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Fixed point free action on groups of odd order

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Abstract

Let A be a finite abelian group that acts fixed point freely on a finite (solvable) group G. Assume that $|G|$ is odd and A is of squarefree exponent coprime to 6. We show that the Fitting length of G is bounded by the length of the longest chain of subgroups of A.

Keywords: Solvable groups; Fixed point free action; Finite groups; Representations

Introduction

Let G be a finite solvable group and A be a finite group acting fixed point freely on G. A long-standing conjecture is that if $(|G|, |A|) = 1$, then the Fitting length $f(G)$ of G is bounded by the length $\ell(A)$ of the longest chain of subgroups of A. By an elegant result due to Bell and Hartley [1], it is known that any finite nonnilpotent group A can act fixed point freely on a solvable group G of arbitrarily large Fitting length with $(|G|, |A|) \neq 1$. We expect that the conjecture is true when the coprimeness condition is replaced by the assumption that A is nilpotent. This question is still unsettled except for cyclic groups A of order pq and pqr for pairwise distinct primes p, q and r [3,4].

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In the present paper we establish the conjecture without the coprimeness condition when A is a finite abelian group of squarefree odd exponent not divisible by 3 and $|G|$ is odd. This improves the bound given in Theorem 3.4 of [6]; as a by-product we also improve a bound given in Theorem 8.5 of [2].

Namely, we shall prove the following:

Theorem A. *Let A be a finite abelian group acting fixed point freely on a finite group G of odd order. If A has squarefree exponent coprime to* 6*, then $f(G) \leqslant \ell(A)$.*

Theorem B. *Let G be a finite* (*solvable*) *group of order coprime to* 6*. If C is a Carter subgroup of G, then $f(G) \leqslant 2(2^{\ell(C)} - 1)$.*

Preliminary remarks. All the groups considered in this paper are finite and solvable. Except for the following, the notation and terminology are as in [2].

Let G be a group.

We denote by $\tilde{G}$ the Frattini factor group of G.

If S is a subgroup of G and $a \in G$, then for any positive integer n, we denote by $[S, a]^n$ the commutator subgroup $[S, a, \dots, a]$ with a repeated n times.

Let K be a group acting on G, that is, there is a homomorphism from K into $\mathrm{Aut}(G)$. We write $(K \text{ on } G)$ to denote this action. If $x \in K$, then we write g^x for the image of $g \in G$ under the automorphism of G which is the image of x in $\mathrm{Aut}(G)$. Let another group L act on K, and let $l \in L$. We write $(K \text{ on } G)^l$ to denote the action of K on G given by $x \to (K \text{ on } G)(x^{l^{-1}})$ for $x \in K$.

Let K be a group acting on groups H and G. We say $(K \text{ on } G)$ and $(K \text{ on } H)$ are weakly equivalent if each nontrivial irreducible section of $(K \text{ on } G)$ is K-isomorphic to an irreducible section of $(K \text{ on } H)$ and vice versa. We write $(K \text{ on } H) \equiv_w (K \text{ on } G)$ if $(K \text{ on } H)$ is weakly equivalent to $(K \text{ on } G)$.

Let K, L, G and H be groups.

(a) If $(K \text{ on } G) \equiv_w (K \text{ on } H)$, then $(L \text{ on } G) \equiv_w (L \text{ on } H)$ for each $L \leqslant K$.

(b) Let L act on K and K act on G and H. If $(K \text{ on } G) \equiv_w (K \text{ on } H)$, then $(K \text{ on } G)^l \equiv_w (K \text{ on } H)^l$ for each $l \in L$.

(c) Let V be a completely reducible kG-module for a field k and let L act on G. Let $l \in L$ and V_l denote the kG-module with respect to $(G \text{ on } V)^l$. Assume that $(G \text{ on } V) \equiv_w (G \text{ on } V)^l$. Let $M \leqslant G$ such that M is $\langle l \rangle$-invariant, and W be the sum of all irreducible kG-submodules of V on which M acts nontrivially. Then $W = W^\# = W_l$ as subspaces where $W^\#$ stands for the sum of all irreducible kG-submodules of V_l on which M acts nontrivially.

Note that W and W_l need not be isomorphic as kG-modules.

Lemma 1. *Let $S\langle\alpha\rangle$ be a group where $S \lhd S\langle\alpha\rangle$, S is an s-group for some prime s, $\Phi(S) \leqslant Z(S)$, $\langle\alpha\rangle$ is cyclic of order p for an odd prime p. Suppose that V is a $kS\langle\alpha\rangle$-module for a field k of characteristic different from s. Then $C_V(\alpha) \neq 0$ if one of the following is satisfied*:

(i) $[Z(S), \alpha]$ *is nontrivial on V.*
(ii) $[S, \alpha]^{p-1}$ *is nontrivial on V and $p = s$.*

Furthermore, if $S\langle\alpha\rangle$ acts irreducibly on V or the characteristic of k is different from p, then we also have $(C \text{ on } C_V(\alpha)) \equiv_w (C \text{ on } V)$ where $C = C_D(\alpha)$ for

$$D = \begin{cases} S & \text{when (i) holds,} \\ [S,\alpha]^{p-1} & \text{when (ii) holds.} \end{cases}$$

Proof. See [2, Proposition 3.10]. □

Lemma 2. *(See Lemma 5.30 in [2].) Let $S \lhd S\langle\alpha\rangle$ where $\langle\alpha\rangle$ is cyclic of prime order and let V be an irreducible $kS\langle\alpha\rangle$-module. If E is an $\langle\alpha\rangle$-invariant subgroup of $Z(S)$ and U is a nonzero $E\langle\alpha\rangle$-submodule of V, then* $\mathrm{Ker}(E \text{ on } V) = \mathrm{Ker}(E \text{ on } U)$.

Lemma 3. *Let $S\langle\alpha\rangle$ be a group such that $S \lhd S\langle\alpha\rangle$ where $\langle\alpha\rangle$ is of prime order p. Suppose that V is a $kS\langle\alpha\rangle$-module for a field k of characteristic different from p, and Ω is an $S\langle\alpha\rangle$-stable subset of V^*. Set $V_0 = \bigcap\{\mathrm{Ker}\, f \mid f \in \Omega - C_\Omega(\alpha)\}$. If there exists a nonzero f in Ω and $x \in S$ such that $f(V_0) \neq 0$ and $[x,a,\alpha] \notin C_S(f)$ for each $1 \neq a \in \langle\alpha\rangle$, then $C_V(\alpha) \nsubseteq V_0$.*

Proof. Since $f(V_0) \neq 0$, it follows that $f \in C_\Omega(\alpha)$ and so $C_S(f)$ is normalized by $\langle\alpha\rangle$. The assumption $[x,a,\alpha] \notin C_S(f)$ for each $1 \neq a \in \langle\alpha\rangle$ yields that $[x,a] \notin C_S(f)$ for each $1 \neq a \in \langle\alpha\rangle$. Then $bxf \notin C_\Omega(\alpha)$ for each $b \in \langle\alpha\rangle$. Set $g = \sum_{b\in\langle\alpha\rangle} bxf$. It is clear that $g \in C_\Omega(\alpha)$ and so $[V,\alpha] \subseteq \mathrm{Ker}\, g$. Since $V = [V,\alpha] \oplus C_V(\alpha)$, either $g = 0$ or $C_V(\alpha) \nsubseteq \mathrm{Ker}\, g$. If the latter holds, then $C_V(\alpha) \nsubseteq V_0$ as claimed, because $V_0 \subseteq \mathrm{Ker}(bxf)$ for each $b \in \langle\alpha\rangle$. Hence we may assume that $g = 0$. Now $0 = x^{-1}g = f + \sum_{1\neq b\in\langle\alpha\rangle}[x,b]f$ and then $f = -\sum_{1\neq b\in\langle\alpha\rangle}[x,b]f$. Since $[x,b,\alpha] \notin C_S(f)$ by the hypothesis, we have $[x,b]f \notin C_\Omega(\alpha)$ for each $1 \neq b \in \langle\alpha\rangle$. Then $f(V_0) = 0$. This contradiction completes the proof. □

The following result is a generalization of Theorem 2.1.A in [5].

Theorem 1. *Let $S\langle\alpha\rangle$ be a group such that $S \lhd S\langle\alpha\rangle$, S is an s-group, $\langle\alpha\rangle$ is cyclic of order p for odd primes s and p with $p \geqslant 5$, $\Phi(\Phi(S)) = 1$, $\Phi(S) \leqslant Z(S)$.*

Suppose that k is a field of characteristic not dividing ps and V is a $kS\langle\alpha\rangle$-module such that $[S,\alpha]^{p-1}$ acts nontrivially on each irreducible submodule of $V|_S$.

Let Ω be an $S\langle\alpha\rangle$-stable subset of V^* which linearly spans V^* and set $V_0 = \bigcap\{\mathrm{Ker}\, f \mid f \in \Omega - C_\Omega(\alpha)\}$. Then $C_V(\alpha) \nsubseteq V_0$ and

$$\big(C_D(\alpha) \text{ on } C_V(\alpha)/C_{V_0}(\alpha)\big) \equiv_w \big(C_D(\alpha) \text{ on } V\big) \quad \text{where}$$

$$D = \begin{cases} [S,\alpha]^{p-1} & \text{when } s = p, \\ S & \text{otherwise.} \end{cases}$$

Proof. Assume that the theorem is false and consider a counterexample with $\dim V + |S\langle\alpha\rangle|$ minimal. Set $X = C_V(\alpha)/C_{V_0}(\alpha)$ and $C = C_D(\alpha)$.

Claim 1. *We may assume that S acts faithfully and $S\langle\alpha\rangle$ acts irreducibly on V and k is a splitting field for all subgroups of $S\langle\alpha\rangle$.*

Put $\bar{S} = S/\operatorname{Ker}(S \text{ on } V)$. By induction applied to the action of $\bar{S}\langle\alpha\rangle$ on V, we get $C_V(\alpha) \nsubseteq V_0$ and $(C_{\overline{D}}(\alpha) \text{ on } X) \equiv_w (C_{\overline{D}}(\alpha) \text{ on } V)$. As $\overline{C} = \overline{C_D(\alpha)} \leqslant C_{\overline{D}}(\alpha)$, we have obtained $(C \text{ on } X) \equiv_w (C \text{ on } V)$. Thus we may assume that S is faithful on V.

Since V is completely reducible as an $S\langle\alpha\rangle$-module, we have a collection $\{V_1, \ldots, V_l\}$ of irreducible $S\langle\alpha\rangle$-submodules of V such that $V = \bigoplus_{i=1}^{l} V_i$. Now $[S,\alpha]^{p-1}$ acts nontrivially on each irreducible constituent of $V_i|_S$ and hence $[S,\alpha]^{p-1}$ acts nontrivially on each V_i for $i = 1, \ldots, l$. It is easy to observe that $\Omega|_{V_i}$ is an $S\langle\alpha\rangle$-stable subset of V_i^* and $\langle\Omega|_{V_i}\rangle = V_i^*$ for each $i = 1, \ldots, l$. If V is not irreducible as an $S\langle\alpha\rangle$-module, we apply induction to the action of $S\langle\alpha\rangle$ on V_i for each i and get $C_{V_i}(\alpha) \nsubseteq (V_i)_0$ and $(C \text{ on } C_V(\alpha)/C_{(V_i)_0}(\alpha)) \equiv_w (C \text{ on } V_i)$. Set $X_i = C_{V_i}(\alpha)/C_{V_i \cap V_0}(\alpha)$. Now $(C \text{ on } X_i) \equiv_w (C \text{ on } V_i)$ since $(V_i)_0 = \bigcap\{\operatorname{Ker} g \mid g \in \Omega_i - C_{\Omega_i}(\alpha)\} \supseteq V_i \cap V_0$. As $V = \bigoplus_{i=1}^{l} V_i$ and $X \cong \bigoplus_{i=1}^{l} X_i$, it follows that $(C \text{ on } X) \equiv_w (C \text{ on } V)$. Therefore we can regard V as an irreducible $S\langle\alpha\rangle$-module.

Claim 2. $[Z(S), \alpha, \alpha] = 1$.

Assume the contrary. Set $S_1 = Z(S)C$. Then S_1 is an $\langle\alpha\rangle$-invariant subgroup of S and $V|_{S_1\langle\alpha\rangle}$ is completely reducible. Note that $C \lhd S_1\langle\alpha\rangle$. Let V_i be an irreducible $S_1\langle\alpha\rangle$-submodule of V and W be a homogeneous component of $V_i|_C$.

Now $Z(S)\langle\alpha\rangle \leqslant C_{S_1\langle\alpha\rangle}(C) \leqslant N_{S_1\langle\alpha\rangle}(W)$. This yields that $V_i|_C$ is homogeneous. We also observe that $\operatorname{Ker}(Z(S) \text{ on } V_i) = \operatorname{Ker}(Z(S) \text{ on } V) = 1$ by applying Lemma 2 to the action of $S\langle\alpha\rangle$ on V.

Since $[Z(S), \alpha] \neq 1$, $[Z(S_1), \alpha]$ is nontrivial on V_i. Applying Lemma 1 to the action of $S_1\langle\alpha\rangle$ on V_i, we obtain $C_{V_i}(\alpha) \neq 0$. If $C_{V_i}(\alpha) \nsubseteq V_0$, it follows that $(C \text{ on } C_{V_i}(\alpha)/C_{V_i \cap V_0}(\alpha)) \equiv_w (C \text{ on } V_i)$ as $V_i|_C$ is homogeneous. This forces that there is an irreducible $S_1\langle\alpha\rangle$-submodule V_i of the completely reducible module $V|_{S_1\langle\alpha\rangle}$ such that $C_{V_i}(\alpha) \subseteq V_0$. Since $0 \neq C_{V_i}(\alpha)$, we have $V_i \cap V_0 \neq 0$. Set $\Omega_i = \Omega|_{V_i}$. Now Ω_i is an $S_1\langle\alpha\rangle$-stable subset of V_i^*, and $(V_i)_0 = \bigcap\{\operatorname{Ker} h \mid h \in \Omega_i - C_{\Omega_i}(\alpha)\} \neq 0$ as $V_i \cap V_0 \subseteq (V_i)_0$. Let $f \in \Omega$ be such that $f((V_i)_0) \neq 0$. Then $f_i = f|_{V_i} \in C_{\Omega_i}(\alpha)$. Consider $\langle f_i\rangle = \{cf_i \mid c \in k\}$, a $C_{Z(S)}(f_i)\langle\alpha\rangle$-submodule of V_i^*. Appealing to Lemma 2 together with $\langle f_i\rangle$ and $C_{Z(S)}(f_i)$, we get $C_{Z(S)}(f_i) = \operatorname{Ker}(C_{Z(S)}(f_i) \text{ on } V_i^*) = 1$. On the other hand, there exists $x \in Z(S)$ such that $[x, \alpha, \alpha] \neq 1$, as $[Z(S), \alpha, \alpha] \neq 1$. It follows that $[x, a, \alpha] \neq 1$ for any $1 \neq a \in \langle\alpha\rangle$, that is $[x, a, \alpha] \notin C_{S_1}(f_i)$, for any $1 \neq a \in \langle\alpha\rangle$. Now Lemma 3 applied to the action of $S_1\langle\alpha\rangle$ on V_i, together with f_i and Ω_i, gives that $C_{V_i}(\alpha) \nsubseteq (V_i)_0$. This is a contradiction as $V_i \cap V_0 \subseteq (V_i)_0$ and $C_{V_i}(\alpha) \subseteq V_0$. Thus we have the claim.

Claim 3. $s \neq p$.

Assume that $s = p$. Since $[S,\alpha]^{p-1} \neq 1$, $[S,\alpha]^{p-3} \neq 1$. Set $S_1 = [S,\alpha]^{p-3}$. We can prove that $[S_1, [S,\alpha]^{p-1}] \leqslant [\Phi(S), \alpha]^{p-3} = 1$ (see [2, 5.37]). Hence $[S,\alpha]^{p-1} \leqslant Z(S_1)$.

We have a collection $\{V_1, \ldots, V_l\}$ of irreducible $S_1\langle\alpha\rangle$-modules such that $V = \bigoplus_{i=1}^{l} V_i$. Fix $i \in \{1, \ldots, l\}$. We notice that $C = C_{[S,\alpha]^{p-1}}(\alpha) \lhd S_1\langle\alpha\rangle$ implying $V|_C$ is completely reducible. In particular, $C \leqslant Z(S_1\langle\alpha\rangle)$ and so $V_i|_C$ is homogeneous.

Set $X_i = C_{V_i}(\alpha)/C_{V_i \cap V_0}(\alpha)$ and assume that $(C \text{ on } X_i) \not\equiv_w (C \text{ on } C_{V_i}(\alpha))$. If $[S,\alpha]^{p-1}$ is trivial on V_i, then C acts trivially on V_i, and this contradicts the assumption. Hence $[S,\alpha]^{p-1}$ is not trivial on V_i. If $V_i \cap V_0 = 0$, then $(C \text{ on } X_i) \equiv_w (C \text{ on } C_{V_i}(\alpha))$, and again we have a contradiction. Hence, $V_i \cap V_0 \neq 0$, and there exists some $f \in \Omega$ such that $f(V_i \cap V_0) \neq 0$. Now $f \in C_\Omega(\alpha)$. Set $f|_{V_i} = f_i$. Now $\langle f_i\rangle = \{cf_i \mid c \in k\}$ is a $C_{[S,\alpha]^{p-1}}(f_i)\langle\alpha\rangle$-submodule of V_i^*. Appealing to Lemma 2, we get $C_{Z(S_1)}(f_i) = \operatorname{Ker}(C_{Z(S_1)}(f_i) \text{ on } V_i^*)$. We also have

$C_{[S,\alpha]^{p-1}}(f_i) \leqslant C_{Z(S_1)}(f_i)$. Thus $C_{[S,\alpha]^{p-1}}(f_i)$ is properly contained in $[S,\alpha]^{p-1}$, that is, there is $1 \neq y \in [S,\alpha]^{p-1} - C_{[S,\alpha]^{p-1}}(f_i)$, and $x \in [S,\alpha]^{p-3}$ such that $y = [x,\alpha,\alpha]$. It follows that $1 \neq [x,a,\alpha] \notin C_{[S,\alpha]^{p-1}}(f_i)$ for any $1 \neq a \in \langle\alpha\rangle$. Now we can apply Lemma 3 to the action of $S_1\langle\alpha\rangle$ on V_i together with $\Omega_i = \Omega|_{V_i}$ and f_i, and obtain that $C_{V_i}(\alpha) \nsubseteq V_0$. As $V_i|_C$ is homogeneous, we already have $(C \text{ on } X_i) \equiv_w (C \text{ on } C_{V_i}(\alpha))$.

Therefore we conclude that $(C \text{ on } C_V(\alpha)/C_{V_0}(\alpha)) \equiv_w (C \text{ on } C_V(\alpha))$. Appealing to Lemma 1 together with V and $S\langle\alpha\rangle$, we also see that $C_V(\alpha) \neq 0$ and $(C \text{ on } C_V(\alpha)) \equiv_w (C \text{ on } V)$ hold. Thus $(C \text{ on } V) \equiv_w (C \text{ on } C_V(\alpha)/C_{V_0}(\alpha))$. Since $[S,\alpha]^{p-1} \neq 1$ and $s = p$, $C \neq 1$. Hence C is nontrivial on V and so is on $C_V(\alpha)/C_{V_0}(\alpha)$. This supplies $C_V(\alpha) \nsubseteq V_0$, a contradiction.

Claim 4. *The theorem follows.*

Now $s \neq p$ and $[\Phi(S),\alpha] = 1$. Then $\Phi(S) \leqslant Z(S\langle\alpha\rangle)$ and so S is a central product of $[S,\alpha]$ and $C_S(\alpha)$. As $C = C_S(\alpha) \lhd S\langle\alpha\rangle$, $V|_C$ is completely reducible. In fact, $V|_C$ is homogeneous, because any homogeneous component is stabilized by $S\langle\alpha\rangle$ as C is centralized by $[S,\alpha]\langle\alpha\rangle$. It follows that $(C \text{ on } C_V(\alpha)/C_{V_0}(\alpha)) \equiv_w (C \text{ on } V)$ if $C_V(\alpha) \nsubseteq V_0$ holds. Hence $C_V(\alpha) \subseteq V_0$. Note that $C_V(\alpha) \neq 0$, because otherwise we would have obtained $s = 2$ as $[S,\alpha]$ is nontrivial on V. Then there exists $0 \neq f \in C_\Omega(\alpha)$ with $f(V_0) \neq 0$. Now $C_{Z(S)}(f) = \mathrm{Ker}(C_{Z(S)}(f) \text{ on } V^*) = 1$ by Lemma 2. If follows that $C_{Z([S,\alpha])}(f) = 1$, as $[C_S(\alpha),[S,\alpha]] = 1$. Then $C_{[S,\alpha]}(f)$ is properly contained in $[S,\alpha]$. Let M be a maximal α-invariant subgroup of $[S,\alpha]$ containing $C_{[S,\alpha]}(f)$. The abelian group $[S,\alpha]/M = [\overline{S,\alpha}]$ forms an irreducible $\langle\alpha\rangle$-module on which $\langle\alpha\rangle$ acts fixed point freely. Thus we have $[\bar{x},a] \neq 0$ for any $0 \neq \bar{x} \in [\overline{S,\alpha}]$. It follows that $[\bar{x},a,\alpha] \neq 0$ for each $1 \neq a \in \langle\alpha\rangle$. Put $\bar{x} = xM$ for $x \in [S,\alpha]$. Then $[x,a,\alpha] \notin M$. In particular, $[x,a,\alpha] \notin C_{[S,\alpha]}(f)$ for each $1 \neq a \in \langle\alpha\rangle$. Recall that $V|_C$ is homogeneous. Then Lemma 3 applied to the action of $S\langle\alpha\rangle$ on V gives that $C_V(\alpha) \nsubseteq V_0$. This contradiction completes the proof of Theorem 1. □

Let V be an irreducible $G\langle\alpha\rangle$-module where $G \lhd G\langle\alpha\rangle$ and $\langle\alpha\rangle$ is cyclic of prime order p. We say V is an ample $G\langle\alpha\rangle$-module if $[G,\alpha]^{p-1}$ acts nontrivially on V. Notice that when $|G|$ is odd, this coincides with the definition of an ample module given in [2].

Theorem 2. *Let $S\langle\alpha\rangle$ be a group such that $S \lhd S\langle\alpha\rangle$, S is an s-group, $\langle\alpha\rangle$ is cyclic of order p for distinct primes s and p, $\Phi(\Phi(S)) = 1$, $\Phi(S) \leqslant Z(S)$. Suppose that V is an irreducible $kS\langle\alpha\rangle$-module on which $[S,\alpha]$ acts nontrivially where k is a field of characteristic different from s. Then*

$$[V,\alpha]^{p-1} \neq 0 \quad \text{and} \quad \bigl(C_S(\alpha) \text{ on } V\bigr) \equiv_w \bigl(C_S(\alpha) \text{ on } [V,\alpha]^{p-1}\bigr)$$

unless p is a Fermat prime, $s = 2$ and $[\tilde{S},\alpha]$ is an irreducible $\langle\alpha\rangle$-module.

Proof. See [2, Proposition 3.10]. □

Now we are ready to prove our key result, which improves Theorem 3.1 in [5] obtained by pursuing the idea in Dade's work [2].

Theorem 3. *Let $G \lhd GA$ and $\langle z\rangle \trianglelefteq A$ of prime order p with $p \geqslant 5$. Suppose that $P_1, \ldots, P_t$ is an A-Fitting chain of G such that $[P_1,z] \neq 1$, P_i is a p_i-group where p_i is an odd prime for each*

$i = 1, \dots, t$, and $t \geqslant 3$. Then there are sections $D_{i_0}, \dots, D_t$ of $P_{i_0}, \dots, P_t$, respectively, forming an A-Fitting chain of G such that z centralizes each D_j for $j = i_0, \dots, t$ where

$$i_0 = \begin{cases} 2 & \text{if } p_1 \neq p, \\ 3 & \text{if } p_1 = p. \end{cases}$$

Proof. Let q be a prime number different from p_t, let $p_{t+1} = q$ and let P_{t+1} stand for the regular $\mathbb{Z}_q[P_t P_{t-1} A]$-module. We shall add P_{t+1} to the given chain and define subspaces E_i of P_i for each $i = 1, \dots, t+1$ as follows: $E_1 = P_1$, $E_i = [X_i, E_{i-1}]$ for $i = 2, \dots, t+1$, where $X_i/\Phi(P_i)$ is the sum of all ample irreducible $E_{i-1}\langle z\rangle$-submodules of $\tilde{P}_i$: It is easy to observe that for each $i = 2, \dots, t+1$, E_i are all $E_{i-1}A$-invariant subgroups of P_i and $\tilde{E}_i$ is a direct sum of ample irreducible $E_{i-1}\langle z\rangle$-submodules.

We now define subgroups F_i of E_i for $i = 1, \dots, t+1$ as follows:

$$\begin{aligned} &F_1 = \{1\}, \\ &F_i = C_{E_i}(z) \quad \text{if } p_i \neq p \text{ and } i \geqslant 2, \\ &F_2 = C_{[E_2,z]^{p-1}}(z) \quad \text{if } p_2 = p, \\ &F_i = \bigl[[E_i, z]^{p-1}, F_{i-1}\bigr] \quad \text{if } p_i = p \text{ and } i \geqslant 3. \end{aligned}$$

It can also be easily seen that for each $i = 2, \dots, t+1$, F_i is $F_{i-1}A$-invariant and is centralized by z.

We next define the sections D_i by $D_i = F_i/\operatorname{Ker}(F_i \text{ on } \tilde{E}_{i+1})$ for $i = 2, \dots, t$ and claim that they form an A-chain each of its sections is centralized by z, as desired.

We proceed from this point by assuming that we can prove the following two claims whose proofs will follow later.

Claim 1. *Assume that $i \geqslant 2$ and $p_i \neq p$. If $E_i \neq 1$, then D_i is a nontrivial F_{i-1}-invariant section such that $(F_{i-1}$ on $\tilde{E}_i) \equiv_w (F_{i-1}$ on $\tilde{D}_i)$.*

Claim 2. *Assume that $i \geqslant 2$ and $p_i = p$. If either $i = 2$ or $D_{i-1} \neq 1$, then* $\operatorname{Ker}(F_i$ *on* $\tilde{E}_{i+1}) = 1$, *$D_i = F_i \neq 1$ and $(F_{i-1}$ on $\tilde{E}_i) \equiv_w (F_{i-1}$ on $\tilde{F}_i)$.*

We first prove the theorem in the case $p_1 \neq p$.

Now $E_1 = P_1$ and $[E_1, z]^{p-1} = [E_1, z] \neq 1$. Then the faithful action of P_1 on $\tilde{P}_2 = [\tilde{P}_2, [E_1, z]] \oplus C_{\tilde{P}_2}([E_1, z])$ forces that $\tilde{E}_2 \neq 0$, that is, $\tilde{P}_2$ contains an irreducible ample $E_1\langle z\rangle$-submodule. If $p_2 \neq p$, we apply Claim 1 to the action of $E_1\langle z\rangle$ on $\tilde{E}_2$ and obtain that D_2 is a nontrivial section of E_2. If $p_2 = p$, we also have $D_2 = F_2 \neq 1$ by Claim 2. Thus we have seen that $D_2 \neq 1$ in any case.

Suppose that $D_{i-1} \neq 1$ for some $i \geqslant 3$. Then $E_i \neq 1$. Appealing again to Claims 1 and 2, respectively, when $p_i \neq p$ and $p_i = p$, we see that D_i is a nontrivial F_{i-1}-invariant section and $(F_{i-1}$ on $\tilde{E}_i) \equiv_w (F_{i-1}$ on $\tilde{D}_i)$ for each $i \geqslant 2$. It follows that $D_{i-1} = F_{i-1}/\operatorname{Ker}(F_{i-1}$ on $\tilde{D}_i)$ normalizes $D_i = F_i/\operatorname{Ker}(F_i$ on $\tilde{E}_{i+1})$ and $\operatorname{Ker}(D_{i-1}$ on $D_i) = 1$ for each $i = 3, \dots, t$.

We also have $\Phi(D_i) \leqslant Z(D_i)$, $\Phi(\Phi(D_i)) = 1$ and $[\Phi(D_i), D_{i-1}] = 1$ for $i = 2, \dots, t$.

It remains to prove that $(D_{i-1}$ on $\tilde{D}_i)$ is weakly D_{i-2}-invariant for $i = 4, \dots, t$. Since $(P_{i-1}$ on $\tilde{P}_i)$ is weakly P_{i-2}-invariant, $(E_{i-1}$ on $\tilde{P}_i)$ is weakly F_{i-2}-invariant by Remark (a), that is, $(E_{i-1} \text{ on } \tilde{P}_i) \equiv_w (E_{i-1} \text{ on } \tilde{P}_i)^x$ for each $x \in F_{i-2}$. Then $X_i/\Phi(P_i) = (X_i/\Phi(P_i))_x$ by Remark (c) and so $(E_{i-1} \text{ on } \tilde{E}_i) \equiv_w (E_{i-1} \text{ on } \tilde{E}_i)^x$. Hence $(E_{i-1} \text{ on } \tilde{E}_i)$ is weakly F_{i-2}-invariant. This gives that $(F_{i-1} \text{ on } \tilde{E}_i)$ is weakly F_{i-2}-invariant, too. As $(F_{i-1} \text{ on } \tilde{E}_i) \equiv_w (F_{i-1} \text{ on } \tilde{D}_i)$ holds, it also follows that $(F_{i-1} \text{ on } \tilde{D}_i)$ is weakly F_{i-2}-invariant by Remark (b). Consequently we have obtained that $(D_{i-1} \text{ on } \tilde{D}_i)$ is weakly D_{i-2}-invariant, proving the theorem when $p_1 \neq p$.

Finally we assume that $p_1 = p$, and consider the chain $P_2, \dots, P_t$. Note that $[P_2, z] \neq 1$, because otherwise $[P_1, z] = 1$ by the three subgroup lemma. Since $p_2 \neq p$, the above argument gives an A-Fitting chain $D_3, \dots, D_t$ whose terms are all centralized by z. This completes the proof of Theorem 3. □

We shall need the following fact in proving Claim 1.

Lemma 4. *Assume $p_i \neq p$ and let W be an irreducible submodule of $\tilde{P}_{i+1}|_{E_i}$. If $\Phi(E_i)$ acts nontrivially on W, then so does $[E_i, z]$.*

Proof. Suppose that W is an irreducible submodule of $\tilde{P}_{i+1}|_{E_i}$ on which $\Phi(E_i)$ acts nontrivially and $[E_i, z]$ acts trivially. Then there exists an E_iA-submodule X of $\tilde{P}_{i+1}$ such that W is isomorphic to an irreducible E_i-submodule of X. Since $X|_{E_i}$ is completely reducible, there is a collection $\{U_1, \dots, U_s\}$ of homogeneous E_i-modules such that $X = \bigoplus_{i=1}^s U_i$. Assume that U_1 is a sum of isomorphic copies of W. Then $\mathrm{Ker}(E_i \text{ on } X) = \bigcap_{a \in A} \mathrm{Ker}(E_i \text{ on } U_1)^a = \bigcap_{a \in A} \mathrm{Ker}(E_i \text{ on } W)^a$.

Put $K = \mathrm{Ker}(\Phi(E_i) \text{ on } X)$. K is an A-invariant normal subgroup of E_i. Furthermore, K is E_{i-1}-invariant because $[\Phi(E_i), E_{i-1}] = 1$. Set $\overline{E}_i = E_i/K$ and $\overline{\overline{E}}_i = \overline{E}_i/\mathrm{Ker}(\overline{E}_i \text{ on } X)$. Note that $E_i' = \Phi(E_i)$ since $C_{E_i/E_i'}(E_{i-1}) = 0$. Now $\overline{E}_i$ is nonabelian, because otherwise $E_i' = \Phi(E_i) = K$, which is not the case. It follows that $V = \overline{E}_i/Z(\overline{E}_i) \neq 0$. Obviously we have $\overline{Z(\overline{E}_i)} \subseteq Z(\overline{\overline{E}}_i)$. On the other hand, if $Z(\overline{\overline{E}}_i) = \overline{\overline{C}} = \overline{C}/\mathrm{Ker}(\overline{C} \text{ on } X)$, then $[\overline{C}, \overline{E}_i] \leqslant \mathrm{Ker}(\overline{E}_i \text{ on } X) \cap \Phi(\overline{E}_i) = 1$, because $\Phi(\overline{E}_i) = \Phi(E_i/K)$ is faithful on X. Therefore $\overline{C} \leqslant Z(\overline{E}_i)$, that is, $Z(\overline{\overline{E}}_i) = \overline{Z(\overline{E}_i)}$.

Also note that $\mathrm{Ker}(\overline{E}_i \text{ on } X) \subset Z(\overline{E}_i)$: Because otherwise there is $\bar{x} \in \mathrm{Ker}(\overline{E}_i \text{ on } X) \setminus Z(\overline{E}_i)$ and so there is $\bar{y} \in \overline{E}_i$ such that $1 \neq [\bar{x}, \bar{y}]$. Now $[\bar{x}, \bar{y}]$ is a nontrivial element of $\Phi(\overline{E}_i)$ acting trivially on X. This contradicts the fact that $\Phi(\overline{E}_i)$ is faithful on X.

Thus $Z(\overline{\overline{E}}_i) = Z(\overline{E}_i)/\mathrm{Ker}(\overline{E}_i \text{ on } X)$. We conclude that $\overline{E}_i/Z(\overline{E}_i)$ and $\overline{\overline{E}}_i/Z(\overline{\overline{E}}_i)$ are $\langle z \rangle$-isomorphic modules. Since $\langle z \rangle$ is trivial on $\overline{\overline{E}}_i$, it is trivial on V also. An application of the three subgroup lemma supplies that $[E_{i-1}, z]$ is also trivial on V. It follows that $[E_{i-1}, z]$ is trivial on each of the $E_{i-1}\langle z \rangle$-composition factors of V. Note that V is a nonzero quotient module of $\tilde{E}_i$. Since $\tilde{E}_i$ is a direct sum of ample irreducible $E_{i-1}\langle z \rangle$-submodules, so is V, that is, $[E_{i-1}, z]^{p-1}$ and hence $[E_{i-1}, z]$ is nontrivial on V, a contradiction completing the proof of Lemma 4. □

Proof of Claim 1. We have $E_{i-1} \neq 1$ as $[E_i, E_{i-1}] = E_i$. Also $\mathrm{Ker}(E_i \text{ on } X_{i+1}/\Phi(P_{i+1})) = \mathrm{Ker}(E_i \text{ on } E_{i+1}) = \mathrm{Ker}(E_i \text{ on } \tilde{E}_{i+1})$. Appealing to Remark (c) together with $V = \tilde{P}_{i+1}$, $G = P_i$, $L = F_{i-1}$ and $M = [E_i, z]$, we see that $\mathrm{Ker}(E_i \text{ on } \tilde{E}_{i+1})$ is F_{i-1}-invariant. This yields that $D_i = F_i/\mathrm{Ker}(F_i \text{ on } \tilde{E}_{i+1})$ is F_{i-1}-invariant, as F_{i-1} normalizes F_i.

We know that $\tilde{E}_i = \bigoplus_{j=1}^l W_{i_j}$ where $W_{i_1}, \dots, W_{i_l}$ are irreducible ample $E_{i-1}\langle z \rangle$-submodules. Set $W_{i_j} = U_j/\Phi(E_i)$ for each $j = 1, \dots, l$. Since $\tilde{P}_{i+1}|_{E_i}$ is completely reducible and E_i is faithful on $\tilde{P}_{i+1}$, there exists at least one irreducible component of $\tilde{P}_{i+1}|_{E_i}$ on which U_j acts nontrivially. Let $\mathfrak{N}_j$ denote the set of all such components of $\tilde{P}_{i+1}|_{E_i}$.

There are two cases: Either

(I) there is at least one N in $\mathfrak{N}_j$ on which $\Phi(E_i)$ acts trivially,

or

(II) there is no N in $\mathfrak{N}_j$ on which $\Phi(E_i)$ acts trivially.

In the latter case, as an immediate consequence of Lemma 4, we have the following:

Let N be an irreducible component of $\tilde{E}_{i+1}|_{E_i}$. Then $N \in \mathfrak{N}_j$ iff $\Phi(E_i)$ acts nontrivially on N.

Thus U_j is trivial on each irreducible component N of $\tilde{P}_{i+1}|_{E_i}$ lying outside $\tilde{E}_{i+1}$, because otherwise $N \in \mathfrak{N}_j$ implying that $\Phi(E_i)$ and hence $[E_i, z]$ is nontrivial on N, a contradiction. It follows that

$$1 = \mathrm{Ker}(U_j \text{ on } \tilde{P}_{i+1}) = \mathrm{Ker}(U_j \text{ on } \tilde{E}_{i+1}) \quad \text{when (II) holds.}$$

Now suppose that $\mathrm{Ker}(U_j \text{ on } \tilde{E}_{i+1}) = 1$ for each $j = 1, \ldots, s$ and $\mathrm{Ker}(U_j \text{ on } \tilde{E}_{i+1}) \neq 1$ for each $j = s+1, \ldots, l$.

For each $j = s+1, \ldots, l$, set $\Omega_j = \{f \in W_{i_j}^* \mid \text{there exists } N \text{ in } \mathfrak{N}_j \text{ on which } \Phi(E_i) \text{ acts trivially and } \mathrm{Ker}(U_j \text{ on } N)/\Phi(E_i) \subseteq \mathrm{Ker}\, f\}$. Now for each N in $\mathfrak{N}_j$ on which $\Phi(E_i)$ acts trivially, $\mathrm{Ker}(U_j \text{ on } N)/\Phi(E_i)$ is proper in W_{i_j} and hence is contained in a maximal subspace M. Therefore $\Omega_j \neq \{0\}$. Also Ω_j is $E_{i-1}\langle z\rangle$-invariant. This yields that $\langle \Omega_j \rangle = W_{i_j}^*$, by the irreducibility of $W_{i_j}^*$ as an $E_{i-1}\langle z\rangle$-module.

Now for each $j = 1, \ldots, l$, we set $K_j = \mathrm{Ker}(U_j \text{ on } \tilde{E}_{i+1})$. Then $K_j\Phi(E_i)/\Phi(E_i) \subseteq (W_{i_j})_0$: If not, then $j \in \{s+1, \ldots, l\}$ and there exist $x \in K_j$, $f \in \Omega_j - C_{\Omega_j}(z)$ such that $f(x\Phi(E_i)) \neq 0$. By the definition of Ω_j, we can find an irreducible submodule N of $\tilde{P}_{i+1}|_{E_i}$ on which U_j is nontrivial, $\Phi(E_i)$ is trivial and $\mathrm{Ker}(U_j \text{ on } N)/\Phi(E_i) \subseteq \mathrm{Ker}\, f$. Then $x \notin \mathrm{Ker}(U_j \text{ on } N)$. As $x \in \mathrm{Ker}(U_j \text{ on } \tilde{E}_{i+1})$, N lies outside $\tilde{E}_{i+1}|_{E_i}$, that is, $[E_i, z]^{p-1} = [E_i, z]$ acts trivially on N. Thus $[U_j, z]$ is trivial on N and so $f \in C_{\Omega_j}(z)$, a contradiction.

Since W_{i_j} is an irreducible $E_{i-1}\langle z\rangle$-module, $W_{i_j}|_{E_{i-1}}$ decomposes into a direct sum of homogeneous E_{i-1}-modules which are permuted transitively by $\langle z\rangle$. Since $[E_{i-1}, z]^{p-1}$ is nontrivial on at least one of these components, it is nontrivial on all of them. It follows that $[E_{i-1}, z]^{p-1}$ acts nontrivially on each irreducible component of $W_{i_j}|_{E_{i-1}}$ for each $j = 1, \ldots, l$.

Let Ω_j denote the whole of $W_{i_j}^*$ when $j \in \{1, \ldots, s\}$. Appealing to Theorem 1 for each $j = 1, \ldots, l$ together with the action of $E_{i-1}\langle z\rangle$ on W_{i_j} and the corresponding Ω_j, we see that $C_{W_{i_j}}(z) \nsubseteq (W_{i_j})_0$ and $(F_{i-1} \text{ on } C_{W_{i_j}}(z)/C_{(W_{i_j})_0}(z)) \equiv_w (F_{i-1} \text{ on } W_{i_j})$.

We shall now observe that for each $j = 1, \ldots, l$, $(F_{i-1} \text{ on } W_{i_j}) \equiv_w (F_{i-1} \text{ on } C_{W_{i_j}}(z))$: If $p_{i-1} = p$ or $[Z(E_{i-1}), z]$ is nontrivial on W_{i_j}, this holds by Lemma 1. Assume that $p_{i-1} \neq p$ and $[Z(E_{i-1}), z] \leqslant K = \mathrm{Ker}(E_{i-1} \text{ on } W_{i_j})$. Since $[E_{i-1}, z]$ is nontrivial on W_{i_j} and p_{i-1} is odd, it can be easily seen that $C_{W_{i_j}}(z) \neq 0$. Put $\overline{E}_{i-1} = E_{i-1}/K$. As $\overline{\Phi(E_{i-1})} = \Phi(\overline{E}_{i-1}) \leqslant Z(\overline{E}_{i-1}\langle z\rangle)$, $\overline{E}_{i-1}$ is a central product of $[\overline{E}_{i-1}, z]\langle z\rangle$ and $C_{\overline{E}_{i-1}}(z)$. Then $C_{\overline{E}_{i-1}}(z) \lhd \overline{E}_{i-1}\langle z\rangle$

and $W_{i_j}|_{C_{\overline{E}_{i-1}}(z)}$ is homogeneous. We have $\overline{F}_{i-1} \leqslant C_{\overline{E}_{i-1}}(z)$ yielding that $(\overline{F}_{i-1} \text{ on } W_{i_j}) \equiv_w (\overline{F}_{i-1} \text{ on } C_{W_{i_j}}(z))$. Thus $(F_{i-1} \text{ on } W_{i_j}) \equiv_w (F_{i-1} \text{ on } C_{W_{i_j}}(z))$.

Now $(F_{i-1} \text{ on } C_{W_{i_j}}(z)) \equiv_w (F_{i-1} \text{ on } C_{W_{i_j}}(z)/C_{(W_{i_j})_0}(z)) \equiv_w (F_{i-1} \text{ on } W_{i_j})$ holds, for each $j = 1, \ldots, l$. Set $L_j = \mathrm{Ker}(C_{U_j}(z) \text{ on } \tilde{E}_{i+1})$. Notice that any nontrivial irreducible F_{i-1}-submodule of $C_{W_{i_j}}(z)/C_{(W_{i_j})_0}(z)$ is F_{i-1}-isomorphic to an irreducible F_{i-1}-submodule of $C_{U_j}(z)/L_j$. Therefore any nontrivial irreducible F_{i-1}-submodule of W_{i_j} is F_{i-1}-isomorphic to an irreducible F_{i-1}-submodule of $C_{U_j}(z)/L_j$. On the other hand, any nontrivial irreducible F_{i-1}-submodule of $C_{U_j}(z)/L_j$ is F_{i-1}-isomorphic to an irreducible F_{i-1}-submodule of $C_{U_j}(z)$ and hence to an irreducible F_{i-1}-submodule of W_{i_j}. This shows that $(F_{i-1} \text{ on } W_{i_j}) \equiv_w (F_{i-1} \text{ on } C_{U_j}(z)/L_j)$ for each $j = 1, \ldots, l$.

As $\tilde{E}_i = \bigoplus_{j=1}^{l} W_{i_j}$ and $C_{\tilde{E}_i}(z) = \bigoplus_{j=1}^{l} C_{W_{i_j}}(z) = \bigoplus_{j=1}^{l} C_{U_j}(z)\Phi(E_i)/\Phi(E_i)$, we have $(F_{i-1} \text{ on } \tilde{E}_i) \equiv_w (F_{i-1} \text{ on } C_{E_i}(z)/\mathrm{Ker}(C_{E_i}(z) \text{ on } \tilde{E}_{i+1}))$. Notice that $D_i = C_{E_i}(z)/\mathrm{Ker}(C_{E_i}(z)$ on $\tilde{E}_{i+1})$. Hence $(F_{i-1} \text{ on } \tilde{E}_i) \equiv_w (F_{i-1} \text{ on } D_i) \equiv_w (F_{i-1} \text{ on } \tilde{D}_i)$, because $[\Phi(D_i), F_{i-1}] = 1$. Since $C_{W_{i_j}}(z) \nsubseteq (W_{i_j})_0$ we have $F_i = C_{E_i}(z) \nsubseteq \mathrm{Ker}(E_i \text{ on } \tilde{E}_{i+1})$ and so $D_i \neq 1$, completing the proof of Claim 1. □

Proof of Claim 2. Suppose that $p_i = p$ for some $i \geqslant 2$. If $i \neq 2$, assume that $D_{i-1} \neq 1$. Now $\mathrm{Ker}([E_i, z]^{p-1} \text{ on } \tilde{E}_{i+1}) = \mathrm{Ker}([E_i, z]^{p-1} \text{ on } \tilde{P}_{i+1}) = 1$. Since $F_i \leqslant [E_i, z]^{p-1}$, we have $\mathrm{Ker}(F_i \text{ on } \tilde{E}_{i+1}) = 1$, that is $D_i = F_i$.

We first consider the case $i = 2$. Then $p_2 = p$ and so $p_1 \neq p$. Since $E_1 = P_1$ and $[E_1, z] \neq 1$, we see that $\tilde{E}_2 \neq 0$. Applying Theorem 2 to the action of $E_1\langle z\rangle$ on each irreducible $E_1\langle z\rangle$-component of $\tilde{E}_2$, we get $[\tilde{E}_2, z]^{p-1} \neq 0$. This yields that $[E_2, z]^{p-1} \neq 1$ and so $F_2 = C_{[E_2,z]^{p-1}}(z) \neq 1$. As $F_1 = 1$, this completes the proof of Claim 2 when $i = 2$.

We next assume that $i > 2$. Now $p_{i-1} \neq p$ and $F_{i-1} = C_{E_{i-1}}(z)$. Since $D_{i-1} \neq 1$, $F_{i-1} \neq 1$ and $\tilde{E}_i \neq 0$. We apply Theorem 2 to the action of $E_{i-1}\langle z\rangle$ on each irreducible $E_{i-1}\langle z\rangle$-component of $\tilde{E}_i$ to get $[\tilde{E}_i, z]^{p-1} \neq 0$ and $(F_{i-1} \text{ on } \tilde{E}_i) \equiv_w (F_{i-1} \text{ on } [\tilde{E}_i, z]^{p-1})$. This gives that $(F_{i-1} \text{ on } \tilde{E}_i) \equiv_w (F_{i-1} \text{ on } [[\tilde{E}_i, z]^{p-1}, F_{i-1}])$ as $[\tilde{E}_i, z]^{p-1} = [[\tilde{E}_i, z]^{p-1}, F_{i-1}] \oplus C_{[\tilde{E}_i,z]^{p-1}}(F_{i-1})$. Now $(F_{i-1} \text{ on } \tilde{E}_i) \equiv_w (F_{i-1} \text{ on } \tilde{F}_i)$ holds, because $[\Phi(E_i), F_{i-1}] = 1$. This finishes the proof of Claim 2. □

Proofs of theorems

Theorem A. *Let A be an abelian group acting fixed point freely on a group G of odd order. If A has squarefree exponent coprime to* 6, *then $f(G) \leqslant \ell(A)$.*

Proof. Set $f = f(G)$. By Lemmas 8.1 and 8.2 in [2], there is an A-Fitting chain of length f in G. Since A is nilpotent, it is a Carter subgroup of any semidirect product of it with a section of G. Thus A acts fixed point freely on any section of this chain.

Hence once the following assertion referring only to A-Fitting chains is proved, the theorem will follow immediately.

Let A be an abelian group of squarefree exponent coprime to 6, *and let $P_1, \ldots, P_t$ be an A-Fitting chain of a finite solvable group G such that P_i has odd order and A acts fixed point freely on P_i for each $i = 1, \ldots, t$. Then $t \leqslant \ell(A)$.*

We shall use induction on t. We may assume that P_1 is an irreducible A-module. As A acts fixed point freely on P_1, there exists $z \in A$ of prime order p such that $[P_1, z] \neq 1$. Then $[P_1, z] =$

P_1 and so $p_1 \neq p$. Also $p \geqslant 5$. Theorem 3 applied to the chain $P_1, \ldots, P_t$ gives us an A-Fitting chain $D_2, \ldots, D_t$ such that z centralizes each D_i, for $i = 2, \ldots, t$. Hence $D_2, \ldots, D_t$ is an $A/\langle z\rangle$-Fitting chain on each of its sections $A/\langle z\rangle$ acts fixed point freely. By induction, it follows that $t - 1 \leqslant \ell(A) - 1$. Then $t \leqslant \ell(A)$, as desired. □

Lemma 5. *Let A be a group acting on a Fitting chain $P_1, P_2, \ldots, P_t$ where each P_i has odd order, in such a way that A centralizes no nontrivial section of any P_i, $i = 1, 2, \ldots, t$. Assume that A is nilpotent of order coprime to 6. Then $t \leqslant 2^{\ell(A)} - 1$.*

Proof. Let $\ell = \ell(A)$. We prove that $t \leqslant 2^{\ell} - 1$ by induction on ℓ.

If $\ell = 0$ the statement is trivial and if $0 < \ell \leqslant 2$, the statement is well known. Therefore we may assume that $\ell \geqslant 3$.

If A is a q-group for some prime number q then the action is coprime. By [5] we have $t \leqslant 2\ell$ and, since $2\ell \leqslant 2^{\ell} - 1$, in this case the statement is proved.

We now suppose that there exist two distinct prime numbers q and r which divide the order of A. Since A is nilpotent, there exist $\alpha, \beta \in A$ of order q and r respectively such that $\langle\alpha\rangle$ and $\langle\beta\rangle$ are normal in A. Set $B = \langle\alpha, \beta\rangle$. Let k be the biggest integer such that α and β centralize $P_1, \ldots, P_k$ and suppose that $[P_{k+1}, \alpha] \neq 1$ (if $[P_{k+1}, \alpha] = 1$ then by hypothesis $[P_{k+1}, \beta] \neq 1$). Therefore A/B acts on $P_1, \ldots, P_k$ and the induction hypothesis gives $k \leqslant 2^{\ell-2} - 1$. If $t - k \leqslant 2$ the statement is proved, since $2^{\ell-2} - 1 + 2 \leqslant 2^{\ell} - 1$. If $t - k \geqslant 3$ then, by Theorem 3 applied to $P_{k+1}, \ldots, P_t$, there are sections $D_{k+3}, \ldots, D_t$ such that each D_i is centralized by α (or, respectively, by β).

Since $A/\langle\alpha\rangle$ and $D_{k+3}, \ldots, D_t$ satisfy the hypothesis, we have $t - (k + 2) \leqslant 2^{\ell-1} - 1$ and therefore $t \leqslant 2^{\ell-1} - 1 + 2^{\ell-2} - 1 + 2 \leqslant 2^{\ell} - 1$. □

Theorem 4. *Let H be a group of order coprime to 6. If a Carter subgroup C of H admits a normal complement G, then $f(G) \leqslant 2^{\ell(C)} - 1$.*

Proof. Set $f = f(G)$. By Lemmas 8.1 and 8.2 in [2], there is a C-Fitting chain $P_1, \ldots, P_f$. Since C is a Carter subgroup of H with $G \cap C = 1$, it centralizes no nontrivial section of G. By Lemma 5, we obtain that $f \leqslant 2^{\ell(C)} - 1$. □

Theorem B. *Let C be a Carter subgroup of a group G. If G has order coprime to 6, then $f(G) \leqslant 2(2^{\ell(C)} - 1)$.*

Proof. Set $f = f(G)$. We use induction on $\ell(C)$. If $\ell(C) = 0$, then $C = 1$, $G = 1$ and so the theorem follows. Assume that $\ell(C) > 0$. Fix a Carter subgroup C of G. There is an integer $k \geqslant 0$ such that $F_k(G) \cap C = 1$ and $F_{k+1}(G) \cap C \neq 1$. Put $\overline{G} = G/F_{k+1}(G)$. Since $\overline{C}$ is a Carter subgroup of $\overline{G}$ and $F_{k+1}(G) \cap C \neq 1$, $\ell(\overline{C}) < \ell(C)$. So by induction

$$f(\overline{G}) = f - k - 1 \leqslant 2\big(2^{\ell(\overline{C})} - 1\big).$$

Now C is a Carter subgroup of $K = CF_k(G)$ and $F_k(G)$ is a normal complement to each Carter subgroup of K. Thus $k = f(F_k(G)) \leqslant 2^{\ell(C)} - 1$ by Theorem 4.

It follows that

$$f = 1 + k + (f - k - 1) \leqslant 1 + 2^{\ell(C)} - 1 + 2\big(2^{\ell(C)-1} - 1\big) = 2\big(2^{\ell(C)} - 1\big). \qquad \square$$

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