# SOME PROBLEMS ON THE GEOMETRY OF CALIBRATED MANIFOLDS

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# ABSTRACT

## SOME PROBLEMS ON THE GEOMETRY OF CALIBRATED MANIFOLDS


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In this thesis, we study three problems on the geometry of calibrated manifolds, which are Riemannian manifolds equipped with a special closed differential form called a calibration. Firstly, we compute the homology of Grasmannian manifold of oriented 3-planes in $\mathbb{R}^6$, namely $G_3^+(\mathbb{R}^6)$, and its special submanifold called SLAG, the set of 3-planes in $G_3^+(\mathbb{R}^6)$ determined by the special Lagrangian calibration on Calabi-Yau 3-fold $\mathbb{C}^3 \cong \mathbb{R}^6$. We make an immediate application of these computations. Secondly, we investigate a related problem on the embedding of oriented closed manifolds into $\mathbb{C}^n$ as special Lagrangian-free (sLag-free). Finally, we study the geography of symplectic 8-dimensional manifolds and obtain certain results on the existence of symplectic 8-manifolds with Spin$(7)$-structure.



Keywords: calibration, special Lagrangian, sLag-free submanifold, Spin$(7)$-structure

# ÖZ

## KALİBRE EDİLMİŞ ÇOK-KATMANLILAR GEOMETRİSİNDE BAZI PROBLEMLER

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Bu tezde kalibrasyon adı verilen özel kapalı bir form ile donatılan Riemann manifoldlarının özel adıyla kalibre edilmiş manifoldların geometrisi üzerine üç problem çalışılmıştır. İlk olarak, $\mathbb{R}^6$ içerisindeki yönlendirilmiş 3-uzayların Grasmann manifoldu $G_3^+\mathbb{R}^6$'nın ve $G_3^+\mathbb{R}^6$'nın özel bir alt manifoldu olan ve $\mathbb{C}^3 \cong \mathbb{R}^6$ Calabi-Yau 3-manifoldu üzerindeki özel Lagrange kalibrasyonu tarafından belirlenen SLAG altmanifoldunun homoloji grupları hesaplanmıştır. Devamında, bu hesapların geometrik bir uygulaması da yapılmıştır. İkinci olarak yaptığımız hesaplarla ilişkili, yönlendirilmiş kapalı manifoldların $\mathbb{C}^n$ içine özel Lagrange-serbest(sLag-serbest) olarak gömülebilmesi problemi incelenmiştir. Son olarak, simplektik 8-manifoldların coğrafyası çalışılıp, $\mathrm{Spin}(7)$ yapısı olan simplektik 8-manifoldların varlığı hakkında belirli sonuçlar elde edilmiştir.

Anahtar Kelimeler: 8-boyutlu çokkatmanlılar, coğrafya, $\mathrm{Spin}(7)$ yapısı

To my wife Elif...

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| | |
|---|---|
| Hol | Holonomy Group |
| SLAG | Set of Special Lagrangians |

# CHAPTER 1

# INTRODUCTORY MATERIALS

In this chapter, we give a brief introduction to certain concepts and tools in the theory of calibrated geometries. These are extensively used in the later chapters.

## 1.1 Holonomy Groups

### 1.1.1 Connections and Holonomy Groups

**Definition 1.1.1.** *Let* $E \to X$ *be a vector bundle over a manifold* $X$*. A linear map* $\nabla : C^\infty(E) \to C^\infty(E \otimes T^*X)$ *is called a connection if* $\nabla$ *satisfies the condition*

$$\nabla(fe) = f\nabla e + e \otimes df,$$

*such that* $e \in C^\infty(E)$ *is a smooth section of* $E$ *and* $f$ *is a smooth function on X. If* $\nabla$ *is such a connection,* $e \in C^\infty(E)$ *and* $v \in C^\infty(E)$ *and* $f, g$ *are smooth functions on* $X$*, we have*

$$\nabla_{fv}(ge) = fg\nabla_v e + f(v \cdot g)e.$$

$v \cdot dg$ *is derivative of* $g$ *by* $v$ *and It is a smooth function on* $X$*.*

Other important concept in geometry is *curvature*. Write $End(E) = E \otimes E^*$, where $E^*$ is the dual vector bundle to $E$. Let $\nabla$ be a connection on E. Then the curvature $R(\nabla)$ of the connection $\nabla$ is a smooth section of the vector bundle $End(E) \otimes \Lambda^2 T^*X$, defined as follows:

$$R(\nabla)(e \otimes v \wedge w) = \nabla_v \nabla_w e - \nabla_w \nabla_v e - \nabla_{[v,w]} e$$

for all $v, w \in C^\infty(TX)$ and $e \in C^\infty(E)$.

We now define the holonomy groups which is related with the curvature. Before defining holonomy, paralel transport is the main part of these concepts.

**Definition 1.1.2** (Paralel Transport)**.** *[1] Let $\gamma : [0,1] \to X$ be a loop at $x \in X$, that is a piecewise-smooth path with $\gamma(0) = \gamma(1) = x$. For each $v \in T_xX$ there exists a unique section $s$ of $\gamma^*(TM)$, which is smooth everywhere where $\gamma$ is smooth, such that $s(0) = v$ and $\nabla_{\dot{\gamma}(t)}s(t) = 0$ for all $t \in [0,1]$. Define $\Gamma_\gamma(v) = s(1)$. Then there exists a well-defined linear map $\Gamma_\gamma : T_xX \to T_xX$ which is called the parallel transport map.*

Now, for any two loops $\alpha, \beta$ at $x \in X$, consider the following loops:

$$\alpha\beta(t) = \begin{cases} \beta(2t) & \text{if } 0 \leq t \leq 1/2 \\ \alpha(2t-1) & \text{if } 1/2 \leq t \leq 1 \end{cases} \quad \text{and } \alpha^{-1}(t) = \alpha(1-t)$$

Then $\Gamma_\alpha \circ \Gamma_\beta = \Gamma_{\alpha\beta}$. If we define $\iota(t) = x,\ t \in [0,1]$, then we have $\Gamma_\iota = Identity$ and $\Gamma_\alpha \circ \Gamma_{\alpha^{-1}} = \Gamma_{\alpha^{-1}} \circ \Gamma_\alpha = Identity$, that is $\Gamma_\alpha^{-1} = \Gamma_{\alpha^{-1}}$. Hence we see that the set of the parallel transports corresponding to the loops based at a fixed point with composition as the binary operation defines a group.

**Definition 1.1.3** (Holonomy Group)**.** *Let $x \in X$. Then,*

$$Hol_x(\nabla) = \{\Gamma_\gamma : \gamma \text{ is a loop based at } x\}$$

*is called holonomy group of $\nabla$ at $x$. Also,*

$$Hol_x^0(\nabla) = \{\Gamma_\gamma : \gamma \text{ is a null-homotopic loop based at } x\}$$

*is called the restricted holonomy group of $\nabla$ at $x$. If $\nabla$ is the Riemannian connection of metric $g$, then we use $Hol_x(g)$ and $Hol_x^0(g)$ instead.*

**Proposition 1.1.4.** *If $X$ is connected, then $Hol_x(g)$ and $Hol_x^0(g)$ are subgroups of $O(n)$ defined up to conjugation and are independent of $x$. We thus use the notation $Hol(g)$ and $Hol^0(g)$ for the subgroups of $O(n)$. If $X$ is simply connected, then $Hol_x(g) = Hol_x^0(g)$*

Holonomy groups of Riemannian metrics are classified by Berger in 1955. [2].

**Theorem 1.1.5.** *Let $(X, g)$ be an oriented, connected, simply connected, irreducible and non-symmetric $n$-dimensional Riemannian manifold. $Hol(g)$ must be equal to one of the followings:*

| $\mathbf{Hol(g)}$ | $\mathbf{dim(M)}$ | ***Type of Manifold*** | ***Properties*** |
|---|---|---|---|
| $SO(n)$ | $n$ | *Orientable Manifold* | |
| $U(n)$ | $2n$ | *Kähler Manifold* | |
| $SU(n)$ | $2n$ | *Calabi-Yau Manifold* | *Ricci-flat* |
| $Sp(n) \cdot Sp(1)$ | $4n$ | *Quaternion-Kähler Manifold* | *Einstein* |
| $Sp(n)$ | $4n$ | *Hyper-Kähler Manifold* | *Ricci-flat* |
| $G_2$ | *7* | *$G_2$-Manifold* | *Ricci-flat* |
| $Spin(7)$ | *8* | *$Spin(7)$-Manifold* | *Ricci-flat* |

There is a correspondence between holonomy groups and the curvature of space. Therefore the holonomy group of a manifold define its geometry.

### 1.2 Calibrated Geometry

Minimal submanifolds form an important class of submanifolds in differential geometry. In their seminal paper [3] seminal paper, Harvey and Lawson gave a new direction to minimal submanifold theory. Harvey and Lawson constructed the theory of calibrated geometry [3]. Calibrated submanifolds are the special kind of minimal submanifold of the Riemannian manifold $X$ defined using a special closed form on $X$ called a *calibration* . Let $V$ be an oriented tangent $k$-plane and vector subspace $V$ of tangent space $T_xX$ to $X$ at some $x \in X$ with $dim(V) = k$. $g|_V$ defines Euclidean metric with oriented tangent $k$-plane on $X$. Combination of $g|_V$ with the orientation on $V$ gives a natural volume form $vol_V$ on $V$, which is a $k$-form on $V$ [1].
A differential $k$-form $\varphi$ is called calibration on $X$ if it is closed (i.e. $d\varphi = 0$) and for every oriented $k$-plane $V$ on $X$ inequality $\varphi|_V \leq vol_V$ holds. $\varphi|_V = \alpha.vol_V$ for some $\alpha \in \mathbf{R}$, and $\varphi|_V \leq vol_V$ if $\alpha \leq 1$. Let $N$ be an oriented submanifold of $X$ with $dim(N) = k$. Then each tangent space $T_xN$ for $x \in N$ is an oriented tangent $k$-plane.

If the restriction of $\varphi$ to tangent plane of $N$ for all $x \in N$ equalş to volume form of

$N$ (i.e. $\varphi|_{T_xN} = vol_{TN}$ ), then $N$ is called $\varphi$-submanifold or calibrated submanifold. Now, we will state the main result about calibrations which is also called *Fundamental Theorem of Calibrations.*

**Theorem 1.2.1.** *Let $(M, g)$ be a Riemannian manifold, and $\varphi$ a calibration on $X$, then all calibrated submanifolds are minimal and homologically volume-minimizing.*

*Proof.* Let $[N] \in H_k(X, \mathbb{R})$ with dimension $N$, and $[\varphi] \in H^k(X, \mathbb{R})$. Then

$$[\varphi].[N] = \int_{x\in N} \varphi|_{T_xN} = \int_{x\in N} vol_{T_xN} = vol(N).$$

Let $N'$ is homologous to $N$ i.e. $[N'] = [N]$.

$$vol(N) = \int_{x\in N} \varphi|_{T_xN} = \int_{x\in N'} \varphi|_{T_xN'} \leq \int_{x\in N} vol_{T_xN} = vol(N').$$

□

There is a close connection between holonomy groups and calibrations. Define $G \subset O(n)$ as a possible holonomy group of Riemannian metric. $G$ may be one of the holonomy group of Berger list. Then $G$ acts on the $k$-forms $\wedge^k(\mathbb{R}^n)$ on the $\mathbb{R}^n$. One can choose with tangent plane at fixed point. Then it is spread on $X$.

Suppose $\varphi_0$ is non-zero $G$-invariant $k$-form on $\mathbb{R}^n$. $\varphi_0$ can be arranged that for each oriented $k$-plane $\mathcal{V} \subset \mathbb{R}^n$ such that $\varphi_0|_{\mathcal{V}} \leq vol|_{\mathcal{V}}$, and that $\varphi_0|_{\mathcal{V}} = vol|_{\mathcal{V}}$ for at least one such $\mathcal{V}$. Let $\mathscr{V}$ be set of containing oriented $k$-planes $\mathcal{V}$ in $\mathbb{R}^n$ with $\varphi_0|_{\mathcal{V}} = vol|_{\mathcal{V}}$ and $dim(\mathscr{V}) = d$. If $\mathcal{V} \in \mathscr{V}$ then $g.\mathcal{V} \in \mathscr{V}$ for all $g \in G$, since $\varphi_0$ is invariant under $G$.

One can apply this idea to Riemannian manifold. Let $(X^n, g, \nabla, G)$ be Riemannian manifold with dimension $n$, and metric $g$ on $X$ with Levi-Civita connection $\nabla$ and holonomy group $G$. There exists a $k$-form $\varphi$ on $X$ with $\nabla\varphi = 0$, corresponding to $\varphi_0$. By proposition, hence $\varphi$ is closed (i.e. $d\varphi = 0$). Also, the condition $\varphi_0|_{\mathcal{V}} \leq vol|_{\mathcal{V}}$ for all oriented $k$-planes $\mathcal{V}$ in $\mathbb{R}^n$ implies that $\varphi|_{\mathcal{V}} \leq vol|_{\mathcal{V}}$ for all oriented tangent $k$-planes in $X$. Thus $\varphi$ is a calibration on $X$.

At every point $x \in X$ there is an $d$-dimensional family $\mathscr{V}_x$ of oriented tangent $k$-planes $U$ with $\varphi|_U = vol|_U$, isomorphic to $\mathscr{V}$. Hence, the set of oriented tangent $k$-planes $U$ in $X$ with $\varphi|_U = vol|_U$ has the $d + n$. This suggests that locally there should exist many $\varphi$-submanifolds $N$ is $X$, so calibrated geometry of $\varphi$ of $(X, g)$ is nontrivial. In the following table we can see the most well-known calibrations coming from the special holonomy.

| **Type of Manifold** | **Calibration** | **Calibrated Planes** |
|---|---|---|
| Kähler Manifold | $\frac{\omega^p}{p!}$ ($\omega$=Kähler form) | Complex p-planes |
| Calabi-Yau Manifold | Re($\Omega$) ($\Omega$=Holomorphic Volume form) | Special Lagrangian n-planes |
| Hyper-Kähler Manifold | $\Phi = \frac{1}{3}(\frac{\omega_I^2}{2} + \frac{\omega_J^2}{2} + \frac{\omega_K^2}{2})$ | Quaternion Lines |
| $G_2$-Manifold | $\phi$ (associative 3-form) | Associative 3-planes |
| $G_2$-Manifold | $*\phi$ (co-associative 4-form) | Co-associative 4-planes |
| $Spin(7)$-Manifold | $\psi$ (Cayley 4-form) | Cayley 4-planes |

### 1.3 $\varphi$-free Submanifolds

Let $(X, \varphi)$ be a calibrated manifold. A $p$-plane $\xi$ is said to be **tangential** to a submanifold $M \subset X$ if $span\xi \subset T_xM$ for some $x \in M$

**Definition 1.3.1.** *A closed submanifold $M \subset X$ is called $\varphi$-**free** if there are no $\varphi$-planes $\xi \in G(\varphi)$ which are tangential to $M$.*

Each submanifold of dimension strictly less than the degree of $\varphi$ is automatically $\varphi$-free. Locally, generic p-dimensional submanifolds are $\varphi$-free. Depending on the calibration, there is an upper bound for the dimension of a $\varphi$-free submanifold.

**Definition 1.3.2.** *The free dimension $fd(\varphi)$ of a calibrated manifold $(X, \varphi)$ is the maximum dimension of a linear subspaces in $TX$ which contains no $\varphi$-planes. Subspaces which satisfy such condition are called $\varphi$-free.*

Hence, the dimension of a $\varphi$-free submanifold can not exceed **fd($\varphi$)**. For all well-known calibrations on manifolds with special holonomy this dimension is computed and shown in the following table (for details see [4].

| Calibration | Free dimension |
|---|---|
| $\omega$=Kähler form | $n$ |
| Re($\Omega$) ($\Omega$=Holomorphic Volume form) | $2n-2$ |
| $\Phi = \frac{1}{3}(\frac{\omega_I^2}{2} + \frac{\omega_J^2}{2} + \frac{\omega_K^2}{2})$ | $3n$ |
| $\phi$ (associative 3-form) | 4 |
| $*\phi$ (co-associative 4-form) | 4 |
| $\psi$ (Cayley 4-form) | 4 |

$\varphi$-free submanifolds are the generalization of totally real submanifolds in complex geometry to calibrated manifolds. They are used to construct Stein-like domains in calibrated manifolds, called as $\varphi$-convex domains.

## 1.4 Spin Structure

**Definition 1.4.1** (Spin Structure)**.** *Let $M^n$ be a manifold. If the structure group of $M$ is double cover of $SO(n)$. In other words, M admits a spin-structure if and only if* $w_1(M) = w_2(M) = 0$

Let $E$ be an oriented $n$-dimensional Riemannian vector bundle over a manifold $X$, and let $p_{SO}(E)$ be its bundle of oriented orthonormal frames. It is naturally defined universal covering homomorphism $\xi_0 : Spin_n \to SO_n$ with kernel $\{-1, 1\} \simeq Z_2$ for $n \geq 3$

**Definition 1.4.2.** *Let $n \geq 3$. Then a spin structure on $E$ is a principal $Spin_n$-bundle $P_{Spin}(E)$ together with a 2-sheeted covering*

$$\xi : P_{Spin}(E) \to P_{SO}(E)$$

*such that $\xi(pg) = \xi(p)\xi_0(g)$ for all $p \in P_{Spin}(E)$ and all $g \in Spin_n$.*

**Theorem 1.4.3.** *Let $p : E \to B$ be a oriented vector bundle. Then there exists a spin structure on $E$ if and only if $w_2 = 0$, and then the distinct spin structures on $E$ are in one-to-one correspondence with the elements of $H^1(B, Z_2)$.*

The spin structures on manifolds would be discussed with vector bundle. Now, the definition is the following;

**Definition 1.4.4.** *An oriented Riemannian manifold is called a* ***spin manifold*** *if its tangent bundle carries spin structure.*

**Lemma 1.4.5.** *Let $M$and $N$ be two manifold which admit spin structure. Then the connected sum $M\sharp N$ also admit spin structure.*

**Definition 1.4.6.** *Let $M$and $N$ be spin manifolds. Then they are called to differentiably equivalent if the map is diffeomorphism which preserves orientations and spin structures.*

Let $\Omega_n^{Spin}$ denote the free abelian group generated by equivalence classes of compact connected $n$- dimensional spin manifold, modulo the subgroup generated by elements $[X_1] + ... + [X_k]$ where $X_1 \amalg ... \amalg X_k$ is spin cobordant to zero. $\Omega_n^{Spin}$ denotes the *n-dimensional spin cobordism group.* By the proposition, the product of two spin manifolds has a uniquely determined spin structure. Given two spin $n$-manifolds $X_1$ and $X_2$, we can form their connected sum $X_1\#X_2$ and equip it with a spin structure so that $X_1\#X_2$ and $X_1 \amalg X_2$ are spin cobordant.

### 1.5 Spin(7) Structure

One can define a $4-$form on $R^8$.

**Definition 1.5.1.** *Let $(x_1, ..., x_8)$ be coordinates on $R^8$ and for the form $dx_{ij...k}$. Define a $4-$form $\Omega_0$ by :*

$$\begin{aligned}\Omega_0 &= dx_{1234} + dx_{1256} + dx_{1278} + dx_{1357} - dx_{1368} - dx_{1458} - dx_{1467} \\ &+ dx_{5678} + dx_{3478} + dx_{3456} + dx_{2468} + dx_{2457} + dx_{2367} + dx_{2358}.\end{aligned}$$

*$\Omega_0$ is a calibrations on $R^8$ and submanifolds calibrated with respect to $\Omega_0$ are called Cayley 4-folds.*

**Definition 1.5.2.** *A $4-$form $\Omega$ on $M$ is admissible if $\Omega|_{T_xM} \in \Lambda_a^4 T^*M$ for all $x \in M$*

**Definition 1.5.3.** *Let $M$ be an oriented $8-$manifold, let $\Omega$ be an admissible $4-$form on $M$ and let $g$ be the metric associated to $\Omega$. We call $(\Omega, g)$ a $Spin(7)$ structure on $M$.*

## 1.6 Symplectic Structure

**Definition 1.6.1.** *[5] A smooth manifold $M^n$ is called a symplectic manifold if there exists a 2-form $\omega$ on $M$ such that*

* $d\omega = 0$ *(i.e. $\omega$ is closed)*
* $\omega$ *is non- degenerate.*

Hence, the top form $\omega^n \neq 0$ and $M$ must be oriented and even dimensional. Moreoever, $(M, \omega_M)$ and $(N, \omega_N)$ be symplectic manifold. A map $f : M \to N$ is called symplectic map if $f^*\omega_N = \omega_M$; also if $f$ diffeomorphism then it is called symplectomorphism.

**Example 1.6.2.** *Euclidean space $\mathbb{R}^{2n}$ with coordinates $(e_1, ..., e_n, f_1, ..., f_n)$ and the form*

$$\omega_0 = \Sigma_{i=1}^n de_i \wedge df_i$$

*is symplectic manifold.*

Darboux's Theorem states that all symplectic manifold is locally isomorphic to previos canonical example Euclidean space with the standard symplectic form [6].

**Definition 1.6.3.** *Let $(M, \omega)$ be a symplectic manifold and $N$ be a submanifold of $M$. Then, $N$ is called a symplectic submanifold if $\omega|_{TN}$ is also symplectic form.*

**Proposition 1.6.4.** *Let $(M_1, \omega_1)$ and $(M_2, \omega_2)$ be symplectic manifolds. Suppose that $j_1$ and $j_2$ of $N$ in $M_1$ and $M_2$, be symplectic embeddings such that the normal bundles of the two embeddings are isomorphic. Then there exists tubular neighborhoods $U_1, U_2$ of $j_1(N), j_2(N)$ and a symplectomorphism $\Theta : U_1 \to U_2$ such that the differential of $\Theta$ induces between the normal bundles the given isomorphism.*

# CHAPTER 2

# RESULTS

In this chapter, we present various results on the topology of $G_3^+\mathbb{R}^6$, the Grassmannian of oriented 3-planes in $\mathbb{R}^6$ and compute its cohomology ring. Then, we compute the homology groups of $SLAG \subset G_3^+\mathbb{R}^6$ which is the contact set of the special Lagrangian calibration in $\mathbb{C}^3 \cong \mathbb{R}^6$. We will give some definitions and theorems from fundamental articles of Harvey and Lawson [7]. Let $G_k^+\mathbb{R}^n$ be a Grassmannian manifold defined by all oriented $k$-dimensional subspaces of $\mathbb{R}^n$. For any $v \in G_k^+\mathbb{R}^n$ there are orthonormal frame $e_1, ..., e_k$ such that $v = e_1 \wedge ... \wedge e_k$. Let $\phi$ be a closed $k$-form on $\mathbb{R}^n$. If $\phi(v_1 \wedge v_2 \wedge \cdots \wedge v_k) \leq 1$ for any orthonormal vectors $v_1, v_2, ..., v_k \in \mathbb{R}^n$, then $\phi$ is called a calibration on $\mathbb{R}^n$. The set

$$\{v_1 \wedge v_2 \wedge ... \wedge v_k \in G_k^+\mathbb{R}^n \mid \phi(v_1 \wedge v_2 \wedge ... \wedge v_k) = 1\}$$

is called contact set or face of calibration $\phi$. $\mathbb{R}^{2n} \simeq \mathbb{C}^n$ associated to the calibrations $\alpha = Re\{dz_1 \wedge ... \wedge dz_n\}$. The submanifolds in this geometry are Lagrangian submanifolds of $\mathbb{C}^n$ which satisfy an additional determinant condition. They are therefore called 'Special Lagrangian' submanifolds. They, of course, have the property of being absolutely area-minimizing.

This chapter is organized as follows. In section §2.1 we deal with the related Stiefel manifolds, in section §2.2 with the Grassmann manifold, in section §2.3 with $SLAG$, in §2.4 with the cohomology ring, and finally in §2.5-2.6 we give some immediate geometric applications of these computations.

## 2.1 Stiefel Manifolds

In order to compute the invariants of Grassmannian manifolds, some knowledge about the related Stiefel manifolds is necessary. That is why we are going to study these manifolds in this section. We start with a simpler one. Namely $V_2\mathbb{R}^5$, the bundle of ordered orthonormal 2-frames in the Euclidean 7-space. We start with the following proposition.

**Proposition 2.1.1.** *The homology of the Stiefel manifold $V_2\mathbb{R}^5$ is the following.*

$$H_*(V_2\mathbb{R}^5;\mathbb{Z}) = (\mathbb{Z},0,0,\mathbb{Z}_2,0,0,0,\mathbb{Z}).$$

*Proof.* $\pi_{012}(V_2\mathbb{R}^5) = (0,0,0)$ since 7-dimensional manifold $V_2\mathbb{R}^5$ is 2-connected. By Hurewicz theorem, one can find $H_{012} = (\mathbb{Z},0,0)$ and $H_3(V_2\mathbb{R}^5;\mathbb{Z}) = \mathbb{Z}_2$.
Next step, the following fibration gives the rest homology classes by using spectral sequences.

$$\begin{array}{ccc} \mathbb{S}^3 & \rightarrow & V_2\mathbb{R}^5 \\ & & \downarrow \\ & & \mathbb{S}^4 \end{array}$$

And,

$$E^{\infty}_{p,q} = F_{p,q}/F_{p-1,q+1}$$

where

$$F_{p,q} := im(H_{p+q}(E^p;G)) \rightarrow H_{p+q}(E;G)).$$

| | | | | | |
|---|---|---|---|---|---|
| q | | | | | |
| 5 | 0 | 0 | 0 | 0 | 0 |
| 4 | 0 | 0 | 0 | 0 | 0 |
| 3 | $\mathbb{Z}$ | 0 | 0 | 0 | $\mathbb{Z}$ |
| 2 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | 0 | 0 |
| 0 | $\mathbb{Z}$ | 0 | 0 | 0 | $\mathbb{Z}$ |
| | 0 | 1 | 2 | 3 | 4 |

$$E^5_{4,0} = E^\infty_{4,0} = F_{4,0}/F_{3,1} \simeq H_4(E = V_2\mathbb{R}^5)/F_{3,1}$$

$$F_{0,4}/F_{-1,5} = F_{0,4}/0 = E^\infty_{0,4} = E^5_{0,4} = 0$$

$$F_{1,3}/F_{0,4} = E^\infty_{1,3} = E^5_{1,3} = 0$$

$$F_{2,2}/F_{1,3} = E^\infty_{2,2} = E^5_{2,2} = 0$$

$$F_{3,1}/F_{2,2} = E^\infty_{3,1} = E^5_{3,1} = 0$$

then $H_4(E)/0 = E^5_{4,0}$. We have

$$E^5_{4,0} = ker\ (E^4_{4,0} \to E^4_{0,3})/Im\ (E^4_{8,-3} \to E^4_{4,0})$$

the sequence is continued until second page $H_4(B) = \mathbb{Z}$
$H_4(E) = E^4_{4,0}$ since $E^4_{4,0}/ker(E^4_{4,0} \to E^4_{0,3}) = 0$. Finally, $H_4(E) = 0$ □

Next, using the information coming out of this proposition, we are going to manage a higher Stiefel manifold $V_3\mathbb{R}^6$. We have the following result on this manifold.

**Proposition 2.1.2.** *The homology of the Stiefel manifold $V_3\mathbb{R}^6$ is the following.*

$$H_*(V_3\mathbb{R}^6; \mathbb{Z}) = (\mathbb{Z}, 0, 0, \mathbb{Z}_2, 0, \mathbb{Z}, 0, 0, 0, 0, 0, 0, \mathbb{Z})$$

*Proof.* We know that $V_3R^6$ is 6-3-1=2 connected, then $\pi_{012} \simeq 0$ and $H_{012} = (\mathbb{Z}, 0, 0)$ and $H_3(V_3\mathbb{R}^6) = \mathbb{Z}_2$ by Hurewicz Theorem . Next step, the following fibration gives the rest homology classes by using spectral sequences.

$$\begin{array}{ccc} V_2\mathbb{R}^5 & \to & V_3\mathbb{R}^6 \\ & & \downarrow \\ & & \mathbb{S}^5 \end{array}$$

| $q$ | | | | | | |
|---|---|---|---|---|---|---|
| 8 | 0 | 0 | 0 | 0 | 0 | 0 |
| 7 | $\mathbb{Z}$ | 0 | 0 | 0 | 0 | $\mathbb{Z}$ |
| 6 | 0 | 0 | 0 | 0 | 0 | 0 |
| 5 | 0 | 0 | 0 | 0 | 0 | 0 |
| 4 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 | $\mathbb{Z}_2$ | 0 | 0 | 0 | 0 | $\mathbb{Z}_2$ |
| 2 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | $\mathbb{Z}$ | 0 | 0 | 0 | 0 | $\mathbb{Z}$ |
| | 0 | 1 | 2 | 3 | 4 | 5 |

The page $E^2$ of fibration is given above.

$$E^9_{4,0} = E^\infty_{4,0} = F_{4,0}/F_{3,1} \simeq H_4(E = V_3\mathbb{R}^6)/F_{3,1}$$

$$F_{0,4}/F_{-1,5} = F_{0,4}/0 = E^\infty_{0,4} = E^9_{0,4} = 0$$
$$F_{1,3}/F_{0,4} = E^\infty_{1,3} = E^9_{1,3} = 0$$
$$F_{2,2}/F_{1,3} = E^\infty_{2,2} = E^9_{2,2} = 0$$
$$F_{3,1}/F_{2,2} = E^\infty_{3,1} = E^9_{3,1} = 0$$

then $H^4(E)/0 = E^9_{4,0}$

$$E^9_{4,0} = ker\ (E^8_{4,0} \to E^8_{-4,7})/Im\ (E^8_{12,-7} \to E^8_{4,0})$$

the sequence is continued until second page $E^2_{4,0} = H_4(B) = 0$

Finally, $H_4(E) = 0$, by using poincare duality, other classes can be found. $\square$

We can summarize the Propositions 2.1.1 and 2.1.2 in terms of cohomology as follows.

**Corollary 2.1.3.** *The cohomology ring of the Stiefel manifolds are the following for which* $\deg x_m = m$.

*1.* $H^*(V_2\mathbb{R}^5;\mathbb{Z}) = \mathbb{Z}[x_7]/(x_7^2) \oplus \mathbb{Z}_2[x_4]/(x_4^2)$

*2.* $H^*(V_3\mathbb{R}^6;\mathbb{Z}) = \mathbb{Z}[x_7, x_{12}]/(x_7^2, x_{12}^2, x_7x_{12}) \oplus \mathbb{Z}_2[x_9]/(x_9^2)$

## 2.2 The Grassmannian Manifold $G_3^+\mathbb{R}^6$

In this section we are going to compute some of the algebraic topological invariants of the Grasmmannian manifold. We start with the following Lemma.

**Lemma 2.2.1.** *The homology group* $H_3(G_3^+\mathbb{R}^6;\mathbb{Z})$ *is nontrivial.*

*Proof.* In order to be able to prove our result, we need to work with the following double fibration.

$$\begin{array}{ccc} & S^3 & \\ & \downarrow & \\ S^2 \longrightarrow & S(E_3\mathbb{R}^6) = E_0 & \xrightarrow{\pi_h} G_3^+\mathbb{R}^6 \\ & \downarrow{\scriptstyle \pi_v} & \\ & G_2^+\mathbb{R}^6 & \end{array}$$

The tautological bundle over $G_3^+\mathbb{R}^6$ is denoted by $E_3\mathbb{R}^6$. The 2-sphere bundle $S(E_3\mathbb{R}^6)$ is obtained by furnishing a with a metric and taking the unit sphere in each fiber. That is how the horizontal fibration obtained. From there one can obtain the vertical fibration with the following procedure. A point in $G_2^+\mathbb{R}^6$ represents a 2- plane which can be extended to a 3-plane by adding a unit vector in the 6-2=4 dimensional orthogonal complement. Another interpretation is through the oriented flag variety $F_{2,3}^+(\mathbb{R}^6)$ with its projection maps. Now, consider the Gysin exact sequence [8] of the vertical fibration.

$$\cdots \longrightarrow H^2(G_2^+\mathbb{R}^6;\mathbb{Z}) \xrightarrow{\cup e_4} H^6(G_2^+\mathbb{R}^6;\mathbb{Z}) \xrightarrow{\pi_v^*} H^6(E_0;\mathbb{Z}) \longrightarrow H^3(G_2^+\mathbb{R}^6;\mathbb{Z}) \longrightarrow \cdots$$

The homology of the Grassmannian $G_2^+\mathbb{R}^6$ is zero at the odd levels as can be seen in [9], this implies that the last term $H^3(G_2^+\mathbb{R}^6;\mathbb{Z})$ in the above vanishes. Now, suppose that $H^6(G_2^+\mathbb{R}^6;\mathbb{Z})$ is zero to raise a contradiction. This will then imply that $H^6(E_0;\mathbb{Z})$ is also zero as the term squeezed in the middle. Next consider the Gysin sequence of the horizontal fibration.

$$\cdots \longrightarrow H^6(E_0;\mathbb{Z}) \longrightarrow H^4(G_3^+\mathbb{R}^6;\mathbb{Z}) \xrightarrow{\cup e_3} H^7(G_3^+\mathbb{R}^6;\mathbb{Z}) \xrightarrow{\pi_h^*} H^7(E_0;\mathbb{Z}) \longrightarrow \cdots$$

Since we arrived at the point that the first term is zero, the middle map is an injection. If one compares the ranks: $\mathrm{rk}H^4 = 1$ and $\mathrm{rk}H^5 = 0$ from the Poincaré polynomial, such an inclusion is not possible. Therefore, the cohomology $H^6(G_3^+\mathbb{R}^6;\mathbb{Z})$ is nonzero. □

Akbulut and Kalafat computed [10] homology groups of various Grassmann bundles. We will use similar technique to compute homology groups of $G_3^+\mathbb{R}^6$. The previous lemma shows that third homology group is non-trivial. Then, this result will guarentee that $H_3(G_3^+\mathbb{R}^6) = \mathbb{Z}_2$. Next, we need the following fiber bundle to find homotopy groups of the Grassmannian.

$$SO(3) \to V_3\mathbb{R}^6 \longrightarrow G_3^+\mathbb{R}^6 \tag{2.1}$$

Exploiting this fibration, the previous Lemma leads to another preliminary result.

**Lemma 2.2.2.** *The preliminary homotopy groups of the Grassmannian are the following.*

$$\pi_{0123}\, G_3^+\mathbb{R}^6 = (0, 0, \mathbb{Z}_2, \mathbb{Z}_2).$$

*Proof.* We apply the homotopy exact sequence to the fiber bundle (2.1), a part of which is as follows.

$$\cdots \to \pi_5\, G_3^+\mathbb{R}^6 \to \mathbb{Z}_2 \to \mathbb{Z}_2 \to \pi_4\, G_3^+\mathbb{R}^6 \to \mathbb{Z} \to \mathbb{Z}_2 \overset{\pi_*}{\twoheadrightarrow} \pi_3\, G_3^+\mathbb{R}^6 \to 0 \to 0 \to$$
$$\pi_2\, G_3^+\mathbb{R}^6 \to \mathbb{Z}_2 \to 0 \to \pi_1\, G_3^+\mathbb{R}^6 \to 0.$$

We use the fact that $\pi_1 V_3\mathbb{R}^6$ is 2-connected and Proposition 2.1.2 for the Stiefel manifold, and higher homotopy groups of $SO(3)$ is the same as of its universal cover which is the 3-sphere. Surjecivity of the map $\pi_*$ reveals that the only option for the 3rd level is $\mathbb{Z}_2$ other than the trivial group. Since the Grassmannian is 1-connected, the Hurewicz homomorphism

$$h : \pi_3\, G_3^+\mathbb{R}^6 \longrightarrow H_3(G_3^+\mathbb{R}^6;\mathbb{Z})$$

is an epimorphism by [11] at this level. This implies that $\mathbb{Z}_2$ is the only nontrivial option for the 3rd homology as well. Since by the previous Lemma 2.2.2 this is nontrivial we arrive at $H_3(G_3^+\mathbb{R}^6;\mathbb{Z}) = \mathbb{Z}_2$. So that surjectivity reveals the homotopy. One can continue to analyse this exact sequence by inserting the homotopy groups of the Stiefel manifold from [12] starting from the 4-th level as we did above. □

On the other hand we have proved following results by using Serre spectral sequences;

**Theorem 2.2.3.** *The homology of the oriented Grassmann manifold $G_3^+\mathbb{R}^6$ is given by:*

$$H_*(G_3^+\mathbb{R}^6; Z) = (\mathbb{Z}, 0, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z} \oplus \mathbb{Z}_2, \mathbb{Z}, 0, 0, 0, \mathbb{Z}).$$

*Proof.* We construct spectral sequence, to calculate homology groups of $G_3^+\mathbb{R}^6$. Firstly, the following holds

$$H_*(SO_3;\mathbb{Z}) = (\mathbb{Z}, \mathbb{Z}_2, 0, \mathbb{Z})$$

$$E^2_{p,q} := H_p(G_3^+\mathbb{R}^6; H_q(SO_3;\mathbb{Z}))$$

$$E^\infty_{p,q} = F_{p,q}/F_{p-1,q+1}$$

where

$$F_{p,q} := im(H_{p+q}(E^p;G)) \to H_{p+q}(E;G))$$

for the fibration at 2.1. There exists also homomorphism maps $d^n_{p,q}$ such that

$$d^n_{p,q} : E^n_{p,q} \to E^n_{p-n,q+n-1}$$

Now, let $G_3^+\mathbb{R}^6 = \mathbb{G}$ and $V_3^+\mathbb{R}^6 = \mathbb{V}$ as a short notation. Then, the second page of fibration is the following:

| q | | | | | | | |
|---|---|---|---|---|---|---|---|
| : | 0 | 0 | 0 | 0 | 0 | 0 | |
| 5 | 0 | 0 | 0 | 0 | 0 | 0 | |
| 4 | 0 | 0 | 0 | 0 | 0 | 0 | |
| 3 | $H_0(\mathbb{G},\mathbb{Z})$ | $H_1(\mathbb{G},\mathbb{Z})$ | $H_2(\mathbb{G},\mathbb{Z})$ | $H_3(\mathbb{G},\mathbb{Z})$ | $H_4(\mathbb{G},\mathbb{Z})$ | .. | |
| 2 | 0 | 0 | 0 | 0 | 0 | 0 | |
| 1 | $H_0(\mathbb{G},\mathbb{Z}_2)$ | $H_1(\mathbb{G},\mathbb{Z}_2)$ | $H_2(\mathbb{G},\mathbb{Z}_2)$ | $H_3(\mathbb{G},\mathbb{Z}_2)$ | $H_4(\mathbb{G},\mathbb{Z}_2)$ | .. | |
| 0 | $H_0(\mathbb{G},\mathbb{Z})$ | $H_1(\mathbb{G},\mathbb{Z})$ | $H_2(\mathbb{G},\mathbb{Z})$ | $H_3(\mathbb{G},\mathbb{Z})$ | $H_4(\mathbb{G},\mathbb{Z})$ | .. | |
| | 0 | 1 | 2 | 3 | 4 | .. | $p$ |

We find the non-zero elements of the table by using Gysin sequence with the fibration

.

Firstly, it can be calculated derivation maps at the fifth page

$$d^5 : E^5_{p,q} \to E^5_{p-5,q+4}.$$

$d^5 = 0$ since either $E^5_{p,q} = 0$ when $p < 0$ or $E^5_{p-5,q+4} = 0$ when $p > 0$. Hence,

$$E^5_{p,q} = E^\infty_{p,q}.$$

Now, $E^5_{0,0} = E^\infty_{0,0} = F_{0,0}/F_{-1,1} = H_0(\mathbb{V}) = \mathbb{Z}$. In other direction, $E^5_{0,0} = ker(E^4_{0,0} \to E^4_{-4,3})/Im(E^4_{4,-3} \to E^4_{0,0}) = ... = E^4_{0,0} = E^3_{0,0} = E^2_{0,0} = H_0(\mathbb{G},\mathbb{Z}) = \mathbb{Z}$.

For the first homology class, we use same method; $E^5_{1,0} = E^\infty_{1,0} = F_{1,0}/F_{0,1} = H_1(\mathbb{V}) = 0$ In other direction, $E^5_{1,0} = ker(E^4_{1,0} \to E^4_{-3,3})/Im(E^4_{5,-1} \to E^4_{1,0}) = ... = E^3_{1,0} = E^2_{1,0} = H_1(\mathbb{G},\mathbb{Z}) = 0$

Second homology class;
$E^\infty_{2,0} = E^5_{2,0} = F_{2,0}/F_{1,1} = H_2(\mathbb{V})/0$ then $E^5_{2,0} = 0$ since $H_2(\mathbb{V}) = 0$ $E^\infty_{2,0} = E^5_{2,0} = ker\ (E^4_{2,0} \to E^4_{-2,3})/Im\ (E^4_{6,-3} \to E^4_{2,0}) \simeq E^4_{2,0}$
$E^4_{2,0} = ker\ (E^3_{2,0} \to E^3_{-1,2})/Im\ (E^3_{5,-2} \to E^3_{2,0}) \simeq E^3_{2,0}$
$E^3_{2,0} = ker\ (E^2_{2,0} \to E^2_{0,1})/Im\ (E^2_{4,-1} \to E^2_{2,0})$
$E^2_{0,1} = \mathbb{Z}_2$ by universal coefficient theorem since $H_0(\mathbb{G}) = \mathbb{Z}$
$E^5_{2,0} = E^4_{2,0} = E^3_{2,0} = ker\ (H_2(\mathbb{G}) \to \mathbb{Z}_2) = 0$ since $E^\infty_{2,0} = E^5_{2,0} = 0$
then the map $H_2(\mathbb{G}) \to \mathbb{Z}_2$ is injective.

And we write the following maps until limiting page to find $Im(E^2_{2,0} \to E^2_{0,1})$.

$$E^3_{0,1} = ker\ (E^2_{0,1} \to E^2_{-1,1})/Im\ (E^2_{2,0} \to E^2_{0,1})$$

$$E^4_{0,1} = ker\ (E^3_{0,1} \to E^3_{-2,3})/Im\ (E^3_{3,-1} \to E^3_{0,1})$$

$E^3_{0,1} = E^4_{0,1}$

$$E^5_{0,1} = ker\ (E^4_{0,1} \to E^4_{-3,4})/Im\ (E^4_{4,-2} \to E^4_{0,1})$$

At the same time, $E^5_{0,1} = E^\infty_{0,1} = F_{0,1}/F_{-1,-2}$ and $0 \subset F_{0,1} \subset F_{1,0} = 0$ So $E^5_{0,1} = 0$ and $E^5_{0,1} = E^4_{0,1} = E^3_{0,1} = 0$ then returned to

$$E^3_{0,1} = ker\ (E^2_{0,1} \to E^2_{-1,1})/Im\ (E^2_{2,0} \to E^2_{0,1})$$

$$E^2_{0,1}/(Im\ (E^2_{2,0} \to E^2_{0,1}) = 0$$

$$Im\ (E^2_{2,0} \to E^2_{0,1}) = \mathbb{Z}_2$$

since $E^2_{0,1} = \mathbb{Z}_2$. Therefore, by using first isomorphism theorem, $E^2_{2,0} = H_2(\mathbb{G}) = \mathbb{Z}_2$ since $Im\ (E^2_{2,0} \to E^2_{0,1}) = \mathbb{Z}_2$ and $ker\ (E^2_{2,0} \to E^2_{0,1}) = 0$ hold.

Third homology class;

$E^\infty_{3,0} = F_{3,0}/F_{2,1} = H_3(E)/F_{2,1} = \mathbb{Z}_2/F_{2,1},\ F_{2,1}/F_{1,2} = E^\infty_{2,1} = \mathbb{Z}_2$, Therefore,

$$F_{1,2} \subset F_{2,1} \subset F_{3,0} = H_3(\mathbb{V}) = \mathbb{Z}_2$$

and $E^\infty_{3,0} \in \{0, \mathbb{Z}_2\}$ since $F_{2,1} \in \{Z_2, 0\}$
Conversely,

$$E^\infty_{3,0} = E^5_{3,0} = ker\ (E^4_{3,0} \to E^4_{-1,3})/Im\ (E^4_{7,-3} \to E^4_{3,0}) \simeq E^4_{3,0}$$

$$E^4_{3,0} = ker\ (E^3_{3,0} \to E^3_{0,2})/Im\ (E^3_{6,-2} \to E^3_{3,0}) \simeq E^3_{3,0}$$

$$E^3_{3,0} = ker\ (E^2_{3,0} \to E^2_{1,1})/Im\ (E^2_{5,-1} \to E^2_{3,0}) \simeq E^2_{3,0} \simeq H_3(\mathbb{G})$$

$H_3(\mathbb{G}) \in \{0, \mathbb{Z}_2\}$ since $E^2_{1,1} = H_1(\mathbb{G}; \mathbb{Z}_2) = 0$ from previous calculations.In lemma 2.2.1, it is proven that $H_3(\mathbb{G})$ is non-trivial. Therefore, $H_3(\mathbb{G}) = \mathbb{Z}_2$

Forth homology class;

$$E^\infty_{4,0} = F_{4,0}/F_{3,1} = H_4(E)/F_{3,1} = 0/F_{3,1} \tag{2.2}$$

$$E^{\infty}_{3,1} = F_{3,1}/F_{2,2} = E^5_{3,1}$$

$$E^{\infty}_{2,2} = F_{2,2}/F_{1,3} = E^5_{2,2}$$

$$E^{\infty}_{1,3} = F_{1,3}/F_{0,4} = E^5_{1,3}$$

$$E^{\infty}_{0,4} = F_{0,4}/F_{-1,5} = E^5_{0,4} = 0$$

$$F_{0,4} = F_{1,3} = F_{2,2} = F_{3,1} = 0$$

then

$$E^{\infty}_{4,0} = 0 = E^5_{4,0},$$

now, it can be calculated other direction;

$$E^5_{4,0} = ker\ (E^4_{4,0} \to E^4_{0,3})/Im\ (E^4_{8,-3} \to E^4_{4,0})$$

$$E^4_{0,3} = ker\ (E^3_{0,3} \to E^3_{-3,5})/Im\ (E^3_{3,1} \to E^3_{0,3})$$

$$E^3_{3,1} = ker\ (E^2_{3,1} \to E^2_{1,2})/Im\ (E^2_{5,0} \to E^2_{3,1})$$

$E^3_{3,1} = 0$ since $E^2_{3,1} = 0$. Then,

$$E^3_{0,3} = ker\ (E^2_{0,3} \to E^2_{-2,4})/Im\ (E^2_{2,2} \to E^2_{0,3}) \simeq \mathbb{Z}$$

since $E^2_{2,2} = 0$ from table. So, $E^4_{0,3} = E^3_{0,3} \simeq \mathbb{Z}$.
Now, $E^5_{4,0} = ker\ (E^4_{4,0} \to E^4_{0,3}) = ker\ (E^4_{4,0} \to \mathbb{Z})$ and $E^5_{4,0} = ker\ (E^4_{4,0} \to \mathbb{Z}) \simeq 0$ from 2.2. To find $E^4_{4,0}$, one shall calculate image of the map $E^4_{4,0} \to E^4_{0,3}$.

$$E^5_{0,3} = ker\ (E^4_{0,3} \to E^4_{-4,6})/Im\ (E^4_{4,0} \to E^4_{0,3}) \tag{2.3}$$

and

$$E^5_{0,3} = E^{\infty}_{0,3} = F_{0,3}/F_{-1,4} = 0 \tag{2.4}$$

since $H_3(\mathbb{V}) = 0$.
Then combine 2.3 and 2.4, it can be concluded $E^4_{4,0}/ker\ (E^4_{4,0} \to E^4_{0,3}) \simeq Im\ (E^4_{4,0} \to E^4_{0,3}) = 0$ then $Im\ (E^4_{4,0} \to E^4_{0,3}) \simeq \mathbb{Z}$

$0 = E^5_{4,0} = E^4_{4,0} = E^3_{4,0} = ker\ (E^2_{4,0} \to E^2_{2,1})/Im\ (E^2_{6,-1} \to E^2_{4,0})$ finally,

$$ker(H_4(\mathbb{G}) \to \mathbb{Z}_2) = 0.$$

Conversely,

it should find $Im\ (E^2_{4,0} \to E^2_{2,1})$

$$E^3_{2,1} = ker\ (E^2_{2,1} \to E^2_{0,2})/Im\ (E^2_{4,0} \to E^2_{2,1})$$

$$E^4_{2,1} = ker\ (E^3_{2,1} \to E^3_{-1,3})/Im\ (E^3_{5,-1} \to E^3_{2,1}) \simeq E^3_{2,1}$$

$$E^5_{2,1} = ker\ (E^4_{2,1} \to E^4_{-2,4})/Im\ (E^4_{6,-2} \to E^4_{2,1}) \simeq E^4_{2,1}$$

and

$$E^5_{2,1} = E^\infty_{2,1} = F_{2,1}/F_{1,2} = 0/0$$

since $F_{1,2} \subset F_{2,1} \subset F_{3,0} = H_3(\mathbb{V}) = 0$. Then, combine previous results,

$$0 = E^\infty_{2,1} = E^5_{2,1} = E^4_{2,1} = E^3_{2,1}$$

$$E^3_{2,1} = ker\ (E^2_{2,1} \to E^2_{0,2})/Im\ (E^2_{4,0} \to E^2_{2,1}) = \mathbb{Z}_2/Im\ (E^2_{4,0} \to E^2_{2,1}) = 0$$

from table. In conclusion, $Im\ (E^2_{4,0} \to E^2_{2,1}) = \mathbb{Z}_2$ Consequently, by first isomorphism theorem, $H_4/\mathbb{Z} \simeq \mathbb{Z}_2$ holds, then $H_4(\mathbb{G}) = \mathbb{Z} \oplus \mathbb{Z}_2$

Fifth homology class;

$$E^\infty_{5,0} = F_{5,0}/F_{4,1} = H_5(E)/F_{4,1} = \mathbb{Z}/F_{4,1} \tag{2.5}$$

$$E^\infty_{4,1} = F_{4,1}/F_{3,2} = E^5_{4,1}$$

$$E^\infty_{3,2} = F_{3,2}/F_{2,3} = E^5_{3,2}$$

$$E^\infty_{1,4} = F_{1,4}/F_{0,5} = E^5_{1,4}$$

$$E^\infty_{0,5} = F_{0,5}/F_{-1,6} = E^5_{0,5} = 0$$

$$E^\infty_{0,4} = F_{0,4}/F_{-1,5} = E^5_{0,4} = 0$$

$$F_{0,5} = F_{1,4} = F_{2,3} = F_{3,2} = 0$$

then

$$E^\infty_{5,0} = 0 = E^5_{5,0},$$

now, it can be calculated other direction;

$$E^5_{5,0} = ker\ (E^4_{5,0} \to E^4_{1,3})/Im\ (E^4_{8,-2} \to E^4_{5,0}) = ker\ (E^4_{5,0} \to E^4_{1,3}) = \mathbb{Z}$$

$$E^4_{5,0} = ker\ (E^3_{5,0} \to E^3_{2,2})/Im\ (E^3_{8,-2} \to E^3_{5,0})$$

$E^45,0 = E^3_{5,0}$ since $E^3_{2,2} = 0$ by using table.

$$E^3_{5,0} = ker\ (E^2_{5,0} \to E^2_{3,1})/Im\ (E^2_{7,-1} \to E^2_{5,0})$$

$E^3_{5,0} = E^2_{5,0} = H_5(\mathbb{G})$ since $E^2_{3,1} = 0$. In 2.5 ,$E^4_{1,3} = 0$ by using table. Then,combine elements are found at 2.5,

$$ker\ (H_5 \to E^4_{1,3}) = \mathbb{Z}$$

$$ker\ (H_5 \to 0) = \mathbb{Z}.$$

Therefore, $H_5(\mathbb{G}) = \mathbb{Z}$.
Sixth homology class,

$$E^5_{6,0} = E\infty_{6,0} = F_{6,0}/F_{5,1} = H_6(E)/F_{5,1} = 0/F_{5,1}$$

$$E^5_{6,0} = E^4_{6,0} = E^3_{6,0} = E^2_{6,0} = H_6(\mathbb{G}) = 0.$$

Seventh homology class,

$$E^5_{7,0} = E^\infty_{7,0} = F_{7,0}/F_{6,1} = H_7(E)/F_{6,1} = 0/F_{6,1}$$

$$E^5_{7,0} = E^4_{7,0} = E^3_{7,0} = E^2_{7,0} = H_7(\mathbb{G}) = 0.$$

Eight homology class,

$$E^5_{8,0} = E\infty_{8,0} = F_{8,0}/F_{7,1} = H_8(E)/F_{7,1} = 0/F_{7,1}$$

$$E^5_{8,0} = E^4_{8,0} = E^3_{8,0} = E^2_{8,0} = H_8(\mathbb{G}) = 0.$$

Nineth homology class,
$H_9(\mathbb{G}) = H_0(\mathbb{G}) = \mathbb{Z}$ by Poincáre duality. □

### 2.3 SLAG in $G_3^+\mathbb{R}^6$

In this section we calculate some invariants of the submanifold SLAG which is defined as follows [1].

**Definition 2.3.1.** *Let $(M, J, g, \Omega)$ be a compact Calabi-Yau manifold with complex dimension $m$ and holonomy $SU(m)$ where $\Omega = \Omega_1 + i\Omega_2$ is a holomorphic volume* *form. The real part $Re(\Omega)$ is calibration on $M$, and the corresponding calibrated submanifolds are called special Lagrangian submanifolds.*

Then, the contact set in $G_3^{+}\mathbb{R}^6$ is denoted by $SLAG$ with the given calibration $Re(\Omega)$ in $\mathbb{R}^6$. The homology of $SLAG$ is the following;

**Theorem 2.3.2.** *The homology of the oriented SLAG manifold in $G_3^{+}\mathbb{R}^6$ is given by:*

$$H_*(SLAG;\mathbb{Z}) = (\mathbb{Z}, 0, \mathbb{Z}_2, 0, 0, \mathbb{Z})$$

*Proof.* We will construct spectral sequence, to able to show the some homology groups of $SLAG$ Firstly, we know the following

$$H_*(SU_3;\mathbb{Z}) = (\mathbb{Z}, 0, 0, \mathbb{Z}, 0, \mathbb{Z}, 0, 0, \mathbb{Z})$$

$$H_*(SO_3;\mathbb{Z}) = (\mathbb{Z}, \mathbb{Z}_2, 0, \mathbb{Z})$$

$$E^2_{p,q} := (SLAG; H_q(SO_3;\mathbb{Z}))$$

Then, second page of spectral sequence is following:
Let $SLAG = X$

| q | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| : | 0 | 0 | 0 | 0 | 0 | 0 | 0 | |
| 5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | |
| 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | |
| 3 | $H_0(X,\mathbb{Z})$ | $H_1(X,\mathbb{Z})$ | $H_2(X,\mathbb{Z})$ | $H_3(X,\mathbb{Z})$ | $H_4(X,\mathbb{Z})$ | $H_5(X,\mathbb{Z})$ | 0 | |
| 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | |
| 1 | $H_0(X,\mathbb{Z}_2)$ | $H_1(X,\mathbb{Z}_2)$ | $H_2(X,\mathbb{Z}_2)$ | $H_3(X,\mathbb{Z}_2)$ | $H_4(X,\mathbb{Z}_2)$ | $H_5(X,\mathbb{Z}_2)$ | 0 | |
| 0 | $H_0(X,\mathbb{Z})$ | $H_1(X,\mathbb{Z})$ | $H_2(X,\mathbb{Z})$ | $H_3(X,\mathbb{Z})$ | $H_4(X,\mathbb{Z})$ | $H_5(X,\mathbb{Z})$ | 0 | |
| | 0 | 1 | 2 | 3 | 4 | 5 | ... | $p$ |

and remember

$$E^{\infty}_{p,q} = F_{p,q}/F_{p-1,q+1}$$

where

$$Fp,q := im(H_{p+q}(E^p;G)) \to H_{p+q}(E;G)).$$

Now, let's find some elements of table.

Firstly, one can calculate derivation maps at the fifth page

$$d^5 : E^5_{p,q} \to E^5_{p-5,q+4},$$

since when $p < 0$, $E^5_{p,q} = 0$ also when $p > 0$, $E^5_{p-5,q+4} = 0$ In conclusion, $E^5_{p,q} = E^\infty_{p,q}$ Now, $E^5_{0,0} = E^\infty_{0,0} = F_{0,0}/F_{-1,1} = H_0(E) = \mathbb{Z}$ In other direction, $E^5_{0,0} = ker(E^4_{0,0} \to E^4_{-4,3})/Im(E^4_{4,-3} \to E^4_{0,0}) = ... = E^4_{0,0} = E^3_{0,0} = E^2_{0,0} = H_0(X,\mathbb{Z}) = \mathbb{Z}$.
Next homology class, one use same method; $E^5_{1,0} = E^\infty_{1,0} = F_{1,0}/F_{0,1} = H_1(E) = 0$ In other direction, $E^5_{1,0} = ker(E^4_{1,0} \to E^4_{-3,3})/Im(E^4_{5,-1} \to E^4_{1,0}) = ... = E^3_{1,0} = E^2_{1,0} = H_1(X,\mathbb{Z}) = 0$

Second homology class,

$E^5_{2,0} = E^\infty_{2,0} = F_{2,0}/F_{1,1} = H_2(E)/F_{1,1} = 0/F_{1,1} = 0$. In other direction, $E^5_{2,0} = ker(E^4_{2,0} \to E^4_{-2,3})/Im(E^4_{6,-3} \to E^4_{2,0}) = ... = E^3_{2,0} = 0$ then,

$$E^3_{2,0} = ker(E^2_{2,0} \to E^2_{0,1})/Im(E^2_{4,-1} \to E^2_{2,0})$$

$$E^3_{2,0} = ker(H_2(X,\mathbb{Z}) \to H_0(X,\mathbb{Z}_2) \simeq \mathbb{Z}_2) = 0$$

by universal coefficient theorem,

$$H_0(X,\mathbb{Z}_2) = H_0(X,\mathbb{Z}) \otimes \mathbb{Z}_2 \oplus Tor(H_{-1}(X,\mathbb{Z}),\mathbb{Z}_2).$$

Third homology class,

$$E^2_{5,2} \to E^2_{3,3} \to E^2_{1,4}$$

$$E^2_{3,3} = H_3(X) = 0$$

since $E^2_{5,2} \simeq E^2_{1,4} \simeq 0$ from second page.

Forth homology class,

$$E^2_{6,2} \to E^2_{3,3} \to E^2_{2,4}$$

$$E^2_{4,3} = H_4(X) = 0$$

since $E^2_{6,2} \simeq E^2_{2,4} \simeq 0$ from second page.

Fifth homology class, $H^5(X, \mathbb{Z}) = H_0(X, \mathbb{Z}) = \mathbb{Z}$ since the SLAG is compact $5-dimensional$ space then Poincare duality holds. Now, by universal coefficient theorem, $H_5(X, \mathbb{Z}) = \mathbb{Z}$ □

Consequently we get the following ring structure.

**Corollary 2.3.3.** *The cohomology ring of the SLAG manifold is the following truncated polynomial ring for which* deg$x_m = m$*.*

$$H^*(SLAG; \mathbb{Z}) = \mathbb{Z}[x_3, x_5]/(2x_3, x_3^2, x_5^2, x_3x_5).$$

## 2.4 The Ring Structure

In this section we are going to compute the cohomology ring of the Grassmann manifold. The cohomological Serre spectral sequence is more appropriate for this task. In the following discussion, $E = E(3, 6)$ will denote the canonical bundle over the Grassmann manifold $G_3^+\mathbb{R}^6$. It is obtained by taking the plane corresponding to a point to produce a vector bundle of rank 3 over our Grassmannian. We collect the related results of [13] here as follows.

**Theorem 2.4.1.** *We have the following relations in the cohomology of the Grassmann manifold $G_3^+\mathbb{R}^6$.*

*(a)* $2^{-1}p_1E(3, 6)$ *is a generator for* $H^4(G_3^+\mathbb{R}^6; \mathbb{Z})$ *and its Poincaré dual is* $[SLAG]$ *which is a generator of* $H_5(G_3^+\mathbb{R}^6; \mathbb{Z})$*.*

*(b)* $4(3\pi)^{-1} * p_1E(3, 6)$ *is a generator for* $H^5(G_3^+\mathbb{R}^6; \mathbb{Z})$ *and its Poincaré dual is* $[G(2, 4)]$ *which is a generator of* $H_4(G_3^+\mathbb{R}^6; \mathbb{Z})$*.*

Using these characteristic classes and integrals we will be able to figure out the generators and relations in our Grassmannian. Now we are ready to compute the cohomology ring.

**Theorem 2.4.2.** *The cohomology ring of the Grassmannian* $G_3^+\mathbb{R}^6$ *is as follows where* $\deg x_m = \deg y_m = m$.

$$H^*(G_3^+\mathbb{R}^6;\mathbb{Z}) = \mathbb{Z}[x_4, x_5]/\langle x_4^2, x_5^2, x_4x_5 - x_5x_4\rangle \oplus \mathbb{Z}_2[y_5, y_6, y_7]/\langle y_5^2, y_6^2, y_7^2, y_5y_6, y_6y_7, y_5y_7\rangle.$$

*Proof.* Torsion generators are obtained through our computations in section §2.2. Their relations are obtained through dimension restrictions. On the other hand the free part requires more attention. We pick the generators in the levels 4 and 5 and assign the following $x_4 := 2^{-1}p_1E(3,6), x_5 := 4(3\pi)^{-1} * p_1E(3,6)$ values. After handling the relations produced by the dimensional restrictions we have to understand the product of the chosen generators. So that we evaluate $x_4x_5$ over the Grassmannian to get its coefficient. The integral,

$$_{G_3^+\mathbb{R}^6}\int x_4x_5 =_{G(2,4)} \int x_4 = 1$$

provides that the product $x_4x_5$ is the generator of the top level. □

Alternatively we suggest to the reader to compute the product structure through the cohomological Serre spectral sequence.

### 2.5 The Geometry

In this section, we analyze the geometry of $SLAG$ in $G_3^+\mathbb{R}^6$. For any vector $v \in G_3^+\mathbb{R}^6$, there are orthonormal vectors $e_1 \wedge ... \wedge e_6$ and there exists $\phi \in \bigwedge^3(\mathbb{R}^6)$.

**Theorem 2.5.1.** *[14] The function* $\Phi : G_3^+\mathbb{R}^6 \to \mathbb{R}$ *defined by* $\Phi(v) = \langle v, \phi\rangle, \ \forall v \in G_3^+\mathbb{R}^6$ *is a degenerate Morse function for almost every form in* $\bigwedge^3(\mathbb{R}^6)$ *where* $\langle , \rangle$ *is the inner product on* $\bigwedge^3(\mathbb{R}^6)$. *The critical submanifolds are* $\Phi^{-1}(1)$ *and* $\Phi^{-1}(-1)$ *with indices* $4$ *and* $0$, *respectively.*

Let $\phi$ be a closed 3-form on Euclidean space $\mathbb{R}^6$. We call a calibration on $\mathbb{R}^6$ [7] if $\phi(v_1\wedge v_2\wedge v_3) \leq 1$ for any $v_i \in \mathbb{R}^6$. The set $\{v_1\wedge v_2\wedge v_3 \in G_3^+\mathbb{R}^6 \mid \phi(v_1\wedge v_2\wedge v_3) = 1\}$

is called the face of calibration $\phi$. Thus the face of $\phi$ is a critical submanifold of $\Phi : G_3^+\mathbb{R}^6 \to \mathbb{R}$ defined by $\Phi$

Now, let $J$ be the complex structure on Eucliedean space $R^6$ and $e_1, ..., e_6$ be the orthonormal basis $R^6$ with $Je_{2i-1} = e_{2i}$. We define 3-form

$$\phi = Re[(e_1 + ie_2) \wedge ... \wedge (e_5 + ie_6)].$$

$\phi$ is called a special Lagrangian calibration on $R^6$ with the face of set $SLAG = SU(3)/SO(3)$.

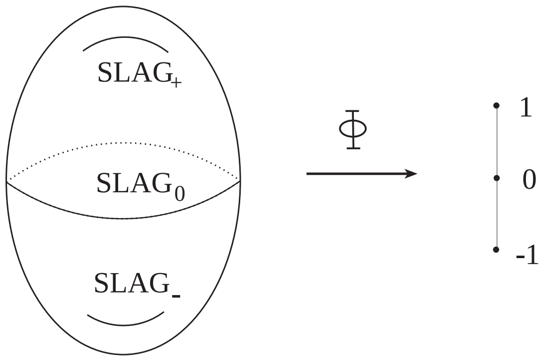


Figure 2.1: The map $\Phi : G_3^+\mathbb{R}^6 \to \mathbb{R}$

### 2.6 An application to the normal bundles

In this section we make an application to embeddings. Let $i : M \longrightarrow \mathbb{R}^6$ be an immersion of a 3-manifold into the Euclidean space. We have the following theorem.

**Theorem 2.6.1.** *Let $M^3$ be a closed, oriented 3-manifold and $i : M \to \mathbb{R}^6$ be an immersion, then the image $g_N(M)$ of the normal bundle under the normal Gauss map $g_N : M \to G_3^+\mathbb{R}^6$ is contractible and the normal bundle of the immersed submanifold is trivial, the immersion is generically special Lagrangian free .*

*Proof.* We are going to use obstruction theory. We shrink the normal Gauss map $g_N : M \to G_3^+\mathbb{R}^6$ skeleton by skeleton. The restriction of $g_N$ to the 0-th and 1-st skeleton of $M$ can be contracted to a point by a homotopy because the image lies in the Grassmannian $G_3^+\mathbb{R}^6$ which is a connected and simply connected space. After this homotopy we obtain a map,

$$g_N : M_{(2)}/M_{(1)} \longrightarrow G_3^+\mathbb{R}^6.$$

Since this map is shrinked over the 1-skeleton, it defines a 2-cochain which is closed by obstruction theory and hence defines a class in the second cohomology of $M$ with $\pi_2$ coefficients,which is computed to be $\mathbb{Z}_2$ in the previous section.

$$\mathfrak{o}_2 \in H^2(M; \{\pi_2\, G_3^+\mathbb{R}^7\}) = H^2(M; \mathbb{Z}_2).$$

The second Stiefel-Whitney class $\omega_2$ of the 3-manifold is equal to this obstruction. Since oriented 3-manifolds are parallelizable, all the characteristic classes vanish, in particular the Stiefel-Whitney classes. The next obstruction is,

$$\mathfrak{o}_3 \in H^3(M; \{\pi_3\, G_3^+\mathbb{R}^7\}) = H^3(M; \mathbb{Z}_2),$$

by the Lemma 2.2.2. Since the 3rd Stiefel-Whitney class is zero this obstruction also vanishes and the normal Gauss map is contractible.

Since the free dimension for the special Lagrangian calibration is 2n-2=4 in this case, a 3-manifold is generically an SL-free submanifold by [15]. □

# CHAPTER 3

# ON SPECIAL LAGRANGIAN FREE SUBMANIFOLDS

In this chapter we prove various results on the topology of special Lagrangian-free (sLag-free) submanifolds in $\mathbb{C}^n$. We begin with the detailed proof of the computation of the free dimension of the special Lagrangian calibration.

**Proposition 3.0.1.** *[3] If $(M, \Omega)$ is a Calabi-Yau manifold of real $2n$-dimension with special Lagrangian calibration $\Omega$, then $fd(\Omega) = 2n - 2$.*

*Proof.* Firstly, we prove that every real hyperplane $P^{2n-1} \subset \mathbb{C}^n$ contains a special Lagrangian $n$-plane and so it is not $sLag$-free. Now, take a unit vector $u \perp P$ and consider the orthogonal decompositon $\mathbb{C}^n = (\mathbb{R}u) \oplus (\mathbb{R}Ju) \oplus P_0$ where $P_0 = P \cap J(P)$.
If $L_0 \subset P_0$ is a Lagrangian subspace of $P_0$, then $L = (RJu) \oplus L_0$ is a Lagrangian subspace of $P$. As claimed, $L$ becomes Special Lagrangian by rotating $L_0$ in $P_0$. Now, $U \subset \mathbb{C}^m$ of $2m - 2$,

$$U \; is \; \Upsilon\text{-free} \iff J(U^{\perp}) \nsubseteq U \iff U \; is \; \text{symplectic} \; i.e. \; \omega^{m-1}|_U \neq 0.$$

For the first equivalence note that if $J(U^{\perp}) \subset U$, then the construction above gives a Special Lagrangian $L \subset U$.
In other direction, given $L \subset U$, $J(L) = L^{\perp} = (L^{\perp} \cap U) \oplus U^{\perp}$ and so $J(U^{\perp}) \subset U$. for the second equivalance, note that $J(U^{\perp}) \subset U$ implies that $J(U^{\perp})$ lies in the null space of $\omega|_U$. Conversely, if $v \in U$ lies in the null space of $\omega|_U$, then $J(span\{v, Jv\}) \subset U^{\perp}$ □

**Definition 3.0.2.** *A closed orientable submanifold $M^k$, $1 \leq k \leq 2n-2$, of a Calabi-Yau manifold $(X^n, Re(\Omega))$ is sLag-free if there are no special Lagrangian n-planes tangential to $M$.*

Our aim is to find the obstructions to embed any closed oriented manifold into the flat Calabi-Yau manifold $\mathbb{C}^n \cong \mathbb{R}^{2n}$ as sLag-free. In order to achieve this, we will use the Gauss map of any embedding and intersection theory.

Now, let $f : M^k \longrightarrow \mathbb{R}^{2n}$ be an embedding of a closed oriented k-dimensional manifold $M^k$ into $\mathbb{R}^{2n}$, where $0 < k \leq 2n-2$. If $k < n$, then this embedding is automatically sLag-free due to the fact that the dimension of a special Lagrangian plane is $n$. Locally, it will also be sLag-free for $k = n$. For $n \leq k$, consider the Gauss map $\mathcal{G}_f : M^k \longrightarrow G_k^+(\mathbb{R}^{2n})$, and let $\mathcal{S} \in G_k(\mathbb{R}^{2n})$ be the subset of k-planes which contains a special Lagrangian n-plane. If $\mathcal{G}_f(M^k) \cap \mathcal{S} = \emptyset$, then this embedding will be sLag-free, too. If the intersection is non-empty, then we will try to find a topological invariant of $M^k$ which will be an obstruction for this embedding to be sLag-free and we plan to do this using this intersection set. Unfortunately, this can easily be done, as far as we know, if the intersection of $\mathcal{G}_f(M^k)$ and $\mathcal{S}$ is a set of points which only occurs if $dim(\mathcal{S}) + dim(\mathcal{G}_f(M^k)) = dim(\mathcal{S}) + k = dim(G_k(\mathbb{R}^{2n})$. Since $\mathcal{G}_f(M^k)$ and $\mathcal{S}$ are closed, generically they will intersect at finitely many points under this condition. By using the weak Whitney embedding theorem and some classical results in differential topology, these intersections can be made transversal. Hence, we can compute the algebraic intersection numbers between them and then try to find conditions on $M^k$ which may make these numbers equal to zero. First, we will find the conditions on $k$ and $n$ when this intersection can generically be at just points. This is clarified after the following result.

**Lemma 3.0.3.** *Let $\mathcal{S} \in G_k(\mathbb{R}^{2n})$ be the subset of k-planes which contain a special Lagrangian n-plane. Then, $dim(\mathcal{S}) + k = dim(G_k(\mathbb{R}^{2n})$ if and only if $(k, n) = (2, 2)$ or $(k, n) = (6, 5)$*

*Proof.* Let

$$\mathcal{S} = \{V^k \in G_k(\mathbb{R}^{2n}) |\ V^k \text{contains a special Lagrangian n-plane}\}$$

It is clear that $k \geq n$. Then, we can define the map $\pi : \mathcal{S} \to SLAG_n$ which maps each k-plane $V^k$ to the special Lagrangian plane contained. If we look at the fibers, i.e. $\pi^{-1}(Q)$ for any special Lagrangian n-plane $Q$ in $\mathbb{C}^n$, we see that any k-plane $V^k \in \pi^{-1}(Q)$ will be of the form $V^k = Q \oplus \Lambda$ where $\Lambda$ is a (k-n) dimensional plane i.e. $\Lambda^{k-n} \in G_{k-n}(\mathbb{R}^n)$.

$$\begin{array}{ccc} \Lambda \cong \mathbb{R}^{k-n} & \longrightarrow & \mathcal{S} \\ & & \downarrow \\ & & SLAG_n \end{array}$$

Hence, this fibration gives us

$$dim(\mathcal{S}) = dim(SLAG_n) + dim(G_{k-n}(\mathbb{R}^n))$$

We have $dim(SLAG_n) = \dfrac{(n^2+n-2)}{2}$ since $SLAG_n \cong SU(n)/SO(n)$. Then, Then, the equation $dim(S) + k = dim(G_k(\mathbb{R}^{2n})$ can be written as

$$\frac{(n^2+n-2)}{2} + (k-n)(2n-k) + k = (2n-k).k$$

which will turn to;

$$-3n^2 + (1+2k)n + 2k - 2 = 0$$

$$n_{1,2} = \frac{-(1+2k) \pm \sqrt{4k^2+28k-23}}{-6}$$

where $n \leq k \leq 2n-2$.

The only integer solutions to this equation are $(k,n) = (1,0),(2,2),(6,5)$. Obviously, $(k,n) = (1,0)$ is not a geometric object.

□

For the case $(k,n) = (6,5)$, we don't have enough tools to compute this intersection number as the dimensions are really big. However, in the low dimensional case, $(k,n) = (2,2)$ we completely classified which closed orientable surfaces can be embedded into $\mathbb{C}^2 \cong \mathbb{R}^4$ as sLag-free.

**Theorem 3.0.4.** *Let $M$ be a closed orientable surface. If $M$ can be embedded into $\mathbb{R}^4 \cong \mathbb{C}^2$ as a sLag-free submanifold, then the Euler Characteristic of $M$, $\chi(M) = 0$.*

We give two proofs of this theorem.

*Proof.* Let $f : M \to \mathbb{R}^4$ be an embedding of an oriented surface $M$ into $\mathbb{R}^4$, and $G_f : M \to G_2^+(\mathbb{R}^4)$ be its corresponding Gauss map. Then we have

$$G_{f*}[M] = \frac{1}{2}\chi(M)[G_2(\mathbb{R}^3)] = -\frac{1}{2}\chi(M)[\mathbb{CP}^1] - \frac{1}{2}\chi(M)[\overline{\mathbb{CP}}^1].$$

$SLAG_2 \cong SU(2)/SO(2)$ and $\widetilde{SLAG}_2$ (the set of special Lagrangian 2-planes with opposite orientation) are 2-dimensional submanifolds of the oriented Grassmannian of 2-planes in $\mathbb{R}^4$, i.e. $G_2^+(\mathbb{R}^4)$ (For their embeddings into $G_2^+(\mathbb{R}^4)$, see [13]). Generically, $G_{f*}[M]$ will intersect with $SLAG_2$ and $\widetilde{SLAG}_2$ at finitely many points. If both of the intersections are empty, then $f$ is a sLag-free embedding of $M$ into $\mathbb{R}^4$. Otherwise, we can count these intersections with sign or compute both of the algebraic intersection numbers between $G_{f*}[M]$ and $SLAG_2$ or $\widetilde{SLAG}_2$. These intersection numbers will give topological conditions on $M$ to make algebraic intersection numbers equal to zero. However, as it can be seen from the equation above, the intersection numbers between the classes $[SLAG_2]$, $[\widetilde{SLAG}_2]$ and $[G_{f*}[M]]$ actually just depends on the Euler Characteristic of $M$, $\chi(M)$. Hence, in order to have a sLag-free embedding of $M$ into $\mathbb{R}^4$, $\chi(M)$ must be equal to zero.

□

Now we give an alternative proof. By using this proof, we can actually see that the converse of the theorem is true, too.

*Proof.* The case $n = 2$ is actually the special case of Special Lagrangian in $\mathbb{C}^n$. Let $\mathbb{C}^2$ have complex coordinates $(z_1, z_2)$, complex structure $J$, and metric $g$, Kähler form $\omega$, and holomorphic 2-form $\Omega$ defined in $\mathbb{C}^2$. Define real coordinates $(x_0, x_1, x_2, x_3)$ on $\mathbb{C}^2 \simeq \mathbb{R}^4$ by $z_0 = x_0 + ix_1$, $z_1 = x_2 + ix_3$. Then,

$$g = dx_0^2 + ... + dx_3^2,\ \omega = dx_0 \wedge dx_1 + dx_2 \wedge dx_3$$

$$Re(\Omega) = dx_0 \wedge dx_2 - dx_1 \wedge dx_3 \text{ and } Im(\Omega) = dx_0 \wedge dx_3 + dx_1 + dx_2.$$

Now, define a different set of complex coordinates $(w_1, w_2)$ on $\mathbb{C}^2 \simeq \mathbb{R}^4$ by $w_1 = x_0 + ix_2$ and $w_2 = x_1 - ix_3$. Then $\Omega - iIm\Omega = dw_1 \wedge dw_2$. But by proposition , a real 2-submanifold $L \subset$ is special Lagrangian if and only if $\Omega|_L = Im\Omega|_L = 0$. Thus, $L$ is special Lagrangian if and oly if $(dw_1 \wedge dw_2)|_L = 0$. But this holds if and olny if

$L$ is holomorphic curve with respect to the complex coordinates $(w_1, w_2)$. There are two different complex structure $J$ and $\bar{J}$ involved in this problem, associated to the two different complex coordinate systems $(z_1, z_2)$ and $(w_1, w_2)$on $\mathbb{R}^4$. the coordinates $(x_0, ..., x_3)$, $J$ and $\bar{J}$ are given by

$$J(\frac{\partial}{\partial x_0}) = \frac{\partial}{\partial x_1},\ J(\frac{\partial}{\partial x_1}) = -\frac{\partial}{\partial x_0},\ J(\frac{\partial}{\partial x_2}) = \frac{\partial}{\partial x_3},\ J(\frac{\partial}{\partial x_3}) = -\frac{\partial}{\partial x_2}$$

$$\bar{J}(\frac{\partial}{\partial x_0}) = \frac{\partial}{\partial x_2},\ \bar{J}(\frac{\partial}{\partial x_1}) = -\frac{\partial}{\partial x_3},\ \bar{J}(\frac{\partial}{\partial x_2}) = -\frac{\partial}{\partial x_0},\ \bar{J}(\frac{\partial}{\partial x_3}) = \frac{\partial}{\partial x_1}$$

The usual complex structure on $\mathbb{C}^2$ is $J$ but a $2-$fold $L$ is $\mathbb{C}^2$ is special Lagrangian if and only if it holomorphic with respect to the alternative complex structure $\bar{J}$. □

As a result, the second proof shows that being sLag-free with respect to the standard complex structure $J$ is equivalent to being totally real with respect to the complex structure $\bar{J}$ for $n = 2$. Since, $h$-principle holds for totally real embeddings (see [16]) and the only obstruction for sLag-free embeddings is $\chi(M)$, we get the following result.

**Corollary 3.0.5.** *A closed orientable surface $M$ can be embedded into $\mathbb{R}^4 \cong \mathbb{C}^2$ as a sLag-free submanifold if and only if the Euler Characteristic of $M$, $\chi(M) = 0$.*

By using the classification of orientable surfaces, we see that only $\mathbb{T}^2 = S^1 \times S^1$ can be embedded into $\mathbb{R}^4 \cong \mathbb{C}^2$ as sLag-free.

# CHAPTER 4

# GEOGRAPHY OF SYMPLECTIC MANIFOLDS WITH SPIN(7)-STRUCTURE

## 4.1 Geography of Symplectic Manifolds Admitting Spin$(7)$ Structure

In their work Arikan, Cho and Salur study 7-dimensional manifolds with both contact and $G_2$ structures. They try to understand $G_2$ manifolds by using the techniques coming from contact geometry. Using the associative calibration they find a contact 1-form on 7-manifolds with $G_2$-structure. There is a similar problem for 8-dimensional case; namely 8-manifolds with both symplectic and $Spin(7)$-structures. In this chapter, we will prove certain results on their existence and try to give some examples of symplectic 8-manifolds with $Spin(7)$-structures. Some of our results depend on the work of Pasquotto where she constructs 8-dimensional symplectic manifolds for every Chern number system which satisfy certain modular equations. Existence of a $Spin(7)$-structure on a spin 8-dimensional manifold also depends if an equation involving Pontryagin classes, which can be expressed in terms of Chern classes in the symplectic case, is satisfied.

The organization of this chapter is as follows. We will start giving the conditions for an 8-manifold to have a $Spin(7)$-structure and then will give some examples of 8-manifolds with $Spin(7)$-structure. Secondly, we will give a short summary of Pasquotte's methods for constructing symplectic 8-manifolds. Finally, we will use Pasquotto's constructions to find a large family of symplectic 8-manifolds which may have a $Spin(7)$-structure.

### 4.1.1 Spin(7)-structure

**Definition 4.1.1** (G-structure)**.** *Let $M$ be a smooth n-manifold and $G \subset GL(n, \mathbb{R}^n)$ be a Lie group. If the structure group of $TM$ can be topologically reduced from $GL(n, \mathbb{R}^n)$ to G, then we say $M$ has a (topological) G-structure.*

Reducing the structure group to $O(n)$ is equivalent to choosing a Riemannian metric on $M$. Furthermore, to reduce it to $SO(n)$, $M$ must be orientable.
Let $M$ be a smooth 8-manifold and let $G = Spin(7) \subset SO(8)$, then we have the following result which gives the necessary and sufficient conditions on $M$ to have a $Spin(7)$-structure.

**Theorem 4.1.2.** *[17] Let $M$ be a differentiable 8-manifold. $M$ carries a $Spin(7)$-structure if and only if $w_1 = w_2 = 0$ and for appropriate choice of orientation on $X$ we have that*

$$p_1(M)^2 - 4p_2(M) + 8\chi(M) = 0.$$

Before the general 8-dimensional manifold, we will construct some simple spaces. In this case, the equation can be defined by combination of Chern numbers as the following.

**Corollary 4.1.3.** *[17] Let $M$ be a complex manifold of dimension $4$. Then $M$ carries a topological $Spin(7)^+$-structure if and only if*

$$c_1[c_1^3 - 4c_1c_2 + 8c_3] = 0$$

*Proof.* The Pontryagin classes of $M$ is even Chern classes of complexified tangent bundle of $M$. Moreoever, Euler class of $M$ is top Chern class of of $M$. We have the following relations between Pontryagin classes and Chern classes of a complex vector bundle $\xi$ ($\xi_{\mathbb{R}}$, underlying real vector bundle)

$$p_k(\xi_{\mathbb{R}}) = (-1)^k \Sigma_{i+j=2k} c_i(\xi)(-1)^j c_j(\xi),$$

$$p_k(\xi_{\mathbb{R}}) = c_k(\xi)^2 - 2c_{k-1}(\xi)c_{k+1}(\xi) + ... \pm 2c_{2k}(\xi)$$

$$p_1 = c_1^2 - 2c_2,\ p_2 = c_2^2 - 2c_1c_3 + 2c_4 \ and \ \chi = c_4.$$

Hence,

$$c_1^4 - 4c_1^2c_2 + 8c_1c_3 = 0$$

□

**Corollary 4.1.4.** *[17] Let $M$ and $N$ be compact spin 4-manifolds. Then the product $X = M \times N$ carries a topological Spin(7)-sturcture if and only if*

$$9\sigma(M)\sigma(N) = 4\chi(M)\chi(N)$$

*In particular, $M \times M$ has a such structure if and only if*

$$3\sigma(M) = \pm 2\chi(M).$$

*where $\sigma$ is the signature of $M^4$ and $\chi$ is the Euler characteristic of $M^4$.*

Proof of this theorem follows from the fact that $c_1^2$ and $c_2$ are topological invariants of a 4-manifold $M^4$ and

$$c_2(M) = \chi, \quad c_1^2(M) = 2c_2(M) + p_1(M) = 2\chi + 3\sigma$$

Let $\mathbb{E}_1 = \mathbb{CP}^2 \# 9\overline{\mathbb{CP}}^2$ be an elliptic surfaces.

**Example 4.1.5.** *Let $\mathbb{E}_n$ denote the simply connected, relatively minimal elliptic surface with topological Euler characteristic $\chi(\mathbb{E}_n) = 12n > 0$ and no multiple fibers. The diffeomorphism type of $E_n$ is unique, the $\sigma(\mathbb{E}_n) = -8n$ and $\mathbb{E}_\ltimes$ may be obtained symplectically by taking the fiber sum of $n$ copies of a rational elliptic surface $\mathbb{E}_1$. In particular, for $n = 2$ $\sigma(\mathbb{E}_{2k}) = -16k$ then $\mathbb{E}_{2k}$ is spin since $\sigma(\mathbb{E}_{2k}) = 0\ (mod 16)$ and $\chi(\mathbb{E}_2) = 24k$ and $3\sigma = 2\chi$ holds. If $\mathbb{E}_{2k}$ is spin then $\mathbb{E}_{2k} \times \mathbb{E}_{2k}$ admits topological $Spin(7)$ structure.*

Next example is general construction on $\mathbb{T}^4$.

**Example 4.1.6.** *Let $0 < a, b \leq 9$ be integers and choose nine distinct points $p_1, ..., p_9$ in $\mathbb{T}^2$. Suppose the $a + b$ tori $\mathbb{T}^2 \times p_i$ $(0 < i \leq a)$ and $p_j \times \mathbb{T}^2$ $(0 < j \leq b)$ in $\mathbb{T}^2 \times \mathbb{T}^2$. $ab$ points $p_j \times p_i$ are intersection points of these tori. Blowing up these points, we obtain $a+b$ disjoint tori in $\mathbb{T}^4 \# ab\overline{\mathbb{CP}}^2$. The first $a$ of these have square $-n$, and the remaining $b$ tori have square $-a$. We reduce all squares to $-9$ by blowing up* *another $a(9-b)+b(9-a)$ times. Then we symplectically sum the final manifold $\mathbb{T}^4\#(9(a+b)-ab)\overline{\mathbb{CP}}^2$ with $a+b$ copies of $\mathbb{CP}^2$ along curves, to obtain a symplectic manifold $S_{a,b}$.*

$$\chi(S_{a,b}) = 12(a+b) - ab$$

$$\sigma(S_{a,b}) = ab - 8(a+b).$$

By Rochlin's Theorem, if $S_{a,b}$ is a spin 4-manifold then $\sigma(S_{a,b}) \equiv 0\ (mod 16)$. Furthermore, $S_{a,b} \times S_{a,b}$ carries $Spin(7)$ structure if and only if $3\sigma(S_{a,b}) = \pm 2\chi(S_{a,b})$. If we solve

$$3(ab - 8(a+b)) = \pm 2(12(a+b) - ab)$$

we get two cases:

- **Case I:** $5ab = 48a + 48b$.
- **Case II:** $ab = 0$ (We just get $\mathbb{T}^4$.)

**Corollary 4.1.7.** *Let $S_{a,b}$ be symplectic manifold. $S_{a,b}\times S_{a,b}$ admits $Spin(7)$ structure if and only if either $ab = 0$ or $5ab = 48a + 48b$.*

**Example 4.1.8** (Building Blocks 5.8)**.** *[18] We will construct a symplectic manifold $Q_2$ that is a torus bundle over-a genus 2 surface and has a symplectic section with square 0. First, we consider the manifold Z described by Thurston in [T]. This manifold is a quotient of $\mathbb{R}^4$ by the action of a discrete group $G$ of symplectomorphisms. The group G is generated by unit translations parallel to the $x_1$-,$x_2$- and $x_3$-axes, together with the map $(x_1, ..., x_4) \to (x_1 + x_2, x_2, x_3, x_4 + 1)$. The standard symplectic form $dx_l \wedge dx_2 + dx_3 \wedge dx_4$ descends to a symplectic form on Z. Projection onto the last two coordinates induces a bundle structure $\pi\ :\ Z \to \mathbb{T}^2$ with torus fibers that are symplectic. We have a section $\phi\ :\ \mathbb{T} \to Z$ given by $\phi(x_3, X_4) = (0, 0, x_3, x_4)$, which is a symplectic embedding, and the image of $\phi$ has a canonical normal framing via the vector field $\frac{\partial}{\partial x_1}$. The manifold Z is parallelizable, by the frame field $(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} + x_4\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4})$. Thus $\sigma(Z) = \chi(Z) = 0$ and Z is spin. Then $Z \times Z$ admits $Spin(7)$-structure.*

**Example 4.1.9** (K3 surface)**.** *[17] Let $V^n(d) \subset \mathbb{CP}^{n+1}$ be the non-singular complex hypersurface with degree $d$. Then,*

$$c_1(V^n(d)) = (n+2-d).g$$

*where $g$ is the canonical generator of $H^2(V^n(d);\mathbb{Z})$. The following holds;*

$$V^n(d) \text{ is spin } \iff n+d \text{ is even.}$$

*Specifically, $V^2(4) = \{(w_0, w_1, w_2, w_3)|w_0^4 + w_1^4 + w_2^4 + w_3^4 = 0\} \subset \mathbb{CP}^3$ is a spin manifold with signature $16$, which is called Kummer (or K3) surface. Thus, $K3 \times K3$ admits $Spin(7)$ structure.*

### 4.1.2 Spin 8-manifolds

Let $M$ be a closed 8-dimensional spin manifold $M$ with signature $\sigma$. According to Thom and Hirzebruch [19] one has the relation

$$\sigma = \frac{1}{45}(7p_2 - p_1^2)[M].$$

Moreoever, using $\hat{A}$-genus (see [17] for details), the following formula is defined for 8-manifold

$$\hat{A}[M] = \frac{1}{5760}(7p_1^2 - 4p_2)[M].$$

Atiyah and Hirzebruch prove that this is integer for a spin manifold. Also, by eliminating $p_2[M]$ between these two linear equations, we obtain the formula

$$\hat{A}[M] = \frac{1}{896}(p_1^2[M] - 4\sigma)$$

where $p_i[M]$'s are Pontryagin numbers. In special case, if a manifold admits an almost complex structure then Pontryagin numbers can be expressed in terms of Chern numbers.

## 4.2 Symplectic Geography

In this section we summarize Pasquotto's results. Details can be found in [20].
The Riemann-Roch theorem gives the relations to be satisfied by a given system of integers to appear as the system of Chern numbers of an almost complex manifold. In dimension 8, the relations are given below:

$$\begin{aligned} -c_4 + c_1c_3 + 3c_2^2 + 4c_2c_1^2 &\equiv 0 \; (mod\, 720) \\ 2c_1^4 + c_1^2c_2 &\equiv 0 \; (mod\, 12) \\ -2c_4 + c_1c_3 &\equiv 0 \; (mod\, 4) \end{aligned}$$

Let a given quintuple of integer numbers $(c_4, c_1c_3, c_2^2, c_1^2c_2, c_1^4)$ satisfy the system of congruence relations given above. Then there exist integers $(a, j, k, m, b)$ such that

$$\begin{aligned} a &= c_4 \\ 720j &= -c_4 + c_1c_3 + 3c_2^2 + 4c_2c_1^2 - c_1^4 \\ 12k &= 2c_1^4 + c_1^2c_2 \\ 4m &= -2c_4 + c_1c_3 \\ b &= c_1^4 \end{aligned}$$

This system is equivalent to the following:

$$\begin{aligned} c_4 &= a \\ c_1c_3 &= 4m + 2a \\ c_1^4 &= b \\ c_1^2c_2 &= 12k - 2b \\ 3c_2^2 &= 720j - a - 4m - 48k + 9b \end{aligned}$$

Pasquotto proves that if $a + m = 0 \; mod\, 3$, then one can construct a symplectic 8-manifold with these Chern numbers. She basically uses certain manifolds as building blocks and applies blow-ups (at a point or along a submanifold) or fiber sum operations until the given modular equations are satisfied. We will give the change of the Chern numbers after blow-ups in the following subsection.

**Chern Numbers of Blown-up at a point**

Let $M$ be an 8-dimensional almost complex symplectic manifold and $\hat{M}$ be a blow up of $X$. These relation is in Pasquotto article [20].

$$c_4[\hat{M}] = c_4[M] + 3$$

$$c_1c_3[\hat{M}] = c_1c_3[M] + 6$$

$$c_2^2[\hat{M}] = c_2^2[M] - 4$$

$$c_1^2c_2[\hat{M}] = c_1^2c_2[M] - 18$$

$$c_1^4[\hat{M}] = c_1^4[X] - 81.$$

where $\hat{M}$ is blow-up of 8-dimensional manifold $X$ at one point. Hence, $(a', b', j', k', m')$ will change as following;

$$a' = a + 3$$

$$4m' = 4m$$

$$720j' = 720j$$

$$12k' = 12k - 180$$

$$b' = b - 81.$$

**Blowing up along a submanifold**

Let $(a', b', j', k', m')$ be the new quintuple after blowing up along symplectic $C$, with genus $g$ and normal bundle $\nu_C$ :

$$a' = a + 4(1 - g)$$

$$4m' = 4m - 4(1 - g)$$

$$720j' = 720j$$

$$12k' = 12k - 144(1 - g) - 36 < c_1(\nu_C), [C] >$$

$$b' = b - 64(1 - g) - 16 < c_1(\nu_C), [C] > .$$

Moreover, $(a', b', j', k', m')$ is following while blowing up along symplectic $4$-dimensional submanifold $M$ with normal bundle $\nu_C$ :

$$a' = a + c_2[M]$$

$$4m' = 4m + c_1^2[M] - 3c_2[M]$$

$$720j' = 720j$$

$$12k' = 12k - 13c_1^2[M] - c_2[M] - 18 < c_1(M)c_1(\nu_M) - 6 < c_1^2(\nu_M), [M] >$$

$$b' = b - 6c_1^2[M] - 8 < c_1(M)c_1(\nu_M), [M] > -3 < c_1^2(\nu_M), [M] > + < c_2(\nu_M), [M] > .$$

### 4.2.1 Symplectic spin manifolds

**Theorem 4.2.1.** *Let $M^8$ be a connected, symplectic manifold which satisfies the quintuple of $(a, j, k, m, b)$ which are related to Chern numbers. If $M$ is $Spin\ manifold$ then*

$$135b - 720k = \ 0\ (mod\ 5760) \tag{4.1}$$

*Proof.* $\hat{A}$-genus was generally defined in the previous section, then the following formula is defined for 8-manifold;

$$\hat{A}[M] = \frac{1}{5760}(7p_1^2 - 4p_2)[M].$$

Atiyah and Hirzebruch prove that this is integer for a spin manifold.

Since, $M$ admits an almost complex structure then $p_i$'s can be written as a combination Chern numbers in the following; $p_1 = c_1^2 - 2c_2$ and $p_2 = c_2^2 - 2c_1c_3 + 2c_4$.

$$7c_1^4 - 28c_1^2c_2 + 24c_2^2 + 8c_1c_3 - 8c_4 \equiv 0\ (mod\ 5760)$$

$$135b - 720k \equiv \ 0\ (mod\ 5760)$$

□

### 4.2.2 Almost complex manifolds with Spin(7)-structure

**Lemma 4.2.2.** *Let $M$ be an almost complex closed spin 8-dimensional manifold. If it has a $Spin(7)$ structure, then*

$$6c_1^4 - 24c_1^2c_2 + 24c_2^2 - 8c_4 \equiv 0(mod\ 5760)$$

*Proof.* The result follows since $M$ is almost complex, $p_1 = c_1^2 - 2c_2$ and $\chi = c_4$ and by the result $6p_1^2[M] \equiv 8\chi[M](mod\, 5760)$ given before. □

Hence, Chern numbers of a symplectic 8-dimensional manifold with $Spin(7)$-structure will satisfy the following modular equations.

$$\begin{aligned} -c_4 + c_1c_3 + 3c_2^2 + 4c_2c_1^2 &\equiv 0 \mod 720 \\ 2c_1^4 + c_1^2c_2 &\equiv 0 \mod 12 \\ -2c_4 + c_1c_3 &\equiv 0 \mod 4 \\ 6c_1^4 - 24c_1^2c_2 + 24c_2^2 - 8c_4 &\equiv 0 \mod 5760 \end{aligned}$$

Then there exist integers $(a, j, k, m, b)$ such that

$$\begin{aligned} a &= c_4 \\ 720j &= -c_4 + c_1c_3 + 3c_2^2 + 4c_2c_1^2 - c_1^4 \\ 12k &= 2c_1^4 + c_1^2c_2 \\ 4m &= -2c_4 + c_1c_3 \\ b &= c_1^4 \end{aligned} .$$

**Theorem 4.2.3.** *Let $M^8$ be a connected, symplectic manifold. Define $(a, j, k, m, b)$-quintuple which are related to Chern numbers. Then if $M$ carries a $Spin(7)$ structure, then*

$$9b - 48k + 32m + 16a = 0 \tag{4.2}$$

*Proof.* In [20] if

$$\begin{aligned} -c_4 + c_1c_3 + 3c_2^2 + 4c_2c_1^2 &= 0\ (mod\, 720) \\ 2c_1^4 + c_1^2c_2 &= 0\ (mod\, 12) \\ -2c_4 + c_1c_3 &= 0\ (mod\, 4) \end{aligned}$$

are satisfied, Pasquotto constructs a symplectic 8-manifold $M$. Furthermore, if $M$ has a $Spin(7)$ structure, then $p_1(M)^2 - 4p_2(M) + 8\chi(M) = 0$ [17]. Since $M$ is symplectic, then one can write Pontryagin classes in terms of Chern classes; $p_1 = c_1^2 - 2c_2$ and $p_2 = c_2^2 - 2c_1c_3 + 2c_4$. Then,

$$
\begin{aligned}
c_1^4 - 4c_1^2 + 8c_1c_3 &= 0 \\
b - 4(12k - 2b) + 8(4m + 2a) &= 0 \\
9b - 48k + 32m + 16a &= 0
\end{aligned}
$$

□

### 4.2.3 Examples

In this subsection we use Pasquotto's results to give examples.

#### 4.2.3.1 Symplectic Sphere Bundles

Let $(B, \omega)$ be a symplectic closed four-dimensional manifold from building blocks which are given above, and $L \to B$ a complex line bundle over $B$. Let $\epsilon$ be a trivial line bundle over $B$. The bundle $\phi : S \to B$ with a fiber $S^2$ over $B$ is obtained by projectifying the complex bundle with rank two $E \oplus \epsilon$. In [21], the top coholology of sphere bundles can be found by Leray-Hirsch theorem. Then one can compute Chern class by quotient subbundle as a complex line bundle.

Let $l$ be the tautological line bundle over $S$, as in the diagram;

$$
\begin{array}{ccc}
l \subset \phi^*(E \oplus \epsilon) & \longrightarrow & E \oplus \epsilon \\
\downarrow & & \downarrow \\
S & \xrightarrow{\phi} & B
\end{array}
$$

consider $c_1(l^*) =: x$, there exists a ring isomorphism , then

$$H^*(S) \xrightarrow{\simeq} H^*(B)/ < x^2 + \pi^* c_1(E)x > .$$

Therefore, Chern classes of $S$ are given by

$$
\begin{aligned}
c_1(S) &= \phi^*(c_1(B) + c_1(E)) + 2x \\
c_2(S) &= \phi^*(c_1(B) \cup c_1(E) + c_1(B)) + 2\phi^* c_1(TB)x \\
c_3(S) &= 2\phi^* c_2(B)x.
\end{aligned}
$$

By these equations of classes, Chern numbers can be calculated as follows:

$$\begin{aligned} c_1^3[S] &= 6c_1^2[B] + 2 < c_1^2(E), [B] > \\ c_1c_2[S] &= 2(c_1^2[B] + c_2[B]) \\ c_3[S] &= 2c_2[B]. \end{aligned}$$

Construct 8-dimensional example as $M = S \times F$, with F a compact Riemann surface of genus $g$. Use product formula, the followings are Chern numbers of $M$;

$$\begin{aligned} c_1^4[M] &= c_1^3[S].\{8.(1-g)\} \\ c_1^2c_2[M] &= (c_1^3[S] + 2c_1c_2[S]).\{2.(1-g)\} \\ c_2^4[M] &= c_1c_2[S].\{4.(1-g)\} \\ c_1c_3[M] &= (c_1c_2[S] + c_3[S]).\{2.(1-g)\} \\ c_4[M] &= c_3[S].\{2.(1-g)\} \end{aligned}$$

**Example 4.2.4.** *Let $C(1) = CP^2\#9\overline{CP^2}$ be an symplectic 4-dimensional elliptic curve which is given in [18]. Then $C(2) = C(1)\#_{C(1)}C(1)$ and inductively $C(n)$ can be constructed.*

$$\chi(C(n)) = 12n \ , \ \sigma(C(n)) = -8n$$

$$c_1^2(C(n)) = 0$$

*since $3\sigma + 2\chi = c_1^2$.*

*Choose $C(n)$ as an admitting trivial line bundle, then, the Chern class of 6-dimensional manifold $S$ on $C(n)$ with the fibers $S^2$ can be calculated as follows;*

$$\begin{aligned} c_1^3[S] &= 0 \\ c_1c_2[S] &= 24n \\ c_3[S] &= 24n \end{aligned}$$

*Secondly, choose $M = S \times S^2$ and use product formula;*

$$\begin{aligned} c_4[M] &= 4.12n \\ c_1c_3[M] &= 4.12n \\ c_2^2[M] &= 8.12n \\ c_1^2c_2[M] &= 4.2.12n \\ c_1^4[M] &= 0 \end{aligned}$$

*Therefore, $M$ admits $Spin(7)$-structure since Chern number of $M$ satisfies the equation 4.2 .*

**Example 4.2.5.** *A is a non-example 4-dimensional building block from [18] with the following Chern numbers ,*

$$c_1^2 = -7 \quad c_2 = 19.$$

*Use same idea and construction for 8-dimensional manifold $A$, but A does not satisfy the equation . Hence, $A$ does not admit $Spin(7)$- structure.*

#### 4.2.3.2 Case $j \neq 0$

Let $M$ be almost complex 8-dimensional manifold which is costructed on symplectic 4-manifold before Pasquotto's article [20]. Then, quintuples $(a, m, k, b, j)$ of $M$ is the following:

$$\begin{aligned} a &= 48n + 12 \\ 4m &= -12 \\ 12k &= -192n - 468 \\ b &= -128n - 208 \end{aligned}$$

The quintuple satisfies the theorem. Now, blow up of $M$ at $x$ points, $y$ copies of $E_-$, z copies of $F_-$, u copies of $X_{n-}$. Denote by $\hat{M}$ the manifold obtains after blow-ups.

If $M$ admits $Spin(7)$ structure, then final manifold $\hat{M}$ satisfies the condition.

$$384n + 96 + 39x + 32y - 32z + (-24n - 18)u = 0$$

There are infinitely many solution, we find some of the solution by MAGMA at appendix A. Some of positive integer solutions $(x, y, z, u, n)$ in the interval $[0, 50]$ is the following quintuple:

$$(2, 1, 1, 15, 4)$$
$$(2, 1, 4, 11, 1)$$
$$(2, 1, 4, 15, 8)$$
$$(2, 1, 7, 15, 12)$$

#### 4.2.3.3 Case $j = 0$

In this case $8$-dimensional manifold cannot be spin manifold since it does not satisfy the equation 4.1.

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# APPENDIX A

## A.1 Some Calculations

Let $(B, \omega)$ be a 4-dimensional symplectic manifold with

$$c_1^2(B) = -7 \quad c_2(B) = 19.$$

then $S$ is symplectic sphere bundle over $B$ with fiber $S^2$ then

$$\begin{aligned} c_1^3[S] &= -42 \\ c_1c_2[S] &= 2.12 \\ c_3[S] &= 2.19 \end{aligned}$$

Choose $M = S \times S^2$ where $S^2$ is a curve with $g = 0$;

$$\begin{aligned} c_4[M] &= 4.19 \\ c_1c_3 &= 4(-7+38) \\ c_2^2 &= 8.(12) \\ c_1^2c_2 &= 4.(-35+38+0) \\ c_1^4 &= 16(-21). \end{aligned}$$

The quantuble $(a, j, k, m, b)$ can be found by Chern numbers. Therefore, the manifold $M$ does not admits spin-7 structure since the quintuble does not satisfy the equation 4.2.

## A.2 Solution Set

In this part, we will give some solutions of the equation. It can be used MAGMA to compute integer solutions of quintuple $[x, y, z, u, n]$. The code of solutions of

$$384n + 96 + 39x + 42y - 32z + (-24n - 18)u = 0$$

is the following;

```
for x,y,z,u,n in [1..15] do
if 384*n+96+39*x+32*y-32*z+(-24*n-18)*u eq 0
then [x,y,z,u,n]; end if;
end for;
```

[ 2, 1, 1, 15, 4 ]
[ 2, 1, 4, 11, 1 ]
[ 2, 1, 4, 15, 8 ]
[ 2, 1, 7, 15, 12 ]
[ 2, 2, 2, 15, 4 ]
[ 2, 2, 5, 11, 1 ]
[ 2, 2, 5, 15, 8 ]
[ 2, 2, 8, 15, 12 ]
[ 2, 3, 3, 15, 4 ]
[ 2, 3, 6, 11, 1 ]
[ 2, 3, 6, 15, 8 ]
[ 2, 3, 9, 15, 12 ]
[ 2, 4, 4, 15, 4 ]
[ 2, 4, 7, 11, 1 ]
[ 2, 4, 7, 15, 8 ]
[ 2, 4, 10, 15, 12 ]
[ 2, 5, 5, 15, 4 ]
[ 2, 5, 8, 11, 1 ]
[ 2, 5, 8, 15, 8 ]
[ 2, 5, 11, 15, 12 ]
[ 2, 6, 6, 15, 4 ]
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## CURRICULUM VITAE

### PERSONAL INFORMATION

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### EDUCATION

| **Degree** | **Institution** | **Year of Graduation** |
|---|---|---|
| M.S. | McMaster University Department of Mathematics | 2012 |
| B.S. | METU Department of Mathematics | 2009 |
| High School | Çubuk Anadolu High School | 2004 |

### PROFESSIONAL EXPERIENCE

| **Year** | **Place** | **Enrollment** |
|---|---|---|
| 2016-... | TUBITAK | Specialist |
| 2015-2016 | THK University | Lecturer |
| 2014-2015 | TUBITAK | PhD. Scholarship Student |
| 2012-2014 | Erkam Academy | General Manager |
| 2010-2012 | McMaster University | Teaching/Research Asistant |
| 2009-2010 | METU | Teaching/Research Asistant |

**PUBLICATIONS**

**National Conference Publications**

31. Ulusal Matematik Sempozyumu, $8$-boyutlu hemen hemen kompleks manifoldların $Spin(7)$ taşımasına göre sınıflandırılması