

Tame concealed algebras and cluster quivers of minimal infinite type

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Received 17 February 2006; received in revised form 30 October 2006

Available online 28 December 2006

Communicated by M. Broué

Abstract

The well-known list of Happel–Vossieck of tame concealed algebras in terms of quivers with relations, and the list of A. Seven of minimal infinite cluster quivers are compared. There is a 1-1 correspondence between the items in these lists, and we explain how an item in one list naturally corresponds to an item in the other list. A central tool for understanding this correspondence is the theory of cluster-tilted algebras.

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MSC: 16G60; 05E99; 16G20

0. Introduction

In the representation theory of finite dimensional algebras there is a famous list of algebras by Happel and Vossieck [16] of the tame concealed algebras in terms of quivers with relations. These are precisely, together with the so-called generalized Kronecker algebras, the algebras A of minimal infinite type having a preprojective component (see Section 1 for definitions). These algebras are useful for testing whether finite dimensional algebras are of finite representation type.

Seven recently produced another list [18] in connection with his work on cluster algebras. Cluster algebras were defined and first studied by Fomin and Zelevinsky [10]. We consider the special case (the case of “no coefficients” and skew-symmetric matrices) where these algebras are determined by a finite quiver Q with no loops and no oriented cycles of length two. We call such a quiver Q a *cluster quiver*. Here we work over an algebraically closed field k .

A central concept in the theory of cluster algebras is the mutation of quivers, and the quivers in the same mutation class determine isomorphic cluster algebras. The Dynkin quivers A_n , D_n , E_6 , E_7 , E_8 and their mutation classes correspond to cluster algebras of finite type [11]. The list produced in [18] gives the underlying graphs of quivers with at least three vertices with the following properties.

- The quivers in the list are not mutation equivalent to a Dynkin quiver
- Whenever a vertex is removed from a quiver on the list, the resulting quiver is mutation equivalent to a Dynkin quiver.

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The first condition says that the cluster algebras defined by these quivers are not of finite type, while the second condition says that a cluster algebra defined by any quiver obtained by removing one vertex is of finite type. The quivers from A. Seven's list in addition to the generalized Kronecker quivers (which have two vertices) are the only quivers satisfying these conditions. Such quivers are here called *minimal infinite* cluster quivers. Here we only consider simply-laced cluster quivers, where we call a (not necessarily Dynkin-) quiver simply-laced if, when considered as a valued quiver, it only has valuations of the form (t, t) .

As opposed to the quivers in A. Seven's list, the quivers of Happel–Vossieck are directed, i.e. they do not contain oriented cycles. However, the quivers of Happel–Vossieck contain also some dotted edges, which correspond to relations. If one replaces the dotted edges in their list with solid ones with direction such that oriented cycles are created, one obtains exactly the simply-laced quivers on Seven's list [18].

The aim of this paper is to explain why these lists are so closely related. For this we use the theory of cluster categories and cluster-tilted algebras from [7,4,5]. This theory was motivated by trying to model the essential ingredients in the definition of a cluster algebra in module theoretical/categorical terms. In particular we explain why and how we can construct one list from the other one. A key point is that by using [5] we see that the quivers of minimal infinite cluster algebras coincide with the quivers of the cluster-tilted algebras of infinite type where all factor algebras obtained by factoring out an ideal generated by a vertex are of finite type. Then we compare the conditions of minimal infinite type for tilted and cluster-tilted algebras, and give a procedure for passing from the first class to the second one, as was done in [8] for tilted algebras of Dynkin type.

For going in the opposite direction we need to remove arrows from the quivers in the list of A. Seven. Actually the list of A. Seven is a list of graphs, and the associated quivers are those where every full cycle is an oriented cycle, except in the case of \widetilde{A}_n , where we must have a non-oriented cycle.

Here we use quadratic forms of quivers with solid and dotted arrows to describe which set of solid arrows should be made dotted, i.e. removed from the quiver of a minimal infinite cluster-tilted algebra in order to recover the quiver for the corresponding tame concealed algebra. We also provide an algorithm for actually finding our desired set of arrows.

The results of the first four sections were announced at the Oberwolfach meeting “Representation Theory of Finite-dimensional Algebras” in February 2005, where we also thank Thomas Brüstle for interesting conversations.

1. General background

In this section we give some relevant background material from the representation theory of finite dimensional algebras, from the theory of cluster algebras, and from the theory of cluster categories and cluster-tilted algebras.

1.1. Finite dimensional algebras

Let Λ be a connected finite dimensional algebra over an algebraically closed field k . Then the category $\text{mod } \Lambda$ of finite dimensional (left) Λ -modules has almost split sequences, and there is an associated translation τ . For some finite dimensional algebras the AR-quiver can have special components called *preprojective* components. These components are defined by the following property: the τ -orbit of each indecomposable Λ -module in the component contains an indecomposable projective, and the component has no oriented cycles [13]. We later use that only a finite number of indecomposable Λ -modules, up to isomorphism, have nonzero maps to a given Λ -module in a preprojective component [13].

Not all algebras have preprojective components. But the hereditary algebras $H = kQ$, where Q is a finite quiver without oriented cycles, have such components. Also the tilted algebras have preprojective components [19]. Recall that an algebra Λ is tilted if $\Lambda = \text{End}_H(T)^{\text{op}}$ where T is a tilting module over a hereditary algebra H , that is, $\text{Ext}_H^1(T, T) = 0$ and there is an exact sequence $0 \rightarrow H \rightarrow T_0 \rightarrow T_1 \rightarrow 0$ with T_0 and T_1 direct summands of finite direct sums of copies of T . The quiver of a tilted algebra is known to have no oriented cycles. A central class of tilted algebras are the *tame concealed algebras*, that is, the algebras of the form $\text{End}_H(T)^{\text{op}}$, where $H = kQ$ for an extended Dynkin quiver Q , and T is a *preprojective tilting module*, that is, T lies in the (unique) preprojective component of H .

The quiver with two vertices a, b and with $r \geq 2$ arrows from a to b is often called a *generalized Kronecker quiver*, and we call a path algebra of such a quiver a *generalized Kronecker algebra*.

1.2. Cluster algebras

We here indicate the definition of a cluster algebra. The general definition from [10] involves “coefficients” and skew-symmetrizable integer matrices. We restrict to the case with no coefficients and skew-symmetric matrices.

Start with a *seed* (\underline{x}, Q) , where $\underline{x} = \{x_1, \dots, x_n\}$ is a transcendence basis for the rational function field $\mathbb{Q}(x_1, \dots, x_n)$ and Q is a finite quiver with no loops or cycles of length two and with vertex set $\{1, 2, \dots, n\}$. For each $i = 1, \dots, n$ one can define new quiver $\mu_i(Q) = Q'$ and a new seed $\mu_i((\underline{x}, Q)) = (\underline{x}', Q')$, where $\underline{x}' = \{x_1, \dots, x'_i, \dots, x_n\}$ is a transcendence basis where the element x_i in \underline{x} is replaced by an element x'_i in $\mathbb{Q}(x_1, \dots, x_n)$, depending on \underline{x} and Q . The new quiver Q' (or seed (\underline{x}', Q')) is called the *mutation* of the quiver Q (or seed (\underline{x}, Q)) in direction i . Continuing this way with mutation of quivers and seeds, the *cluster algebra* $\mathcal{A}(Q)$ is defined to be the subalgebra of $\mathbb{Q}(x_1, \dots, x_n)$ generated by all elements appearing in the n -element subsets $\underline{x}, \underline{x}', \dots$. These subsets are called *clusters*. The quivers obtained by a sequence of mutations of the quiver Q are said to be mutation equivalent to Q .

A major result in [11] is that there is only a finite number of seeds if and only if the quivers occurring in seeds are mutation equivalent to a Dynkin quiver.

1.3. Cluster categories and cluster-tilted algebras

Let $H = kQ$ again be a finite dimensional hereditary k -algebra, and let $\mathcal{D}^b(H)$ denote the bounded derived category of finitely generated H -modules. Denote by $\tau: \mathcal{D}^b(H) \rightarrow \mathcal{D}^b(H)$ the equivalence such that for C an indecomposable object in $\mathcal{D}^b(H)$ we have an almost split triangle $\tau C \rightarrow B \rightarrow C \rightarrow$, see [12]. We then also have an equivalence $F = \tau^{-1}[1]$, where $[1]$ denotes the shift functor in $\mathcal{D}^b(H)$. The cluster category \mathcal{C}_H , introduced and investigated in [7], is by definition the factor category $\mathcal{D}^b(H)/F$, which is a triangulated category [15]. See also [9] for an alternative approach to the cluster category of type A_n .

A central concept is the notion of a (cluster-)tilting object T in \mathcal{C}_H , where $\text{Ext}_{\mathcal{C}_H}^1(T, T) = 0$ and T is maximal with this property, that is, if $\text{Ext}_{\mathcal{C}_H}^1(T \oplus X, T \oplus X) = 0$, then X is a direct summand of a finite direct sum of copies of T . It is shown that any tilting H -module induces a tilting object in \mathcal{C}_H , and by possibly changing H up to derived equivalence, all tilting objects in \mathcal{C}_H are obtained this way.

A study of the closely related cluster-tilted algebras $\Gamma = \text{End}_{\mathcal{C}_H}(T)^{\text{op}}$, where T is a tilting object in \mathcal{C}_H , was initiated in [4]. A useful result is that if Γ is cluster-tilted and e is a vertex in the quiver, then also $\Gamma/\Gamma e \Gamma$ is cluster-tilted [5]. As a consequence we have the technique of shortening of paths from [6], namely if $a \rightarrow b \rightarrow c$ is a path in a quiver with non-zero composition and no other arrows leaving or entering b , then we can replace this path by $a \rightarrow c$, and the new quiver is still the quiver of a cluster-tilted algebra. Assume the quiver Q has only single arrows. We say that a (not necessarily oriented) cycle is *full* if the subquiver generated by the cycle contains no further arrows. For an arrow $\alpha: j \rightarrow i$ in the quiver of a cluster-tilted algebra we say that a path from i to j is a *shortest path* if it does not go through any oriented cycle, and together with α gives a full oriented cycle. A shortest path from i to j is necessarily involved in a relation from i to j . For finite representation type we have that any full cycle is oriented, and that the homomorphism space between two vertices (more precisely between the corresponding projective modules) is at most one-dimensional. For finite type there are at most two shortest paths from i to j corresponding to α . One shortest path gives rise to a zero-relation and two shortest paths give rise to a commutativity relation [6].

2. Preliminaries

In this section we discuss the two classification theorems which we compare in this paper.

We start with discussing the work of Happel and Vossieck [16]. A basic connected finite dimensional algebra Λ is said to be of minimal infinite type if it is of infinite type and for each vertex e in the quiver of Λ we have that $\Lambda/\Lambda e \Lambda$ is of finite type. There is no general classification theorem of algebras of finite type in the sense that we have a list of all of them. There is, however, the following useful criterion for when an algebra is of minimal infinite type.

Theorem 2.1 ([16]). *An algebra is of minimal infinite type and has a preprojective component if and only if it is a tame concealed algebra or it is a generalized Kronecker algebra.*

Actually a list of all basic connected tame concealed algebras in terms of quivers with relations is provided in [16]. The above result has been even more useful because of this list.

Since we have seen that the tilted algebras always have a preprojective component, a special case of [Theorem 2.1](#) can be formulated as follows:

Corollary 2.2. *The minimal infinite type tilted algebras are the tame concealed algebras and the generalized Kronecker algebras.*

As we have seen, cluster algebras (with “no coefficients”) are determined by finite quivers without loops or cycles of length two, i.e. cluster quivers. The quivers associated with cluster algebras of finite type are the mutation (equivalence) classes of the Dynkin quivers. There is also here no known list of the class of quivers obtained this way. But Seven [18] has given a list of minimal infinite cluster quivers. These are quivers with the property that if any vertex is removed, then we get into the mutation class of a Dynkin quiver. Actually his list is a list of graphs, where the quivers associated with these graphs are obtained by choosing the directions of edges such that each cycle becomes oriented. There is however one exception, if the underlying graph is \widetilde{A}_n , then the possible orientations are the ones which do not give an oriented cycle.

Note that it is a consequence of A. Seven’s result that all minimal infinite cluster quivers are mutation equivalent to a quiver without oriented cycles.

It was observed in [18] that the lists are very closely related, and they give in fact the same quivers if the dotted edges in the Happel–Vossieck list are replaced by solid arrows with direction such that oriented cycles are created. In the remaining part of this paper we will give an explanation for why this is the case, using the theory of cluster-tilted algebras. We will also give a procedure for how to construct one list from the other one, and explain why it works.

3. Interplay

In this section we discuss the theory behind the relationship between the two lists. For this it is useful to note the following interpretation, using [4,5], of the quivers appearing in A. Seven’s list.

Theorem 3.1. *Let Q be a cluster quiver. Then Q is minimal infinite if and only if Q is the quiver of a basic cluster-tilted algebra Λ of infinite type with the property that $\Lambda/\Lambda e\Lambda$ is of finite type for each vertex e in the quiver.*

Proof. In [5] it was shown that for a finite connected quiver Q without oriented cycles, the quivers in the mutation class of Q coincide with the quivers of the cluster-tilted algebras coming from the cluster category \mathcal{C}_H for $H = kQ$. We also know that a connected cluster-tilted algebra is of finite type if and only if the corresponding algebra $H = kQ$ is of finite type [4], that is, if and only if Q is a Dynkin quiver. Recall also from Section 1.3 that for any cluster-tilted algebra Γ with a vertex e in the quiver of Γ we have that $\Gamma/\Gamma e\Gamma$ is again a cluster-tilted algebra.

Assume Q is a minimal infinite cluster quiver. Then it follows that a cluster-tilted algebra Γ with quiver Q is of infinite type and whenever a vertex is removed from Q the reduced quiver Q' is the quiver of a cluster-tilted algebra of finite type. Hence Q' is in the mutation class of a disjoint union of Dynkin quivers. It is clear that also the converse holds. This finishes the proof. \square

While tilted and cluster-tilted algebras are quite different with respect to homological properties, they are closely connected through the fact that they are constructed from a common tilting module. [Theorem 3.1](#) shows that the list of A. Seven gives a list of underlying graphs of quivers of cluster-tilted algebras with properties similar to properties of the tilted algebras appearing in the Happel–Vossieck list. This motivates the following:

Theorem 3.2. *Let T be a tilting module over a finite dimensional hereditary k -algebra $H = kQ$, and let Λ be the tilted algebra $\text{End}_H(T)^{\text{op}}$ and Γ the cluster-tilted algebra $\text{End}_{\mathcal{C}_H}(T)^{\text{op}}$. Assume that Λ is of infinite type. Then Λ is of minimal infinite type if and only if Γ is of minimal infinite type.*

Proof. This is obvious if Λ has at most two simple modules. Assume therefore H has at least 3 simples and that the tilted algebra Λ is of minimal infinite type, so that Λ is tame concealed. Since Λ is a factor algebra of Γ , it follows

that Γ is also of infinite type. Consider the diagram

$$\begin{array}{ccc}
 \text{mod } H & \xrightarrow{\text{Hom}_H(T, \cdot)} & \text{mod } \Lambda \\
 \downarrow & & \downarrow i \\
 \mathcal{C}_H & \xrightarrow{\text{Hom}_{\mathcal{C}_H}(T, \cdot)} & \text{mod } \Gamma
 \end{array}$$

where i is the natural inclusion functor obtained from Λ being a factor algebra of Γ . Since Λ is tame concealed, we can assume that the tilting module T is preprojective. Then all but a finite number of indecomposable H -modules are in the subcategory $\text{Fac } T$ of $\text{mod } \Lambda$, whose objects are the factors of finite direct sums of copies of T . Denote by $(\text{Fac } T)_0$ the subcategory of $\text{Fac } T$ where the H -modules X in $(\text{Fac } T)_0$ have the property that $\text{Ext}^1(T, \tau^{-1}X) \simeq D\text{Hom}(X, \tau^2T) = 0$. Also all but a finite number of indecomposable H -modules are in $(\text{Fac } T)_0$, and the same holds when considering $(\text{Fac } T)_0$ as a subcategory of \mathcal{C}_H . Since $\text{Hom}_{D^b(H)}(T, \tau^{-1}X[1]) = 0$ for X in $(\text{Fac } T)_0$, it follows that for an H -module X in $(\text{Fac } T)_0$, we have $\text{Hom}_{\mathcal{C}_H}(T, X) = \text{Hom}_H(T, X) \sqcup \text{Hom}_{D^b(H)}(T, \tau^{-1}X[1]) = \text{Hom}_H(T, X)$, so $\text{Hom}_{\mathcal{C}_H}(T, \cdot)|_{(\text{Fac } T)_0}$ has its image in $\text{mod } \Lambda$, and is the same subcategory as the image of $\text{Hom}_H(T, \cdot)|_{(\text{Fac } T)_0}$. Since only a finite number of indecomposable Γ -modules are not in this image, it follows that only a finite number of indecomposable Γ -modules are not Λ -modules. In particular, for each vertex e in the quiver of Γ , there is only a finite number of indecomposable $\Gamma/\Gamma e\Gamma$ -modules which are not $\Lambda/\Lambda e\Lambda$ -modules. Since $\Lambda/\Lambda e\Lambda$ is of finite type, it follows that $\Gamma/\Gamma e\Gamma$ is of finite type.

Assume now that the cluster-tilted algebra Γ is of minimal infinite type. The corresponding tilted algebra Λ is of infinite type by assumption, and $\Lambda/\Lambda e\Lambda$ is of finite type since it is a factor algebra of the finite type algebra $\Gamma/\Gamma e\Gamma$. \square

Remark. The assumption that the tilted algebra Λ is of infinite type can not be dropped. For if a tilting module T over a tame hereditary algebra H has both a nonzero preprojective and a nonzero preinjective direct summand, then $\Lambda = \text{End}_H(T)^{\text{op}}$ is of finite type [14], while the cluster-tilted algebra $\Gamma = \text{End}_{\mathcal{C}_H}(T)^{\text{op}}$ is of infinite type since H is of infinite type.

4. From tilted to cluster-tilted algebras

In this section we use the previous results to show how and why the list of A. Seven can be obtained from the Happel–Vossieck list.

We start with giving a procedure for passing from the quiver with relations for a tame concealed algebra $\Lambda = \text{End}_H(T)^{\text{op}}$ to the quiver of the cluster-tilted algebra $\Gamma = \text{End}_{\mathcal{C}_H}(T)^{\text{op}}$, in our earlier notation. The Happel–Vossieck list gives the quivers together with a defining finite set of relations for the tame concealed algebras. All of these relations have the following in common: they are of the form $\sum_{i=1}^r \sigma_i$ where $1 \leq r \leq 3$ and $\sigma_1, \dots, \sigma_r$ are **all** the paths from a vertex i to a vertex j and such that any vertex different from i, j is a vertex on at most one of the paths σ_t . It is clear that each relation of this form is minimal. Recall that a relation ρ is *minimal* if whenever $\rho = \sum_{i=1}^n \alpha_i \rho_i \beta_i$, where ρ_1, \dots, ρ_n are relations, then for some t , both α_t and β_t are scalars. We then have the following.

Proposition 4.1. *Let $\Lambda = \text{End}_H(T)^{\text{op}}$ be a tame concealed algebra, given as a quiver Q and a set of defining relations $\{\rho_t\}$ from the Happel–Vossieck list. Then the quiver of the cluster-tilted algebra $\Gamma = \text{End}_{\mathcal{C}_H}(T)^{\text{op}}$ is obtained from Q by adding an arrow from the vertex j to the vertex i if and only if one of the defining relations ρ_t involves paths from i to j .*

We leave out our original proof of this result, since there is now a generalization to any tilted algebra by Assem, Brüstle and Schiffler [1].

The quivers of the cluster-tilted algebras $\text{End}_{\mathcal{C}_H}(T)^{\text{op}}$ are the quivers of the minimal infinite cluster-tilted algebras by Theorem 3.2, which coincide with the minimal infinite cluster quivers by Theorem 3.1. A classification of these is what Seven gave in [18]. Hence we get the following consequence:

Theorem 4.2. *By starting with the Happel–Vossieck list and for any defining relations adding an arrows with direction such that oriented cycles are created, we get the minimal infinite cluster quivers and hence the list of A. Seven.*

5. Properties of minimal infinite tilted and cluster-tilted algebras

Our next goal is to show which set of arrows we have to remove from a minimal infinite cluster quiver in order to obtain the quiver with relations for a minimal infinite tilted algebra. It turns out that this set is uniquely defined for each minimal infinite cluster quiver.

In this section we give some necessary conditions on the set of arrows which should be removed from the quiver of a minimal infinite cluster-tilted algebra. We also discuss quadratic forms associated with signed graphs, and their relationship to Tits forms in our context. Then the goal will be completed in the next section, by proving a result on quadratic forms of signed graphs.

Let T be a preprojective tilting module over a tame hereditary algebra H . We investigate the passage from the tilted algebra $\Lambda = \text{End}_H(T)^{\text{op}}$ to the corresponding cluster-tilted algebra $\Gamma = \text{End}_{C_H}(T)^{\text{op}}$ more carefully. In particular, we would like information saying which arrows appearing in the quiver Q of Γ are new ones.

Proposition 5.1. *With the above terminology, let S be the set of new arrows obtained when passing from Λ to Γ . Then S contains exactly one arrow from each full oriented cycle of Q , and no other arrows.*

Proof. For each new arrow α we have exactly one, two or three full oriented cycles on which α lies. All arrows except α on these cycles occur in the quiver of Λ , and there are no other full cycles. \square

It will be useful to look at quadratic forms associated with minimal infinite cluster quivers. We consider here quadratic forms in n variables x_1, \dots, x_n of the form $q(x_1, \dots, x_n) = \sum_{i=1}^n x_i^2 + \sum_{i,j} a_{ij} x_i x_j$ ($i < j$). If q is positive semidefinite, then the elements z of \mathbb{Z}^n with $q(z) = 0$ are called *radical vectors*.

Let Λ be a finite-dimensional algebra with quiver Q without oriented cycles and with vertices $1, \dots, n$, such that there are no paths from j to i whenever $i < j$. Then the corresponding Tits form is given by $a_{ij} = s - t$ where t is the number of arrows from i and j and s is the dimension of the space of minimal relations from i to j . The Tits form of a tilted algebra of an extended Dynkin quiver Q is known to be isomorphic to the one given by Q , and is in particular positive semi-definite (see [17]).

A crucial property of a tame concealed algebra is that there is a positive sincere radical vector for the Tits form, and amongst the tilted algebras of extended Dynkin type, the tame concealed ones are exactly the ones with this property (see [17]). The corresponding coordinates are given in the Happel–Vossieck list.

Associated with any quadratic form there is a signed graph, i.e. a graph with two kinds of edges which we call *solid* or *dotted*: If $a_{ij} < 0$ we have $-a_{ij}$ solid edges between i and j , and if $a_{ij} > 0$ we have a_{ij} dotted edges between i and j . Conversely, there is a quadratic form associated with a signed graph in an obvious way (see [17]).

When passing from the quiver of a tame concealed algebra Λ to the quiver Q of the corresponding cluster-tilted algebra Γ , with S denoting the set of additional arrows, we associate with Q and S the signed graph obtained from Q by making exactly the edges in S dotted. We denote by q_S the associated quadratic form, which then clearly coincides with the Tits form t_Λ . Hence we have the following useful information:

Proposition 5.2. *Let $\Lambda = \text{End}_H(T)^{\text{op}}$ be a tame concealed algebra, and Q the quiver of the corresponding cluster-tilted algebra $\text{End}_C(T)^{\text{op}}$, with S the additional set of arrows. Then the quadratic form q_S is isomorphic to the quadratic form of an extended Dynkin quiver mutation equivalent to Q and has a sincere positive radical vector.*

Motivated by Proposition 5.2 we call a set of arrows of a minimal infinite cluster quiver *admissible* if S contains exactly one arrow from each full oriented cycle, and no other arrows. The strategy is to consider the quadratic form q_S associated with Q and S , and show that only one choice of admissible set will give a positive sincere radical vector. Here one is using that one can show that q_S is positive semidefinite, as we do in Section 6.

We denote by Λ_S the algebra whose quiver is obtained by removing the arrows in S from Q , and with relations given as follows: For each arrow $\beta: j \rightarrow i$ in S , consider the sum of all shortest paths from i to j . This coincides with the description of the tame concealed algebras in terms of quivers with relations for the “correct” choice of S . The only oriented cycles for Q , not containing properly an oriented cycle, are those created by the arrows $\alpha: j \rightarrow i$ corresponding to relations on paths from i to j . Since all paths from i to j not going through any cycle are involved in the original relation, it follows that all oriented cycles of the above type are full. Hence the quiver of Λ_S has no oriented cycles. Consider the relation ρ associated with β . We want to show that it is minimal for Λ_S . Assume that we have $\rho = \sum_{t=1}^r \alpha_t \rho_t \beta_t$, where for each t we have that ρ_t is a relation associated with an arrow in S and either α_t or β_t

is a non-trivial path. Let σ_1 be a path occurring for ρ . Then for some t , we have $\sigma_1 = \alpha_i \psi_i \beta_i$ where ψ_i is a path in ρ_i . Since we have an arrow from v to u , where ψ_i starts in u and ends in v , where $(u, v) \neq (i, j)$, we get a contradiction to σ_1 being a shortest path. Hence we have the following:

Proposition 5.3. *Let Q be a minimal infinite cluster quiver, S an admissible set of arrows and Λ_S the associated algebra. Then the quiver of Λ_S has no oriented cycles and the quadratic forms q_S and t_S coincide.*

So dealing with the radical vectors of q_S for an admissible set S corresponds to dealing with the radical vectors of the Tits form $t_S = t_{\Lambda_S}$ of the algebra Λ_S . We shall in the next section work with a more general choice of arrows S .

6. From cluster-tilted to tilted algebras

Let Q be a minimal infinite cluster quiver, with associated cluster-tilted algebra Γ . Denote by Q_0 the set of vertices of Q . The aim of this section is to show the following: For the quiver Q there is a unique choice of an admissible set of arrows S such that the Tits form t_S of the induced algebra Λ_S has a positive sincere radical vector, or equivalently, such that the quadratic form q_S of the signed graph of Γ associated with S has a positive sincere radical vector. An algorithm for finding this unique set S is also given. The result is seen as a consequence of a more general result about quadratic forms associated with signed graphs coming from minimal infinite cluster quivers.

The following sign change operation at a vertex allows us to obtain a large number of isomorphic quadratic forms on an undirected graph. Recall that two quadratic forms q and q' on \mathbb{Z}^{Q_0} are called isomorphic if there is an isomorphism A on \mathbb{Z}^{Q_0} such that $q(A(v)) = q'(v)$ for any v in \mathbb{Z}^{Q_0} .

Let Σ be a signed graph and let i be a vertex in Σ . We denote by $r_i(\Sigma)$ the graph obtained from Σ by changing the signs of the edges connected to the vertex i . If $\Sigma = \Sigma(q)$ is the graph of a quadratic form q , then we denote by $r_i(q)$ the quadratic form whose graph is $r_i(\Sigma)$.

Let us note the following obvious property of the sign change operation.

Proposition 6.1. *Suppose that C is a full cycle in the (undirected) graph Σ . Then the parity of the number of dotted edges in C is the same both in Σ and $r_i(\Sigma)$ for any vertex i in Σ (thus r_i preserves the parity of the number of dotted edges in any cycle).*

We will mostly be interested in quadratic forms whose (signed) graphs have the same underlying undirected unsigned graph, such as those that can be obtained from each other by sign changes. It will be convenient to use the following notation. Let Q be a quiver and S a set of arrows of Q . We denote by q_S the quadratic form whose graph $\Sigma(q_S)$ is the underlying (undirected) graph of Q with the following sign assignment: any edge whose corresponding arrow belongs to S is dotted, and the rest of the edges are solid.

Our next result characterizes a class of signed graphs that can be obtained from each other by a special sequence of signs changes. It is the main technical lemma that we use to prove the main theorem in this section. Note that if a minimal infinite cluster quiver contains a full non-oriented cycle, it must be of the form \widetilde{A}_n , and it comes from the same tame concealed algebra, so that in this case there is nothing to prove.

Lemma 6.2. *Suppose that Q is a simply-laced cluster quiver which does not contain any non-oriented full cycle. Let S and S' be two different sets of arrows of Q with the following property: for any full cycle C , the parity of the number of arrows of C contained in S is the same as the parity of the number of arrows contained in S' . Then the graph $\Sigma(q_{S'})$ of the quadratic form $q_{S'}$ can be obtained from the graph $\Sigma(q_S)$ of q_S by a sequence of sign changes such that a vertex is used at most once and not all vertices are used.*

Proof. We prove this by induction on the number, say n , of vertices of Q :

For $n = 2$, the underlying graph of the quiver Q has a single edge, which could be either solid or dotted, so we may assume that q_S corresponds to the solid one and $q_{S'}$ to the dotted one. It is obvious that sign change at any vertex transforms $\Sigma(q_S)$ to $\Sigma(q_{S'})$.

Let us now assume that the lemma holds for quivers with $n - 1$ vertices or less. Suppose that Q has n vertices. Let us consider a connected subquiver Σ obtained by removing a vertex, say j , from Q (the existence of such a vertex leaving a connected subquiver is easily seen). Since Σ has less than n vertices, by the induction argument we have the

following: the restriction of q_S to Σ can be transformed to the restriction of $q_{S'}$ to Σ as described in the lemma. Thus there is a sequence of mutually different vertices i_1, \dots, i_m in Σ with $m < n - 1$ such that

$$q_{S'}|_{\Sigma} = r_{i_m} \dots r_{i_1}(q_S|_{\Sigma}).$$

We claim that either

$$q_{S'} = r_{i_m} \dots r_{i_1}(q_S), \tag{1}$$

or

$$q_{S'} = r_j r_{i_m} \dots r_{i_1}(q_S). \tag{2}$$

We need to show that if (1) does not hold, then (2) does. So let us assume that $q_{S'}$ is not equal to $r_{i_m} \dots r_{i_1}(q_S)$. Denote this last expression by q_m , i.e. $q_m = r_{i_m} \dots r_{i_1}(q_S)$. Note that, by our induction assumption, the forms $q_{S'}$ and q_m agree on the subquiver Σ , i.e. (the signs of) the graphs of $q_{S'}$ and q_m on Σ are the same. Since we assume $q_{S'}$ is not equal to q_m , there is an edge e (in the underlying graph of Q) such that e has opposite signs in $\Sigma(q_{S'})$ and $\Sigma(q_m)$. Since $q_{S'}$ and q_m agree on the subquiver Σ (obtained by removing the vertex j), the vertex j must be one of the end points of e . We will show that, like e , all of the edges containing j have opposite signs in $q_{S'}$ and q_m (thus $q_{S'} = r_j(q_m)$, i.e. (2) holds). For this let us assume that there is an edge f , connected to j such that f has the same signs in $q_{S'}$ and q_m . Let us also denote the remaining vertices of e and f by r and s respectively. We may also assume that e, f have the property that the vertices r, s are such that the length of a shortest possible walk is minimal. Let us denote by P a shortest walk connecting r and s in Σ . We note that j is not connected to any vertex on P other than r and s because of our assumption. Thus the graph, say P_j , induced by the path P and the vertex j is a full cycle (which is oriented in Q because we assumed that all full cycles in Q are oriented. Note also that e and f are the only edges in P_j which are not contained in Σ). We also note that the parity of the dotted edges in P_j is different in $q_{S'}$ and q_m ; the difference is because e has different signs in $q_{S'}$ and q_m and any other edge in P_j has the same sign. This gives a contradiction because sign change at a vertex preserves the parity of dotted edges in cycles (Proposition 6.1), thus P_j has the same parity of dotted edges in $q_m (= r_{i_m} \dots r_{i_1}(q_S))$ as in q_S , thus as in $q_{S'}$, by our assumption. This completes the proof of the lemma. \square

Let us now give an algebraic description of the sign change operation:

Proposition 6.3. *Suppose that q is a quadratic form. Then $r_k(\Sigma(q))$ is the graph of the form q with respect to the basis (variables) obtained from the basis (variables) for $\Sigma(q)$ by changing x_k to $-x_k$ and keeping the other elements the same.*

Proof. Let us assume without loss of generality that $k = 1$. Let $\{y_i\}$ be the dual basis such that $y_1 = -x_1$ and $y_i = x_i$ for $i \neq 1$. We note that we have:

$$q = a_{1,1}x_1^2 + \sum a_{1,i}(x_1)x_i + \sum a_{i,j}x_i x_j, \quad 2 \leq i \leq j,$$

or equivalently

$$q = a_{1,1}(-x_1)^2 + \sum -a_{1,i}(-x_1)x_i + \sum a_{i,j}x_i x_j, \quad 2 \leq i \leq j$$

thus

$$q = a_{1,1}y_1^2 + \sum -a_{1,i}y_1 x_i + \sum a_{i,j}y_i y_j, \quad 2 \leq i \leq j.$$

Then the statement follows from our definitions. \square

Corollary 6.4. *If q is positive semi-definite of corank 1, then the coordinates of the radical vector of q with respect to the basis corresponding to the graph $r_i(\Sigma(q))$ is the same as the one for $\Sigma(q)$ with the exception that the i th coordinate is the negative of the one for $\Sigma(q)$.*

The following basic fact gives interesting examples of signed graphs where each full cycle has an odd number of dotted edges.

Proposition 6.5. *If q is a positive-definite form, then any full cycle in $\Sigma(q)$ has an odd number of dotted edges.*

Proof. Suppose that $\Sigma(q)$ contains a full cycle C which has exactly an even number of dotted edges. To prove the proposition, it is enough to show that the restriction of q to C is positive semi-definite. By Lemma 6.2, the form q is isomorphic to the form q' such that the underlying (undirected and unsigned) graph of $\Sigma(q'|_C)$ is the same as C without dotted edges. Let u be the vector such that for any vertex of C the corresponding coordinate of u is 1 and the rest are 0. Then $q'(u) = 0$, so q' is not positive definite, thus q is not positive definite either. \square

The classification of positive-definite quadratic forms is the well-known Cartan–Killing classification: any positive-definite quadratic form is isomorphic to one given by a Dynkin graph. Similarly positive-semidefinite quadratic forms of corank 1 are classified by the extended Dynkin graphs [17].

If Q is a finite type cluster quiver, then the form q_S is positive-definite. More precisely, the following follows from [2, Thm 1.1].

Proposition 6.6. *Suppose that Q is a finite type cluster quiver (note that Q does not have any non-oriented full cycles) and let S be a set of arrows of Q such that S contains exactly an odd number of arrows from each oriented cycle. Then the form q_S is positive definite.*

We now show the main result of this section.

Theorem 6.7. *Let Q be a minimal infinite cluster quiver. For any set S of arrows such that S contains exactly an odd number of arrows from each oriented cycle and no arrows from any non-oriented cycle, the corresponding quadratic form q_S is isomorphic to the one defined by an extended Dynkin quiver which is mutation equivalent to Q . Furthermore, there is a unique set S_+ where the form q_{S_+} has a sincere positive radical vector.*

Proof. We can assume that Q has at least three vertices, since for two vertices the set S would be empty. It follows from Proposition 5.2 and Lemma 6.2 that for any S as in the theorem, q_S is isomorphic to the quadratic form given by an extended Dynkin quiver mutation equivalent to Q . Let S_+ be a set such that Λ_{S_+} is tame concealed, hence q_{S_+} has a sincere positive radical vector. Now take S' as in the theorem such that $S' \neq S_+$. Then there is a sequence of vertices i_1, \dots, i_m as in Lemma 6.2 such that $q_{S'} = r_{i_m} \dots r_{i_1}(q_{S_+})$, ($m \geq 1$). Since a sign change at a vertex changes only the corresponding coordinate of the radical vector to its negative (Corollary 6.4) and each vertex is used once (and not all vertices are used), exactly the coordinates corresponding to the vertices i_1, \dots, i_m will be negative in the radical vector of $q_{S'}$ (and the remaining ones will be positive). This completes the proof. \square

In view of the discussion in Section 5, we have the following reformulation of Theorem 6.7.

Theorem 6.8. *Let Q be a minimal infinite cluster quiver. Then, for any admissible set of arrows S , the Tits form t_S of the algebra Λ_S is isomorphic to the quadratic form defined by the extended Dynkin quiver which is mutation equivalent to Q . Furthermore, there is a unique admissible set S_+ such that t_{S_+} has a positive sincere radical vector. Also, for any admissible S , the algebra Λ_S is of minimal infinite representation type if and only if $S = S_+$.*

Note that if we are given a minimal infinite cluster quiver Q , and we find a set of arrows S , with exactly one arrow from each oriented cycle and no other arrows, such that the quadratic form of the associated signed graph has a positive sincere radical vector, then we know by Theorem 6.7 that this is the correct choice for obtaining the quiver of the associated tame concealed algebra.

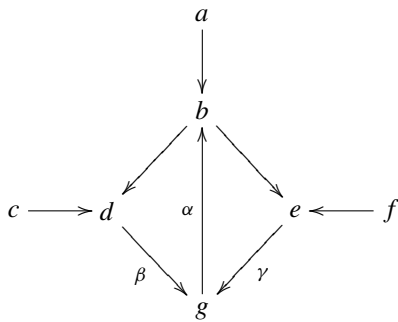
More systematically, as a consequence of the proof of Theorem 6.7, we have the following simple procedure for finding the desired set S_+ .

Algorithm 6.9. To find S_+ in a minimal infinite cluster quiver Q :

- (1) Take any set of arrows S which contains exactly an odd number of arrows from any oriented full cycle,
- (2) compute the radical vector of the quadratic form q_S ,
- (3) apply sign change operation at all vertices whose corresponding coordinates in the radical vector of q_S is negative,
- (4) the set of arrows corresponding to the dotted edges is exactly S_+ .

We end this section by an example and a conjecture. The following seems to be a general fact: If we remove an admissible S such that Λ_S is not of minimal infinite type, then Λ_S is of finite type. To give an example consider the

following minimal infinite cluster quiver:



The choice of S giving Λ_S of minimal infinite type is $S = \{\alpha\}$. Another admissible set is $S' = \{\beta, \gamma\}$. In this case $\Lambda_{S'}$ is of global dimension 3, and hence not tilted, and of finite representation type.

We make the following conjecture.

Conjecture 6.10. *In the set-up of Theorem 6.7, for any admissible set S , the dimension vectors of the indecomposable representations of Q_S are exactly the positive roots of the corresponding Tits form t_S (here x is a root if $t_S(x) = 1$). Furthermore if $S \neq S_+$, then the algebra Λ_S is of finite representation type.*

7. The cluster-tilted algebras of minimal infinite cluster quivers

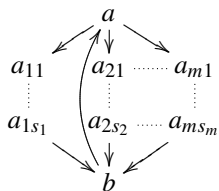
In [6] it was shown that a cluster-tilted algebra of finite type (over an algebraically closed field k) is uniquely determined by its quiver, and has relations of a nice form. In this final section we show that similar results hold for the cluster-tilted algebras investigated in this paper. Recall that all minimal infinite cluster-tilted algebras of rank >2 are endomorphism algebras of preprojective tilting modules over tame hereditary algebras.

Theorem 7.1. *Let T be a preprojective tilting module over a tame hereditary algebra H , and $\Gamma = \text{End}_{C_H}(T)^{\text{op}}$ the associated (basic) minimal infinite cluster-tilted algebra. For each arrow $\alpha: j \rightarrow i$ lying on an oriented cycle in the quiver of Γ , we consider all the shortest paths $\sigma_1, \dots, \sigma_r$ from i to j . Then there is a minimal relation $\sigma_1 + \dots + \sigma_r$, and $r \leq 3$. Furthermore, the set of relations obtained this way is a generating set of minimal relations.*

Proof. We will use the idea that a path passing through an oriented cycle is a zero-path. To see this, we use that such a path corresponds to an endomorphism $X \rightarrow X$ of a preprojective indecomposable module X . In the derived category $\text{Hom}_{D^b(H)}(X, FX)$ is clearly zero, so the claim holds.

We first show that for every arrow $\alpha: j \rightarrow i$ lying on a cycle, there is a relation of the prescribed form.

In case there are two or more shortest paths from i to j , we claim that the subquiver generated by these paths has the following shape (\star):



To prove the claim, we apply Lemma 2.14 in [6] and note that the proof goes through in our setting.

We note that there are minimal infinite cluster quivers of this form with three “arms” from a to b (i.e. $m = 3$), but that it is evident that $m \geq 4$ would give an algebra which is not of minimal infinite type, since there would be too many arrows meeting in a , i.e. after removing the vertex b , there would still four arrows all starting in a .

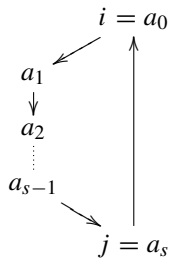
We first discuss quivers which are not of the form (\star). In this case it is clear by the above that for any arrow $\alpha: j \rightarrow i$ lying on a cycle in Q , there is a vertex e such that e does not lie on any of the shortest paths from i to j .

Consider the factor algebra $\Gamma/\Gamma e \Gamma$. This is a cluster-tilted algebra (by [5]) of finite representation type. Thus, using the main result from [6] there is a minimal relation ρ in the factor algebra involving the shortest paths from i

to j . Also from [6] it follows that there are at most two such shortest paths. Note that this can also be observed from the Happel–Vossieck list, or the A. Seven list. Also it is shown in [6] that if there are two such shortest paths ρ_1 and ρ_2 , then the corresponding relation is $\rho_1 - \rho_2$. It is clear from inspection of the Happel–Vossieck list, that we get an isomorphic algebra by changing the relation to $\rho_1 + \rho_2$.

We now claim that ρ is also a minimal relation for Γ . To prove the claim, we first observe that ρ is a relation for Γ . For since ρ is a relation for the factor algebra $\Gamma/\Gamma e\Gamma$, there is a sum of paths ρ' for Γ , all going through the vertex e , such that $\rho + \rho'$ is a relation for Γ . We claim that any such path from i to j which is not a shortest path must pass through an oriented cycle, thus it is itself a 0-relation. Consequently ρ is a relation for Γ .

To prove this claim, assume there is a path $\psi: i \rightarrow a_1 \rightarrow \dots \rightarrow a_s = j$ which is not shortest and does not pass through an oriented cycle. We can assume that the path has minimal length among all paths from i to j with this property. Consider the subquiver:



If there are further arrows going downwards, say $a_x \rightarrow a_y$ with $y > x + 1$, this would contradict the minimality of the length of ψ , so no such arrows exist. If there are additional arrows going upwards, choose such an arrow starting in a vertex a_y with y maximal. It is clear that the quiver has a factor quiver which is a non-oriented cycle, and can hence not be the quiver of a cluster-tilted algebra of finite representation type. This is a contradiction, so all paths which are not the shortest through an oriented cycle.

To show that ρ is minimal for Γ , assume $\rho = \alpha_1\rho_1\beta_1 + \dots + \alpha_n\rho_n\beta_n$, where ρ_1, \dots, ρ_n are minimal relations for Γ . If a path going through e occurs for some $\alpha_i\rho_i\beta_i$, then this path goes through a cycle, and is hence itself a 0-relation, which is not minimal. Since no such path occurs for ρ , these paths can be removed on the right hand side. Hence for some i we have that α_i and β_i are constant, using that ρ is a minimal relation for $\Gamma/\Gamma e\Gamma$. Then we conclude that ρ is minimal also for Γ .

We now consider quivers of minimal infinite cluster-tilted algebras which have a quiver of the form (\star) . Then for all arrows except the one $b \rightarrow a$, we can use the same technique as in the previous case, to conclude that there is a minimal relation as prescribed. For the arrow $b \rightarrow a$ it is clear that the corresponding tilted algebra is the algebra with quiver obtained by removing the arrow $b \rightarrow a$, and there is a relation as prescribed.

Next we need to prove that all the minimal relations give rise to arrows. More precisely, let ρ be a minimal relation involving paths from i to j in the cluster-tilted algebra. We need to show that there exists an arrow from j to i . For this the following lemma due to Assem, Brüstle, Schiffler and Reiten, Todorov is useful, and gives a simplification of our original proof.

Lemma 7.2. *Let A be any cluster-tilted algebra, and let S_1, S_2 be simple A -modules. Then $\dim_k \text{Ext}_A^1(S_1, S_2) \geq \dim_k \text{Ext}_A^2(S_2, S_1)$.*

Recall that the number of arrows from the vertex corresponding to S_1 to the vertex corresponding to S_2 is given by $\dim_k \text{Ext}_A^1(S_1, S_2)$ and that $\dim_k \text{Ext}_A^2(S_2, S_1)$ is the dimension of the space of minimal relations involving paths from the vertex corresponding to S_2 to the vertex corresponding to S_1 , by [3]. Thus the proof of the theorem is finished by the above lemma. \square

Remark: Actually, the last part of the proof of the theorem, namely that minimal relations give rise to arrows, can also be seen directly from studying the quivers and relations on the Happel–Vossieck list. But for a few of the quivers on this list this is a rather cumbersome procedure. We therefore included the more general lemma above.

Using the description of the relations given in Theorem 7.1, it is easy to see that the algebras Λ_S , for an admissible set S , as defined in Section 5, are obtained by restriction of the relations from the cluster-tilted algebra, in the case of minimal infinite type.

Acknowledgments

This work started when the third author A. Seven visited NTNU in spring 2005 as a LieGrits postdoctoral fellow. He thanks the coauthors I. Reiten and A. Buan for their kind hospitality. He also thanks the members of the algebra group at NTNU for many helpful discussions.

The first and second authors were supported by a (Storforsk) grant from the Norwegian Research Council. The third author was supported by EU-LieGrits postdoctoral fellowship.

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