Stationary Lifshitz black holes of R^2-corrected gravity theory

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Abstract

In this short note, I present a generalization of a set of *static* D-dimensional ($D \geq 3$) Lifshitz black holes, which are solutions of the gravitational model obtained by amending the cosmological Einstein theory with the addition of only the curvature-scalar-squared term and that are described by two parameters, to a more general class of exact, analytic solutions that involves an additional parameter which now renders them *stationary*. In the special $D = 3$ and the dynamical exponent $z = 1$ case, the parameters can be adjusted so that the solution becomes identical to the celebrated BTZ black hole metric.

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In recent years a flurry of activity has been seen on the extensions of the celebrated AdS/CFT correspondence to diverse areas of physics, such as non-relativistic, and more specifically, condensed matter systems (see [1, 2] and the references that cite these e.g. [3, 4] for a sample). Among works in this vein, [5] is worthy of mentioning, since it was there that the Lifshitz spacetimes, which have ever since enjoyed a growing interest as gravity duals of theories with nontrivial scaling properties [(and with fewer symmetries) compared to their cousins which are conformal and dual to AdS spacetimes], were first introduced into the literature. On the other hand, black hole solutions which asymptote to these Lifshitz spacetimes, and thus which are simply referred to as "Lifshitz black holes", are needed when describing finite temperature aspects of these non-relativistic systems. However, there aren't that many exact analytic Lifshitz black holes around, and among those that have been found so far (see e.g. [6–11] and the references therein), there are only *static* ones and, perhaps surprisingly, there are no known *stationary* ones at all. This is the main motivation of the present work. Here, I find a class of stationary D-dimensional exact analytic spacetimes, which are solutions of the R^2-corrected gravity theory, and discuss the conditions which allow these to be interpreted as Lifshitz black holes.

The action of the gravity theory I consider is

$$I = \int d^D x \sqrt{-g} \left(R + 2\Lambda + \alpha R^2 \right) , \tag{1}$$

where Λ is the cosmological constant, and the field equations that follow from the variation of (1) are

$$R_{ab} - \frac{1}{2} g_{ab} R - \Lambda g_{ab} + \alpha \Big(2 R R_{ab} - 2 \nabla_a \nabla_b R + g_{ab} (2 \Box R - \frac{1}{2} R^2) \Big) = 0 \, , \tag{2}$$

with $\Box \equiv \nabla_c \nabla^c$.

Aside from these, I also demand the modified scaling transformations

$$t \mapsto \lambda^z t \, , \quad \vec{x} \mapsto \lambda \vec{x} \, , \quad r \mapsto r/\lambda \, ,$$

where $\vec{x}$ denotes a $(D-2)$-dimensional vector, that are respected by the *static* Lifshitz spacetimes to hold as usual. I also want to keep the invariance under time and space translations, and spatial rotations intact; however instead of separately asking for spatial parity (P) and time reversal (T) invariance, I go for the weaker PT invariance.[1] These assumptions lead to the following *stationary* spacetimes

$$ds^2 = -\frac{r^{2z}}{\ell^{2z}} \, dt^2 + 2\omega \, \frac{r^{z+1}}{\ell^{z+1}} \, dt \, d\phi + \frac{r^2}{\ell^2} \, d\phi^2 + \frac{\ell^2}{r^2} \, dr^2 + \frac{r^2}{\ell^2} \, d\vec{x}^2 \, , \tag{3}$$

[1] To be precise, I also do not include the transformation $r \mapsto -r$ to the allowed P transformations, or the translation $r \mapsto r - r_0$ to the spatial translations, in accordance with the nomenclature on the Lifshitz spacetimes in the literature. Thus, e.g. I think of PT as PT: $(t, r, \phi, \vec{x}) \mapsto (-t, r, -\phi, -\vec{x})$ throughout.

where $0 \leq r < \infty$ and $d\vec{x}^2 \equiv \sum_{i=1}^{D-3} dx_i^2$, which obviously reduce to the usual static Lifshitz spacetimes when the 'rotation parameter' ω is set to zero. For later convenience, I have chosen $x_{D-2} \equiv \phi$ here and separated it from the remaining x_i ($1 \leq i \leq D-3$). Clearly $\ell > 0$ sets the length scale in this geometry, and the parameter ω, just like the 'dynamical exponent' z, is dimensionless.

One finds that (3) is a solution to (2) for generic values of the parameters z and ω in any $D \geq 3$, provided that the coupling constant α and the cosmological constant Λ are chosen as

$$\alpha = \frac{1}{8\Lambda}\,, \qquad \Lambda = \frac{2D^2 + 3(z-1)^2 + 2D(2z-3)}{8\ell^2} + \frac{(z-1)^2}{8\ell^2(1+\omega^2)}\,. \tag{4}$$

However, there is even more to the story: Provided that the coupling constants in the action (1) are chosen precisely as in (4), the following metric

$$ds^2 = -\frac{r^{2z}}{\ell^{2z}}\, h(r)\, dt^2 + \frac{r^2}{\ell^2}\Big(d\phi + \omega\, \frac{\ell^2}{r^2}\, dt\Big)^2 + \frac{\ell^2}{r^2}\, \frac{dr^2}{h(r)} + \frac{r^2}{\ell^2}\, d\vec{x}^2\,, \text{ where} \tag{5}$$

$$h(r) \equiv c + k\, \frac{\ell^{2(1+z)}}{r^{2(1+z)}} + M^-\, \frac{\ell^{p_-}}{r^{p_-}} + M^+\, \frac{\ell^{p_+}}{r^{p_+}}\,, \text{ with} \tag{6}$$

$$c \equiv \frac{4\,\ell^2\,\Lambda}{2z^2 + (D-2)(2z+D-1)}\,, \quad k \equiv \frac{2\,\omega^2}{D^2 - 7D + 14 - 2z(D-3)}\,, \text{ and} \tag{7}$$

$$p_\pm \equiv \frac{1}{2}\Big(3z + 2(D-2) \pm \sqrt{z^2 + 4(D-2)(z-1)}\Big)\,, \tag{8}$$

turns out to be a solution of the field equations (2) for any dimension $D \geq 3$. Note that the coefficients c and k are completely determined by z and ω, whereas the integration constants $M^\pm$ are left as free parameters.

Before proceeding any further, it must be stated that the static version of the metrics (3) and (5) [i.e. those with vanishing ω for which $c = 1$, $k = 0$, the relation (4) for the constants in the action (1) and the metric function $h(r)$ in (6) are simplified accordingly] were first presented in Sec. 2 of [8]. However, with the turning on of the parameter ω, the stationary metrics (3) and (5) (with the accompanying equations (4) and (6)-(8), respectively) are clearly more general than the solutions given in [8].

To note a rather appealing feature of these solutions and to give an interesting example, let me quickly consider the conformal limit $z = 1$ and, for simplicity, set $D = 3$. One finds that the metric (5) becomes identical to the BTZ metric [12] when one sets $M^+ = 0, M^- = -M < 0$ and $\omega = -j/2$:

$$ds^2 = \Big(M - \frac{r^2}{\ell^2}\Big)\, dt^2 - j\, dt\, d\phi + \frac{r^2}{\ell^2}\, d\phi^2 + \frac{dr^2}{-M + \frac{r^2}{\ell^2} + \frac{j^2\ell^2}{4r^2}}\,,$$

which can be further brought to the canonical form when one takes $\phi = \theta\ell$ first and later sets $J = j\ell$.

Let me also note that the curvature scalars of the metrics (3) and (5) are both given by $R = -4\Lambda$ precisely. As discussed earlier in [8] and [13], this allows for the casting of the action (1) into the form

$$I = \frac{1}{8\Lambda}\int d^D x\,\sqrt{-g}\,(R + 4\Lambda)^2\,,$$

and this theory cannot be mapped into a scalar-tensor theory by a conformal transformation of the metric. Thus one is really dealing with an authentic gravity theory here [8].

To have $h(r)$ real, it follows from the expression of $p_\pm$ that $z^2 + 4(D-2)(z-1) > 0$, which further implies that[2]

$$z < z_- \equiv 4 - 2D - 2\sqrt{(D-1)(D-2)} < 0 \quad \text{or} \quad z > z_+ \equiv 4 - 2D + 2\sqrt{(D-1)(D-2)} > 0\,.$$

However, if one is also to demand that the metric (5) describes a black hole solution, then one can consider the branch $z > z_+ > 0$ since then one can show, for the powers of r in the metric function $h(r)$, that $p_+ \geq p_- \geq z_+ > 0$ and $2(z+1) > 0$. Hence for $z > z_+$, the solution (5) represents a black hole with an event horizon located at $r = r_+$, that is found by the largest positive real root of $h(r)$ i.e. $h(r_+) = 0$, provided the remaining parameters ω (thus c and k) and $M^\pm$ also play along appropriately.

Obviously, a general analysis with generic values of z and ω given a specific D is a complicated problem. However, it should be possible in principle to "exorcise" other exotic solutions such as extremal black holes given the generic solution (5) by "playing around" with the parameters $M^\pm$ and/or z and ω. An example in this vein is obtained when one considers the special case where the critical value for the dynamical exponent z is reached and one sets $z = z_+$ exactly. The solution now reads

$$ds^2 = -\frac{r^{2z_+}}{\ell^{2z_+}}\,f(r)\,dt^2 + \frac{r^2}{\ell^2}\Big(d\phi + \omega\,\frac{\ell^2}{r^2}\,dt\Big)^2 + \frac{\ell^2}{r^2}\,\frac{dr^2}{f(r)} + \frac{r^2}{\ell^2}\,d\vec{x}^2\,, \text{ where} \tag{9}$$

$$f(r) \equiv c_+ + k_+\,\frac{\ell^{2(1+z_+)}}{r^{2(1+z_+)}} + \frac{\ell^p}{r^p}\Big(M_1 + M_2\ln\left(r/\ell\right)\Big)\,, \text{ with} \tag{10}$$

$$s \equiv c_+\Big|_{z=z_+}\,,\quad k_+ \equiv k\Big|_{z=z_+}\,,\quad \text{and}\quad p \equiv p_+\Big|_{z=z_+}\,, \tag{11}$$

[2] In the discussion that follows, the points $z = z_-$ and $z = z_+$ are excluded from the allowed range of the dynamical exponent z for a good reason: At these critical values, the metric function $h(r)$ given in (6) is no longer valid and needs to be modified. This is discussed in detail for the $z = z_+$ case in the paragraph containing Eqs. (9)-(11) below.

and M_1 and M_2 are the new free parameters. Note that this is very similar in form to the previous one (5)-(8), but now the metric function $f(r)$ asymptotes to a constant in a slower fashion when one approaches the boundary at $r \to \infty$.

In the discussion above, I had to set $z < z_- < 0$ or $z > z_+ > 0$ so that the metric function $h(r)$ in (6) is real in the first place. One cannot help but wonder whether this really means that the region $z \in (z_-, z_+)$ is completely 'forbidden' for the dynamical exponent z? Working out the field equations for the special value $z = 0$ indicates that this is certainly not the case! For $z = 0$, the solution is

$$ds^2 = -g(r)\, dt^2 + \frac{r^2}{\ell^2}\Big(d\phi + \omega\, \frac{\ell^2}{r^2}\, dt\Big)^2 + \frac{\ell^2}{r^2}\, \frac{dr^2}{g(r)} + \frac{r^2}{\ell^2}\, d\vec{x}^2\,, \text{ where} \tag{12}$$

$$g(r) \equiv c_0 + k_0\, \frac{\ell^2}{r^2} + \frac{\ell^{D-2}}{r^{D-2}}\Big(M_c \cos\big(\sqrt{D-2}\,\ln(r/\ell)\big) + M_s \sin\big(\sqrt{D-2}\,\ln(r/\ell)\big)\Big)\,, \text{ with} \tag{13}$$

$$c_0 \equiv c\Big|_{z=0} \text{ and } k_0 \equiv k\Big|_{z=0}\,, \tag{14}$$

and M_c and M_s are the two integration constants.

It is clearly of paramount importance to compute the conserved charges and to study the thermodynamical properties of the solutions presented here to better understand their geometrical and physical aspects. This should also be relevant for their dual condensed matter systems. In [13], the solutions (5)-(8) with vanishing ω were studied using the background Killing charge definition developed in [14], and they were shown to have both their energy $E = 0$ and their entropy $S = 0$. One would expect this result to remain intact, i.e. that both $E = 0$ and $S = 0$, which, taken together in the first law of black hole thermodynamics, also implies that the angular momentum J vanishes as well: $J = 0$. This should make all the more sense when one considers the remark on the vanishing of the action I that follows from $R = -4\Lambda$ at the first place.

As a concrete and illustrative example along these lines, let me now concentrate on the $D = 4$, $z = 3/2$ case. This particular choice is sort of motivated by historical reasons: The corresponding solution (5)-(8) together with $M^+ = 0$ and $\omega = 0$ was the first ever member of the black hole solutions generalized here and was first given in [15]. In this case, the cosmological constant Λ and the metric function $h(r)$ in (6) read

$$\Lambda = \frac{132 + 131\omega^2}{32\ell^2(1+\omega^2)} \text{ and } h(r) = \frac{132 + 131\omega^2}{132(1+\omega^2)} - 2\omega^2\, \frac{\ell^5}{r^5} + M^-\, \frac{\ell^3}{r^3} + M^+\, \frac{\ell^{11/2}}{r^{11/2}}\,,$$

respectively. Now the background Killing vector that yields the energy is $\bar{\xi}^a = -(\partial/\partial t)^a$, and the one that gives the angular momentum is $\bar{\varsigma}^a = (\partial/\partial\phi)^a$, in the notation of [13]. However, there seems to be a nontrivial ambiguity in the choice of the relevant background. If one naively, but somehow more intuitively, chooses the background to be the usual *static* Lifshitz spacetime with $z = 3/2$

$$ds^2 = -\frac{r^3}{\ell^3}\, dt^2 + \frac{r^2}{\ell^2}\, d\phi^2 + \frac{\ell^2}{r^2}\, dr^2 + \frac{r^2}{\ell^2}\, dx^2\,, \tag{15}$$

which was the background that was originally employed in [13] anyway, one surprisingly and unexpectedly finds the divergent, unphysical results $E \to \infty$ and $J \to \infty$. The background that leads to the aforementioned predictions, i.e. $E = 0$ and $J = 0$ result, is obtained when one sets $M^{\pm} \to 0$ but leaves ω on, i.e. $\omega \neq 0$, in $h(r)$ above and uses this 'undressed' $h(r)$ in (5) with $z = 3/2$.

Obviously, the former (15) and the latter backgrounds are quite different from each other both locally and structurally. This difference may partially account for the discrepancy between the two energy and angular momentum computations, but I believe this is not the proper place for a discussion on this point, and that the issue itself is beyond the scope of the present work. Obviously, a more detailed and robust calculation of the conserved charges using other gravitational charge definitions would be of great value in studying the thermodynamics of the solutions presented here. Other open problems worthy of further study involve an analysis regarding the stability of these solutions and a detailed discussion, similar to the one in [5], on the implications of these geometries on the condensed matter systems that they are related to by holographic techniques.

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