# Gauge Covariance and Spin Current Conservation in the Gauge Field Formulation of Systems with Spin-Orbit Coupling

M. S. Shikakhwa

Physics Program, Middle East Technical University Northern Cyprus Campus, Kalkanlı, Güzelyurt, via Mersin 10, Turkey

> S. Turgut and N. K. Pak Department of Physics, Middle East Technical University,

> > TR-06800, Ankara, Turkey

The question of gauge-covariance in the non-Abelian gauge-field formulation of two spacedimensional systems with spin-orbit coupling relevant to spintronics is investigated. Although, these are generally gauge-fixed models, it is found that for the class of gauge fields that are spacetime independent and satisfy a U(1) algebra, thus having a vanishing field strength, there is a residual gauge freedom in the Hamiltonian. The gauge transformations assume the form of a spacedependent rotation of the transformed wave functions with rotation angles and axes determined by the specific form of the gauge-field, i.e., the spin-orbit coupling. The fields can be gauged away, reducing the Hamiltonian to one which is isospectral to the free-particle Hamiltonian, and giving rise to the phenomenon of persistent spin helix reported first by B. A. Bernevig *et al.* [Phys. Rev. Lett. **97**, 236601 (2006)]. The investigation of the global gauge transformations leads to the derivation of a continuity equation where the component of the spin-density along given directions, again fixed by the specific form of the gauge field, is conserved.

# I. INTRODUCTION

Spintronics[1] is an emerging direction in solid state physics that is rapidly growing. An important ingredient in spintronics is the issue of the generation and manipulation of spin current. An important class of systems where this can be realized is the quasi-two-dimensional electron (or hole) systems, where the generation of the spin current might be achieved via the spin-orbit coupling mechanism[2]. The most popular models with various spin-orbit forms are the Rashba (R)[3] and Dresselhaus (D)[4] couplings or a combination of these (R-D). The spin current, its definition and conservation received considerable attention in the past. Many works are devoted to the investigation of the question of what is the correct (or, more precisely, the most convenient) expression of the spin current that one should use. This is because, unlike the charge (or matter) current, the spin current is not conserved in the presence of a magnetic field and/or spin-orbit coupling, and the continuity equation contains a nonzero right-hand side:

$$\frac{\partial S^a}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{J}_0^a \neq 0$$

where

$$S^{a} = \phi^{\dagger} \sigma^{a} \phi$$
,  $J_{0}^{a} = \frac{\hbar}{2mi} \phi^{\dagger} \sigma^{a} \nabla \phi + \text{c.c.}$  (1)

are the spin density and the spin current respectively. This led to works suggesting alternative definitions of the spin current[5–9] that are conserved, other than the one given in Eq. (1), which is sometimes called the "natural definition" and merely generalizes the probability current density by "plugging" in a Pauli matrix in its definition.

An approach for investigating various aspects of models of non-relativistic spin one-half particles with spin-orbit coupling, based on expressing the spin-orbit coupling as an SU(2) gauge field was introduced in Ref. 10 building on the idea introduced in Ref. 11, and generated some interest[12–16]. In these models — in contrast to particle physics models— the gauge field, being just the electric field, is physical and directly observable. Therefore, a gauge transformation corresponds to switching from a given physical configuration to another, which is physically different. In the works in Refs. 10–12, a term quadratic in the gauge field, which —generally speaking— breaks the gauge-symmetry of the Hamiltonian, was absorbed into the definition of the electrostatic potential present in the Hamiltonian, and thus the Hamiltonian analyzed was taken as gauge-symmetric. For spin-orbit couplings derived as  $O\left(\frac{1}{c^2}\right)$  limit of the Dirac Lagrangian, this gauge symmetry is, therefore, an  $O\left(\frac{1}{c^2}\right)$  symmetry. For spin-orbit couplings in quasi-two-dimensional electron systems, such as R and D spin-orbit couplings mentioned above, the origin of these couplings are different, and it might not be well-justified to ignore this gauge symmetry-breaking term. For example, in Refs. 14–16, this gauge-symmetry breaking term was kept. However, the issue of SU(2) gauge transformations and gauge-symmetry was not fully analyzed in any of these works. It is the aim of this work to address this point

and investigate some of its consequences. We carry out the analysis within a Hamiltonian rather than a Lagrangian framework, which is, we believe, more convenient for condensed-matter applications.

We also note that while the natural framework for the derivation of the conserved spin-current is Noether's theorem, there is no published work that applies this theorem to the free Pauli Lagrangian to extract the conserved current. We do this in the appendix.

# II. SPIN-ORBIT COUPLING AND SU(2) GAUGE SYMMETRY

### A. SU(2) gauge Symmetry as a Unitary Transformation of the Hamiltonian

Let the Hamiltonian H (time-independent) of a physical system having eigenstates  $\phi_n$  transforms under some unitary transformation U as  $H \to H' = UHU^{-1}$ , then  $\phi_n$  transform as  $\phi_n \to \phi'_n = U\phi_n$ , so that  $\phi'_n$  are now eigenstates of H' with the same eigenvalues. In the case of Abelian gauge transformations, the Hamiltonian is a function of some gauge field A,

$$H = H(\boldsymbol{A}) = \frac{1}{2m} \left( \boldsymbol{p} - \frac{e}{c} \boldsymbol{A} \right)^2 + V(\boldsymbol{x}) \quad .$$
<sup>(2)</sup>

A gauge transformation is defined as a simultaneous transformation of the wavefunction  $\phi_n \to \phi'_n = U\phi_n = \exp(\frac{-ie}{\hbar c}\Lambda(\boldsymbol{x}))\phi_n$  and the gauge field  $\boldsymbol{A} \to \boldsymbol{A}' = \boldsymbol{A} + \boldsymbol{\nabla}\Lambda$ . The transformation of the gauge field so defined guarantees that  $H' = H(\boldsymbol{A}') = UH(\boldsymbol{A})U^{-1}$ , so that  $\phi'_n = U\phi_n$  is an eigenstate of H'. In the non-Abelian SU(2) case, the Hamiltonian is a function of the non-Abelian gauge field  $\boldsymbol{W}$  with components  $W_i = W_i^a \tau_a$ , where  $\tau_a$  are  $2 \times 2$  matrices satisfying the algebra  $[\tau_a, \tau_b] = i\epsilon_{abc}\tau_c$  (a, b, c = 1, 2, 3) (e.g.,  $\tau_a = \sigma_a/2$  where  $\sigma_a$  are Pauli spin matrices),

$$H = H(\boldsymbol{W}) = \frac{1}{2m} \left(\boldsymbol{p} - g\boldsymbol{W}\right)^2 + V(\boldsymbol{x}) \quad .$$
(3)

The spinor wavefunction transforms as  $\phi_n \to \phi'_n = U\phi_n = \exp(-i\Lambda(\mathbf{x}) \cdot \boldsymbol{\tau})\phi_n$ , and the gauge field transforms simultaneously  $\mathbf{W} \to \mathbf{W}'$ . The transformation law for the gauge field can be derived by demanding that it should —as in the Abelian case— correspond to a transformation of the Hamiltonian  $H(\mathbf{W}') = UH(\mathbf{W})U^{-1}$ . Since

$$UH(\boldsymbol{W})U^{-1} = \frac{1}{2m}U(\boldsymbol{p} - g\boldsymbol{W})U^{-1} \cdot U(\boldsymbol{p} - g\boldsymbol{W})U^{-1} + V(\boldsymbol{x})$$
(4)

$$= \frac{1}{2m} \left( \boldsymbol{p} - g \boldsymbol{W'} \right) \cdot \left( \boldsymbol{p} - g \boldsymbol{W'} \right) + V(\boldsymbol{x}) \quad , \tag{5}$$

it can be seen that we need  $U(\mathbf{p} - g\mathbf{W})U^{-1} = (\mathbf{p} - g\mathbf{W}')$ , which gives the following transformation law for  $\mathbf{W}$ :

$$\boldsymbol{W}' = U\boldsymbol{W}U^{-1} + \frac{i\hbar}{g}U\boldsymbol{\nabla}U^{-1}$$
(6)

This is the well-known transformation law for a non-Abelian gauge field, usually derived within a Lagrangian formalism within the framework of relativistic field theory models as a transformation that leaves the action functional invariant[17]. The above "Hamiltonian" derivation, on the other hand, ascribes a different meaning to gaugetransformation: A Hamiltonian  $H(\mathbf{W})$  that transforms under Eq. (6) to another Hamiltonian  $H(\mathbf{W'})$  is unitarilyequivalent to the original one by  $H(\mathbf{W'}) = UH(\mathbf{W})U^{-1}$ , in which case the Schrödinger equation is gauge-covariant. We will also use the term gauge-covariant or gauge-symmetric for a Hamiltonian that after a gauge transformation transforms to a form that is unitarily equivalent to its form before gauge transformation. The Hamiltonian, Eq. (3), is gauge-covariant by construction.

### B. SU(2) Gauge Field Formalism and Gauge Transformations

The spin-orbit interaction emerges upon considering the  $O\left(\frac{1}{c^2}\right)$  expansion of the Dirac Hamiltonian for a spin one-half particle subject to a scalar potential  $V(\boldsymbol{x})$ :

$$H = \frac{\boldsymbol{p}^2}{2m} + V(\boldsymbol{x}) + \frac{e\hbar}{4m^2c^2}\boldsymbol{\sigma} \cdot (\boldsymbol{E} \wedge \boldsymbol{p}) = \frac{\boldsymbol{p}^2}{2m} + V(\boldsymbol{x}) + \frac{e\hbar}{4m^2c^2}\boldsymbol{p} \cdot (\boldsymbol{\sigma} \wedge \boldsymbol{E})$$

where  $\nabla \wedge E = 0$  was assumed. Let us now define the SU(2) gauge field  $W_i^a$  by

$$-gW_i^a \equiv \frac{e\hbar}{2mc^2}\epsilon_{iaj}E_j \quad . \tag{7}$$

Using W, the Hamiltonian can be expressed as

$$H = \frac{\boldsymbol{p}^2}{2m} - \frac{g}{m} \boldsymbol{p} \cdot \boldsymbol{W} + V(\boldsymbol{x}) = \frac{\boldsymbol{p}^2}{2m} - \frac{g}{m} \boldsymbol{W} \cdot \boldsymbol{p} + V(\boldsymbol{x}) \quad .$$
(8)

Completing the square, we can put this into the form

$$H = H(\boldsymbol{W}) = \frac{(\boldsymbol{p} - g\boldsymbol{W})^2}{2m} - \frac{g^2}{2m} \boldsymbol{W} \cdot \boldsymbol{W} + V(\boldsymbol{x})$$
(9)

where  $W_i = W_i^a \tau_a$  is the *i*th component of the field  $\boldsymbol{W}$ . The above is the Hamiltonian of a spin one-half particle coupled to the SU(2) gauge field  $\boldsymbol{W}$ . In most of the works that use this model, the term quadratic in the gauge field, i.e.,  $-\frac{g^2}{2m}\boldsymbol{W}\cdot\boldsymbol{W}$ , which generally breaks the SU(2) gauge symmetry, is absorbed into the potential  $V(\boldsymbol{x})$  and so, in a sense, it is "put under the carpet", so that the Hamiltonian becomes gauge-invariant [10, 12]. This work, on the other hand, is based on keeping this term explicit in the Hamiltonian, and analyzing the model accordingly. In fact, important consequences of keeping this term explicitly in the Hamiltonian were noted in Ref. 16.

Many of the popular models for a two-dimensional electron gas with spin-orbit couplings, like the Rashba (R) coupling[3],

$$H = \frac{\boldsymbol{p}^2}{2m} + \frac{\alpha}{\hbar} \left( p_y \sigma_x - p_x \sigma_y \right) + V(\boldsymbol{x}) \quad , \tag{10}$$

the Dresselhaus (D) coupling[4]

$$H = \frac{\boldsymbol{p}^2}{2m} + \frac{\beta}{\hbar} \left( p_x \sigma_x - p_y \sigma_y \right) + V(\boldsymbol{x}) \tag{11}$$

and the R-D coupling

$$H = \frac{\boldsymbol{p}^2}{2m} + \frac{\alpha}{\hbar} \left( p_y \sigma_x - p_x \sigma_y \right) + \frac{\beta}{\hbar} \left( p_x \sigma_x - p_y \sigma_y \right) + V(\boldsymbol{x})$$
(12)

can be cast in the form in Eq. (9) with the term quadratic in the gauge field present. Note that, these are effective Hamiltonians that describe the motion of electrons inside a two-dimensional semiconductor heterostructure. The spin-orbit coupling terms given above originate from the strong electric fields associated with the confining potentials of these structures and the molecular potentials, which are rapidly varying on the microscopic scale. However, various approximations (e.g., the envelope wave function approximation) are usually employed [18] for eliminating the degree of freedom perpendicular to the 2D confinement plane and averaging out the molecular potentials on the microscopic scale; the effective Hamiltonians given in Eqs. (10-12) are obtained after such approximations. In these Hamiltonians, m is the effective mass and  $V(\mathbf{x})$  is an external potential that may be applied on the structure, which is always slowly varying on microscopic length scales. For this reason, even though the spin-orbit terms originate from a physical electric field, these fields do not appear elsewhere in the effective Hamiltonian. Thus, we consider the gauge field  $\mathbf{W}$  in these models to be arbitrary and independent of the external potential  $V(\mathbf{x})$  as it may have come from any origin. Because of the same reason, the quadratic term in  $\mathbf{W}$  is not usually negligible in these effective Hamiltonians.

We will study the motion in 2D and therefore we require W to have x and y-components only, and to satisfy the Coulomb gauge-fixing condition

$$\partial_i W_i = 0 \quad . \tag{13}$$

If the spin-orbit coupling is related to a physical electric field E by Eq. (7), then the Coulomb condition is equivalent to the Maxwell equation  $\nabla \wedge E = 0$ ; the spin-orbit gauge theory is a gauge-fixed theory[19]. While, as we have mentioned above, we consider the gauge field in our model to have come from any origin, we are going to assume that the Coulomb condition is satisfied for such cases. However, it is possible in some cases to find residual gauge transformations that respect this gauge condition. A closely related question in relativistic non-Abelian gauge theories is the Gribov ambiguity[20].

In this subsection, we are going to derive the class of gauge transformations that respect the Coulomb gauge-fixing condition, Eq. (13), and under which the Hamiltonian, Eq. (9), is gauge-covariant (in the sense discussed in the last

paragraph of Sec. II A), and investigate the consequences and the physical meaning of these transformations. Even though, the Hamiltonian given in Eq. (9) will be our main concern in this article, for the sake of completeness, the rules will be derived for the general time-dependent gauge transformations. In the general case, the electron may also be subjected to (time-dependent) electromagnetic fields. For this reason, consider the Hamiltonian

$$H = \frac{1}{2m} \left( \boldsymbol{p} - \frac{e}{c} \boldsymbol{A} \right)^2 - \frac{g}{m} \boldsymbol{W} \cdot \left( \boldsymbol{p} - \frac{e}{c} \boldsymbol{A} \right) + V(\boldsymbol{x}, t) + g W_0$$
(14)

$$=\frac{1}{2m}\left(\boldsymbol{p}-\frac{e}{c}\boldsymbol{A}-g\boldsymbol{W}\right)^{2}-\frac{g^{2}}{2m}\boldsymbol{W}\cdot\boldsymbol{W}+V(\boldsymbol{x},t)+gW_{0}$$
(15)

where  $W_0 = W_0^a \tau_a$  is the scalar counterpart of gauge field  $\boldsymbol{W}$ . If H is the original Hamiltonian written for a given heterostructure (in other words, no SU(2) gauge transformation has been applied yet), then this term is the same as the Zeeman term, i.e., we have  $gW_0^a = -2\mu_B B^a$  where  $\boldsymbol{B} = \boldsymbol{\nabla} \wedge \boldsymbol{A}$  is the magnetic field and  $\mu_B$  is the Bohr magneton. Of course, this is a gauge specific relation. Such a simple relation between  $W_0$  and the actual magnetic field  $\boldsymbol{B}$  will not hold when an SU(2) gauge transformation has already been carried out. The comments that we have made for the  $\boldsymbol{W}$  field can be repeated for the scalar component  $W_0$ . Just like the SU(2) gauge transformation alters the "physical electric field" that gave way to the spin-orbit coupling, the transformation also alters the "physical magnetic field" which is associated with the  $W_0$  term. The electromagnetic potentials V and  $\boldsymbol{A}$  remain invariant under these transformations. For this reason, in order to be able to study the SU(2) gauge transformations for the Hamiltonians of the form given in Eq. (15), it is necessary to take  $W_0$  and  $\boldsymbol{A}$  to be independent. Hence, no specific relation between  $\boldsymbol{A}$  and  $W_0^a$  should be assumed.

It is known that the Hamiltonian in Eq. (15) has the U(1) symmetry[10]. We only need to analyze its SU(2) symmetry. Using the covariance of the Schrödinger's equation  $i\hbar\partial_t\psi = H\psi$ , the gauge transformation  $\psi' = U\psi$  implies that the Hamiltonian changes as  $H' = UHU^{-1} - i\hbar U\partial_t U^{-1}$  and hence, if the Hamiltonian remains covariant, the following relations must be satisfied

$$\boldsymbol{W}' = U\boldsymbol{W}U^{-1} + \frac{i\hbar}{g}U\boldsymbol{\nabla}U^{-1} \quad , \tag{16}$$

$$W_0' = U W_0 U^{-1} - \frac{i\hbar}{g} U \partial_t U^{-1} \quad , \tag{17}$$

$$\boldsymbol{W}' \cdot \boldsymbol{W}' = U \boldsymbol{W} \cdot \boldsymbol{W} U^{-1} \quad . \tag{18}$$

Obviously, only the quadratic term of the Hamiltonian, namely  $-\frac{g^2}{2m^2} \boldsymbol{W} \cdot \boldsymbol{W}$ , breaks the full gauge symmetry of the Hamiltonian. If H is to be gauge-covariant under the gauge transformation, we should have

$$g^{2}\boldsymbol{W'}\cdot\boldsymbol{W'} = \left(\hbar^{2}\left(\boldsymbol{\nabla}U\right)\cdot\boldsymbol{\nabla}U^{-1} + i\hbar g\left[U\boldsymbol{W}\cdot\boldsymbol{\nabla}U^{-1} - \left(\boldsymbol{\nabla}U\right)\cdot\boldsymbol{W}U^{-1}\right]\right) + g^{2}U\boldsymbol{W}\cdot\boldsymbol{W}U^{-1} \quad . \tag{19}$$

Gauge-covariance then, dictates that the first bracket above should vanish

$$\hbar^{2} (\boldsymbol{\nabla} U) \cdot \boldsymbol{\nabla} U^{-1} + i\hbar g \Big[ U \boldsymbol{W} \cdot \boldsymbol{\nabla} U^{-1} - (\boldsymbol{\nabla} U) \cdot \boldsymbol{W} U^{-1} \Big] = 0 \quad .$$
<sup>(20)</sup>

Similarly, forcing W' to respect the Coulomb gauge condition we have

$$i\hbar g \boldsymbol{\nabla} \cdot \boldsymbol{W}' = -\hbar^2 \Big\{ (\boldsymbol{\nabla} U) \cdot \boldsymbol{\nabla} U^{-1} + U \boldsymbol{\nabla}^2 U^{-1} \Big\} + i\hbar g \Big[ U \boldsymbol{W} \cdot \boldsymbol{\nabla} U^{-1} + (\boldsymbol{\nabla} U) \cdot \boldsymbol{W} U^{-1} \Big] = 0 \quad .$$
(21)

Adding the above two equations leads to

$$\left(\boldsymbol{\nabla} - \frac{2ig}{\hbar}\boldsymbol{W}\right) \cdot \boldsymbol{\nabla} U^{-1} = 0 \quad . \tag{22}$$

What is remarkable is that the Eq. (22) is in fact equivalent to the previous two conditions, namely Eqs. (20) and (21). This can be shown by inserting into Eqs. (20) and (21) the following expressions

$$igU\mathbf{W} \cdot \nabla U^{-1} = \frac{\hbar}{2}U\nabla^2 U^{-1} ,$$
  
$$ig\nabla U \cdot \mathbf{W}U^{-1} = -\frac{\hbar}{2}(\nabla^2 U)U^{-1}$$

TABLE I: Some special spin-orbit couplings that has vanishing determinant  $det(W_i^a) = 0$ .

$W_i^a = \left(\begin{array}{cc} -\alpha & \alpha \\ -\alpha & \alpha \end{array}\right)$	$\boldsymbol{W} = (-\alpha \tau_x + \alpha \tau_y, -\alpha \tau_x + \alpha \tau_y)$
$W_i^a = \begin{pmatrix} \alpha & \alpha \\ -\alpha & -\alpha \end{pmatrix}$	$\boldsymbol{W} = (\alpha \tau_x + \alpha \tau_y, -\alpha \tau_x - \alpha \tau_y)$
$W_i^a = \left(\begin{array}{cc} \alpha & \beta \\ \alpha & \beta \end{array}\right)$	$\boldsymbol{W} = (\alpha \tau_x + \beta \tau_y, \alpha \tau_x + \beta \tau_y)$
$W_i^a = \left(\begin{array}{cc} \alpha & \alpha \\ \beta & \beta \end{array}\right)$	$\boldsymbol{W} = (\alpha \tau_x + \alpha \tau_y, \beta \tau_x + \beta \tau_y)$

where the former is a re-expression of Eq. (22) and the latter is obtained from the hermitian conjugation of the former. Hence, Eq. (22) is the differential equation that gives the class of gauge transformations U that respect the Coulomb's gauge condition, and under which the Hamiltonian is gauge-covariant. The equation is valid for time-dependent transformations and in the presence of electromagnetic fields. However, for the sake of simplicity, in the rest of the article the Hamiltonian is taken to be as in Eq. (9) (i.e.,  $\mathbf{A} = 0$  and H is time independent) and only time-independent transformations are discussed.

Rather than attempting to find all of the solutions of Eq. (22) systematically, which is an involved task, we start by noting that a class of solutions of the above equation that one can easily *guess* is

$$U = \exp\left(-\frac{2ig}{\hbar}\boldsymbol{W}\cdot\boldsymbol{x}\right) \tag{23}$$

valid for W satisfying the following conditions:

$$[W_1, W_2] = 0 , (24)$$

$$x_j \partial_i W_j = 0 \quad (i = 1, 2) . \tag{25}$$

The first condition above means that the field W is now Abelian, i.e., U(1), although it is still a 2 × 2 matrix. It is straightforward to check that the above  $U^{-1}$  satisfies both conditions (20) and (21). We now move on to try to find explicitly various fields W satisfying the above two conditions so as to identify spin-orbit couplings that lead to gauge-covariant Hamiltonians.

The condition in Eq. (25) is satisfied trivially by restricting our fields to those that are space-time independent. This amounts then to considering fields that lead to a vanishing field strength tensor, i.e.,

$$F_{ij} = \partial_i W_j - \partial_j W_i - \frac{ig}{\hbar} \left[ W_i, W_j \right] = 0$$
<sup>(26)</sup>

To this end, we first analyze condition (24) more closely. Writing  $W_i = W_i^a \tau_a$ , and taking a to run over 1 and 2 only (i.e., assuming that  $W_i^3 = 0$ ), this condition can be expressed as

$$[W_1, W_2] = i\tau_3 \left( W_1^1 W_2^2 - W_1^2 W_2^1 \right) = 0 \quad , \tag{27}$$

which can be also written in terms of a determinant

$$\begin{vmatrix} W_1^1 & W_1^2 \\ W_2^1 & W_2^2 \end{vmatrix} = 0 \quad .$$
<sup>(28)</sup>

So, our task reduces to the trivial task of finding sets of  $2 \times 2$  constant matrices with vanishing determinants. The possibilities are obviously infinite! We list a few of these in Table I.  $\alpha$  and  $\beta$  are constants, and the notation is self-explanatory.

The first two fields in Table I are of special importance as they represent the R-D spin-orbit coupling with constant coupling coefficients in the special cases of  $\alpha = \pm \beta$ , respectively, a model that was studied extensively in the literature. Here, they appear just as two members of an infinite set of possibilities. Other field configurations have no physical realizations as far as we know.

To gain a deeper insight on the meaning of the gauge transformation, Eq. (23), we first investigate its effect on the gauge field itself W;

$$W \to W' = \frac{i\hbar}{g} U \nabla U^{-1} + U W U^{-1} = -2U W U^{-1} + U W U^{-1} = -U W U^{-1} = -W$$
 (29)

Our gauge transformation, under which the Hamiltonian is gauge-covariant amounts merely to reversing the direction of the gauge field, i.e., the electric field generating the spin-orbit coupling, which is a satisfying result. To investigate the effect of the corresponding phase transformation on the wave function, we note that Eq. (23) can be expressed as

$$U = \exp\left(-i\boldsymbol{\eta}\cdot\boldsymbol{\tau}\right) = I\cos\frac{\eta}{2} - i\hat{\boldsymbol{n}}\cdot\boldsymbol{\sigma}\sin\frac{\eta}{2}$$
(30)

where

$$\eta^a = \frac{2g}{\hbar} W_i^a x^i \tag{31}$$

with both a and i running over 1 and 2 only,  $\eta = |\boldsymbol{\eta}|$  and  $\hat{\boldsymbol{n}} = \boldsymbol{\eta}/\eta$ . The gauge transformation U is, therefore, a rotation about the axis  $\hat{\boldsymbol{n}}$  with a *space-dependent* angle  $\eta$ . To consider a specific example, consider  $W_i^a = \begin{pmatrix} \alpha & \beta \\ \alpha & \beta \end{pmatrix}$  (or  $\boldsymbol{W} = (\alpha \tau_x + \beta \tau_y, \alpha \tau_x + \beta \tau_y)$ ), which is the third entry in Table I above, for which we have  $\eta = (x + y)\sqrt{\alpha^2 + \beta^2}$  and  $\hat{\boldsymbol{n}} = (\alpha \hat{\boldsymbol{i}} + \beta \hat{\boldsymbol{j}})/\sqrt{\alpha^2 + \beta^2}$ . Thus

$$U = I \cos\left((x+y)\sqrt{\alpha^2 + \beta^2}\right) - i\hat{\boldsymbol{n}} \cdot \boldsymbol{\sigma} \sin\left((x+y)\sqrt{\alpha^2 + \beta^2}\right) \quad . \tag{32}$$

Here, the space-dependence of the rotation angle in the arguments of the trigonometric functions above is evident.

It is a well-established fact that the vanishing of the field strength tensor of any gauge-field is a necessary and sufficient condition for the existence of a gauge transformation that takes this field to zero[17]. Therefore, it should be possible to transform all the gauge fields under consideration to zero. Finding the transformation that achieves this is an easy task. One immediately checks that for

$$U = \exp\left(-\frac{ig}{\hbar}\boldsymbol{W}\cdot\boldsymbol{x}\right) \tag{33}$$

one has:

$$\boldsymbol{W} \to \boldsymbol{W}' = \frac{i\hbar}{g} U \boldsymbol{\nabla} U + U \boldsymbol{W} U^{-1} = -U \boldsymbol{W} U^{-1} + U \boldsymbol{W} U^{-1} = 0 \quad . \tag{34}$$

Obviously, the above transformation can also be brought to the form of a rotation about some axis with a positiondependent angle, just as was done with the gauge transformation, Eq. (30). The work in Ref. 21 considered a specific field; the one that results from the R-D coupling in the special cases  $\alpha = \pm \beta$  (the first and second entries in Table I). The gauge field was gauged away using a transformation identical to the one above leaving a free particle Hamiltonian with the same spectrum as that of the spin-orbit coupled one. This fact was employed to account for the appearance of a persistent spin helix (PSH) in this model: One can imagine a free particle that enters a region where the spin-orbit coupling is turned on, which corresponds to a gauge transformation that is the inverse of the one given in Eq. (34), so the particle is subject to a position-dependent rotation about some axis that is similar to the one discussed above. The particle will propagate with its spin rotating so that its projection along the rotation axis is conserved, a phenomenon that was called the PSH[21]. Here, we are saying that it is possible to gauge away any space-time independent field satisfying the condition (27). In other words, any Hamiltonian with a gauge-field satisfying the condition (27) is in fact unitarily equivalent to the free particle Hamiltonian, and we can have the phenomenon of PSH in all these cases.

There is a fine but important detail here, which was not noted, or at least not discussed in the literature. The spin-orbit coupled Hamiltonian, Eq. (9) is not in fact covariant under the transformation, Eq. (34). The reason is the quadratic term  $g^2 \mathbf{W} \cdot \mathbf{W}$ , which is non-zero in one Hamiltonian and vanishes in the other one. Gauge-covariance, on the other hand, as we have noted earlier, requires that it transforms as  $g^2 \mathbf{W} \cdot \mathbf{W} \rightarrow g^2 U \mathbf{W} \cdot \mathbf{W} U^{-1} = g^2 \mathbf{W} \cdot \mathbf{W}$ ? Fortunately, for space-time independent fields, this quadratic term is just a constant. Thus, the gauge-transformed Hamiltonian is gauge-covariant up to a constant, and is unitarily equivalent to a free particle Hamiltonian up to a constant;

$$H\left(\boldsymbol{W'}=0\right) = UH\left(\boldsymbol{W}\right)U^{-1} + \frac{g^2}{2m}\boldsymbol{W}\cdot\boldsymbol{W} = UH\left(\boldsymbol{W}\right)U^{-1} + \text{constant} = H_0.$$
(35)

In the work in Ref. 14, where the R-D spin-orbit coupling in the special case  $\alpha = \beta$  =constant was considered, the gauge field was gauged away using exactly the same transformation above. However, there, the transformation of the constant quadratic term was ignored altogether. Being just a constant, the result is the same. However, strictly

speaking, the transformation of this term should be considered. Our treatment here illuminates this point. It also suggests that there exist —in principle— a wide class of spin-orbit coupling forms that lead to essentially the same result, i.e., the unitary-equivalence with the free particle Hamiltonian, and consequently the emergence of the PSH. Only a few of these are realized physically, the others are —so far— theoretical. Yet, it is highly possible that physical systems with such couplings might be realized in the near future.

At this point, we can come back to the question of finding solutions to Eq. (22). Indeed, from Eq. (35), it is obvious that under the transformation

$$U_{\boldsymbol{W}'} = \exp\left(\frac{ig}{\hbar}\boldsymbol{W}'\cdot\boldsymbol{x}\right) \tag{36}$$

the free-particle Hamiltonian (for W' independent of space and time such that  $det(W_i^{a'}) = 0$ ) transforms as:

$$U_{\boldsymbol{W}'}H(0)U_{\boldsymbol{W}'}^{-1} = H(\boldsymbol{W}') + \frac{g^2}{2m}\boldsymbol{W}'\cdot\boldsymbol{W}'.$$
(37)

Therefore, we can immediately write down the following transformation:

$$U_{\boldsymbol{W}'}U_{\boldsymbol{W}}H\left(\boldsymbol{W}\right)U_{\boldsymbol{W}'}^{-1}U_{\boldsymbol{W}'}^{-1} = H\left(\boldsymbol{W'}\right) + \frac{g^2}{2m}\boldsymbol{W'}\cdot\boldsymbol{W'} - \frac{g^2}{2m}\boldsymbol{W}\cdot\boldsymbol{W}.$$
(38)

where we have denoted with  $U_{\mathbf{W}}$  the transformation (33). If, now, the condition  $\mathbf{W} \cdot \mathbf{W} = \mathbf{W}' \cdot \mathbf{W}'$  (or, equivalently  $W_i^a W_i^a = W_i^a W_i^{a'}$ ) is satisfied, then the above equation reduces to

$$U_{\boldsymbol{W}'}U_{\boldsymbol{W}}H\left(\boldsymbol{W}\right)U_{\boldsymbol{W}}^{-1}U_{\boldsymbol{W}'}^{-1} = H\left(\boldsymbol{W'}\right)$$
(39)

which immediately means that the unitary transformation

$$U = U_{\mathbf{W}'}U_{\mathbf{W}} = \exp\left(\frac{ig}{\hbar}\mathbf{W}'\cdot\mathbf{x}\right)\exp\left(-\frac{ig}{\hbar}\mathbf{W}\cdot\mathbf{x}\right) \quad . \tag{40}$$

induces a gauge transformation  $W \longrightarrow W'$  under which the Hamiltonian is gauge-covariant, i.e.,

$$\frac{\left(\boldsymbol{p}-g\boldsymbol{W}'\right)^2}{2m} - \frac{g^2}{2m^2}\boldsymbol{W}'\cdot\boldsymbol{W}' + V(\boldsymbol{x}) = U\left(\frac{\left(\boldsymbol{p}-g\boldsymbol{W}\right)^2}{2m} - \frac{g^2}{2m^2}\boldsymbol{W}\cdot\boldsymbol{W} + V(\boldsymbol{x})\right)U^{-1} \quad , \tag{41}$$

Therefore, all fields W independent of space and time such that  $\det(W_i^a) = 0$  with the same value of the product  $W \cdot W$  are related by gauge transformations of the form (40) that are symmetries of the Hamiltonian. The gauge transformation (23) found earlier is just one of these transformations as can be seen easily by substituting  $W' \longrightarrow -W$  in (40). Moreover, although laborious, it is straightforward to show that this U satisfies Eq. (22). Thus, infinite classes of solutions for this equation have been constructed. Physically, the above results say that different spin-orbit couplings corresponding to various electric fields configurations are related —if they satisfy certain conditions— by a gauge-transformation, and their corresponding Hamiltonians are unitarily equivalent, thus having the same spectrum.

# III. GLOBAL GAUGE-SYMMETRY AND CONSERVED SPIN CURRENTS

We turn now to the investigation of the SU(2) global phase invariance of the Hamiltonian, Eq. (9). When  $\boldsymbol{W}$  is space-time independent satisfying the commutation relations in Eq. (27), H is invariant under the gauge transformation  $U = \exp\left(-\frac{2ig}{\hbar}\boldsymbol{W}\cdot\boldsymbol{x}\right)$ , which is actually an Abelian symmetry as we have noted earlier. If we replace  $\boldsymbol{x}$  with a constant vector  $\frac{2g}{\hbar}\boldsymbol{l}$ , we will have the Hamiltonian invariant under the global phase transformation

$$U = \exp\left(-i\boldsymbol{W}\cdot\boldsymbol{l}\right) \quad . \tag{42}$$

Obviously, this is just a rotation in the spin space. To put it in a more convenient form, we again write

$$\boldsymbol{W} \cdot \boldsymbol{l} = W_i^a \tau_a l^i = \xi^a \tau_a$$

with

$$\xi^a = W_i^a l^i , \quad \xi = |\boldsymbol{\xi}| , \quad \hat{\boldsymbol{n}} = \boldsymbol{\xi}/\boldsymbol{\xi}$$
(43)

so that

$$U = \exp\left(-i\boldsymbol{\xi}\cdot\boldsymbol{\tau}\right) = \exp\left(-i\boldsymbol{\xi}\hat{\boldsymbol{n}}\cdot\frac{\boldsymbol{\sigma}}{2}\right) \ . \tag{44}$$

The invariance of H under the above transformation immediately implies the conservation of the operator  $\boldsymbol{\sigma} \cdot \hat{\boldsymbol{n}}$  (in the Heisenberg picture), which means that if  $\phi$  is the wavefunction, then  $\int \phi^{\dagger} \boldsymbol{\sigma} \cdot \hat{\boldsymbol{n}} \phi$  is constant in time. Therefore, we expect the density  $\boldsymbol{S} \cdot \hat{\boldsymbol{n}} = \phi^{\dagger} \boldsymbol{\sigma} \cdot \hat{\boldsymbol{n}} \phi$  to satisfy a continuity equation with a conserved current of the form

$$\frac{\partial}{\partial t} \left( \boldsymbol{S} \cdot \hat{\boldsymbol{n}} \right) + \partial_i \left( \boldsymbol{J}_i \cdot \hat{\boldsymbol{n}} \right) = 0 \tag{45}$$

The above relation then means that the spin density  $S \cdot \hat{n}$  is conserved, and thus a particle polarized in this direction will not feel a torque and thus will have a long life time, a property of great value in spintronics [22, 23]. Since  $\hat{n}$  is determined by  $W_i^a$ , i.e., by the specific spin-orbit coupling present, then for different spin-orbit couplings, we will have different conserved spin densities.

It is interesting to see how the above continuity equation emerges from the Schrödinger equation in the conventional sense. For the Hamiltonian in Eq. (9), with W being any arbitrary SU(2) gauge field in the Coulomb gauge, one gets after some algebra:

$$\frac{\partial S^a}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{J}^a = \frac{-ig}{2m} \epsilon^{abc} \left[ \phi^{\dagger} \sigma^b W_i^c \left( \partial_i \phi \right) - \left( \partial_i \phi^{\dagger} \right) W_i^b \sigma^c \phi \right]$$
(46)

where the current is  $J^a = J_0^a + J^{a'}$ ;  $J_0^a$  is the bare spin current defined earlier in Eq. (1), and  $J'^a$  is defined as

$$\boldsymbol{J}^{\prime a} = -\frac{g}{2m} \phi^{\dagger} \boldsymbol{W}^{a} \phi \quad . \tag{47}$$

Obviously the spin current is not conserved due to the presence of spin-orbit coupling, which —just as in the case of a magnetic field— breaks the SU(2) global phase invariance. The above continuity equation can be put into an alternative form where the current is covariantly conserved. To do this, we note that the right-hand side of Eq. (46) can be brought to the form  $-\frac{g}{\hbar}\epsilon^{abc} W^b \cdot J^c_W$  with

$$\boldsymbol{J}_{\boldsymbol{W}}^{c} \equiv -\frac{i\hbar}{2m} (\phi^{\dagger} \sigma^{a} \boldsymbol{D} \phi - (\boldsymbol{D} \phi)^{\dagger} \sigma^{a} \phi)$$

where  $D = \nabla - \frac{ig}{\hbar} W$  is the covariant derivative. Therefore, Eq. (46) can be cast into the form:

$$\frac{\partial}{\partial t} S^{a}[\boldsymbol{W}] + \boldsymbol{D}^{ac} \cdot \boldsymbol{J}_{\boldsymbol{W}}^{c} = 0$$
(48)

where  $D^{ac} \equiv \delta^{ac} \nabla + \frac{g}{\hbar} \epsilon^{abc} W^b$ . This is the covariant continuity equation, and was written down in Ref. 10 directly from the Lagrangian of the theory rather than from the equations of motion. To derive the continuity equation, Eq. (45), from the above equation for the special cases when the fields W satisfy the commutation relations (24), we multiply both sides of Eq. (48) by  $\hat{n}^a$  to get

$$\frac{\partial}{\partial t}\hat{n}^{a}S^{a}[\boldsymbol{W}] + \hat{n}^{a}\boldsymbol{D}^{ac}\cdot\boldsymbol{J}_{\boldsymbol{W}}^{c} = 0, \qquad (49)$$

We need to show that  $\epsilon^{abc} \hat{n}^a W_i^b J_{\boldsymbol{w},i}^c = 0$  so as to reduce the covariant derivative to the ordinary derivative. Expressing  $n^a$  in terms of  $\xi$  and using Eq. (43) we get

$$\frac{g}{\hbar}\epsilon^{abc}n^a W^b_i J^c_{\boldsymbol{W},i} = \frac{g}{\hbar}\frac{l_j}{\xi}\epsilon^{abc} W^a_j W^b_i J^c_{\boldsymbol{W},i} = 0$$

where we have noted that  $det(W_i^a) = 0$  is equivalent to  $\epsilon^{abc}W_j^aW_i^b = 0$  for i, j = 1, 2. Thus, Eq. (49) immediately reduces to the continuity equation, Eq. (45).

Again, we apply the above results to explicitly find the conserved spin and current densities for some of the field examples that we have presented in Table I. For  $(W^a) = \begin{pmatrix} -\alpha & \alpha \\ -\alpha & \alpha \end{pmatrix}$ , so that  $\hat{\boldsymbol{n}} = -\frac{1}{2}(\hat{\boldsymbol{i}} - \hat{\boldsymbol{i}}) = \hat{\boldsymbol{n}}_1$ , we get

$$\frac{\partial}{\partial t} \left( \boldsymbol{S} \cdot \hat{\boldsymbol{n}}_1 \right) + \partial_i \left( \boldsymbol{J}_i \cdot \hat{\boldsymbol{n}}_1 \right) = 0$$
(50)

The above result which corresponds to the R-D coupling with  $\alpha = \beta = \text{constant}$  was reported in Ref. 22, without the use of gauge-field formalism, however. In our case, we have re-derived it within the context of a more general

theoretical formalism, and extended it to other couplings. For  $(W_i^a) = \begin{pmatrix} \alpha & \beta \\ \alpha & \beta \end{pmatrix}$ , we have  $\mathbf{S} \cdot \hat{\mathbf{n}}_2$  satisfying the above continuity equation, with  $\hat{\mathbf{n}}_2 = (\alpha \hat{\mathbf{i}} + \beta \hat{\mathbf{j}})/\sqrt{\alpha^2 + \beta^2}$ . As for  $(W_i^a) = \begin{pmatrix} \alpha & \alpha \\ \beta & \beta \end{pmatrix}$  we have  $\mathbf{S} \cdot \hat{\mathbf{n}}_3$  conserved with  $\hat{\mathbf{n}}_2 = (\alpha \hat{\mathbf{i}} + \beta \hat{\mathbf{j}})/\sqrt{\alpha^2 + \beta^2}$ .

 $\hat{\boldsymbol{n}}_3 = (\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}})/\sqrt{2}.$ 

We again note that the natural framework for the derivation of the continuity equation is Noether's theorem. The infinitesimal version of the transformation, Eq. (42), is a global continuous symmetry of the Lagrangian of the theory, and Noether's theorem demands the existence of a conserved charge and current related by a continuity equation, which can be shown to be just the continuity equation, Eq. (45). We give the details of this derivation as well in the appendix.

Since the free particle Hamiltonian and the spin-orbit-coupled Hamiltonian with fields of the type given in Table I are related (up to a constant) by the gauge transformation, Eq. (34), it should be possible to get the continuity equations, Eq. (45), for these classes of fields from the free-particle continuity equation by a gauge transformation. Obviously, it will be sufficient to show how one can relate the free-particle continuity equation to the continuity equation, Eq. (48), through a gauge transformation. For this purpose, we first express both the bare and the covariant continuity equations in the adjoint representation, so we define

$$\widetilde{S}[\boldsymbol{W}] \equiv S^{a}[\boldsymbol{W}]\tau_{a} \quad , \qquad \widetilde{J}_{i}[\boldsymbol{W}] \equiv J^{a}_{i\boldsymbol{W}}[\boldsymbol{W}]\tau_{a} \quad , \tag{51}$$

and express the covariant continuity equation, Eq. (48), as

$$\frac{\partial}{\partial t}\widetilde{S}[\boldsymbol{W}] + \widetilde{D}_{i}[\boldsymbol{W}]\widetilde{J}_{i}[\boldsymbol{W}] = 0 \quad , \tag{52}$$

where  $\widetilde{D}_i[\mathbf{W}] = \partial_i - \frac{ig}{\hbar}[W_i^a \tau_a, \ldots]$  is the covariant derivative in the adjoint representation, and the dependence on the gauge field W was now shown explicitly. Under a gauge transformation,  $W \to W'$ , given by Eq. (6), the above equation transforms to

$$\frac{\partial}{\partial t}\widetilde{S}[\mathbf{W}'] + \widetilde{D}_i[\mathbf{W}']\widetilde{J}_i[\mathbf{W}'] = 0 \quad .$$
(53)

Now, the free-particle continuity equation can be expressed also in the adjoint representation as

$$\frac{\partial}{\partial t}\widetilde{S}[0] + \widetilde{D}_i[0]\widetilde{J}_i[0] = 0 \tag{54}$$

with  $\widetilde{D}_i[0] = \partial_i - \frac{ig}{\hbar}[0,\ldots]$ . Under the (inverse of) the gauge transformation given by Eq. (34), we have

$$0 \to \boldsymbol{W} = \frac{i\hbar}{g} U^{-1} \boldsymbol{\nabla} U, \tag{55}$$

so that  $\widetilde{D}_i[0] = \partial_i - \frac{ig}{\hbar}[0,\ldots] \to \widetilde{D}_i[\mathbf{W}] = \partial_i - \frac{ig}{\hbar}[\mathbf{W},\ldots]$ , and Eq. (54) transforms immediately to Eq. (52).

#### SUMMARY AND CONCLUSIONS IV.

The question of gauge-covariance in the two-dimensional Hamiltonian of a spin one-half particle subject to spinorbit coupling formulated in a non-Abelian gauge field language was considered. Such a Hamiltonian contains a term that is quadratic in the gauge field, which generally breaks the gauge symmetry. Moreover, with the gauge field representing a physical quantity, namely the electric field, the theory is a gauge-fixed one. The conditions for the existence of residual gauge symmetry in the Coulomb gauge were investigated, and a condition for its existence, namely Eq. (22), was derived. A class of gauge fields and their corresponding gauge transformations that satisfy this condition were found. They turn out to be fields whose components are any  $2 \times 2$  space-time independent Abelian matrices, and thus are just gauges having a vanishing field strength tensor. The corresponding gauge transformation that are symmetries of the Hamiltonian are seen to correspond to rotations in the spin space of the particle with space-dependent rotation angles, the specific form of which and of the rotation axes being determined by the explicit

form of the gauge field . Gauging away the gauge field leads us to see the phenomenon of persistent spin helix (PSH) discussed in the literature[21]. The global version of the admissible gauge transformations, which also form a symmetry of the Hamiltonian, lead to a continuity equation for the projection of the spin density along the rotation axis, which is fixed by the specific form of the gauge field. The spin along these axes is not subject to any torque and is, therefore, long-living; a property that is important in spintronic applications and was reported in the literature for the specific case of a special case of the R-D coupling in Ref. 22. Our re-derivation adds to this —as well as to the PSH derivation— in two ways: First, it comes within a general framework, based on the idea of gauge-covariance of the Hamiltonian of two-dimensional systems with spin-orbit coupling. Second; the phenomena reported in the above two references were derived for the specific cases of R-D coupling with constant equal coefficients only. In our case, these two cases appear as only two special cases of wider theoretical possibilities. The R-D couplings are the only couplings that are realized physically so far.

Finally, an important point that deserves further attention is the existence of solutions to Eq. (22) other than the Abelian space-independent ones found here. Indeed, it would be interesting to find such solutions; this issue is under current investigation.

# Acknowledgment

This work was supported by the Campus Research Fund of Middle East Technical University Northern Cyprus Campus under project BAP-Fen 11.

## Appendix

In this section, Noether's theorem is applied for deriving the expression for the spin current, Eq. (1), and establishing the continuity equation, Eq. (45). For this purpose, we first look at the case of bare spin current. Consider the case where there is no magnetic field and no spin-orbit coupling, i.e., the gauge field is zero. The Pauli Lagrangian density for the electrons is then

$$\mathcal{L} = \frac{i\hbar}{2} \left( \phi^{\dagger} \dot{\phi} - \dot{\phi}^{\dagger} \phi \right) - \frac{\hbar^2}{2m} \nabla \phi^{\dagger} \cdot \nabla \phi - V(\boldsymbol{x}) \phi^{\dagger} \phi \quad , \tag{56}$$

where  $\phi$  is the two-component spinor wave function. This Lagrangian density has the global SU(2) symmetry, i.e., it is invariant under the transformation

$$\phi \to \exp(-i\boldsymbol{\alpha} \cdot \boldsymbol{\sigma})\phi \approx \phi - i\boldsymbol{\alpha} \cdot \boldsymbol{\sigma}\phi \quad ,$$

where  $\boldsymbol{\alpha}$  is an infinitesimal vector that is independent of position and time. Noether's theorem[17] then implies the conservation of current density  $\mathcal{J}^{\mu}$  (i.e., the continuity equation  $\partial_{\mu}\mathcal{J}^{\mu} = 0$  is satisfied) where

$$\mathcal{J}^{\mu} = \delta \phi^{\dagger} \frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \phi^{\dagger})} + \frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \phi)} \delta \phi \quad .$$
(57)

It is straightforward to check that the time and position components of the conserved current are given by

$$\mathcal{J}^0 = \hbar \alpha_a S^a \quad , \tag{58}$$

$$\mathcal{J} = \hbar \alpha_a J_0^a \quad , \tag{59}$$

where  $S^a$  and  $J_0^a$  are as given in Eq. (1).

We now turn to the derivation of the continuity equation, Eq. (45) by applying Noether's theorem to the Pauli Lagrangian coupled to the  $2 \times 2$ -matrix Abelian gauge field  $\boldsymbol{W}$ . Note that the gauge fields we are considering here are space-time-independent Abelian class of fields satisfying Eq. (27). For this purpose, we write down the Pauli Lagrangian for a spin one-half particle with spin-orbit coupling encoded in a coupling to the gauge field  $\boldsymbol{W}$ ,

$$\mathcal{L} = \frac{i\hbar}{2} (\phi^{\dagger} \dot{\phi} - \dot{\phi}^{\dagger} \phi) - \frac{\hbar^2}{2m} \left( (\boldsymbol{D}\phi)^{\dagger} \cdot \boldsymbol{D}\phi \right) + \frac{g^2}{2m} \phi^{\dagger} \boldsymbol{W} \cdot \boldsymbol{W}\phi - V(\boldsymbol{x})\phi^{\dagger}\phi \quad , \tag{60}$$

where  $D = \nabla - \frac{ig}{\hbar} W$ . The infinitesimal version of the global gauge transformation, Eq. (42) is

$$\phi \to U\phi \simeq (1 - i\boldsymbol{w} \cdot \boldsymbol{l})\phi = (1 - i\xi\hat{n}^a\tau_a)\phi \tag{61}$$

where  $\boldsymbol{w}$  and  $\xi^a$  are now *infinitesimal*;  $\boldsymbol{\xi}$  and  $\hat{\boldsymbol{n}}$  are defined by Eq. (43). Note that  $\hat{\boldsymbol{n}}$  is *not arbitrary* but depends on the particular  $\boldsymbol{W}$ . Since  $\boldsymbol{w}$  and  $\boldsymbol{W}$  still satisfy the algebra (27), the above transformation is a symmetry of the Lagrangian. The variation in the fields is

$$\delta\phi = -i\xi\hat{n}^a\tau_a\phi \quad , \qquad \delta\phi^\dagger = i\phi^\dagger\xi\hat{n}^a\tau_a \quad . \tag{62}$$

Noether's theorem then dictates the existence of a conserved current given by Eq. (57). It is straightforward to show that the associated continuity equation is

$$\xi \hat{n}^a \partial_t S^a + \xi \hat{n}^a \partial_i \left( J^a_{i\mathbf{W}} \right) = 0 .$$
<sup>(63)</sup>

Since  $\xi$  is arbitrary (but not  $\hat{n}$ ), the continuity equation, Eq. (45) follows.

- [1] I. Žutić, J. Fabian and S. Das Sarma, Rev. Mod. Phys. 76, 323 (2004).
- [2] see for example R. Winkler, Spin-Orbit Coupling Effects in Two-Dimensional Electron and Hole Systems, Springer-Verlag, Berlin, 2003.
- [3] E. I. Rashba, Sov. Phys. Solid State 2, 1109 (1960).
- [4] G. Dresselhaus, Phys. Rev. **100**, 580 (1955).
- [5] Q-F. Sun and X. C. Xie, Phys. Rev. B 72, 245305 (2005).
- [6] J. Shi, P. Zhang, D. Xiao and Q. Niu, Phys. Rev. Lett. 96, 076604 (2006).
- [7] R. Shen, Y. Chen, Z. D. Wang and D. Y. Xing, Phys. Rev. B 74, 125313 (2006).
- [8] Y. Li and R. Tao, Phys. Rev. B 75, 075319 (2007).
- [9] P-Q. Jin and Y-Q. Li, Phys. Rev. B 77, 155304 (2008).
- [10] P-Q. Jin, Y-Q. Li and F-C. Zhang, J. Phys. A: Math. Gen. 39, 7115 (2006).
- [11] J. Fröhlich and U. M. Studer, Rev. Mod. Phys. 65, 733 (1993).
- [12] I. V. Tokatly, Phys. Rev. Lett. **101**, 106601 (2008).
- [13] C. A. Dartora and G. G. Cabrera, Phys. Rev. B 78, 012403 (2008).
- [14] S-H. Chen and C-R. Chang, Phys. Rev. B 77, 045324 (2008).
- [15] J-S. Yang, X-G. He, S-H. Chen and C-R. Chang, Phys. Rev. B 78, 085312 (2008).
- [16] E. Medina, A. Lopez and B. Berche, Eur. Phys. Lett. 83, 47005 (2008).
- [17] S. Weinberg, The Quantum Theory of Fields, Vol.1, Foundations, Cambridge University Press, Cambridge, 1995; and Vol.2, Modern Applications, Cambridge University Press, Cambridge, 1996.
- [18] G. Bastard, Wave Mechanics Applied to Semiconductor Heterostructures, Les Editions de Physique, Les Ulis, 1998.
- [19] B. W. A. Leurs, Z. Nazario, D. I. Santiago and J. Zaanen, Ann. Phys. 323, 907 (2008).
- [20] V. Gribov, Nucl. Phys. B139, 1 (1978).
- [21] B. A. Bernevig, J. Orenstein and S-C. Zhang, Phys. Rev. Lett. 97, 236601 (2006).
- [22] J. Schliemann, J. C. Egues and D. Loss, Phys. Rev. Lett. 90, 146801 (2003).
- [23] K. C. Hall, W. H. Lau, K. Gündoğdu, M. E. Flatté and T. F. Boggess, Appl. Phys. Lett. 83, 2937 (2003).