dividing $|G|$. Suppose for every proper subgroup H of G which is not a q-group nor a q'-group that p divides $|H|$. If q^a is the q-part of $|G|$ and $p > q^a - 1$ or if $p = q^a - 1$ and a Sylow p-subgroup of G is abelian then no primes but p and q divide $|G|$.*

Keywords: TI-subgroups, normal complement, Frobenius group, p-nilpotent, p-closed

ÖZ

BAZI ALTGRUP YERLEŞME ÖZELLİKLERİNİN BİR SONLU GRUBUN YAPISI ÜZERİNE ETKİLERİ

Kızmaz, M.Yasir
Doktora, Matematik Bölümü
Tez Yöneticisi : Prof. Dr. Gülin Ercan

Ağustos 2018 , 56 sayfa

Sonlu bir G grubunun kendisinden farklı eşlenikleriyle kesişimleri sıradan olan H altgrubuna TI-altgrup denir. H'nin TI-altgrup olma özelliği taşıyan bir Hall π-altgrubu olduğunu varsayalım. Frobenius'un meşhur bir teoremi gösteriyor ki, eğer H öz normalleştiren bir alt grupsa, G grubunun bir normal π-tamlayanı vardır. Bu durumda H, Frobenius tamlayanı diye adlandırılır ve böyle gruplara da Frobenius grup denir.

Bu tezin ilk ana sonucu, Frobenius teoreminin bir genellemesi olarak elde ettiğimiz aşağıdaki teoremdir.

Teorem. *H, G'nin bir TI-altgrubu ve $N_G(H)$'nin bir Hall altgrubu olsun. O zaman H'nin G içinde bir normal tamlayanı olması için gerek ve yeter koşul H'nin $N_G(H)$ içinde bir normal tamlayanı olmasıdır. Eğer H, G de normal değilse ve H'nin $N_G(H)$ içinde bir normal tamlayanı varsa, H bir Frobenius tamlayanıdır.*

Yukarıdaki kurguda, G'nin bir Frobenius grup olması gerekmese de; teoremin ikinci kısmı, H'nin, bir Frobenius grubun içine Frobenius tamlayanı olarak gömülebileceğini garanti eder.

Tezin diğer bir katkısı ise, Gow'a ait bir sonucu (Bkz. Teorem 1.0.1) π-ayrışabilir gruplara genelleyen aşağıdaki teoremdir. Burada Hall π-altgrubu TI olma özelliği taşıyan π-ayrışabilir grupların yapısının çok sınırlı olduğu gösterilmektedir.

Teorem. *π kümesi H'nin mertebesini bölen asalların kümesi olmak üzere; H, π-ayrışabilir bir grup olan G'nin normal olmayan bir TI-altgrubu olsun. Aynı zamanda H'nin $N_G(H)$'ye ait bir Hall altgrubu olduğunu varsayalım. O zaman aşağıdaki özellikler sağlanır.*

$a)$ *G'nin π-uzunluğu 1'e eşittir ve $G = O_{\pi'}(G)N_G(H)$ olur;*

$b)$ *G'nin öyle bir H-değişmez kesiti vardır ki H'nin bu kesit üzerindeki etkisi Frobenius'tur. $O_{\pi'}(G)$ çözülebilir olduğunda; G'nin bu kesiti, G nin bir ana kesiti olarak seçilebilir.*

$c)$ *G'nin çözülebilir olması için gerek ve yeter koşul $O_{\pi'}(G)$'nin çözülebilir olması ve H'nin $SL(2,5)$'e izomorf olan bir altgrup içermemesidir.*

Son olarak, Isaacs'e ait aşağıdaki iki ayrı çözülebilirlik teoremine ([5], Teorem 1 and Teorem 2) karakter teoriden bağımsız, sadece transfer teori ve çizge teorisi kullanarak alternatif ispatlar sunacağız.

Theorem. *Her Sylow p-altgrubu döngüsel ve her p'-altgrubu abelyen olan sonlu bir grup ya p-kapalıdır, ya da p-nilpotenttir.*

Theorem. *$p \neq 2$ ve q, sonlu G grubunun mertebesini bölen asallar olsunlar. G'nin q-altgrup ya da q'-altgrup olmayan her altgrubunun mertebesinin p'ye bölündüğünü kabul edelim. q^a, $|G|$ nin q-kısmı olmak üzere $p > q^a - 1$ veya $p = q^a - 1$ ve Sylow p-altgruplarının abelyen olsun. Bu durumda G nin mertebesini yalnız p ve q asalları bölebilir.*

Anahtar Kelimeler: TI-altgrup, normal tamlayan, Frobenius grup, p-nilpotent, p-kapalı

To my mother

&

my father

ACKNOWLEDGMENTS

I would like to thank everyone who has supported me during my studies.

Foremost, I would like to express my sincere gratitude to Prof. Dr. Gülin Ercan. In the first two years of my graduate study, while I was taking courses on various fields of mathematics, and was eager to ask questions, she let me ask uncountably "finite" questions and her kind and supportive attitude helped me get interested in finite group theory. After taking a few Ph.D. courses from her, I realized that she was the best choice for me as an advisor; and fortunately, she accepted to mentor me. I am deeply thankful that her encouraging attitude has not changed after we started working together and she always let me freely choose the topics I wanted to study, she fed my curiosity by giving her full attention every time I needed her to consult.

I wish to thank Prof. Dr. İsmail Şuayip Güloğlu for his invaluable guidance. I have to mention that every time I came up with a false conjecture, he immediately constructed a counterexample and suggested a better hypothesis.

I am grateful to all my friends who have been by my side throughout my Ph.D. I especially want to mention some of them.
Dr. Şükran Gül, with whom I like brainstorming on problems related to group theory, especially p-groups.;
Kübra Dölaslan, who has the ability to solve any kind of problem related to group theory or life, by her own very creative ways. She always use this ability of her very generously for her friends.
Fuat Erdem, whose companionship has been valuable both in the department and in many conferences.

I want to acknowledge TÜBİTAK for financially supporting me during my Ph.D.

Above all, I am indebted to my family for their pure love and unconditional support.

TABLE OF CONTENTS

LIST OF SYMBOLS

$\lvert G : H\rvert = \lvert\{gH \mid g \in G\}\rvert$	the index of H in G
$C_G(H) = \{g \in G \mid g^{-1}xg = x \text{ for all } x \in H\}$	the centralizer of H in G
$N_G(H) = \{g \in G \mid g^{-1}Hg = H\}$	the normalizer of H in G
$Stab_G(\alpha)$	the stabilizer of α in G
$[H, K]$	the commutator subgroup of H and K
G'	the commutator subgroup of G
$Z(G)$	the center of G
$\Phi(G)$	the Frattini subgroup of G
$Foc_G(H)$	the focal subgroup of H in G
$Syl_p(G)$	the set of Sylow p-subgroups of G
$O_\pi(G)$	the largest normal π-subgroup of G
$O^\pi(G)$	the smallest normal subgroup of G with p-power index
H^g	$g^{-1}Hg$
$H \rtimes K$	the semidirect product of H by K

CHAPTER 1

INTRODUCTION

In the theory of finite groups, one of the questions of particular interest is how the embedding properties of certain types of subgroups can influence the structure of the whole group. The present thesis contributes to this kind of research by studying the impact of different kinds of embeddings of some special subgroups. More specifically, this work has two main targets considered in chapters 3 and 4; the first of which is to extend some results due to Gow and Frobenius by weakening their assumptions, and the second of which is to give alternative proofs for two solvability theorems due to Isaacs. Unlike the original proofs, we do not appeal to character theory.

We shall, from now on, assume that all groups under discussion are finite, and firstly give a precise description of our results presented in Chapter 3.

A nontrivial subgroup H of a group G is called a TI-subgroup if $H \cap H^x = H$ or trivial for any $x \in G$. Clearly, any normal subgroup is also a TI-subgroup but it is not very interesting beside normality. Hence we mostly consider nonnormal TI-subgroups. Being a TI-subgroup is a subgroup embedding property which forms the central concept of Chapter 3 of this thesis. Historicaly our results begin with Frobenius groups. In one of his celebrated works Frobenius proved that a self normalizing TI-subgroup H of a group G has a normal complement in G, by using his theory of induced characters. In this case, H is called a Frobenius complement. Moreover, such groups G are named as Frobenius groups after him and they are well studied. It is now over 100 years old and no purely group theoretical proof is still known. From another point of view, in case a group H acts on a group K by automorphisms in such a way that every nonidentity element of H acts on K fixed point freely, the semidirect product KH is a Frobenius group with complement H where the action of H on K is called *Frobenius*. Note also that we call H as a Frobenius complement if there exists a group on which H has a Frobenius action. The structure of Frobenius complements were investigated by Burnside and Zassenhaus: Sylow subgroups of Frobenius complements are cyclic or generalized quaternion, moreover Frobenius complements are solvable unless they contain subgroups isomorphic to $SL(2, 5)$. Notice that Frobenius complements are Hall subgroups of Frobenius groups, and so there are two natural questions to ask for a given Hall subgroup H which is also a nonnormal TI-subgroup of a group G:

Question A *Under what conditions is H a Frobenius complement ?*

Question B *Under what conditions does H have a normal complement in G ?*

In 1975, Gow obtained a partial answer to Question A, namely he proved the following:

Theorem 1.0.1. *(Gow) Let H be a Hall subgroup of the solvable group G such that H is also a nonnormal TI-subgroup of G. Then H has an irreducible representation on some elementary abelian section of G on which each of its nonidentity elements acts without fixed points.*

In the present thesis, we extend his result to π-separable groups as a more general answer to Question A by proving

Theorem 1.0.2. *Let H be a nonnormal TI-subgroup of the π-separable group G where π is the set of primes dividing the order of H. Further assume that H is a Hall subgroup of $N_G(H)$. Then the following hold:*

$a)$ *G has π-length 1 where $G = O_{\pi'}(G)N_G(H)$;*

$b)$ *there is an H-invariant section of G on which the action of H is Frobenius. This section can be chosen as a chief factor of G whenever $O_{\pi'}(G)$ is solvable;*

$c)$ *G is solvable if and only if $O_{\pi'}(G)$ is solvable and H does not involve a subgroup isomorphic to $SL(2, 5)$.*

It is apparent that the first statement of part (b) extends the result of Gow to π-separable groups and says additionally that the section under consideration can be chosen as a chief factor of G whenever $O_{\pi'}(G)$ is solvable. By parts (a) and (c) we obtain a further determination of the structure of G.

Clearly in (c) we give a necessary and sufficient condition for a group G, satisfying the hypothesis of the theorem, to be solvable. Notice that, under the hypothesis of above theorem, we have $G = O_{\pi'}(G)N_G(H)$. On the other hand, by Schur-Zassenhaus theorem (see 2.1.25), H has a complement in $N_G(H)$, say Q, and hence the equality $G = O_{\pi'}(G)HQ$ holds. This need not be a semidirect product as $O_{\pi'}(G) \cap Q$ may not be trivial.

Recall that a double Frobenius group is defined to be a group K such that $K = (A \rtimes B) \rtimes C$ where AB and BC are Frobenius groups. Here B is a nonnormal TI-subgroup of K and $K = AN_K(B)$ as $N_K(B) = BC$. The structure of the group G, in somehow, resembles the structure of a double Frobenius group. More precisely, we obtained the following theorem showing that G has a factor group containing double Frobenius groups under some additional hypothesis.

Theorem 1.0.3. *Assume that the hypothesis of Theorem 1.0.2 hold. Assume further that H is of odd order with $[O_{\pi'}(G), H] = O_{\pi'}(G)$ and that $O_{\pi'}(G)$ is solvable with $Q \nleq O_{\pi'}(G)'$. Set $\overline{G} = G/O_{\pi'}(G)'$. Then*
$a)$ $\overline{G} = (\overline{O}_{\pi'}(G) \rtimes \overline{H}) \rtimes \overline{Q}$;
$b)$ $\overline{Q}$ is an abelian group acting faithfully on $\overline{H}$;
$c)$ $\overline{O_{\pi'}(G)}[\overline{H}, \beta]\langle\beta\rangle$ is a double Frobenius group for every element $\beta \in \overline{Q}$ of prime order.

A next question is to determine the structure of $O_{\pi'}(G)$. It seems that it may be really difficult to answer as examples of similar questions show.

In proving the above theorem we need to obtain some new technical results which will be given below. The following proposition is quite important as it gives a sufficient condition which guarantees the conjugacy of Hall π-subgroups. Namely we prove that there is a single conjugacy class of Hall π-subgroups whenever Hall π-subgroups are TI-subgroups. This condition also guarantees that any π-subgroup is contained in a Hall π-subgroup. It should be noted that a group G may have nonisomorphic Hall π-subgroups. In this case, it is not possible that they are conjugate in G. On the other hand, even if all Hall-π subgroups are conjugate in G, it might be the case that some of π-subgroups of G are not contained in any Hall π-subgroups.

Proposition 1.0.4. *Let G be a group containing a Hall π-subgroup H which is also a TI- subgroup. Then any π-subgroup of G is contained in a conjugate of H. In particular, the set of all Hall π-subgroups of G forms a single G-conjugacy class.*

The next theorem guarantees that the subgroup H in the hypothesis contains a Hall subgroup which has a normal complement in G under some additional assumption.

Theorem 1.0.5. *Assume that the hypothesis of Theorem 1.0.2 holds. Assume further that a Sylow 2-subgroup of H is abelian and Q is a complement of H in $N_G(H)$. Then $C_H(Q)$ is a Hall subgroup of G having a normal complement in G.*

This theorem is obtained by using the result below which is of independent interest too, especially for researchers studying coprime action.

Proposition 1.0.6. *Let A be a group acting coprimely on G by automorphisms. Assume that Sylow subgroups of G are cyclic. Then,*
$a)$ $C_G(A)$ is a Hall subgroup of G;
$b)$ $G = [G, A] \rtimes C_G(A)$;
$c)$ the group $[G, A]$ is cyclic.

We are now ready to present the most important result of this thesis as a full answer to Question B by finding a necessary and sufficient condition for a Hall TI-subgroup to have a normal complement. It generalizes the classical result of Frobenius which asserts that for a Frobenius group G with complement H, the set $N = (G - \bigcup_{g \in G} H^g) \cup \{1\}$ is a normal subgroup of G with $G = NH$. Namely, we prove the following.

Theorem 1.0.7. *Let H be a TI-subgroup of G which is also a Hall subgroup of $N_G(H)$. Then H has a normal complement in $N_G(H)$ if and only if H has a normal complement in G. Moreover, if H is nonnormal in G and H has a normal complement in $N_G(H)$ then H is a Frobenius complement.*

In this framework we may state the following open question:

Conjecture. *Let H be a TI-subgroup of G which is also a Hall π-subgroup of $N_G(H)$. Then $G/O^\pi(G) \cong N_G(H)/O^\pi(N_G(H))$.*

Recall that this dissertation has two targets the second of which is to give character free proofs to the following two solvability theorems due to Isaacs.

Theorem A *Let G be a finite group having a cyclic Sylow p-subgroup. Assume that every p'-subgroup of G is abelian. Then G is either p-nilpotent or p-closed.*

Theorem B *Let G be a finite group and let $p \neq 2$ and q be primes dividing $|G|$. Suppose for every proper subgroup H of G which is not a q-group nor a q'-group that p divides $|H|$. If q^a is the q-part of $|G|$ and $p > q^a - 1$ or if $p = q^a - 1$ and a Sylow p-subgroup of G is abelian then no primes but p and q divide $|G|$.*

It should be noted that the lack of a character-free proof for a theorem would always be very impressive to the group theorists. We achieve these alternative proofs using some basic transfer theoretical facts in reducing the structure of a minimal counterexample to a hypothetical simple group. Then, the structure of the commuting graph of involutions of the group G lead us to the final contradiction that G is 2-closed. In this sense, our proofs are important as they do not use the complex machinery of character theory.

We close this chapter by an outline of the thesis:

Chapter 2 includes all necessary preparation from general group theory, group action, especially coprime action, and transfer theory. Most of them are well known results which will be referred throughout the thesis.

Section 3.1 contains key propositions which will be needed to prove the main results in Chapter 3 while Section 3.2 includes some technical lemmas pertaining to the proofs. In Section 3.3 we present the results obtained as an answer to Question A. We close Chapter 3 by Section 3.4 including a generalization of Frobenius' theorem.

Chapter 4 is devoted to the character free proofs of two solvability theorems of Isaacs. It begins with Section 4.1 consisting of graph theoretical preparation which we need in the proofs. Sections 4.2 and Section 4.3 includes, in turn, the proof of Theorem A and the proof of Theorem B.

CHAPTER 2

BACKGROUND MATERIAL

This chapter is devoted to a review of general group theoretical facts, and to some crucial results related to group action, especially coprime action and Frobenius action, and transfer theory. They will be referred throughout the thesis.

2.1 General group theoretical part

Definition 2.1.1. *Let G be a group. We define the* ***commutator subgroup*** *G' of G by*

$$G' = \langle [g, h] = g^{-1}h^{-1}gh \mid g, h \in G \rangle.$$

Lemma 2.1.2. *Let G be a group and $N \lhd G$. Then G/N is abelian if and only if $G' \leq N$.*

Lemma 2.1.3. [7, pages 113,114] *Let G be a group and $g, h, k \in G$. Then the following identities hold:*

(i) $[g, h][h, g] = 1$.

(ii) $[gh, k] = [g, k]^h[h, k]$.

(iii) $[g, hk] = [g, k][g, h]^k$.

Definition 2.1.4. *Let G be a group and let $H, K \leq G$ be subgroups. Then the commutator of H and K, denoted by $[H, K]$, is defined to be*

$$[H, K] = \langle [h, k] \mid h \in H, k \in K \rangle.$$

Since the generators of $[H, K]$ are the inverses of the generators of $[K, H]$, we have $[H, K] = [K, H]$.

Note that the multiple commutator group $[A_1, A_2, \dots, A_{n-1}, A_n]$ is recursively defined to be $[[A_1, A_2, \dots A_{n-1}], A_n]$ for $n > 2$ where $\{A_i \mid i = 1, \dots, n\}$ is a collection of subgroups of G. Similarly, we define $[G, G, \dots, G]_n = [[G, G \dots, G]_{n-1}, G]$ for $n > 2$.

Lemma 2.1.5. [7, Lemma 4.1] *Let G be a group and let H, K be subgroups of G. Then $[H, K] \trianglelefteq \langle H, K \rangle$.*

Lemma 2.1.6. [7, Lemma 4.9] ***(Three subgroup lemma)*** *Let X, Y, Z be subgroups of an arbitrary group G, and suppose that $[X, Y, Z] = 1$ and $[Y, Z, X] = 1$. Then $[Z, X, Y] = 1$.*

Definition 2.1.7. *A group G is called **solvable** if it has a normal subgroup series $1 = N_0 \leq N_1 \leq N_2 \dots \leq N_k = G$ such that N_{i+1}/N_i is abelian for $i = 0, \dots, k-1$.*

Corollary 2.1.8. *For a solvable group G, we have $H' < H$ for each nontrivial subgroup of H of G.*

Lemma 2.1.9. [9, Theorem 5.46] *Let N be a minimal normal subgroup of a solvable group G. Then N is an elementary abelian p-group.*

Lemma 2.1.10. [9, Theorem 5.45 (ii)] *Let G be a group and $N \lhd G$. Then G is solvable if and only if both N and G/N are solvable.*

Definition 2.1.11. *Let G be a group. We define the **center** of G by*

$$Z(G) = \{g \in G \mid gx = xg \ \ for \ all \ x \in G\}.$$

Definition 2.1.12. *A group G is called **nilpotent** if it has a normal subgroup series $1 = N_0 \leq N_1 \leq N_2 \dots \leq N_k = G$ such that $N_{i+1}/N_i \leq Z(G/N_i)$ for $i = 0, \dots, k-1$.*

Definition 2.1.13. *Let G be a group and p be a prime. We call G a p**-group** if each element of G is of order p^k for some nonnegative integer k.*

Theorem 2.1.14. [7, Theorem 1.22] *Let G be a nilpotent group and let H be a proper subgroup of G. Then $H < N_G(H)$.*

Theorem 2.1.15. [7, Theorem 1.26] *Let G be a group. Then the following are equivalent.*

$a)$ G is nilpotent.

$b)$ $N_G(H) > H$ for every proper subgroup H of G.

$c)$ Every maximal subgroup of G is normal.

$d)$ Every Sylow subgroup of G is normal.

$e)$ G is the internal direct product of its nontrivial Sylow subgroups.

Corollary 2.1.16. *Let G be a p-group. Then G is nilpotent.*

Lemma 2.1.17. [9, Theorem 5.9 (i)] *A group G is nilpotent if and only if $G/Z(G)$ is nilpotent.*

Lemma 2.1.18. *A group G is nilpotent if and only if $[G, G, \ldots, G]_n = 1$ for some integer n.*

Corollary 2.1.19. *Let G be a nilpotent group and $1 \neq N \lhd G$. Then $N \cap Z(G) \neq 1$.*

Definition 2.1.20. *For a group G, the intersection of all of maximal subgroups of G is called the **Frattini subgroup** of G, which is denoted by $\Phi(G)$.*

Lemma 2.1.21. [7, Lemma 4.5] *Let $N \trianglelefteq P$, where P is a p-group. Then P/N is elementary abelian if and only if $\Phi(P) \leq N$.*

Theorem 2.1.22. [7, Corollary X.19] ***(NC-Theorem)*** *Let G be a group and H be a subgroup of G. Set $N = N_G(H)$ and $C = C_G(H)$. Then $C \lhd N$ and N/C is isomorphic to a subgroup of $Aut(H)$.*

Lemma 2.1.23. ***(Dedekind Rule)*** *Let A, B and C be subgroups of a group G. If $A \leq C$ then $A(B \cap C) = AB \cap C$.*

Proof. Let $x \in A(B \cap C)$. Then $x = as$ where $a \in A$ and $s \in B \cap C$. It follows that $as \in AB$. Since $A \leq C$, as is also an element of C. Thus we get $A(B \cap C) \subseteq AB \cap C$.

Now let $x = ab \in AB \cap C$ where $a \in A$ and $b \in B$. Since $ab \in C$ and $A \leq C$, $b \in C$. It follows that $b \in B \cap C$ and $x \in A(B \cap C)$. Therefore $AB \cap C \subseteq A(B \cap C)$, and hence we have the desired equality. $\square$

Definition 2.1.24. *Let G be a group and H be a subgroup of G. We say that H has **a complement** in G, if there exists a subgroup K of G such that $G = HK$ and $H \cap K = 1$. Moreover, such K is called **a normal complement** whenever $K \trianglelefteq G$.*

Theorem 2.1.25. [7, Theorem 3.8 and 3.12] ***(Schur-Zassenhaus Theorem)*** *Let G be a group and $N \trianglelefteq G$ such that $gcd(|N|, |G : N|) = 1$. Then there exists a complement of N in G. Moreover, any two complements of N are conjugate by an element of N whenever N or G/N is solvable.*

Theorem 2.1.26. [7, Corollary X.7] *Let G be a cyclic group. Then for each positive divisor d of $|G|$, there is a unique subgroup of G of order d.*

Theorem 2.1.27. [7, Lemma X.14] *Let G be a cyclic group. Then $Aut(G)$ is an abelian group of order $\varphi(|G|)$, where φ is Euler's totient function.*

Lemma 2.1.28. [7, Lemma X.15] *Let $N \lhd G$, where G is a group, and let C be characteristic in N. Then $C \lhd G$.*

Lemma 2.1.29. *Let G be a group and $\mathfrak{F} = \{N_i \lhd G \mid i \in I\}$ be a family of normal subgroup of G where I is an finite index set. Then the group $G/(\bigcap\limits_{i\in I} N_i)$ is isomorphic to a subgroup of $\prod\limits_{i\in I} G/N_i$.*

Definition 2.1.30. *A group G is called p-closed if it has a unique Sylow p-subgroup P, that is, $P \trianglelefteq G$.*

Lemma 2.1.31. *All subgroups of a p-closed group G are p-closed.*

2.2 Group action

Definition 2.2.1. *Let G be a group and Ω be a set. Let "." be a operation such that $g.x \in \Omega$ for all $g \in G$ and $x \in \Omega$. We say that the group G **acts** on the set Ω if the following are satisfied:*
a) $e.x = x$ for all $x \in \Omega$ where e is the identity element of G.
b) $g_1(g_2.x) = (g_1g_2).x$ for all $g_1, g_2 \in G$ and $x \in \Omega$.

One can observe that each element of G induces a permutation on the set Ω. As a consequence, we have a homomorphism $\varphi : G \to Sym(X)$. Conversely, if we have a homomorphism $\varphi : G \to Sym(X)$ then we may obtain a group action by setting

$$g.x = \varphi(g)(x)$$

for all $g \in G$ and $x \in \Omega$.

Thus existence of such a homomorphism can be given as a equivalent definition of a group action.

It should be noted that we also use notation x^g to denote $g.x$ when convenient.

Let G be act on Ω. The set $O_x = \{g.x \mid g \in G\}$ is called **an orbit of** G for each element $x \in \Omega$. On the other hand, the subgroup $Stab_G(x) = \{g \in G \mid g.x = x\}$ of G is called **the stabilizer of** x in G. It is well known as the **orbit-stabilizer theorem** (see [8], Corollary 4.10) that

$$|G : Stab_G(x)| = |O_x|.$$

It is easy to see that the set of all distinct orbits constitutes a partition of Ω. Then one can obtain

$$|\Omega| = \sum_{x \in \Omega_0} |G : Stab_G(x)|$$

where Ω_0 is the set of representatives of each orbits. The action of G on Ω is called **transitive** if there is a single orbit, that is, for each $x, y \in \Omega$, there exists $g \in G$ such that $g.x = y$.

Theorem 2.2.2. *Let G be a group acting on a set Ω and let H be a subgroup of G such that the action of H on Ω is transitive. Then the equality*

$$Stab_G(\alpha)H = G$$

holds for all $\alpha \in \Omega$.

Proof. Let $g \in G$ and $\alpha \in \Omega$. Clearly $g\alpha \in \Omega$ and there exists $h \in H$ such that $hg\alpha = \alpha$. Then $hg \in Stab(\alpha)$ and so $g \in h^{-1}Stab(\alpha)$. Since g is arbitrary, the result follows. $\square$

Definition 2.2.3. *Let A and G be groups. We say that A **acts on** G **via automorphisms** if there is a homomorphism φ from A to $Aut(G)$. In that case, we denote $\varphi(a)(g)$ as g^a for all $a \in A$ and $g \in G$.*

In this case, $ker(\varphi) = C_A(G) = \{a \in A \mid g^a = g,\ \forall g \in G\}$ and $A/C_A(G)$ is isomorphic to a subgroup of $Aut(G)$. We say that A **acts faithfully** on G if $C_A(G) = 1$.

We also note that $C_G(A) = \{g \in G \mid g^a = g \; for \; all \; a \in A\}$ forms a subgroup of G, which is called **the subgroup of fixed points of** A **in** G.

Clearly, we can form the semidirect product $\Gamma = G \rtimes_{\varphi} A$ and consider $C_A(G)$, $[G, A]$ and $C_G(A)$ as subgroups of Γ.

Lemma 2.2.4. [7, Lemma 4.20] *Let A act on G via automorphisms. Then $[G, A]$ is the unique smallest A-invariant normal subgroup of G such that the induced action of A on the factor group is trivial.*

Lemma 2.2.5. *Let Γ be a group given by $\Gamma = G \rtimes A$. Then $N_{\Gamma}(A) = C_G(A)A$.*

Proof. Clearly, $N_{\Gamma}(A) = N_{\Gamma}(A) \cap \Gamma = N_{\Gamma}(A) \cap GA$. By Lemma 2.1.23, $N_{\Gamma}(A) = A(G \cap N_{\Gamma}(A)) = AN_G(A)$. On the other hand, $[N_G(A), A] \leq G \cap A = 1$, and so $N_G(A) \leq C_G(A)$. This yields that $N_G(A) = C_G(A)$, and hence $N_{\Gamma}(A) = C_G(A)A$ as desired. $\square$

Lemma 2.2.6. *Let A act on G via automorphisms. A subgroup H of G is A-invariant if and only if $[H, A] \leq H$. Moreover, if H and K are both A-invariant subgroups of G then $[H, K]$ is also A-invariant. In particular, $[H, A]$ is A-invariant whenever H is A-invariant.*

Lemma 2.2.7. *Let A act on G via automorphisms and N be an A-invariant normal subgroup of G. Then the equality $[G, A]N/N = [G/N, A]$ holds.*

Definition 2.2.8. *We say that A **acts on** G **coprimely** if A acts on G via automorphisms and $gcd(|A|, |G|) = 1$.*

Theorem 2.2.9. [7, Lemma 3.24] ***(Glauberman)*** *Let A act on G coprimely. Assume that A or G is solvable. Further assume that the group GA acts on Ω such that the action of G on Ω is transitive. Then A fixes at least one point of Ω. Moreover, for any two fixed points α, β of A in Ω, there exists $c \in C_G(A)$ such that $\alpha^c = \beta$.*

Corollary 2.2.10. [7, Theorem 3.23] *Let A act on G coprimely. Assume that A or G is solvable. Then for each prime p, the following hold:*

$a)$ There exists an A-invariant Sylow p-subgroup of G.
$b)$ If S and T are A-invariant Sylow p-subgroups of G then there exists $c \in C_G(A)$ such that $S^c = T$ whenever A or G is solvable.

Corollary 2.2.11. [7, Corollary 3.23] *Let A act on G coprimely. Assume that A or G is solvable. Then every A-invariant p-subgroup is contained in an A-invariant Sylow p-subgroup of G.*

Theorem 2.2.12. [7, Theorem 3.26] *Let A act on G coprimely. Assume that A or G is solvable. Then any two elements of $C_G(A)$ which are conjugate by an element of G are also conjugate by an element of $C_G(A)$.*

Corollary 2.2.13. [7, Corollary 3.28] *Let A act on G via automorphisms and let N be an A-invariant normal subgroup of G. Assume that $gcd(|A|, |N|) = 1$ and that A or N is solvable. Writing $\overline{G} = G/N$, we have $C_{\overline{G}}(A) = \overline{C_G(A)}$.*

Corollary 2.2.14. [7, Corollary 3.29] *Let A act on G coprimely. Then the action of A on G is trivial if and only if the induced action of A on $G/\Phi(G)$ is trivial.*

Lemma 2.2.15. [7, Lemma 3.32] *Let A act on G coprimely. Assume that A or G is solvable and let $P \in Syl_p(G)$ be A-invariant. Then $P \cap C_G(A) \in Syl_p(C_G(A))$.*

Theorem 2.2.16. [7, Lemma 4.28] *Let A act on G coprimely. Assume that A or G is solvable. Then the equality $G = [G, A]C_G(A)$ holds.*

Lemma 2.2.17. [7, Lemma 4.29] *Let A act on G coprimely. Then $[G, A, A] = [G, A]$.*

Theorem 2.2.18. [7, Theorem 4.34] ***(Fitting)*** *Let A act on an abelian group G coprimely. Then the equality $G = [G, A] \times C_G(A)$ holds.*

Corollary 2.2.19. *Let A act coprimely on a cyclic p-group P. Then either $[P, A] = P$ or $[P, A] = 1$, equivalently, $C_P(A) = P$ or $C_P(A) = 1$.*

Proof. By Theorem 2.2.18, we have $P = [P, A] \times C_P(A)$. Since P is cyclic, it has a unique subgroup of order p. This forces that either $[P, A] = 1$ or $C_P(A) = 1$, that is, $[P, A] = P$. □

2.3 Dihedral and Quaternion groups and their automorphism groups

Definition 2.3.1. *A group G is called **dihedral** if it contains a nontrivial cyclic subgroup C of index 2 such that each element of $G \setminus C$ is an involution.*

Lemma 2.3.2. *A dihedral group G of order greater than 4 has a unique cyclic subgroup C of index 2. In this case, C is also a characteristic subgroup of G.*

Proof. Let G be a dihedral group of order greater than 4. Then G has a cyclic subgroup C of index 2 where the set $G \setminus C$ consists of involutions. Assume $D = \langle y \rangle$ is another cyclic subgroup of G of index 2. We have $|y| > 2$ as $|G| > 4$. Since $y \notin G \setminus C$, we obtain that $y \in C$, and hence $D = C$ as desired. The uniqueness of C yields that C is characteristic in G. $\square$

Remark 2.3.3. The dihedral group $G = \{1, a, b, c\}$ of order 4 is isomorphic to Klein 4-group, that is, $\mathbb{Z}_2 \times \mathbb{Z}_2$. $Aut(G)$ acts on the set $G \setminus \{1\}$, and hence it is isomorphic to a subgroup of S_3. One can easily check that the cycles $\alpha = (a, b, c)$ and $\beta = (a, b)$ induce automorphisms on G, and hence $Aut(G) \cong S_3$.

Lemma 2.3.4. [7, Lemma 2.14 (a)] *Let G be a group with a cyclic subgroup C of index 2 and $i \in G \setminus C$ be an involution. Then for each $g \in C$, $g^i = g^{-1}$ if and only if G is a dihedral group.*

Lemma 2.3.5. [7, Lemma 2.14 (b)] *Let G be a group and i, j be two involutions in G. Set $D = \langle i, j \rangle$. Then D is a dihedral group with cyclic subgroup $C = \langle ij \rangle$ of index 2.*

Definition 2.3.6. *Let G be a 2-group with a cyclic subgroup C of index 2. Then G is called a* ***quaternion group*** *if there exists $y \in G \setminus C$ of order 4 such that $x^y = x^{-1}$ for each $x \in C$.*

Lemma 2.3.7. *Let G be a quaternion group. Then the following hold;*
$a)$ G has a unique involution z, which is contained in $Z(G)$.
$b)$ The quotient $G/\langle z \rangle$ is a dihedral group.

Proof. Let $C = \langle x \rangle$ be a cyclic subgroup G of index 2 and $y \in G \setminus C$ of order 4 such that $x^y = x^{-1}$.

(a) Since C is a cyclic 2-group, it has a unique involution, say z. We claim that z is the only involution of G, that is, there is no involution in the set $G \setminus C$. Suppose not and pick $a \in G \setminus C$. Then $a = x^n y$ for some $n \in \mathbb{Z}$ as the set $G \setminus C$ is equal to the right coset Cy. Now,

$$a^2 = (x^n y)(x^n y) = y(y^{-1}x^n y)(x^n y) = yx^{-n}x^n y = y^2 \neq 1.$$

Consequently, z is the unique involution in G, and hence $z \in Z(G)$.

(b) Set $\overline{G} = G/\langle z\rangle$. Since $z \in C$, we clearly have $|\overline{G} : \overline{C}| = 2$ and $\overline{C}$ is cyclic. As y^2 is an involution, we also have $y^2 = z$ by part (a). Thus, $\overline{y}$ is an involution lying in the set $\overline{G} \setminus \overline{C}$ such that $\overline{x}^{\overline{y}} = \overline{x^y} = \overline{x}^{-1}$. Hence $\overline{G}$ is a dihedral group by Lemma 2.3.4. □

Theorem 2.3.8. *Let G be a dihedral 2-group with $|G| > 4$ or a quaternion group with $|G| > 8$. Then $Aut(G)$ is a 2-group.*

Proof. Suppose first that G is a dihedral group of order greater than 4 and let C be a cyclic subgroup of index 2. Then C is a characteristic subgroup of G by Lemma 2.3.2.

Let $\alpha \in Aut(G)$ of odd order. Note that α induces the trivial automorphism on G/C as $G/C \cong \mathbb{Z}_2$. On the other hand, α also induces the trivial homomorphism on C as $|Aut(C)| = 2^{k-1}$ by Theorem 2.1.27, where $|C| = 2^k$. Then $[G, \alpha, \alpha] = 1$, and hence $[G, \alpha] = 1$ by Lemma 2.2.17. Thus, $\alpha = 1$ as required.

Suppose next that G is a quaternion group of order greater than 8 and let z be the unique involution in G. Notice that $\langle z\rangle$ is a characteristic subgroup of G. Pick $\alpha \in Aut(G)$ of odd order. Then $\overline{G} = G/\langle z\rangle$ is a dihedral group by Lemma 2.3.7. Since $|\overline{G}| > 4$, we have $[\overline{G}, \alpha] = 1$ by the above argument. Since $[\langle z\rangle, \alpha] = 1$, we get $[G, \alpha, \alpha] = 1$, and hence $[G, \alpha] = 1$ by Lemma 2.2.17. □

Corollary 2.3.9. *Let G be a quaternion group. Then $Aut(G)$ is solvable.*

Proof. By the above theorem, we only need to consider the case that G is the quaternion group of order 8. Let z be the unique involution of G. Then $Aut(G)$ acts on $G/\langle z\rangle$ via automorphisms, where $G/\langle z\rangle$ is a dihedral group of order 4, that is, $G/\langle z\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ by Lemma 2.3.7. Let K be the kernel of this action. Then we have $[G, K] \leq \langle z\rangle$ and $Aut(G)/K$ is isomorphic to a subgroup of $Aut(\mathbb{Z}_2 \times \mathbb{Z}_2) = S_3$, which is solvable. Therefore, it is sufficient to show that K is solvable by Lemma

2.1.10. Let $1 \neq k \in K$ of odd order. As $[k, \langle z \rangle] = 1$, we get $[G, k] = [G, k, k] = 1$ by Lemma 2.2.17. This contradicts the fact that $[G, k] \neq 1$ as $k \in Aut(G)$. It follows that K is a 2-group, and so $Aut(G)$ is solvable as desired. $\square$

2.4 Hall subgroups and p-nilpotency

Definition 2.4.1. *A subgroup H of a group G is called a* ***Hall subgroup*** *if $gcd(|H|, |G : H|) = 1$. Moreover, we call H a Hall π-subgroup to emphasize that $|H|$ is a π-number for a prime set π.*

Note that Hall π-subgroups may not exist for a prime set π in general. The following theorem guarantees the existence of Hall subgroups in solvable groups.

Theorem 2.4.2. [2, Theorem 4.1] ***(Hall)*** *Let G be a solvable group and let π be a set of primes. Then G has a Hall π-subgroup. Moreover, any two Hall π-subgroups of G are conjugate in G and each π-subgroup of G is contained in a Hall π-subgroup of G.*

Definition 2.4.3. *Let G be a group and π be a set of primes. We say that G is* ***π-separable****, if there exists a normal subgroup series $1 = N_0 \leq N_1 \leq \ldots \leq N_k = G$ such that N_{i+1}/N_i is either π-group or π'-group for $i = 0, \ldots, k-1$.*

Lemma 2.4.4. *Let G be a π-separable group. Then all subgroups of G are π-separable.*

Proof. Suppose that G is π-separable group, that is, there is a normal subgroup series

$$1 = N_0 \leq N_1 \leq \ldots \leq N_k = G$$

such that N_{i+1}/N_i is either π-group or π'-group for $i = 0, \ldots, k-1$. Let H be a subgroup of G. Set $H_i = N_i \cap H$ for each i. Clearly each H_i is normal in H, and hence

$$1 = H_0 \leq H_1 \leq \ldots \leq H_k = H$$

is a normal subgroup series in H.

Now consider the group N_{i+1}/N_i. We have

$$(N_{i+1}/N_i) \cap (HN_i/N_i) = (H \cap N_{i+1})N_i/N_i$$

which is a subgroup of N_{i+1}/N_i. Observe that

$$(H \cap N_{i+1})N_i/N_i \cong H \cap N_{i+1}/(H \cap N_{i+1} \cap N_i) = H \cap N_{i+1}/H \cap N_i = H_{i+1}/H_i.$$

It then follows that H_{i+1}/H_i is isomorphic to a subgroup of N_{i+1}/N_i. Thus, each H_{i+1}/H_i is either π-group or π'-group, and hence H is π-separable. $\square$

Theorem 2.4.5. [7, Theorem 3.20] *Let G be a π-separable group. Then G has a Hall π-subgroup.*

In a group G, $O_\pi(G)$ is defined to be **the largest normal π-subgroup** of G.

Theorem 2.4.6. [7, Theorem 3.13] ***(Hall-Higman)*** *Let G be a π-separable group such that $O_{\pi'}(G) = 1$. Then*

$$C_G(O_\pi(G)) \leq O_\pi(G).$$

Definition 2.4.7. *Let G be a π-separable group. The* ***π-length*** *of G is the minimum possible number of π-factors that are π-groups in any normal series of G in which each factor is either π-group or π'-group.*

Definition 2.4.8. *Let G be a group and let π be a prime set. We say that G has a normal* ***π-complement*** *if G has a normal Hall π'-subgroup. In the case where G has a normal π-complement for $\pi = \{p\}$ then G is called a* ***p-nilpotent*** *group.*

Lemma 2.4.9. *All subgroups of a p-nilpotent group are p-nilpotent.*

Proof. Let G be a p-nilpotent group, that is, G has a normal Hall p'-subgroup N. Let $H \leq G$. Then $H \cap N$ is a normal π-subgroup of H. Since G/N is p-group, $HN/N \cong H/H \cap N$ is also p-group. It follows that $H \cap N$ is a normal Hall p'-subgroup of H, and hence H is p-nilpotent. $\square$

Lemma 2.4.10. *If G is a group which is p-nilpotent for each prime p-dividing its order, then G is nilpotent.*

Proof. Let G be a group which is p-nilpotent for each prime p dividing $|G|$ and let P be a Sylow p-subgroup of G. We claim first that $P \lhd G$. Let us proceed by induction on the order of G. We may assume that G is not a p-group. Let q be a prime dividing the order G such that $q \neq p$. Then by the hypothesis, G has a normal Hall q'-subgroup N, which contains P. Note that N is a proper group satisfying the hypothesis and so $P \trianglelefteq N$ by induction. Then P is a characteristic subgroup of N, and hence $P \lhd G$. Since p is arbitrary, we see by Theorem 2.1.15 that G is nilpotent. □

Theorem 2.4.11. [8, Lemma 8.10] ***(Frattini argument)*** *Let G be a group and $N \trianglelefteq G$. Assume that H is a Hall π-subgroup of N such that any Hall π-subgroup of N is conjugate to H. Then the equality $G = N_G(H)N$ holds.*

Proof. It follows as a corollary of Theorem 2.2.2. □

2.5 Transfer and fusion

Definition 2.5.1. *Let G be a group and $H \leq G$. A* ***right transversal set*** *for H in G is a set constructed by choosing exactly one element from each right coset of H in G.*

Definition 2.5.2. *Let G be a group and $H \leq G$. For each right transversal T for H in G, we define the action of G on T by $t.g = s$ for any $t, s \in T$ and $g \in G$ where $Htg = Hs$.*

It is trivial to check that the "dot operation" is really an action. Notice that $tg(t.g)^{-1} \in H$ for any $t \in T$ and $g \in G$. We call this action as a "**dot action**".

Definition 2.5.3. *Let G be a group and $H \leq G$. Let $T = \{t_i \mid i = 1, 2 \ldots, n\}$ be a right transversal for H in G. The map $V : G \to H$ defined by*

$$V(g) = \prod_{i=1}^{n} t_i g(t_i.g)^{-1}$$

is called ***a pretransfer map*** *from G to H.*

Definition 2.5.4. *Let G be a group and $H \leq G$. The map $v : G \to H/H'$ defined by $v(g) = V(g)H'$ where V is a pretransfer map from G to H is called* ***the transfer map*** *from G to H/H'.*

Theorem 2.5.5. [7, Theorem 5.1] *Let G be a group and let $H \leq G$. Then the transfer map v from G to H/H' is independent of the choice of the right transversal used to define it.*

Notice that v is also independent of an ordering of elements of T. Thus, we may write $V(g) = \prod_{t \in T} tg(t.g)^{-1}$ instead of $V(g) = \prod_{i=1}^{n} t_i g(t_i.g)^{-1}$.

Theorem 2.5.6. [7, Theorem 5.2] *Let G be a group and $H \leq G$. Then the transfer map v from G to H/H' is a homomorphism.*

Lemma 2.5.7. [7, Lemma 5.5] *Let G be a group and $H \leq G$. Let V be a pretransfer map from G to H constructed by using the right transversal T. Fix an element $g \in G$ and let T_0 be the set of orbit representatives of the dot action of $\langle g \rangle$ on the set T and let n_t denote the length of the orbit represented by t for $t \in T_0$. Then the following hold:*

a) $tg^{n_t}t^{-1} \in H$ *for all* $t \in T_0$.
b) $V(g)H' = (\prod_{t \in T_0} tg^{n_t}t^{-1})H'$.

Lemma 2.5.8. [7, Lemma 5.11] *Let H be a Hall subgroup of G and let v be the transfer map from G to H/H'. Then $v(G) = v(H)$, and so $|H : H \cap Ker(v)| = |G : ker(v)|$.*

Definition 2.5.9. *Let G be a group and let $H \leq K \leq G$. We say that K **controls** G**-fusion** in H if any two elements of H which are conjugate by an element of G are also conjugate by an element of K.*

Theorem 2.5.10. ***(Wielandt)*** *Let G be a group having a nilpotent Hall π-subgroup. Then any two nilpotent Hall π-subgroups of G are conjugate in G.*

Proof. We proceed by induction on the order of G. Let H and K be nilpotent Hall π-subgroups of G. Let $P \in Syl_p(H)$ and $Q \in Syl_p(K)$. Then there exists $x \in G$ such that $P = Q^x$ and hence $P \leq H \cap K^x$. Since both H and K^x are nilpotent, we get $H, K^x \leq N_G(P)$. If $N_G(P) < G$ then H and K^x are conjugate in $N_G(P)$ by the inductive hypothesis. Thus, we may assume that $P \lhd G$. In this case H/P and K^x/P are conjugate in G/P by induction, that is, $H/P = (K^x/P)^y = K^{xy}/P$ for some $y \in G$. It follows that $H = K^{xy}$, which concludes the proof. $\square$

Remark 2.5.11. What Wielandt proved is more general than what we stated. He proved that any two Hall π-subgroups of G are conjugate under the hypothesis of the above theorem. Yet, we shall not need this stronger form.

Now by using the Wielandt's Theorem, we state [7, Lemma 5.12] in a stronger form.

Lemma 2.5.12. *Let G be a group and let H be a nilpotent Hall π-subgroup of G. Then $N_G(H)$ controls G-fusion in $C_G(H)$.*

Proof. Let $x, x^g \in C_G(H)$. Then x is contained in both $C_G(H)$ and $C_G(H^{g^{-1}})$, and hence $H, H^{g^{-1}} \leq C_G(x)$. There exists $y \in C_G(x)$ such that $H^y = H^{g^{-1}}$ by Theorem 2.5.10. We have $yg \in N_G(H)$ and $x^{yg} = x^g$, completing the proof. $\square$

We also give [7, Theorem 5.18] in a general setting.

Theorem 2.5.13. *Let H be an abelian Hall π-subgroup of G and let v be the transfer map from G to H/H'. Then*

$$ker(v) \cap H \cap Z(N_G(H)) = 1.$$

Proof. Let $x \in ker(v) \cap H \cap Z(N_G(H))$. Since $x \in ker(v)$, we get

$$1 = v(x) = \prod_{t \in T_0} t x^{n_t} t^{-1}$$

by Lemma 2.5.7(b). Note that $x^{n_t} \in H$ as $x \in H$ and $tx^{n_t}t^{-1} \in H$ by Lemma 2.5.7(a). Since H is abelian, we have $H \leq C_G(H)$. Then by Lemma 2.5.12, $N_G(H)$ controls G-fusion in H, and hence x^{n_t} and $tx^{n_t}t^{-1}$ are also conjugate by an element of $N_G(H)$. However, $x^{n_t} \in Z(N_G(H))$ and hence, $x^{n_t} = tx^{n_t}t^{-1}$. Thus, $1 = v(x) = \prod_{t \in T_0} tx^{n_t}t^{-1} = \prod_{t \in T_0} x^{n_t} = x^{|G:H|}$. Since x is a π-element and $|G : H|$ is a π'-number, we obtain that $x = 1$, which completes the proof. $\square$

Corollary 2.5.14. *Let H be an abelian Hall π-subgroup of G. Then*

$$G' \cap H \cap Z(N_G(H)) = 1.$$

Corollary 2.5.15. ***(Burnside)*** *Let H be Hall π-subgroup of G. Assume that $H \leq Z(N_G(H))$. Then G has a normal π-complement.*

Proof. By Theorem 2.5.13, $ker(v) \cap H \cap Z(N_G(H)) = ker(v) \cap H = 1$. Since $|G : ker(v)| = |H : ker(v) \cap H| = |H|$ by Lemma 2.5.8, we get $G = ker(v)H$. Then $ker(v)$ is the desired normal π-complement. □

Corollary 2.5.16. *Let P be a cyclic Sylow p-subgroup of G. Then either G is p-nilpotent or $P \leq G'$.*

Proof. Assume that $P \nleq G'$ and let $\alpha \in N_G(P)$ of p'-order. Note that $[P, \alpha] < P$ as $[P, \alpha] \leq G'$. We have $[P, \alpha] = 1$ by Corollary 2.2.19. On the other hand, P is the unique Sylow p-subgroup of $N_G(P)$ and centralized by p-elements of $N_G(P)$. Thus, $P \leq Z(N_G(P))$, and hence G is p-nilpotent by Burnside theorem (see 2.5.15). □

Corollary 2.5.17. *Let P be a cyclic Sylow p-subgroup of G where p is the smallest prime dividing the order of G. Then G is p-nilpotent.*

Proof. If $P \cong C_{p^k}$ then $|Aut(P)| = \phi(p^k) = p^{k-1}(p-1)$. By Theorem 2.1.22, $|N_G(P)/C_G(P)|$ divides $p^{k-1}(p-1)$. Since p is the smallest prime dividing the order of G, we get $q = p$. This contradicts the fact that $|N_G(P)/C_G(P)|$ is a p'-number. Therefore $N_G(P) = C_G(P)$, that is, $P \leq Z(N_G(P))$. Thus G is p-nilpotent by Burnside theorem (see 2.5.15). □

Corollary 2.5.18. *Let G be a group whose Sylow subgroups are all cyclic. Then G is solvable.*

Proof. We proceed by induction on the order G. Let $P \in Syl_p(G)$ where p is the smallest prime dividing the order of G. Then G has a normal p-complement, say N by Corollary 2.5.17. Since N satisfies the hypothesis, we see that N is solvable by induction. On the other hand, $G/N \cong P$, and hence G/N is also solvable. As a consequence, G is solvable by Lemma 2.1.10. □

Definition 2.5.19. *Let G be a group and let π be a set of primes. We define $A^\pi(G)$ as the smallest normal subgroup of G such that $G/A^\pi(G)$ is an abelian π-group. Similarly, we define $O^\pi(G)$ as the smallest normal subgroup of G such that $G/O^\pi(G)$ is a π-group.*

Definition 2.5.20. *Let H be a subgroup of G. We define the focal subgroup H of G by*

$$Foc_G(H) = \langle \{x^{-1}y \mid x \in H \; and \; y \in x^G \cap H\} \rangle.$$

Now we state ([7], Theorem 5.21) in a stronger form.

Theorem 2.5.21. ***(Focal Subgroup Theorem)*** [7, Theorem 5.21]
Let H be a Hall π-subgroup of G and let v be the transfer map from G to H/H'. Then

$$Foc_G(H) = H \cap G' = H \cap A^{\pi}(G) = H \cap ker(v).$$

Proof. We first show that

$$Foc_G(H) \leq H \cap G' \leq H \cap A^{\pi}(G) \leq H \cap ker(v).$$

Since $Foc_G(H)$ is generated by some commutators, it is contained in G'. It is clear that $Foc_G(H) \leq H$, and so $Foc_G(H) \leq H \cap G'$. Since $G' \leq A^{\pi}(G)$ by Lemma 2.1.2, we also get $H \cap G' \leq H \cap A^{\pi}(G)$. As $v : G \to H/H'$ and H/H' is abelian, $G/ker(v)$ is an abelian π-group. Thus we have $A^{\pi}(G) \leq ker(v)$, and hence $H \cap A^{\pi}(G) \leq H \cap ker(v)$.

It remains to show that $H \cap ker(v) \leq Foc_G(H)$ to conclude the proof. Let $x \in H \cap ker(v)$ and let V be a pretransfer map from G to H. Then we have

$$H' = v(x) = V(x)H' = (\prod_{t \in T_0} t x^{n_t} t^{-1})H'$$

by Lemma 2.5.7(b). It follows that $(\prod\limits_{t \in T_0} t x^{n_t} t^{-1}) \in H'$. Notice that both $t x^{n_t} t^{-1}$ and x^{n_t} are elements of H, and hence $t x^{n_t} t^{-1} x^{-n_t} \in Foc_G(H)$. Now

$$H' = (\prod_{t \in T_0} t x^{n_t} t^{-1} x^{-n_t} x^{n_t})H' = (\prod_{t \in T_0} t x^{n_t} t^{-1} x^{-n_t}) \prod_{t \in T_0} (x^{n_t})H' \leq Foc_G(H).$$

Since $(\prod\limits_{t \in T_0} t x^{n_t} t^{-1} x^{-n_t}) \in Foc_G(H)$, we get $\prod\limits_{t \in T_0} (x^{n_t}) = x^{|G:H|} \in Foc_G(H)$. It then follows that $x \in Foc_G(H)$ as x is a π-element and $|G : H|$ is a π'-number. $\square$

Theorem 2.5.22. *Let G be a group and let $P \in Syl_p(G)$. Then G is p-nilpotent if and only if P controls G-fusion in itself.*

Proof. First assume that N is the normal p-complement of G. Let $x, x^g \in P$ for some $g \in G$. Then $g = nh$ for some $n \in N$ and $h \in P$. It follows that $Nx^g = Nx^h$, and hence $x^{-g}x^h \in P \cap N = 1$. Thus $x^g = x^h$. As a consequences, P controls G-fusion itself.

Conversely, assume that P controls G-fusion in itself. Let $O = O^p(G)$, $K = O \cap P$ and $A = A^p(O)$. Since A is characteristic in O and O is normal in G, we have $A \lhd G$. Moreover, G/A is a p-group as G/O and O/A are both p-groups. Therefore $O = A$. Notice that $K \in Syl_p(O)$. Then by the focal subgroup theorem (see 2.5.21), $Foc_O(K) = K \cap A = K$. Now assume that $K \neq 1$. Since P-controls G-fusion in itself, if two elements of K are conjugate by an element O then they are also conjugate by an element of P. It then follows that $Foc_O(K) \leq Foc_P(K)$. As $K \lhd P$, we get $K = Foc_O(K) \leq Foc_P(K) = [P, K] < K$. This contradiction shows that $K = 1$ and O is the desired normal p-complement.

□

2.6 Frobenius groups and TI-subgroups

Definition 2.6.1. *Let A be a nontrivial group acting on a group G via automorphisms. We say that the action of A on G is* ***Frobenius*** *(or A acts Frobeniusly on G) if $C_G(a) = 1$ for each nonidentity element a of A.*

Lemma 2.6.2. *Let A be a group acting on G via automorphism. Then A acts on G Frobeniusly if and only if $C_A(g) = 1$ for each nonidentity element $g \in G$.*

Proof. Suppose first that the action of A on G is Frobenius, that is, $C_G(a) = 1$ for each nonidentity element $a \in A$. Let g be an arbitrary nonidentity element of G. If $1 \neq x \in C_A(g)$ then $1 \neq g \in C_G(x)$, which is not the case. Thus $C_A(g) = 1$ for all $1 \neq g \in G$.

Now assume that $C_A(g) = 1$ for each nonidentity element $g \in G$. Let a be an arbitrary nonidentity element of A. If $1 \neq y \in C_G(a)$ then $1 \neq a \in C_A(y)$, which is not the case. Thus, $C_G(a) = 1$ for all $1 \neq a \in A$, that is, A acts Frobeniusly on G. □

Lemma 2.6.3. [7, Lemma 6.1] *Let a group A have Frobenius action on a group G. Then we have $|G| \equiv 1 \ mod \ |A|$. In particular, a Frobenius action is also a coprime action.*

Proof. For each nonidentity $x \in G$, we have $|A/C_A(x)| = |A|$ due to the Frobenius action by Lemma 2.6.2. Hence the result follows. $\square$

Corollary 2.6.4. [7, Corollary 6.1] *Let a group A have Frobenius action on a group G. Suppose that N is an A-invariant normal subgroup of G. Then the induced action of A on G/N is also Frobenius.*

Proof. Let a be a nonidentity element of A. By the above lemma, A acts on G coprimely. Thus, we have $C_{G/N}(a) = C_G(a)N/N = N$ by Corollary 2.2.13, which proves the claim. $\square$

Theorem 2.6.5. [7, Theorem 6.3] *Let a group A have Frobenius action on a nontrivial group G. If $|A|$ is even then A contains a unique involution and G is abelian.*

Definition 2.6.6. *Let G be a group and $H \leq G$. We say that H is a TI-**subgroup** of G if for every $g \in G$, $H \cap H^g = H$ or $H \cap H^g = 1$, that is, $H \cap H^g = 1$ for all $g \in G \setminus N_G(H)$.*

Lemma 2.6.7. *Let G be a group and H be a TI-subgroup of G. Then the following hold.*
$a)$ H^g is a TI-subgroup of G for each $g \in G$.
$b)$ If $K \leq G$ then $H \cap K$ is a TI-subgroup of K.
$c)$ Every characteristic subgroup of H is also a TI-subgroup of G.

Proof. $a)$ Assume that $H^g \cap H^{gx} \neq H^g$. Then we get

$$(H^g \cap H^{gx})^{g^{-1}} = H \cap H^{gxg^{-1}} \neq H^{gg^{-1}} = H.$$

Since H is a TI-subgroup, we get $H \cap H^{gxg^{-1}} = 1$, and so $H^g \cap H^{gx} = 1^g = 1$. It follows that H^g is also a TI-subgroup.

$b)$ Let K be a subgroup of G and set $S = H \cap K$. Suppose that $S \cap S^k \neq 1$ for some $k \in K$. Then we get $H \cap H^k \neq 1$, as $S \cap S^k \leq H \cap H^k$. It follows that $H^k = H$, and hence $S^k = (H \cap K)^k = H^k \cap K = H \cap K = S$. Thus, S is a TI-subgroup of K.

c) Let C be a characteristic subgroup of H and let $x \in G \setminus N_G(C)$. Then we have $C \lhd N_G(H)$. It follows that $x \notin N_G(H)$, and hence $C \cap C^x \leq H \cap H^x = 1$. Therefore, C is a TI-subgroup of G as desired. $\square$

A subgroup H of G is called **self normalizing** if $N_G(H) = H$.

Definition 2.6.8. *Let G be a group and H be a proper subgroup of G. Then G is called a Frobenius group with complement H if H is a self normalizing TI-subgroup of G.*

Theorem 2.6.9 (Frobenius). *Let G be a Frobenius group with complement H. Then the set $N = (G \setminus \bigcup_{g \in G} H^g) \cup \{1\}$ is a normal subgroup of G. Moreover, $G = NH$ and $N \cap H = 1$.*

Definition 2.6.10. *Let G be a Frobenius group with complement H. The normal subgroup $N = (G \setminus \bigcup_{g \in G} H^g) \cup \{1\}$ of G is called a **Frobenius kernel** of G.*

Lemma 2.6.11. *Let G be a Frobenius group with complement H. Then H is a Hall subgroup of G.*

Proof. Let X be a set of representatives of the double coset (H, H) in G. Then we have $|G| = \sum\limits_{x \in X} |HxH|$. Note that if $x \in H$ then $|HxH| = |H|$ and otherwise,

$$|HxH| = |H(xHx^{-1})x| = |H(H^{x^{-1}})| = |H||xHx^{-1}|/|(H \cap H^x)| = |H|^2.$$

It follows that $|G| = |H| + k|H|^2$ for some k. Then we have $|G : H| = 1 + k|H|$, and hence $gcd(|H|, |G : H|) = 1$ as desired. $\square$

By the above lemma, we may consider Theorem 2.6.9 as a normal π-complement theorem where π is the prime set of H.

Lemma 2.6.12. *Let G be a Frobenius group with complement H and kernel N. Then the conjugation action of H on N is a Frobenius action.*

Proof. Let h be a nonidentity element of H and pick $n \in C_N(h)$. It then follows that $h^n = h$, and hence $1 \neq h \in H \cap H^n$. We obtain now that $H = H^n$ as H is a TI-subgroup. Then $n \in N_G(H) \cap N = H \cap N = 1$, which forces that $n = 1$. Thus we get $C_N(h) = 1$ for all $1 \neq h \in H$, that is, the action of H on N is Frobenius as required. $\square$

The following lemma is the dual of the above lemma.

Lemma 2.6.13. *Let H be a group acting Frobeniusly on the group N. Then the semidirect product $G = N \rtimes H$ is a Frobenius group with complement H and kernel N.*

Proof. First we observe that $N_G(H) = H$. By Lemma 2.2.5, we have $N_G(H) = C_N(H)H$. Pick $1 \neq h \in H$. It follows that $C_N(H) \leq C_N(h) = 1$ due to Frobenius action, and hence $N_G(H) = H$ as required.

Suppose next that $H \cap H^g \neq 1$ for some $g \in G$. Now $g = hn$ for some $h \in H$ and $n \in N$. Then we have $H \cap H^n \neq 1$, and hence $h^n \in H$ for some $h \neq 1$. It follows that $h^{-1}h^n = [h, n] \in H \cap N = 1$. Thus we get $n = 1$ due to the Frobenius action. This gives $H \cap H^g = H$, and hence H is a self normalizing TI-subgroup of G. As a result, G is a Frobenius group with complement H.

It remains to show that N is the Frobenius kernel of G. Obviously

$$N \subseteq (G \setminus \bigcup_{g \in G} H^g) \cup \{1\}$$

and hence it is enough to observe that they have the same cardinality. By a simple counting argument, we get

$$|(G \setminus \bigcup_{g \in G} H^g) \cup \{1\}| = |G| - (|H| - 1)|G : H| - 1 + 1 = |G : H| = |N|.$$

This completes the proof. $\square$

The above lemma leads to an equivalent definition of a Frobenius complement. Namely, a group H is said to be a Frobenius complement if it has a Frobenius action on a group N.

Theorem 2.6.14. [7, Theorem 6.7] *Let G be a group and let $1 < N \lhd G$. Suppose that $C_G(n) \leq N$ for each nonidentity element $n \in N$. Then G is a Frobenius group with kernel N.*

Proof. Let p be a prime dividing the order of N and let $P \in Syl_p(G)$. Since $1 \neq P \cap N \trianglelefteq P$, $Z(P) \cap (P \cap N) = Z(P) \cap N \neq 1$ by Corollary 2.1.19. Let now $1 \neq z \in Z(P) \cap N$. Then $P \leq C_G(z) \leq N$, and hence p is not a divisor of $|G : N|$. This shows that $gcd(|N|, |G : N|) = 1$, and so there exists a complement H in G by Schur-Zassenhaus theorem (see 2.1.25).

Notice that for every $1 \neq n \in N$, we have $C_H(n) = 1$ as $C_G(n) \leq N$. Thus, the conjugation action of H on N is Frobenius by Lemma 2.6.2, and hence G is a Frobenius group by Lemma 2.6.13 as claimed. $\square$

Definition 2.6.15. *Let G be a group.* ***A partition*** *Ω of G is a collection of subgroups of G such that $G = \bigcup_{H \in \Omega} H$ and for every distinct pair $H, K \in \Omega$, we have $H \cap K = 1$.*

Lemma 2.6.16. *[7, Lemma 6.8] Let A be a group acting on an abelian group U via automorphisms. Suppose that A has a partition Ω and U has an element whose order is not a divisor of $|\Omega| - 1$. Then there exists $H \in \Omega$ such that $C_U(H) \neq 1$.*

Now we state a weaker version of [7, Theorem 6.9].

Lemma 2.6.17. *Let A be an elementary abelian p-group. If A is a Frobenius complement then A is a cyclic group of order p.*

Proof. Suppose that a noncyclic elementary abelian p-group A has a Frobenius action on a group N. Then A contains a subgroup B which is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$. Let Ω be set of all subgroups of B whose order is p. Clearly Ω is a partition of B.

Due to the Frobenius action, B acts coprimely on N, and hence N has a B-invariant Sylow r-subgroup R for any prime r dividing the order of N by Corollary 2.2.10 . Now $Z(R)$ is also B-invariant. Since $|\Omega| - 1 = p$ which is coprime to r, we get $C_{Z(R)}(H) \neq 1$ for some $H \in \Omega$ by the previous lemma. This contradiction shows that A is cyclic of order p. $\square$

Corollary 2.6.18. *Let A be a Frobenius complement and let P be a Sylow p-subgroup of A for a prime p dividing $|A|$. Then P has a unique subgroup of order p.*

Proof. We may suppose that A has a Frobenius action on a group N. As $Z(P) \neq 1$, let U be a subgroup of $Z(P)$ of order p. If V is another subgroup of order p then

$UV \cong \mathbb{Z}_p \times \mathbb{Z}_p$, and the action of UV on N is also Frobenius. This is not possible by Lemma 2.6.17. Thus U is the unique subgroup of P of order p. $\square$

Theorem 2.6.19. [7, Theorem 6.9] *Let P be a p-group having a unique subgroup of order p. Then P is either cyclic or $p = 2$ and P is generalized quaternion.*

Corollary 2.6.20. [7, Corollary 6.17] *Let A be a Frobenius complement. Then each Sylow subgroup of A is cyclic or generalized quaternion.*

Corollary 2.6.21. *Let A be a Frobenius complement of odd order. Then A is solvable.*

Proof. This follows directly from Corollary 2.5.18. $\square$

In general, a Frobenius complement need not to be solvable. The following theorem provides a necessary and sufficient condition for a Frobenius complement to be solvable.

Theorem 2.6.22. [4, Theorem 16.7 (d)] *Let A be a Frobenius complement. Then A is solvable if and only if A does not contain a subgroup isomorphic $SL(2,5)$.*

Theorem 2.6.23 ([6], Theorem 8.22)**.** *Let H be a Hall subgroup of G and suppose that whenever two elements of H are conjugate in G, they are already conjugate in H. Assume that for every elementary subgroup E of G, if $|E|$ divides $|H|$, then E is conjugate to a subgroup of H. Then H has a normal complement in G.*

Definition 2.6.24. *Let Γ be a group and $N \trianglelefteq \Gamma$. Assume that N has a complement G in Γ, where G is a Frobenius group with complement H and kernel K. Then Γ is called* ***a double Frobenius group*** *if NK is also a Frobenius group.*

We close this chapter by giving a proof of a famous result due to Thompson. He proved in 1959 that Frobenius kernels are nilpotent. We first consider solvable Frobenius kernels (see [7], Theorem 6.22).

Lemma 2.6.25. *Let N be a solvable Frobenius kernel. Then N is nilpotent.*

Proof. Let N be a minimal counterexample to the theorem. Now there exists a group A acting Frobeniusly on N. Without loss of generality, we may assume that A is of prime order. Note that $Z(N) = 1$, because otherwise $N/Z(N)$ is nilpotent by inductive hypothesis, which is not the case by Lemma 2.1.17.

Let M be a minimal normal A-invariant subgroup of N. Then M is an abelian p-subgroup of N by Lemma 2.1.9. Due to coprime action of A on N, there is an A-invariant Sylow q-subgroup Q of N for any prime q dividing $|N|$ by Corollary 2.2.10. If $MQ \neq N$ then MQ is nilpotent by induction, and hence $[M, Q] = 1$. Since q is arbitrary, we see that elements of M are centralized by each Sylow q-subgroup of N where $p \neq q$. Let $P \in Syl_p(G)$. Since $M \lhd P$, $Z = Z(P) \cap M \neq 1$ by Corollary 2.1.19, and hence we get $1 < Z \leq Z(N)$. This contradictions shows that $N = MQ$.

Since AQ is a Frobenius group, it has a partition Ω consisting of Q and $|Q|$ conjugates of A. Then $|\Omega| - 1 = |Q|$ which is coprime $|M|$. Then some member of Ω fix an nontrivial element in M. Since A and its conjugates acts without nontrivial fixed points, we get $C_M(Q) \neq 1$, and hence $Z(N) \neq 1$. This contradiction completes the proof. □

Theorem 2.6.26. ***(Thompson)*** [7, Theorem 6.22] *Let G be a group and let $P \in Syl_p(G)$ where p is an odd prime. If for every nontrivial characteristic subgroup X of P, $N_G(X)$ is p-nilpotent then so is G.*

Theorem 2.6.27. [7, Theorem 6.24] *Frobenius kernels are nilpotent.*

Proof. Let N be a Frobenius kernel and let A be a group having Frobenius action on N. We claim that N is solvable and proceed by induction on $|N|$. We may assume that $|N|$ has an odd prime divisor p.

We may choose $P \in Syl_p(N)$ such that P is A-invariant by Corollary 2.2.10. Let X be a nontrivial characteristic subgroup of P then X is A-invariant. Set $M = N_N(X)$. Note that M is also an A-invariant subgroup of N. It follows that M/X is also a Frobenius kernel by Corollary 2.6.4, and hence M/X is solvable by induction. Since X is a p-group, we obtain that M is solvable by Lemma 2.1.10. Then we have M is nilpotent by Lemma 2.6.25. Since X is arbitrary, we get N is p-nilpotent by Theorem 2.6.26.

Let H be the normal Hall p'-subgroup of N. Then by induction applied to H, H is solvable, and hence N is solvable as N/H is a p-group. It follows that N is nilpotent by Lemma 2.6.25. $\square$

CHAPTER 3

FINITE GROUPS HAVING NONNORMAL TI-SUBGROUPS

In the present chapter, the structure of a finite group G having a nonnormal TI-subgroup H which is also a Hall π-subgroup is studied. As a generalization of a result due to Gow, we prove that H is a Frobenius complement whenever G is π-separable. This is achieved by obtaining the fact that Hall TI-subgroups are conjugate in a finite group. We also prove two theorems about normal complements one of which generalizes a classical result of Frobenius.

3.1 Key propositions

We first obtain a sufficient condition which guarantees the conjugacy of Hall π-subgroups. This condition also guarantees that any π-subgroup is contained in a Hall π-subgroup. It should be noted that a group G may have nonisomorphic Hall π-subgroups. In this case, it is not possible that they are conjugate in G.

On the other hand, even if all Hall-π subgroups are conjugate in G, it might be the case that some π-subgroups of G are not contained in any Hall π-subgroups. For example, let $G = S_5$ and $H \in Hall_{\{2,3\}}(G)$. It can be easily checked that any $Hall_{\{2,3\}}$ subgroup of G is isomorphic to S_4 and they are conjugate in G. Set $K = \langle(1,2,3)(4,5)\rangle$. Then K is a $\{2,3\}$-subgroup which is not contained in any conjugate of H as S_4 has no element of order 6.

Proposition 3.1.1. *Let G be a group containing a Hall π-subgroup H which is also a TI-subgroup. Then any π-subgroup of G is contained in a conjugate of H. In particular, the set of all Hall π-subgroups of G forms a single G-conjugacy class.*

Proof. Let K be a π-subgroup of G which is not contained in any conjugate of H. Let $P \in Syl_p(K)$ for a prime p dividing the order of K. It should be noted that Sylow p-subgroups of H are also Sylow p-subgroups of G, and hence there exists $x \in G$ such that $P \leq H^x$. Set $T = H^x \cap K$. Then T is a TI-subgroup of K by Lemma 2.6.7(b). Note that T is nontrivial as P is contained in T. Pick an element n from $N_K(T)$. Notice that H^x is also a TI-subgroup of G by Lemma 2.6.7(a). Then

$$1 \neq T = T^n \leq H^x \cap H^{xn}$$

and so n normalizes H^x, which forces that $n \in H^x$, because otherwise the π-group $H^x\langle n\rangle$ contains H^x properly. It follows that T is a self normalizing TI-subgroup of K, and hence T is a Hall subgroup of K by Lemma 2.6.11.

Let now q be a prime dividing $|K{:}T|$ and pick $Q \in Syl_q(K)$. A similar argument as above shows that $Q \leq H^y \cap K$ for some $y \in G$. Set $S = H^y \cap K$. Clearly, the group S is also a self normalizing TI-subgroup of K. If $T \cap S^k \neq 1$ for some $k \in K$, then $H^x \cap H^{yk} \neq 1$, and hence $H^x = H^{yk}$ since H^x is also a TI-subgroup by Lemma 2.6.7(a). This forces the equality $T = S^k$ which is not possible as q is coprime to the order of T. Thus we have $T \cap S^k = 1$ for all $k \in K$. As a consequence we get

$$S \subseteq (K - \bigcup_{k \in K} T^k) \cup \{1\}$$

and hence

$$\bigcup_{k \in K} S^k \subseteq (K - \bigcup_{k \in K} T^k) \cup \{1\}$$

A simple counting argument shows that

$$|K| - \frac{|K|}{|S|} + 1 \leq \frac{|K|}{|T|},$$

and so,

$$1 < 1 + \frac{1}{|K|} \leq \frac{1}{|S|} + \frac{1}{|T|}.$$

This inequality is possible only when $|S| = 1$ or $|T| = 1$, which contradicts the fact that both S and T are nontrivial. Thus, K is contained in a conjugate of H, in particular, if K is a Hall π-subgroup of G then $K = H^g$ for some $g \in G$, completing the proof. $\square$

The following proposition will be used in the proof of a normal complement theorem, namely Theorem 3.3.8. It should be noted that it is of independent interest too, as it provides some new information on the influence of a coprime action.

Proposition 3.1.2. *Let A be a group acting coprimely on G by automorphisms. Assume that Sylow subgroups of G are cyclic. Then,*
$a)$ *$C_G(A)$ is a Hall subgroup of G;*
$b)$ *$G = [G, A] \rtimes C_G(A)$;*
$c)$ *the group $[G, A]$ is cyclic;*
$d)$ *A is abelian if the action of A on G is faithful.*

Proof. $a)$ Let p a prime dividing the order of $C_G(A)$. We may choose an A-invariant Sylow p-subgroup P of G by Corollary 2.2.10. Then $C_P(A) \in Syl_p(C_P(A))$ by Lemma 2.2.15 and $P = [P, A] \times C_P(A)$ by Theorem 2.2.18. As $C_P(A)$ is nontrivial and P is cyclic, we get $C_P(A) = P$. Hence p is coprime to the index of $C_G(A)$ in G.

$b)$ By Theorem 2.2.16, we have $G = [G, A]C_G(A)$, and so it suffices to show that $C = C_G(A) \cap [G, A] = 1$. Assume the contrary and let p the smallest prime dividing the order of C. Let $P \in Syl_p(C)$. Notice that $C = C_{[G,A]}(A)$, and hence C is a Hall subgroup of $[G, A]$ by part $a)$. It follows that P is also a Sylow subgroup of $[G, A]$.

Now we claim that P controls $[G, A]$-fusion in itself. Let $x, x^g \in P$ for some $x \in P$ and $g \in [G, A]$. Since $P \leq C$ and C controls $[G, A]$-fusion in C by Theorem 2.2.12, there exists $c \in C$ such that $x^g = x^c$. On the other hand, we obtain that P has a normal p-complement in C by Corollary 2.5.17, and hence P controls C-fusion in P by Theorem 2.5.22. It follows that there exists $t \in P$ such that $x^c = x^t$, which proves the claim.

Thus, $[G, A]$ has a normal p-complement N, that is, $[G, A] = NP$ by Theorem 2.5.22. This leads to
$$[G, A, A] = [NP, A] = [N, A] \leq N < [G, A]$$
which is not possible by Lemma 2.2.17. Thus, $C = C_G(A) \cap [G, A] = 1$ as claimed.

$c)$ It is enough to show that $K = [G, A]$ is nilpotent because all Sylow subgroup of G are cyclic. We can assume that K is nontrivial. Note that A acts on K fixed point freely by $b)$, that is, $C_K(A) = 1$. Due to coprimeness, we may choose an

A-invariant Sylow p-subgroup P of K for an arbitrary prime p dividing the order of K by Corollary 2.2.10. Clearly, $N_K(P)$ is also an A-invariant subgroup of K so that $L = N_K(P)A$ acts on P via automorphisms. Note that $Aut(P)$ is abelian by Theorem 2.1.27, and hence $L/C_L(P)$ is abelian by Theorem 2.1.22. It follows that

$$[N_K(P), A] \leq L' \leq C_L(P)$$

by Lemma 2.1.2. Since

$$C_{N_K(P)}(A) = N_K(P) \cap C_K(A) = 1,$$

we get $[N_K(P), A] = N_K(P)$ by Theorem 2.2.16. Then $P \leq Z(N_K(P))$ and so K is p-nilpotent by Corollary 2.5.15. As p is arbitrary, K is nilpotent by Lemma 2.4.10.

$d)$ Suppose that the action of A on G is faithful, that is, $C_A(G) = 1$. Set $B = C_A([G, A])$. Since $[G, B] \leq [G, A]$, we get $[G, B, B] = 1$. Due to coprimeness, it is obtained that $[G, B] = 1$ by Lemma 2.2.17, and hence $B \leq C_A(G) = 1$. It follows that A is isomorphic to a subgroup of the automorphism of the cyclic group $[G, A]$, which is abelian by Theorem 2.1.27. Therefore, A is abelian as desired. □

3.2 Some technical lemmas

In this section, we present four lemmas which will be frequently used in proving the main results of this chapter. The first one is a generalization of the fact that a self normalizing TI-subgroup is also a Hall subgroup (see Lemma 2.6.11).

Lemma 3.2.1. *Let H be a TI-subgroup of a group G. Then H is a Hall subgroup of G if and only if H is a Hall subgroup of $N_G(H)$.*

Proof. One direction is trivial to show. Let H be a Hall subgroup of $N_G(H)$, and let p be a prime dividing both $|H|$ and $|G : H|$. Pick $P \in Syl_p(H)$, $Q \in Syl_p(G)$ such that $P < Q$. Since $P < N_Q(P)$ by Theorem 2.1.15, we may choose $x \in N_Q(P) - P$. As $H \cap H^x$ is nontrivial, we have $x \in N_G(H)$. This forces that $x \in H$ since H is a Hall subgroup of $N_G(H)$. Then $P\langle x\rangle$ is a p-subgroup of H containing P properly. This contradiction completes the proof. □

Remark 3.2.2. It should be noted that the lemmas below can be proven by using the conjugacy part of Schur-Zassenhaus theorem and by Feit-Thompson's odd order theorem. Here, we present proofs without appealing to the odd order theorem. Although Lemma 3.2.3 is well known, Lemma 3.2.4 and Lemma 3.2.5 are both new.

Lemma 3.2.3. *Let G be a group and N be a normal subgroup of G with a complement H. If $(|H|, |N|) = 1$ and H is a TI-subgroup of G then for each prime p dividing the order of N, there is an H-invariant Sylow p-subgroup of N.*

Proof. By the Frattini argument (see 2.4.11), $N_G(P)N = G$ for any $P \in Syl_p(N)$. Clearly, we have $K = N \cap N_G(P) \trianglelefteq N_G(P)$ and $|N_G(P) : K| = |G : N|$. Then $|K|$ is coprime to $|N_G(P) : K|$. By the existence part of Schur-Zassenhaus theorem (see 2.1.25), $N_G(P) = KU$ where $|U| = |G : N| = |H|$. It follows by Proposition 3.1.1 that $U = H^g \leq N_G(P)$ for some $g \in G$, that is, H normalizes $P^{g^{-1}}$ as required. □

Lemma 3.2.4. *Let N be a normal subgroup of G and H be a TI-subgroup of G with $(|H|, |N|) = 1$. Then $N_{\overline{G}}(\overline{H}) = \overline{N_G(H)}$ where $\overline{G} = G/N$.*

Proof. It is clear that $\overline{N_G(H)} \leq N_{\overline{G}}(\overline{H})$. Let X be the full inverse image of $N_{\overline{G}}(\overline{H})$ in G. Then $N_G(H)$ is contained in X and $HN \trianglelefteq X$. By Proposition 3.1.1, every complement of N in HN is a conjugate of H. Then the Frattini argument (see 2.4.11) implies the equality $N_X(H)N = X$. As $N_X(H) = N_G(H)$, we have $X = N_G(H)N$. Then $\overline{X} = \overline{N_G(H)}$ as claimed. □

It should be noted that a homomorphic image of a TI-subgroup need not be a TI-subgroup of the image. Therefore, the following lemma provides a sufficient condition for which the image of a TI-subgroup is also a TI-subgroup.

Lemma 3.2.5. *Let H be a TI-subgroup of G and N be a normal subgroup of G with $(|N|, |H|) = 1$. Then HN/N is a T.I subgroup of G/N.*

Proof. Set $T = HN \cap H^gN$ for $g \in G$. Then

$$T = T \cap HN = T \cap H^gN.$$

As $N \leq T$, we have

$$T = N(T \cap H) = N(T \cap H^g).$$

Note that $(|N|, |T \cap H|) = 1$ and $T \cap H$ is a TI-subgroup of T. Then the complements $T \cap H$ and $T \cap H^g$ are conjugate by an element of N by Proposition 3.1.1, that is,

$$(T \cap H)^n = T \cap H^n = T \cap H^g$$

for some $n \in N$. Thus,

$$T \cap H^n = T \cap H^n \cap H^g = 1 \text{ or } H^n = H^g.$$

If the former holds, then $T = N$. If the latter holds, then $T = HN$.

So, we have either $T = HN$ or $T = N$. As a result, HN/N is a TI-subgroup of G/N because $HN/N \cap H^g N/N = T/N$ is either trivial or equal to HN/N. $\square$

3.3 π-separable groups having nonnormal Hall TI-subgroups

In this section, we give a sufficient condition for a Hall TI-subgroup to be a Frobenius complement in π-separable groups. The following theorem of Gow provides a sufficient condition for a Hall TI-subgroup to be a Frobenius complement in solvable groups.

Theorem 3.3.1. *(Gow) Let G be a solvable group and let H be a Hall subgroup of G, which is a nonnormal TI-subgroup of G. Then H has an irreducible representation on some elementary abelian section of G in which each of its nonidentity elements acts without fixed points.*

Now we obtain an extension of his result to π-separable groups as a more general answer to Question A posed in Chapter 1. Namely, we prove the following:

Theorem 3.3.2. *Let H be a nonnormal TI-subgroup of the π-separable group G where π is the set of primes dividing the order of H. Further assume that H is a Hall subgroup of $N_G(H)$. Then the following hold:*
$a)$ G has π-length 1 where $G = O_{\pi'}(G)N_G(H)$;
$b)$ there is an H-invariant section of G on which the action of H is Frobenius. This section can be chosen as a chief factor of G whenever $O_{\pi'}(G)$ is solvable;
$c)$ G is solvable if and only if $O_{\pi'}(G)$ is solvable and H does not involve any subgroup isomorphic to $SL(2, 5)$.

Remark 3.3.3. Parts (a) and (b) of Theorem 3.3.2 not only generalize Theorem 3.3.1 but also determine the structure of the group G under the weaker hypothesis that H is a Hall subgroup of $N_G(H)$. Yet, it turns out to be equivalent to assuming that H is a Hall subgroup of G. (See Lemma 3.2.1). Here is an example showing that the condition of π-separability is indispensable.

Example 3.3.4. Let $K = A_5$ and H be a Sylow 5-subgroup of K. H is a TI-subgroup of K as H is of prime order and is a Hall subgroup of $N_K(H)$. The only H-invariant subgroups of K are $N_K(H)$, H and the trivial subgroup where $|N_K(H)| = 10$. Hence, it is easy to see that both (a) and the first part of Theorem 3.3.2 (b) fail to be true.

We next present an example which shows that the second part of Theorem 3.3.2 (b) is not true in case $O_{\pi'}(G)$ is nonsolvable even if G satisfies other hypotheses.

Example 3.3.5. Let $N = SL(2, 2^7)$ and $\alpha \in Aut(N)$ of order 7 which arises from the nontrivial automorphism of the field with 2^7 elements. Set $G = N\langle\alpha\rangle$. Then $\langle\alpha\rangle$ is a nonnormal Hall TI-subgroup of G. Since N is simple, $\langle\alpha\rangle$ does not have Frobenius action on any chief factor of G, and so the second part of Theorem 3.3.2 (b) fails to be true.

Proof of Theorem 3.3.2. $a)$ Note that H is a Hall subgroup of G by Lemma 2.1. Then we have $O_\pi(G) = 1$ as H is a nonnormal TI-subgroup of G. This forces that $O_{\pi'}(G) \neq 1$ since G is π-separable. Set $\overline{G} = G/O_{\pi'}(G)$. Notice that $\overline{H}$ is a T.I subgroup of $\overline{G}$ by Lemma 3.2.5.
Assume that $\overline{H}$ is not normal in $\overline{G}$. Then $O_\pi(\overline{G})$ is trivial. On the other hand $O_{\pi'}(\overline{G})$ is also trivial, and so $\overline{G}$ is trivial which is not the case. Thus, $\overline{H}$ is normal in $\overline{G}$, and so the lower π-series of G is as follows;

$$1 < O_{\pi'}(G) < HO_{\pi'}(G) \leq G.$$

Now $\overline{G} = N_{\overline{G}}(\overline{H}) = \overline{N_G(H)}$ by Lemma 3.2.4, and hence we have $G = O_{\pi'}(G)N_G(H)$ as claimed.

$b)$ Let G be a minimal counterexample to part (b). Suppose that $H \leq K < G$. Clearly K is π-separable by Lemma 2.4.4 and H is a TI-subgroup of K by Lemma2.6.7(b).

If H is not normal in K, then we get an H-invariant section of K on which the action of H is Frobenius by induction. This leads to a contradiction as each section of K is also a section of G. Thus we have $H \lhd K$.

Let now R be an H-invariant subgroup of $O_{\pi'}(G)$ with $RH \neq G$. Then we have $H \leq RH < G$, and hence $H \lhd RH$ by the previous argument. It then follows that $[H, R] \leq R \cap H = 1$. In particular, If $HO_{\pi'}(G) \neq G$ then $[H, O_{\pi'}(G)] = 1$. As $O_{\pi}(G) = 1$, we obtain $C_G(O_{\pi'}(G)) \leq O_{\pi'}(G)$ by Theorem 2.4.6. It follows that $H \leq O_{\pi'}(G)$, which is a contradiction. Hence the equality $G = O_{\pi'}(G)H$ holds. Lemma 3.2.3 guarantees the existence of an H-invariant Sylow p-subgroup P of $O_{\pi'}(G)$ for any prime p dividing the order of $O_{\pi'}(G)$. If $P \neq O_{\pi'}(G)$, then $[P, H] = 1$. Since p is arbitrary, we obtain $[H, O_{\pi'}(G)] = 1$ which is impossible. Thus, $G = PH$ with $[P, H] \neq 1$ where $P = O_{\pi'}(G)$.

Note that $[P, H]$ is H-invariant as P is H invariant by Lemma 2.2.6. If $[P, H] \neq P$, then $[P, H] = [P, H, H] = 1$ by Lemma 2.2.17, which is not the case. Thus we obtain that $[P, H] = P$, and hence

$$P/P' = [P, H]P'/P' = [P/P', H]$$

by Lemma 2.2.7. Notice that

$$P/P' = [P/P', H] \times C_{P/P'}(H)$$

by Theorem 2.2.18, and hence $C_{P/P'}(H) = 1$. We also observe that H is a TI-subgroup of $\overline{G} = (P/P')H$ by Lemma 3.2.5, and

$$N_{\overline{G}}(H) = C_{P/P'}(H)H = H$$

by Lemma 2.2.5, that is, H is a self normalizing TI-subgroup of $\overline{G}$. As a consequence, $\overline{G}$ is a Frobenius group, completing the proof of the first part of (b).

Suppose next that $O_{\pi'}(G)$ is solvable. Set $L = [O_{\pi'}(G), H]$. Note that $L \neq 1$ by Theorem 2.4.6, and hence $L' < L$ by Corollary 2.1.8. It is almost routine to check that action of H on L/L' is Frobenius by using Lemma 3.2.5 as in the previous paragraph. Observe that L is normalized by both $N_G(H)$ and $O_{\pi'}(G)$ by Lemma 2.2.6, and so L is normal in $G = O_{\pi'}(G)N_G(H)$. Therefore, we observe that any chief factor of G between L and L' is a chief factor on which the action of H is Frobenius.

$c)$ Assume that $O_{\pi'}(G)$ is solvable and H does not involve any subgroup isomorphic to a $SL(2,5)$. As H is a Frobenius complement by part (b), H must be solvable by Theorem 2.6.22. Set $\overline{G} = G/O_{\pi'}(G)$. Now

$$\overline{G} = \overline{O_{\pi'}(G)N_G(H)} = \overline{N_G(H)}$$

by part (a). $N_G(H)$ has a normal π-complement by Schur-Zassenhaus theorem (see 2.1.25), say Q. Then we get

$$\overline{G} = \overline{N_G(H)} = \overline{H} \rtimes \overline{Q}.$$

Therefore, it suffices to show that $\overline{Q}$ is solvable by Lemma 2.1.10.

Note that $\overline{Q}$ acts faithfully on $\overline{H}$ as $C_{\overline{G}}(\overline{H}) \leq \overline{H}$ by Theorem 2.4.6. Due to coprimeness, for each $p \in \pi$, there exists a $\overline{Q}$-invariant Sylow p-subgroup of $\overline{H}$ by Corollary 2.2.10. Let Ω be set of all $\overline{Q}$-invariant Sylow subgroups of $\overline{H}$. Notice that $\bigcap\limits_{X \in \Omega} C_{\overline{Q}}(X) = 1$ as the action of $\overline{Q}$ on $\overline{H}$ is faithful. It follows now that $\overline{Q} \cong \overline{Q}/(\bigcap\limits_{X \in \Omega} C_{\overline{Q}}(X))$ is isomorphic to a subgroup of $\prod\limits_{X \in \Omega} \overline{Q}/C_{\overline{Q}}(X)$ by Lemma 2.1.29. Since each member of Ω is either cyclic or generalized quaternion by Corollary 2.6.20, the group $\overline{Q}/C_{\overline{Q}}(X)$ is solvable for all $X \in \Omega$ by Corollary 2.3.9 and Theorem 2.1.27. Thus $\prod\limits_{X \in \Omega} \overline{Q}/C_{\overline{Q}}(X)$ is solvable, and hence $\overline{Q}$ is solvable as desired. $\square$

Remark 3.3.6. Under the hypothesis of Theorem 3.3.2, we have $G = O_{\pi'}(G)N_G(H)$. On the other hand, by Schur-Zassenhaus theorem (see 2.1.25), H has a complement in $N_G(H)$, say Q. Set $O = O_{\pi'}(G)$. Then we have the equality $G = OHQ$. Note that this need not be a semidirect product as $O \cap Q$ may not be trivial. From now on, we write G as OHQ whenever the hypothesis holds.

The structure of G, in somehow, resembles the structure of a double Frobenius group. More precisely, the following theorem shows that under some additional hypothesis, there is a factor group of G containing double Frobenius groups.

Theorem 3.3.7. *Assume that the hypothesis of Theorem 3.3.2 hold. Assume further that H is of odd order with $[O,H] = O$ and that O is solvable with $Q \nleq O'$. Set $\overline{G} = G/O'$. Then*

$a)$ $\overline{G} = (\overline{O} \rtimes \overline{H}) \rtimes \overline{Q}$;

b) $\overline{Q}$ *is an abelian group acting faithfully on* $\overline{H}$*;*

c) $\overline{O}[\overline{H},\beta]\langle\beta\rangle$ *is a double Frobenius group for every element* $\beta \in \overline{Q}$ *of prime order.*

Proof. $a)$ Note that $\overline{Q}$ normalizes $\overline{O} \rtimes \overline{H}$, and hence, it is sufficient to show that $\overline{O} \cap \overline{Q} = 1$. We have $[\overline{O},\overline{H}] = \overline{[O,H]} = \overline{O}$ by Lemma 2.2.7. Since $\overline{O} = [\overline{O},\overline{H}] \times C_{\overline{O}}(\overline{H})$ by Theorem 2.2.18, we obtain that $C_{\overline{O}}(\overline{H}) = 1$. Pick $x \in \overline{Q} \cap \overline{O}$. Then $[x,\overline{H}] \leq \overline{H} \cap \overline{O} = 1$, and hence $x \in C_{\overline{O}}(\overline{H}) = 1$. This proves (a).

$b)$ Since $\overline{HO}/\overline{O} = O_{\pi}(\overline{G}/\overline{O})$, the action of $\overline{QO}/\overline{O}$ on $\overline{HO}/\overline{O}$ is faithful by Theorem 2.4.6. Then $\overline{Q}$ acts faithfully on $\overline{H}$, and so the action of $\overline{Q}$ on $[\overline{H},\overline{Q}]$ is also faithful. Corollary 2.6.20 implies that each Sylow subgroup of $\overline{H}$ is cyclic because H is a Frobenius complement of odd order. By Proposition 3.1.2, we get $[\overline{H},\overline{Q}]$ is cyclic. It follows that $\overline{Q}$ is abelian as $\overline{Q}$ is isomorphic to a subgroup of $Aut([\overline{H},\overline{Q}])$.

$c)$ Notice that $\overline{Q}$ is nontrivial as $Q \nleq O'$. Let $\beta \in \overline{Q}$ of prime order. Notice that $[\overline{H},\beta]$ is nontrivial since $\overline{Q}$ acts faithfully on $\overline{H}$. The group $\overline{OH}$ is Frobenius as $\overline{H}$ is a self normalizing TI-subgroup of $\overline{OH}$. Then $\overline{O}[\overline{H},\beta]$ is also a Frobenius group. By Proposition 3.1.2(b), β acts fixed point freely on $[\overline{H},\beta]$, and so $\overline{O}[\overline{H},\beta]\langle\beta\rangle$ is a double Frobenius group as claimed. □

Theorem 3.3.8. *Assume that the hypothesis of Theorem 3.3.2 holds. Assume further that a Sylow* 2*-subgroup of* H *is abelian and* Q *is a complement of* H *in* $N_G(H)$*. Then* $C_H(Q)$ *is a Hall subgroup of G having a normal complement in G.*

Proof. H is a Frobenius complement by Theorem 3.3.2(b), and so every Sylow subgroups of H is cyclic as Sylow 2-subgroups of H are abelian by Corollary 2.6.20. It follows by Proposition 3.1.2 that $C_H(Q)$ is a Hall subgroup of H. Then $C_H(Q)$ is also a Hall subgroup of G. Set $N = O[H,Q]Q$. Clearly N is a group. Proposition 3.1.2 yields that $C_H(Q)$ and $[H,Q]$ have coprime orders, and hence $C_H(Q)$ and N have coprime orders. Then $C_H(Q) \cap N = 1$. We also observe that $C_H(Q)$ normalizes N and $C_H(Q)N = G$ as $G = OHQ$. Consequently, N is the desired normal complement for $C_H(Q)$ in G. □

3.4 A generalization of Frobenius' theorem

In this section, we give a full answer to Question B stated in Chapter 1, by finding a necessary and sufficient condition for a Hall TI-subgroup to have a normal complement. Namely we prove,

Theorem 3.4.1. *Let H be a TI-subgroup of G which is also a Hall subgroup of $N_G(H)$. Then H has a normal complement in $N_G(H)$ if and only if H has a normal complement in G. Moreover, if H is nonnormal in G and H has a normal complement in $N_G(H)$ then H is a Frobenius complement.*

This result appears to be a nice application of Theorem 3.3.2 and Proposition 3.1.1. The proof uses the important result known as Brauer-Suzuki theorem (see 2.6.23).

Proof of Theorem 3.4.1. Assume that H has a normal complement in G, say N. Then

$$N_G(H) = G \cap N_G(H) = NH \cap N_G(H) = H(N \cap N_G(H))$$

by Dedekind rule (see 2.1.23). Then $N \cap N_G(H)$ is the desired normal complement of H in $N_G(H)$. Assume now that H has a normal complement Q in $N_G(H)$. Then $N_G(H) = QH$ and $[Q, H] \leq H \cap Q = 1$. We show first that H controls G-fusion in H: To see this, let x and x^g be elements of H for a nonidentity element $x \in H$ and for some $g \in G$. Now $x \in H \cap H^{g^{-1}}$, and so $H = H^{g^{-1}}$ as H is a TI-subgroup, that is, $g \in N_G(H)$. Then $g = sh$ for $s \in Q$ and $h \in H$. Since $[Q, H] = 1$, we have $x^g = x^{sh} = x^h$ establishing the claim. Note that by Lemma 3.2.1, H is a Hall π-subgroup of G for the prime set π of H. Then every π-subgroup is contained in a conjugate of H by Proposition 3.1.1. Now appealing to Brauer-Suzuki theorem (see 2.6.23), we see that G has a normal π-complement. This proves the first claim of the theorem.

Finally assume that H has normal complement in $N_G(H)$ and $N_G(H) < G$. By the argument above, G has a normal π-complement where π is the prime set of H. It follows that G is π-separable, and hence H is a Frobenius complement by part (b) of Theorem 3.3.2. $\square$

Theorem 3.4.1 can be regarded as a generalization of the classical result of Frobenius which asserts the following;

Corollary 3.4.2 (Frobenius)**.** *Let H be a proper subgroup of G with $H \cap H^x = 1$ for all $x \in G - H$. Then the set $N = (G - \bigcup_{g \in G} H^g) \cup \{1\}$ is a normal subgroup of G where N is a complement for H in G.*

No character free proof of Frobenius theorem is known. Yet under additional assumption that H is solvable, a character free proof is well known. We shall prove Theorem 3.4.1 without using character theory when H is assumed to be solvable.

Proof of Theorem 3.4.1 (by assuming H is solvable.) We will only show that if H has a normal complement in $N_G(H)$ then H has a normal complement in G since the rest of the proof in the previous version is already character free. We may assume that H is not normal in G. We proceed by induction on the order of G. As in the previous version, we can obtain that H controls G-fusion in H and H is a Hall π-subgroup of G. Then $A^\pi(G) \cap H = H'$ by Theorem 2.5.21. Set $A^\pi(G) = A$. Note that $H' < H$ by the hypothesis, and hence $A < G$.

If $H' = 1$ then A is the desired normal complement. Then we may assume that $H' \neq 1$. Now $N_G(H') \geq N_G(H)$ as H' is a characteristic subgroup of H. Assume $N_G(H') \neq N_G(H)$ and pick $x \in N_G(H') \setminus N_G(H)$. It follows that $1 \neq H' \leq H \cap H^x$, which is a contradiction. Thus, $N_G(H') = N_G(H)$ and $N_G(H')$ has a normal π-complement. Then $N_A(H')$ has a normal π-complement. Since H is Hall π-subgroup and $A \lhd G$, H' is also a Hall π-subgroup of A. Lemma 2.6.7(c) implies that H' is a TI-subgroup of G, and hence H' is a TI-subgroup of A by Lemma 2.6.7(b). As a result the pair (H', A) satisfies the hypothesis of the claim, and hence A has a normal complement N by induction applied to A.

Note that $N \lhd G$ as N is characteristic subgroup of A. Moreover, $G = HA = H(H'N) = HN$. Since N is a π'-group, $H \cap N = 1$ and the result follows.

One can pose the following open question;

Conjecture 3.4.3. *Let H be a TI-subgroup of G which is also a Hall π-subgroup of $N_G(H)$. Then $G/O^\pi(G) \cong N_G(H)/O^\pi(N_G(H))$.*

CHAPTER 4

CHARACTER FREE PROOFS FOR TWO SOLVABILITY THEOREMS DUE TO ISAACS

4.1 Introduction

Main purpose of this chapter is to provide character free proofs for the following two results below due to Isaacs ([5], Theorem 1 and Theorem 2).

Theorem A *Let G be a finite group having a cyclic Sylow p-subgroup. Assume that every p'-subgroup of G is abelian. Then G is either p-nilpotent or p-closed.*

Theorem B *Let G be a finite group and let $p \neq 2$ and q be primes dividing $|G|$. Suppose for every proper subgroup H of G which is not a q-group nor a q'-group that p divides $|H|$. If q^a is the q-part of $|G|$ and $p > q^a - 1$ or if $p = q^a - 1$ and a Sylow p-subgroup of G is abelian then no primes but p and q divide $|G|$.*

Indeed, the original proofs of both theorems by Isaacs too do not involve any character theoretical arguments when $p \neq 3$. We present a more elementary proof of Theorem A that shortens the proof of the case $p \neq 3$ and handles the case $p = 3$ in a character free way by using graph theoretical methods. We also provide a proof for Theorem B in case where $p = 3$ using the same graph theoretical arguments. Yet, we shall not cover the case $p \neq 3$ for Theorem B because we made no improvement in this case.

The following example shows that in Theorem A the condition that P is cyclic is unavoidable.

Example 4.1.1. Let $G = A_5$ and $p = 2$. Then every p'-subgroup of G is abelian and Sylow p-subgroup of G is Klein 4-group. Yet G is simple.

Another example shows that the condition that every p'-subgroup is abelian in Theorem A is also indispensable.

Example 4.1.2. Let $G = S_4$ and $p = 3$. Then Sylow p-subgroup of G is cyclic but some of p'-subgroups of G are not abelian. Yet G is neither p-closed nor p-nilpotent.

4.2 Preliminaries

Definition 4.2.1. *A **graph** is an ordered pair $\Gamma = (V, E)$ where V is the set of vertices of Γ and E is the set of edges of Γ which consists of unordered pairs $\{u, v\}$ where u and v are distinct elements of V.*

Let Γ be a graph and $u, v \in V$. We say that u and v are **adjacent** if and only if $\{u, v\} \in E(\Gamma)$. We write $u \sim v$ whenever u and v are adjacent.

An **automorphism** σ of Γ is a permutation of the set V, that is $\sigma \in Sym(V)$, such that $u^\sigma \sim v^\sigma$ if and only if $u \sim v$ for all $u, v \in V$. The set of all distinct automorphisms of Γ forms a group which is called the **automorphism group of** Γ and denoted by $Aut(\Gamma)$.

Let $u \in V$. Then the **degree of** u, denoted by $deg(u)$, is defined to be the number of the vertices in V which are adjacent to u. Γ is called n**-regular** if $deg(u) = n$ for all $u \in V$.

Lemma 4.2.2. *Let Γ be a graph and $\sigma \in Aut(\Gamma)$. Then $deg(u) = deg(u^\sigma)$ for all $u \in V$. In particular, if $G \leq Aut(\Gamma)$ acts on V transitively then Γ is n-regular for some $n \in \mathbb{N}$.*

Let Γ be a graph and $V = \{v_1, v_2, \ldots v_n\}$. The **adjacency matrix** A of Γ is a $n \times n$ matrix where

$$A_{ij} = \begin{cases} 1, & v_i \sim v_j \\ 0, & otherwise. \end{cases}$$

Let $u, v \in V$. **A walk of length** r from u to v is a sequence of vertices satisfying

$$u = u_0 \sim u_1 \sim \ldots \sim u_{r-1} \sim u_r = v.$$

Theorem 4.2.3. [1, Lemma 8.1.2] *Let Γ be a graph with $V = \{v_1, v_2, \ldots v_n\}$. Let A be the adjacency matrix of Γ. Then $(A^r)_{ij}$ is equal to the number of the walks of length r from v_i to v_j for any pair i, j.*

The proposition below, a special case of the well known “friendship problem" in graph theory (see [3]), will be needed in proving the case $p = 3$ in both Theorem A and Theorem B.

Proposition 4.2.4. *Let Γ be an n-regular graph with the property that for any two distinct vertices there is exactly one common neighbour. Then Γ is a triangle.*

Proof. Let m denote the number of vertices of Γ. We shall simply count in two ways the number of the triples (x, y, z) where x and y are distinct vertices and z is the unique common neighbour of x and y. Since the pair (x, y) uniquely determines z, we have $m(m-1)$ such triples. On the other hand, for each z the number of corresponding pairs (x, y) is $n(n-1)$, and hence the count is equal to $mn(n-1)$. It follows that $m = n^2 - n + 1$.

Assume that $n > 2$ and let A be the adjacency matrix of Γ. Then $A^2 = (n-1)I + J$ where J is the matrix where all entries are 1. Note that $AJ = nJ$ since Γ is n-regular. We pick now a prime divisor p of $n-1$ and regard A and J as matrices over the field $\mathbb{Z}_p$. Then we have $A^2 = J$ and $AJ = nJ = J$ as $n \equiv 1 \; mod \; p$, and hence $A^p = J$. In characteristic p, $tr(A^p) = tr(A)^p$ and $tr(A) = 0$. Thus, we obtain that $0 \equiv tr(A^p) \equiv tr(J) \equiv n^2 - n + 1 \equiv 1 \; mod \; p$ which is a contradiction. Thus we have $n = 2$, that is, Γ is a triangle as claimed. $\square$

The following lemma ([5], Lemma 2) will also be needed.

Lemma 4.2.5. *Let H be an abelian group with a collection $\{K_i \mid i \in I\}$ of proper subgroups such that $H = \bigcup_{i \in I} K_i$ and $K_i \cap K_j = 1$ for $i \neq j$. Then H is an elementary abelian p-group for some prime p.*

4.3 Proof of Theorem A

Assume that the theorem is false and let G be a minimal counterexample to the theorem. Note that each proper subgroup of G satisfies the hypothesis and hence is either p-nilpotent or p-closed. Pick a Sylow p-subgroup P of G and let K be the subgroup of P of order p. There exists a normal p-complement Q in $N_G(P)$ by Schur-Zassenhaus theorem (see 2.1.25). That is, $N_G(P) = PQ$. Since G is not p-nilpotent, $N_G(P)$ is not p-nilpotent by Corollary 2.5.15. So we clearly have $[P, Q] \neq 1$. We shall derive a contradiction over a series of steps.

(1) $O_p(G) = 1$.

Assume the contrary. It is easy to see that the group $G/O_p(G)$ satisfies the hypothesis of the theorem and hence is either p-nilpotent or p-closed. If it is p-closed then $P/O_p(G) \lhd G/O_p(G)$ and so $P \lhd G$, which is not the case. Thus, $G/O_p(G)$ is p-nilpotent, that is, $G = NP$ where $N/O_p(G)$ is the normal Hall p'-subgroup of $G/O_p(G)$.
Note that $G/N \cong P/P \cap N$ is cyclic, and so $G' \leq N$ by Lemma2.1.2. Since $G' \leq N$, P is not contained in G'. It follows that G is p-nilpotent by Corollary 2.5.16. This forces that $O_p(G) = 1$ as desired.

(2) $N_G(K) = N_G(P)$ *is a maximal subgroup.*

The containment $N_G(P) \leq N_G(K)$ holds due to the fact that K is a characteristic subgroup of P. Since $N_G(P)$ is not p-nilpotent, $N_G(K)$ is not p-nilpotent by Lemma 2.4.9. As G has no nontrivial normal p-subgroup by (1), $N_G(K)$ is proper in G. It follows that $N_G(K)$ is p-closed, and hence $N_G(K) \leq N_G(P)$. Then we have $N_G(K) = N_G(P)$.

Let now M be a maximal subgroup of G containing $N_G(P)$. Since $N_G(P)$ is not p-nilpotent, M is not p-nilpotent, and so M is p-closed. Thus, $N_G(P) = M$ as desired.

(3) G *is simple.*

Suppose that G has a nontrivial proper normal subgroup N. Then N is either p-nilpotent or p-closed and accordingly N has either a nontrivial characteristic p'-subgroup or a nontrivial characteristic p-subgroup. In particular, we have either $O_{p'}(G) \neq 1$ or $O_p(G) \neq 1$. Notice that $O_p(G) = 1$ by (1), and so $V = O_{p'}(G) \neq 1$. The group G/V satisfies the hypothesis of the theorem, and hence it is either p-nilpotent or p-closed. If the former holds then $G = NP$ where N/V is the normal Hall p'-subgroup of G. It follows that N is also a p'-subgroup as both N/V and V are also p'-subgroup. Yhis forces that $N \cap P = 1$, and hence G is p-nilpotent, which is not the case.

Therefore G/V is p-closed, and hence PV is a normal subgroup of G. Now

$$G = N_G(P)PV = N_G(P)V = QPV$$

by the Frattini argument (see 2.4.11). Note that the group VQ is abelian by the hypothesis, and so $[V, Q] = 1$. Thus, we get $[V, P, Q] = [Q, V, P] = 1$. This yields $[P, Q, V] = 1$ by the three subgroup lemma (see 2.1.6). Note that $[P, Q] \lhd PQ$ by Lemma 2.1.5. Since $[P, Q]$ is also centralized by V, we get $[P, Q] \lhd QPV = G$. It follows that $[P, Q] = 1$ by (1), which is a contradiction. This contradiction shows that G is simple.

(4) $|R| \equiv 1 \ mod\ p$ for each $R \in Syl_r(G)$ where $r \neq p$, in particular, we have $|G : P| \equiv 1 \ mod\ p$.

Let $R \in Syl_r(G)$ for a prime $r \neq p$ and $S \in Syl_p(N_G(R))$. Assume that $S = 1$. Then $N_G(R)$ is abelian by the hypothesis, and hence G is r-nilpotent by Corollary 2.5.15. This is not possible by (3), and hence $S \neq 1$.

Note that

$$C_R(S) \leq Z(N_G(R)) \cap R$$

as p'-elements of $N_G(R)$ act trivially on R. Since $Z(N_G(R)) \cap R \cap G' = 1$ by Corollary 2.5.14 and $G = G'$, we get $C_R(S) = 1$ whence $|R| \equiv 1 \ mod\ p$.

(5) Q is a Hall subgroup of G such that $C_Q(P) \neq 1$ and $|N_G(Q)|$ is divisible by p.

Since $|G : N_G(P)| \equiv |G : P| \equiv 1 \ mod\ p$, we have $|Q| \equiv 1 \ mod\ p$. If Q acts faithfully on P, then $|Q| \leq p-1$ by Theorem 2.1.27. It follows that $Q=1$ which is impossible. Thus $C_Q(P) \neq 1$ as claimed. Since both P and Q centralize $C_Q(P)$, we have $C_Q(P) \leq Z(N_G(P))$. Then we get

$$C_G(C_Q(P)) \geq PQ = N_G(P).$$

Since $N_G(P)$ is a maximal subgroup of G by (2), we have either $C_G(C_Q(P)) = G$ or $C_G(C_Q(P)) = N_G(P)$. If the former holds then $1 \neq C_Q(P) \leq Z(G)$, which is not possible by (3). Then we obtain $C_G(C_Q(P)) = N_G(P)$.

Let $R_0 \in Syl_r(Q)$ and $R \in Syl_r(G)$ such that $R_0 \leq R$ for a prime r dividing $|Q|$. Notice that $N = N_G(R_0)$ satisfies the conclusion of the theorem and hence is a solvable group. We can pick a Hall p'-subgroup H of N such that $R \leq H$. On the other hand $C_Q(P) \leq Q \leq N$ as Q is abelian by the hypothesis. Then $C_Q(P) \leq H^n$ for some $n \in N$ by Theorem 2.4.2. Now $H^n \leq C_G(C_Q(P)) = N_G(P)$ as H is abelian. In particular, $R^n \leq N_G(P) = PQ$ whence $R = R_0$. Therefore Q is a Hall subgroup of G as desired.

One can also observe that $|N_G(Q)|$ is divisible by p because otherwise G has normal π-complement for the prime set π of $|Q|$ by Burnside theorem (see 2.5.15).

(6) $p=3$ and Q is an elementary abelian group of order 4.

Recall that $|N_G(Q)|$ is divisible by p by (5), and pick $\alpha \in N_G(Q)$ of order p. We can observe that $\alpha \notin P$: If not, $[Q, \alpha] \leq P \cap Q = 1$ and so $C_P(Q) \neq 1$. It follows that $[P, Q] = 1$ by Corollary 2.2.19, which is impossible. Therefore, $P, P^{\alpha}, \dots, P^{\alpha^{p-1}}$ are all distinct.

Set $Z = C_Q(P)$. For distinct $i, j \in \{0, 1, \dots, p-1\}$, $Z^{\alpha^i} \cap Z^{\alpha^j}$ centralizes both $N_G(P)^{\alpha^i}$ and $N_G(P)^{\alpha^j}$, two distinct maximal subgroups of G. The fact that $Z(G) = 1$ implies $Z^{\alpha^i} \cap Z^{\alpha^j} = 1$.

Let $|Z| = c$. Then c^2 divides $|Q|$ as $Z \oplus Z^{\alpha} \leq Q$. Letting $|Q| = c^2 t$ we have $p \,|\, (c^2 t - 1)$ by (4). Notice also that due to the faithful action of Q/Z on P, $|Q/Z| = ct \,|\, (p-1)$ by Theorem2.1.27. If $p=2$ then $Q=Z$, and hence $[P,Q]=1$, which is impossible. Then p is an odd prime and hence Q/Z is cyclic.

Notice that $p \mid (\frac{p-1}{ct} + c)$ since $p \mid (p - 1 + c^2 t)$ and $ct \mid (p - 1)$. As both $\frac{p-1}{ct}$ and c divide $p - 1$, we get $\frac{p-1}{ct} + c < 2p$ and so $\frac{p-1}{ct} + c = p$. Now $\frac{p-1}{ct} + ct > p - 1$, that is, the product of $\frac{p-1}{ct}$ and ct is smaller than their sum, and hence we have either $ct = 1$ or $p - 1 = ct$. Since $c > 1$, $ct = p - 1$. The equation $\frac{p-1}{ct} + c = p$ gives that $c = p - 1$ and $t = 1$. That is $|Q| = (p-1)^2$. We get $Q = \bigcup Z^{\alpha^i}$ as

$$|\bigcup Z^{\alpha^i}| = p(p-2) + 1 = |Q|.$$

It follows by Lemma 4.2.5 that the group Q is elementary abelian and hence $|Q/Z| = q = p - 1$. As a consequence $p = 3$ and Q is an elementary abelian group of order 4.

(7) Final contradiction.

Let I be the set of all involutions in G. Note that Q is a Sylow 2-subgroup of G which is elementary abelian of order 4 by the previous step. If $N_G(Q)$ does not act transitively on $I \cap Q$, then one of the involutions in Q is contained in $Z(N_G(Q)) \cap Q \cap G'$, which is impossible by Corollary 2.5.14. Thus $N_G(Q)$ acts transitively on $I \cap Q$ and hence the action of G on I by conjugation is also transitive.

Let Γ be the graph where the vertex set of Γ is I and two distinct vertices are adjacent if and only if they commute in G. We claim that for any two distinct vertices $i, j \in I$ there is a unique vertex which is adjacent to both i and j: To see this we assume first that i and j commute. Then $k = ij$ commutes with both i and j. Since $\{1, i, j, k\}$ is a Sylow 2-subgroup, k is unique as claimed.

Assume next that $[i, j] \neq 1$. Then the group $D = \langle i, j \rangle$ is a nonabelian dihedral group by Lemma 2.3.5 and hence $|D|$ is divisible by p by the hypothesis. Without loss of generality we may assume that $K \leq D$. Then i and j are both contained in $N_G(K) = N_G(P)$. Clearly

$$Z = C_Q(P) \leq Z(N_G(P)).$$

Since $[K, Q] \neq 1$, we have $Z = Z(N_G(P))$. Then the unique involution in Z commutes with both i and j. Let l be another involution commuting with both i and j. Then $[D, l] = 1$ which implies that $[K, l] = 1$. Thus we have $[P, l] = 1$ by Corollary 2.2.19. Now $l \in Z$ establishing the claim.

Since the action of G on I is transitive, the graph Γ is n-regular for some n and so $|I| = 3$ by Proposition 4.2.4. As a consequence G has a unique Sylow 2-subgroup. This contradiction completes the proof. $\square$

We give next a proof of Theorem B when $p = 3$, the only case in the original proof where the character theory is used.

4.4 Proof of Theorem B

Assume the theorem is false. As in the original proof due to Isaacs, the structure of a minimal counterexample G to the theorem is as follows:
Let $P \in Syl_p(G)$ and $Q \in Syl_q(G)$. Then

1. G is a simple group with $p = 3$ and $q = 2$.
2. Q is an elementary abelian 2-group of order 4.
3. $|N_G(P)| = 4|P|$ and $|C_G(P)| = 2|P|$.
4. G acts transitively on the set of involutions I by conjugation.
5. P is an abelian TI-subgroup.

Now let i, j be two involutions in G. If i and j commute then ij is the unique involution which commutes with both i and j. If $[i, j] \neq 1$, then $D =< i, j >$ is a nonabelian dihedral group by Lemma 2.3.5. Since D is not a q-group, $|D|$ is divisible by p. Let K be a subgroup of D of order p. Without loss of generality we may assume that $K \leq P$.

Note that if $g \in G$ normalizes K then $1 \neq K \leq P \cap P^g$, and hence $P^g = P$ due to the fact that P is an TI-subgroup of G. It follows that both i and j are elements of $N_G(P)$. Then i and j both commute with the unique involution c contained in $C_G(P) \leq N_G(P)$. If t is another involution which commutes with both i and j, then

t centralizes K and $t \in N_G(P)$. As $t \neq c$, the Sylow 2-subgroup $\{1, t, c, tc\}$ acts trivially on K. $|N_G(P)| = 4|P|$ implies that

$$K \leq Z(N_G(P)) \cap P \cap G = Z(N_G(P)) \cap P \cap G'$$

which is a contradiction by Corollary 2.5.14. One can consider the commuting graph Γ of involutions of G and easily see as in the proof of Theorem A that Γ is an n-regular graph so that for any two distinct vertices there is exactly one common neighbour. Now Proposition 4.2.4 yields that $|I| = 3$. As a consequence $Q \lhd G$, which is the final contradiction. $\square$

REFERENCES

[1] C. Godsil and G. Royle. *Algebraic Graph Theory*, volume 207 of *Graduate Texts in Mathematics.* volume 207 of Graduate Texts in Mathematics. Springer, 2001.

[2] D. Gorenstein. *Finite groups*, volume 301. American Mathematical Soc., 2007.

[3] C. Huneke. The friendship theorem. *The American Mathematical Monthly*, 109(2):192–194, 2002.

[4] B. Huppert. *Character theory of finite groups*, volume 25. Walter de Gruyter, 1998.

[5] I. M. Isaacs. Two solvability theorems. *Pacific J. Math.*, 23:281–290, 1967.

[6] I. M. Isaacs. *Character Theory of Finite Groups*. AMS Chelsea Publishing Series. AMS Chelsea Pub., 2006.

[7] I. M. Isaacs. *Finite Group Theory*, volume 92. Am. Math. Soc, 2008.

[8] I.M. Isaacs. *Algebra: a graduate course*, volume 100. American Mathematical Soc., 1994.

[9] A. Machì. *Groups: An Introduction to Ideas and Methods of the Theory of Groups*. Springer Science & Business Media, 2012.

CURRICULUM VITAE

PERSONAL INFORMATION

Name, Surname: M. Yasir KIZMAZ
Nationality: Turkish (TC)
Date and Place of Birth: May 11, 1989, Iğdır
Marital Status: Single
E-mail: yasir@metu.edu.tr

EDUCATION

Degree	Institution	Year of Graduation
B.S.	METU, Mathematics	2012

WORK EXPERIENCE

Year	Place	Enrollment
2012-2018	METU, Department of Mathematics	Research Assistant

FOREIGN LANGUAGES

English (fluent)

AWARDS AND SCHOLARSHIPS

- 2012-2017, TÜBİTAK 2211-Domestic Doctorate Fellowship
- B.S. in Mathematics with high honours, METU, 2012

CONFERENCES

- Advances In Group Theory And Applications 2014, Italy
- Zassenhaus Group Theory Conference 2015 (contributed talk)

- Finite Groups And Their Automorphisms 2017 (contributed talk)

PUBLICATIONS

- M. Y. Kızmaz (2018) Character free proofs for two solvability theorems due to Isaacs, Communications in Algebra, 46:6, 2631-2634
- M.Y. Kızmaz, *A sufficient condition for fixed points of a coprime action to have a normal complement.* Archiv der Mathematik, 1-3.
- M.Y. Kızmaz, *Finite Groups Having Nonnormal T.I. Subgroups* (Accepted in International Journal of Algebra and Computation)
- M.Y. Kızmaz, *On The Number Of Topologies On A Finite Set* (Accepted in Algebra and Discrete Mathematics)