SUBSPACES OF NUCLEAR FRÉCHET SPACES

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ABSTRACT

This work is a general survey of subspaces of nuclear Fréchet spaces. Subspaces of a nuclear power series space of finite type, subspaces of a nuclear Fréchet smooth sequence space of infinite type and subspaces of the space of rapidly decreasing sequences are considered.

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CHAPTER I INTRODUCTION and PRELIMINERIES

1. INTRODUCTION:

The investigation of subspaces of nuclear smooth sequence spaces of infinite and finite types is the subject of the paper of M.S. Ramanujan and T.Terzioğlu (11). There are also two important papers of Dubinsky (4,5) in which he tried to find out all infinite dimensional subspaces of nuclear power series spaces of finite and infinite types. The concept of smooth sequence spaces of finite and infinite types were introduced as a generalization of the notion of power series spaces in (13).

In chapter II, we start out with the theorem of B. Mitiagin and G. Henkin which is: A space E which is a subspace of a nuclear finite type power series space a quotient space of another must be finite type power series space. In (11) there is a particular result which is: If a G_1-space is isomorphic to a subspace of $\Lambda_1(\alpha)$ then it is isomorphic to power series space of finite type. At the end of the same chapter, we consider infinite type power series subspaces of a finite type power series space.

Chapter III contains an exact characterization of subspaces of (s), the space of rapidly decreasing sequences. Vogt (18) has shown that a nuclear Frêchet space E is isomorphic to a subspace of (s) if and only if there exist a sequence $(\| \|_k)$ of norms defining the topology of E such that there is a C>0 with

$$\forall k \; \exists j \text{ with } \quad \|x\|_k^2 \leq C \, \|x\|_1 \, \|x\|_j \quad x \in E$$

In the last chapter we discuss G_∞-subspaces of G_∞-spaces, most of the results could be found in (11).

2. PRELIMINARIES:

By a locally convex space, we shall mean a locally convex, Hausdorff topological vector space over the field of real or complex numbers.

We say that E is a Frêchet space, in case E is complete, metrizable locally convex space and by a subspace we shall always mean an infinite dimensional, closed subspace.

Let U and V be absolutely convex and closed neighborhoods of O in E with $V \prec U$ (i.e for some $\rho > o$ $V \subset \rho U$). The n-dimensional Kolmogorov diameter of V with respect to U is

$$d_n(V,U) = \inf \inf \{t>o : V \subset tU+L\}$$

where the first infimum runs over all subspaces L of E with dim $(L) \leq n$. By the diametral dimension $\Delta(E)$ of E, we mean the set of all scalar sequences (t_n) such that for any absolutely convex and closed neighborhood U of E there is another such neighborhood V of E with $V \prec U$ and $\lim_n t_n d_n(V,U) = 0$ (13).

The following theorem of Mitiagin provides us with the notion of nuclearity; one can find different but equivalent definitions in Pietsch (10).

Theorem: The following conditions on E are equivalent:

(i) E is nuclear

(ii) $\forall U$ $\exists V$ such that $\sum_{n=1}^{\infty} d_n(V,U) < \infty$

(iii) $\exists k \geq 1$ such that $((n+1)^k)_{n=o}^{\infty} \varepsilon \Delta(E)$

(iv) $\forall k \geq 1$ such that $((n+1)^k)_{n=o}^{\infty} \varepsilon \Delta(E)$.

We recall that, every closed subspace of a nuclear space is nuclear. For permanence properties of the class of nuclear spaces see (10).

Definition 1: Let w be the set of all real or complex sequences, then $K\subset w$ is called a Köthe set if:

(1) $\forall k\in N$ there is $a\in K$ with $a_k>0$

(2) If a,b are two elements of K there is a $c\in K$ with $a_n\leqq c_n$, $b_n\leqq c_n$ for all n.

The space of all sequences $x=(x_n)$ such that $P_a(x)=\sum_{n=0}^{\infty}|x_n|a_n<\infty$ for every $a\in K$ is called the Köthe space generated by K and denoted by $\lambda(K)$. The seminorms $p_a(\ )$ $a\in K$, defines a locally convex Hausdorff topology on $\lambda(K)$.

Notation: $a_n=O(b_n)$ means $|a_n|\leq \rho\ |b_n|$ for all $n\in N$ and for some $\rho>0$.

$a_n=o(b_n)$ means $|a_n|\leqq \alpha_n|b_n|$ for some sequence (α_n) with $\lim \alpha_n=0$.

A countable Köthe set $A=\{(a_n^k)\}$ is called a G_∞- set and the corresponding köthe space $\lambda(A)$ a G_∞-space or a smooth sequence space of infinite type if A satisfies:

(1) $a_n^1=1$ and $a_n^k\leqq a_{n+1}^k$ for each k and n

(2) $\forall k\ \ \exists j$ with $(a_n^k)^2=0\ (a_n^j)$.

A countable Köthe set $B=(b_n^k)$ is called a G_1-set and $\lambda(B)$ is a G_1-space or a smooth sequence space of finite type if:

(1) $0<b_{n+1}^k<b_n^k<1$ for each k and n

(2) $\forall k\ \exists j$ with $(b_n^k) = O((b_n^j)^2)$.

The nuclearity of a G_∞-space is equivalent to $\Sigma \frac{1}{a_n^2} < \infty$ and the nuclearity of a G_1-space is equivalent to $B \subset \ell_1$ (13).

Let $\alpha = (\alpha_n)$ be an exponent sequence; that is: nondecreasing sequence of non-negative numbers. α is said to be a nuclear exponent sequence of finite type if:

$\Sigma k^{-\alpha_n} < \infty$ for all $k = 1,2,\ldots$

and a nuclear exponent sequence of infinite type if:

$\Sigma k^{-\alpha_n} < \infty$ for some $k=1,2,\ldots$

Let $\alpha = (\alpha_n)$ be a non-decreasing sequence of non-negative real numbers. <u>The power series space of infinite type</u> $\Lambda_\infty(\alpha)$ is the G_∞-space generated by the power set

$\{(k^{\alpha_n}) \; : \; k=1,2,\ldots\}$.

If we take $\alpha_n = \log n$ then $\Lambda_\infty(\log n)$ gives the space of rapidly decreasing sequences which will be denoted by (s).

The <u>power series space of finite type</u> $\Lambda_1(\alpha)$ G_1-space generated by the Köthe set

$\{((\frac{k}{k+1})^{\alpha_n}) \; : \; k=1,2,\ldots\}$.

<u>Theorem</u>: Every G_∞-space is equal to the intersection of a family of power series space of infinite type or similarly every G_1-space is equal to the intersection of a family of power series space of finite type.

Proof: Let P be a G_∞-set, assume $p=(p_n)\geq 1$. For each sequence in P define $\alpha_n^p = \log p_n$, then we get $\Lambda_\infty(\alpha_n^p)$. For each $k\geq 1$, find a j with

$$k^{\alpha_n^p} = 0(p_n^{2j}) .$$

Using the definition of a G_∞-set, we can find $p' \varepsilon P$ such that

$$(p_n)^{2j} = 0\ (p_n')$$

Thus $$k^{\alpha_n^p} = 0\ (p_n') .$$

This completes the first part of the theorem; the proof of the second part can be done similarly.

A locally convex Hausdorff space E is said to be stable if ExE is isomorphic to E.

A nuclear G_∞-space $\lambda(A)$ is stable if and only if for each k there is a j such that $a_{2n}^k = 0\ (a_n^j)$ and $\Lambda_\infty(\alpha)$ is stable if and only if $\alpha_{2n} = 0\ (\alpha_n)$ (15).

Definition 2: A sequence (x_n) in a nuclear Fréchet space E is a basis if for each $x \varepsilon E$ there is a unique sequence (t_n) of scalars such that $x= \Sigma t_n x_n$.

clearly if (x_n) is a basis and (d_n) a sequence of non-zero scalars then $(d_n x_n)$ is again a basis.

A basis (x_n) in E is called equicontinuous, if for each continuous seminorm $\|\cdot\|$ on E, there is a continuous seminorm $|.|$ on E such that

$$\text{Sup } |f_n(x)|\ \|x_n\| \leq |x| \qquad \text{for all x in E.}$$

(x_n) is called an absolutely equicontinuous basis, if for each continuous seminorm $\|\cdot\|$ on E there is a continuous semi-norm $|\cdot|$ on E with

$$\sum_{n=1}^{\infty} |f_n(x)| \, \|x_n\| \leq |x| \quad \text{for all } x \text{ in } E,$$

where (f_n) is the sequence of coefficient functionals of (x_n) (10).

Basis Theorem (10): Each equicontinuous basis in a nuclear locally convex space E is absolute.

The following corollary is a consequence of basis theorem.

Corollary: If E is complete nuclear space with an equicontinuous basis (x_n) with coefficient functionals f_n, then E is isomorphic to the Köthe space $\lambda(P)$, where

$$P = \{(\|x_n\|_i) : i \in I\} \quad \text{and} \quad (\|\ \|_i)_{i\in I} \text{ is a}$$

complete system of seminorms on E. (i.e. the sets,

$U_i = \{x \in E : \|x\|_i \leq 1\}$, $i \in I$ form a base of neighborhoods for the topology of E.)

A sequence (x_n) is called a basic sequence, if there is a closed subspace Y of E such that (x_n) is a basis of Y; and (x_n) is a complemented basic sequence (briefly CBS), if the subspace Y is complemented in X.

Let (x_n) be a basis in E. A seminorm $\|\cdot\|$ on E is said to be (x_n) - normal if $\|f_j(x)\, x_j\| \leq \|x\|$ for every $x \in E$ and every j.

Lemma: Suppose that E is a nuclear Fréchet space with basis (x_n), (f_n) is the sequence of coefficient functionals of the basis, and (y_n) is a CBS in E. Then there exist positive integers k_n with $\lim k_n = \infty$, such that $f_{k_n}(y_n) \neq 0$ for all n and such that for any (x_n)- normal seminorm $|\cdot|$ there is a seminorm $\|\cdot\|$ satisfying the condition

$$a_n|x_{k_n}| \leq |y_n| \leq a_n\|x_{k_n}\|$$

where

$$a_n = |f_{k_n}(y_n)| \quad \text{for } n = 1,2,\ldots$$

The proof of the above lemma is given in (2).

Two bases (x_n) and (y_n) of Fréchet spaces E and F are equivalent if there is an isomorphism $T : E \longrightarrow F$ such that $Tx_n = y_n$ $n\varepsilon N$.

The bases (x_n) and (y_n) are said to be semi-equivalent if there exist scalars $d_n \neq 0$ such that the map which sends x_n to $d_n y_n$ is an isomorphism. Finally the bases (x_n) and (y_n) are quasi-equivalent if there is a permütation π of N such that (x_n) and $(y_{\pi(n)})$ are semi-equivalent. It is know that all bases of a nuclear smooth sequence space are quasi-equivalent (1).

Let (x_n) be a basis for E and $0 = P_o < P_n < P_{n+1}$ $n\varepsilon N$ and

$$y_n = \sum_{i=P_{n-1}+1}^{P_n} t_i x_i \quad n\varepsilon N$$

where (t_i) is the sequence of scalars and $y_n \neq 0$ $n\varepsilon N$, then (y_n) is called a block basic sequence with respect to (x_n).

A basis (x_n) in a nuclear Fréchet space is said to be regular if there is a fundamental system of norms $(\|\ \|_k)$ such that

$$\frac{\|x_n\|_{k+1}}{\|x_n\|_k} \leq \frac{\|x_{n+1}\|_{k+1}}{\|x_{n+1}\|_k} \quad \forall n,k\varepsilon N$$

where by a fundamental system of norms we mean the sequence of norms $(\|\ \|_k)_{k\in N}$ such that

$\|x\|_k \leq \|x\|_{k+1}$ $x\in E$ and $\lim_n x_n = x$ in the topology of E if and only if $\lim_n \|x_n - x\|_k = 0$ $k\in N$.

Throughout we will have the following conventions. The Köthe spaces considered are all nuclear. All the Fréchet Spaces considered are assumed to admit a continuous norm and therefore the space w of all sequences of scalars is excluded from our considerations.

A Fundamental Inequality:

The basis theorem states that every basis of a nuclear Fréchet space is equivalent to the canocical basis of a Köthe space $\lambda(A)$ where A is defined as in the corollary of basis theorem. So if we take a basic sequence (y_n) in $\lambda(A)$, then Y is isomorphic to the Köthe space generated by the Köthe set $\{(\|y_n\|_k) : k = 1,2,..\}$, where $(\|\ \|_k)$ is a fundamental sequence of norms on $\lambda(A)$.

Considering the nuclearity of $\lambda(A)$ we may take

$$\|y_n\|_k = \max_i |t_i^n| a_i^k$$

where t_i^n is the i-th cordinate of y_n in its expansion in terms of the canonical basis (e_i) of $\lambda(A)$. From Dubinsky's lemma we define

$$q(k,n) = q^k(y_n) = \max \{q: \|y_n\|_k = |t_q^n| a_q^k \}$$

So considering inequalities

$$\| y_n \|_k \geq | t^n_{q(k+j,n)} | \, a^k_{q(k+j,n)} \quad \text{and} \quad \| y_n \|_{k+j} \geq | t^n_{q(k,n)} | \, a^{k+j}_{q(k,n)}$$

we get

$$\frac{a^k_{q(k+j,n)}}{a^{k+j}_{q(k+j,n)}} \leq \frac{\| y_n \|_k}{\| y_n \|_{k+j}} \leq \frac{a^k_{q(k,n)}}{a^{k+j}_{q(k,n)}} .$$

Let α_n be a nuclear exponent sequence of finite type and consider power series space $\Lambda_1(\alpha)$ generated by the Köthe set $\{e^{-\alpha_n/k} : k=1,2,\ldots\}$ and set $\lambda_n = e^{-\alpha_n}$. So if (y_n) is a basic sequence in $\Lambda_1(\alpha)$, then our fundamental inequality becomes

$$(\lambda_{q(k+j,n)})^{\frac{j}{k(k+j)}} \leq \frac{\| y_n \|_k}{\| y_n \|_{k+j}} \leq (\lambda_{q(k,n)})^{\frac{j}{k(k+j)}} . \quad (11) .$$

<u>Corollary:</u> If (y_n) is ablock basic sequence with respect to a regular basis of a nuclear Fréchet space, then (y_n) is also a regular basis of Y .

<u>Proof:</u> Assume that the nuclear Fréchet space is $\lambda(A)$. From basis theorem and regularity we can also assume (a^k_n / a^{k+1}_n) is decreasing.

(y_n) is block basic sequence, Thus

$$y_n = \sum_{p_{n-1}+1}^{p_n} t^n_i \, e_i$$

where $0 = p_o < p_1 < p_2 < \ldots$, so we have $p_{n-1} < q(k,n) \leq p_n$ and

$q(k+1,n) \leq q(k,n+1)$.

Therefore from fundamental inequality and above inequalities we obtain

$$\frac{\| y_n \|_k}{\| y_n \|_{k+1}} \geq \frac{a^k_{q(k+1,n)}}{a^{k+1}_{q(k+1,n)}} \geq \frac{a^k_{q(k,n+1)}}{a^{k+1}_{q(k,n+1)}} \geq \frac{\| y_{n+1} \|_k}{\| y_{n+1} \|_{k+1}}$$

Hence

$$\frac{\| y_n \|_{k+1}}{\| y_n \|_k} \leq \frac{\| y_{n+1} \|_{k+1}}{\| y_{n+1} \|_k} \qquad \forall n,k \in N.$$

CHAPTER II SUBSPACES OF FINITE TYPE POWER SERIES SPACES

In this section we take a given power series space of finite type and try to determine if it is possible for this space to have a subspace isomorphic to power series space of infinite type, and find out exactly which of such spaces may appear as subspaces. We also consider G_1 and $\Lambda_1(\beta_n)$ subspaces of power series space of finite type.

The first result in this direction was due to R. Rolewicz (12), who showed how to embed $\Lambda_\infty(\alpha)$ as a subspace of $\Lambda_1(\alpha)$ when $\alpha_n = n$. Actually he has shown that the space of entire functions, in one-complex variable, is isomorphic to a subspace of the space of functions analytic in the interior of a disk.

More recently V.P. Zahariuta (19) has proved that every linear continuous map from a finite type power series space to an infinite type power series space is compact. Hence, no power series space of infinite type can contain a subspace isomorphic to power series space of finite type.

Later, Dubinsky (4) interchanged the words "Infinite" and "finite" in Zahariuta's statement and proved exactly the opposite results that is: For every nuclear power series space X of finite type, there exist a nuclear power series space Y of infinite type and a linear, continuous, non-compact map $T:Y \longrightarrow X$.

1. B. Mitiagin and G. Henkin's Theorem:

Notation A: Let (β_n) be a nuclear exponent sequence and s be any positive number, define:

$$F_s = \{(\xi_n) : \|\xi\|_s = \Sigma |\xi_n|^2 s^{2\beta_n} < \infty\} .$$

So if we take $s_1 < s_2$ $\| \ \|_{s_1} \leq \| \ \|_{s_2}$ and moreover F_{s_2} is a dense subset of F_{s_1}. Let $s_1 < s < s_2$ then $\log s_1 < \log s < \log s_2$ where $0<\delta(s)<1$ then $s = s_1^{\delta(s)} \cdot s_2^{1-\delta(s)}$

where $$\delta(s) = \frac{\log s/s_2}{\log s_1/s_2}$$

$$\|\xi\|_s = (\Sigma|\xi_n|^2 s^{2\beta_n})^{1/2} \leq \Sigma|\xi_n| s^{\beta_n} = \Sigma|\xi_n|^{\delta(s)} s_1^{\delta(s)\beta_n} \cdot |\xi_n|^{1-\delta(s)} s_2^{(1-\delta(s))\beta_n}$$

$$\leq \|\xi\|_{s_1}^{\delta(s)} \cdot \|\xi\|_{s_2}^{1-\delta(s)}$$

If $s_1 < s < s_2$ then $\|\xi\|_s \leq \|\xi\|_{s_1}^{\delta(s)} \|\xi\|_{s_2}^{(1-\delta(s))}$.

F_s can be identified with the space of all $\phi = (\phi_n)$ such that

$$\|\phi\|'_s = \left(\Sigma \frac{|\phi|^2}{s^{2\beta_n}}\right)^{1/2} < \infty ,$$

and identification is done by $\phi(\xi) = \Sigma\phi_n\xi_n$. And similarly if $s_1 < s < s_2$ implies $\| \ \|'_s \leq \| \ \|'^{\delta(s)}_{s_1} \ \| \ \|'^{(1-\delta(s))}_{s_2}$

Notice that if $\Lambda_1(\beta_n)$ is nuclear then

$$\Lambda_1(\beta_n) \cong \bigcap_{a \leq s < 1} F_s$$

where a is any number between 0 and 1. Let us take a=1/2. In (9) Mitiagin and Henkin has proved the following theorem.

<u>Theorem 1</u>: Let $\Lambda_1(\alpha_n)$ and $\Lambda_1(\beta_n)$ be both nuclear finite type power series spaces. Let X be isomorphic to a subspace of $\Lambda_1(\alpha)$ and to a quotient space of $\Lambda_1(\beta_n)$. Then X is isomorphic to some $\Lambda_1(\gamma_n)$.

<u>Proof</u>: First notice that X being a factor space of a Fréchet space is complete.

Assume E_t is defined as F_s and $\Lambda_1(\alpha_n)$ is nuclear, so

$$\Lambda_1(\alpha_n) = \bigcap_{\frac{1}{2} \leq t < 1} E_t .$$

For $\frac{1}{2} \leq t<1$, $\tilde{X}_t$ is the completion of X under

$$\| \xi \|_t = (\Sigma \mid \xi_n \mid^2 t^{2\alpha_n})^{1/2} . \quad \text{So} \quad X = \bigcap_{\frac{1}{2} \leq t<1} \tilde{X}_t$$

$\exists$ continuous, onto map $j : \Lambda_1(\beta_n) \longrightarrow X$ since X is Fréchet space j is also open. We have $F_1 \subset \Lambda_1(\beta_n)$; let

$j(F_1) = X_1 \subset X = \bigcap_{\frac{1}{2} \leq t<1} \tilde{X}_t$, on the other hand, choose $t_n \uparrow 1$

$\frac{1}{2} \leq t_n < 1$ take $x, y \in X_1$ and define $(x,y)_1 = \Sigma \dfrac{(x,y)t_n}{2^n\, k(t_n)^2}$

It is easy to show that this sum is well defined and defines a scalar product on X_1. Here $k(t_n)$ is coming from

$\| j(z) \|_{t_n} \leq k(t_n) \|z\|_{s(t_n)}$ (since j is continuous). So

$\| x \|_{t_n} \leq 2^{n/2} k(t_n) \| x \|_1$.

Now $X_1 \longrightarrow \tilde{X}_t$ is continuous and $\tilde{X}_t \longrightarrow \tilde{X}_{1/2}$ is s-nuclear (14).

So if T is the imbedding of X_1 into $\tilde{X}_{1/2}$, then T is s-nuclear, that is, T can be written in the form

$$Tx = \Sigma \lambda_n (x, e_n)_1 f_n$$

where (e_n) is orthonormal basis for X_1 and (f_n) is orthonormal basis for $\tilde{X}_{1/2}$ and (λ_n) is a rapidly decreasing sequence.

$\lambda_n = \| e_n \|_{1/2}$

Define $e_k' : \tilde{X}_{1/2} \longrightarrow R$ by

$$e_k'(x) = \frac{1}{\lambda_k^2} (x, e_k)_{1/2} \qquad e_k = \lambda_k f_k$$

Then $\qquad \| e_k' \|'_{1/2} = \frac{1}{\lambda_k}$

If $x \epsilon X_1 \qquad e_k'(x) = \frac{1}{\lambda_k^2} (x, e_k)_{1/2} = \frac{1}{\lambda_k^2} \lambda_k (x, f_k)_{1/2}$

$= \frac{1}{\lambda_k} (x, f_k)_{1/2} = (x, e_k)_1$

Since $X_1 \subset \tilde{X}_{1/2}$.

So we obtain $\qquad \| e_k' \|'_{1/2} = \frac{1}{\lambda_k}$

$$\| e'_k \|'_1 = 1 .$$

Let $\phi \varepsilon \tilde{X}'_{1/2}$; $j : F_{s(1/2)} \longrightarrow X_{1/2}$ has norm $k(1/2)$ and let $s(1/2) < \tau s(1/2) < 1$ from above explanations

$$\| j\phi \|_{\tau s(1/2)} \leq (\| j\phi \|'_{s(1/2)})^{\varepsilon(\tau)} . (\| j \phi \|'_1)^{1-\varepsilon(\tau)}$$

where $\tau s(1/2) \longrightarrow 1$ as $\varepsilon(\tau) \to 0$

$\tau s(1/2) \longrightarrow s(1/2)$ as $\varepsilon(\tau) \longrightarrow 1$

j is open therefore $\forall s$ $\exists\, t(s)$ such that

$$X \cap s(E_{t(s)}) \subset \tilde{k}(s) \ j\ (s(F_s))$$

where $s(F_s)$ is the unit ball of F_s,

$$\| \phi \|'_{t(s)} \leq \tilde{k}(s) \ \| j \phi \|'_s$$

Let $s > s(1/2)$ $\tau s(1/2) = s$ then

$$\| \phi \|'_{t(s)} \leq \tilde{k}(s) \ (\| j\phi \|'_{s(1/2)})^{\varepsilon(\frac{s}{s(1/2)})} . (\| j\phi \|'_1)^{1-\varepsilon(\frac{s}{s(1/2)})}$$

Let $\phi = e'_k$

$$\| e'_k \|'_{t(s)} \leq k_1(s) \lambda_k^{-\varepsilon(\frac{s}{s(1/2)})}$$

$$\| e_k \|_t \leq k_2(t) \ \| e_k \|_{1/2}^{\delta(t)} \ \| e_k \|_1^{1-\delta(t)} \quad \text{if } \frac{1}{2} < t < 1$$

$$\leq k_2(t) \ \lambda_k^{\delta(t)}$$

given t choose s such that $\varepsilon(\frac{s}{s(1/2)}) > \frac{1}{2}\, \delta(t)$ so we get

get

$$|e'_k(x)| \ \| e_k \|_t \leq (\| e'_k \|'_{t(s)}) \ \| x \|_{t(s)} \ k_2(t) \ \lambda_k^{\delta(t)}$$

$\leq k_1(s)\, k_2(t)\ \lambda_k^{\delta(t)/2}\ .\ \|x\|_{t(s)}$

so $\Sigma\ |e_k'(x)|\ \|e_k\|_t \leq k_1(s)\, k_2(s)\ \Sigma\ \lambda_k^{\delta(t)}\ \|x\|_{t(s)}$

conclusion follows from the basis theorem.

Remark: Although, there is noting about a basis for X, in the statement of theorem 1, we obtain a basis for X in the proof.

Corollary: A complemented subspace of a nuclear power series space of finite type is itself isomorphic to a power series space of finite type.

Remark: It is not known whether or not the above corollary is true for Power series space of infinite type.

2. G_1-subspaces of $\Lambda_1(\alpha)$:

In (11) M.S. Ramanujan and T.Terzioğlu have proved the following theorem.

Theorem 2: Let $\lambda(Q)$ be a G_1-space such that $\lambda(Q)$ is isomorphic to a supspace of $\Lambda_1(\alpha)$. Then $\lambda(Q)$ is isomorphic to power series space of finite type.

Proof: Assume there is a basic sequence (y_n) in $\Lambda_1(\alpha)$ such that $Y \cong \lambda(Q)$. Then (y_n) is quasi-equivalent to the canonical basis of $\lambda(Q)$ (1). Therefore there is a permutation π and $d_n > 0$ such that $(d_n \|y_n\|_k)$ and $(q^j_{\pi(n)})$ are equivalent. Hence we find m and j_o with $d_n \|y_n\|_1 = 0\,(q^m_{\pi(n)})$ and $q^m_n = 0((q^{j_o}_n)^2)$. If k_o is chosen so that $q^{j_o}_{\pi(n)} = 0(d_n \|y_n\|_{k_o})$

then $\dfrac{\| y_n \|_1}{\| y_n \|_{k_o}} = 0\,(q^{j_o}_{\pi(n)})$

Now, from fundamental Inequality

$$(\lambda_{q(k+j,n)})^{\frac{j}{k(k+j)}} \leq \frac{\| y_n \|_k}{\| y_n \|_{k+j}}$$

Let us take $k=1$ $k+j=k_o$ and the fact that $q^j_n<1$ $\forall n,j$, we have

$$\exp(-\alpha_{q(ko,n)}) = 0\,(q^{j_o}_{\pi(n)}) \;\ldots\ldots\; (i)$$

For each fixed j, find r and k_j such that $q^j_n = 0((q^r_n)^2)$ and $q^r_{\pi(n)} = 0(d_n \| y_n \|_k)$ for $k \geq k_j$. Then from fundamental inequality again, we have

$$q^r_{\pi(n)} = 0\,(d_n \| y_n \|_{2k} \; \exp\,(-\alpha_{q(k,n)}\;/2k))$$

since $(d_n \| y_n \|_{2k})$ is dominated by some $(q^s_{\pi(n)})$, it is a bounded sequence. Hence for each j we can find a k_j such that $q^j_{\pi(n)} = 0\,(\exp(-\alpha_{q(k,n)}/\,k))$ for each $k \geq k_j$:...(ii) claim $\lambda(0)$ is isomorphic to $\Lambda_1(\beta)$ where $\beta_n = -\log\, q^{j_o}_n$ as in (i).

Given $j \geq j_o$ we find $k \geq k_j$ and $k > k_o$ such that

$$(q^j_{\pi(n)})^k = 0\,(\exp(-\alpha_{q(k,n)}))$$

consider $\alpha_{q(k,n)} \geq \alpha_{q(k_o,n)}$,

From (i) and above boundeness condition we obtain

$$(q^j_{\pi(n)})^k = 0\,(q^{j_o}_{\pi(n)})$$

Hence for each j there is an integer k with

$$q_n^j = 0\ (\exp\ (-\beta_n)^{1/k})$$

This completes the proof.

3. $\Lambda_\infty(\beta)$ subspaces of $\Lambda_1(\alpha)$:

Following major technical results about $\Lambda_\infty(\beta)$ subspaces of $\Lambda_1(\alpha)$ were due to Ed. Dubinsky (4).

Lemma 3: Let (a_n^k) be an infinite matrix of positive numbers satisfying the following conditions

$$0<a_n^k< a_n^{k+1} \qquad \text{for all } n,k\varepsilon\ N$$

$$\frac{a_n^{k+1}}{a_n^k} < \frac{a_{n+1}^{k+1}}{a_{n+1}^k} \qquad \text{for all } n,k\varepsilon\ N$$

Given numbers $t_1, \ldots, t_p$ we define for $k\varepsilon$ N,

$$q^k(t_1,\ldots,t_p)=\max \{q: \max_{1\leq i\leq p} |t_i|\ a_i^k = |t_q|a_q^k\}$$

Then if $0<q<\ldots<q^m\leq p$ are integers, it is possible to choose numbers $t_1, \ldots, t_p$ with $t_q1 \neq 0$ but otherwise arbitrary, $t_i=0$ for $i \neq q^1,\ldots,q^m$ and

$$(1)\ \ldots|t_{q^k}|\ \frac{a_{q^k}^{k+1}}{a_{q^{k+1}}^{k+1}} < |t_{q^{k+1}}| < |t_{q^k}|\frac{a_{q^k}^k}{a_{q^{k+1}}^k}\ ,\ k=1,2,\ldots,m-1.$$

Moreover, if any such chorce is made, then

$$q^k(t_1,\dots,t_p)=q^k \qquad k=1,\dots,m.$$

<u>Proof</u>: Set $t_i=0$ for $i \neq q^1,\dots,q^m$ and $t_{q^1} \neq 0$. Then we choose $t_{q^{k+1}}$, $k+1=2,\dots,m$ inductiuely to satisfy the inequality (1). This is possible since $\frac{a_n^{k+1}}{a_n^k} < \frac{a_{n+1}^{k+1}}{a_{n+1}^k}$ is given and from the definition of $q^k(t_1,\dots,t_p)$ we already have $q^{k+1}(t_1,\dots,t_p) \geqq q^k(t_1,\dots,t_p)$.

Let $\rho_o,\dots,\rho_m$ be any sequence of integers such that

$$0 < \rho_o < \rho_1 < \dots < \rho_m \quad , \quad m \leqq p.$$

If we can show that, there exist scalars $t_1,\dots,t_p$ such that $q^k(t_1,\dots,t_p) = q^\ell$ for $\rho_{\ell-1} < k < \rho_\ell$, $\rho_\ell = 1,2,\dots, m$. Then we are done.

By hypothesis we have

$$\max_{\rho_k < k \leq \rho_{k+1}} |t_{q^k}| \frac{a_{q^k}^{k+1}}{a_{q^{k+1}}^{k+1}} < |t_{q^{k+1}}| < \min_{\rho_{k-1} < k \leqq \rho_k} |t_{q^k}| \frac{a_{q^k}^k}{a_{q^{k+1}}^k} \quad \dots\dots(2)$$

Now fix $\ell=1,\dots,m-1$ and consider $k+1$ with $\ell<k+1 \leqq m$ by (2) we have

$$|t_{q^{k+1}}| \, a_{q^{k+1}}^k < |t_{q^k}| \, a_{q^k}^k \qquad \text{for} \quad \rho_{k-1} < k \leqq \rho_k$$

claim this inequality holds for $\rho_{\ell-1} < k \leq \rho_\ell$. Let us take $k=\rho_k$ and the fact that $\rho_\ell \leqq \rho_k$ we obtain for $\rho_{\ell-1} < k \leq \rho_\ell$

$$\frac{|t_{q^{k+1}}|}{|t_{q^k}|} \leqq \frac{a_{q^k}^{\rho_k}}{a_{q^{k+1}}^{\rho_k}} \leqq \frac{a_{q^k}^{\rho_\ell}}{a_{q^{k+1}}^{\rho_\ell}} \leqq \frac{a_{q^k}^{k}}{a_{q^{k+1}}^{k}}$$

Thus $|t_{q^{k+1}}|\ a_{q^{k+1}} < |t_{q^k}|\ a^k_{q^k}$ for $\rho_{\ell-1} < k \leq \rho_\ell$

Repeating this inequality with k+1 replaced successively by $k,\ldots,\ell+1$ we obtain

$$q^k(t_1,\ldots,t_p) \leq q^\ell \qquad \text{for} \quad \rho_{\ell-1} < k < \rho_\ell$$

For $\ell = m$ inequality follows immediately from the fact that $|t_i| \geqq 0$ for all i and $|t_i| \neq 0$ iff $i=q^1,\ldots,q^m$.
To obtain the reverse inequality we argue analogously.

<u>NOTATION B:</u> Let

α be Nuclear exponent sequence of finite type

β " " " " " Infinite "

(y_n)" basic sequence in $\Lambda_1(\alpha)$ whichis of the form

$$y_n = \sum_{i=1}^{p_n} t_i^n\ e_i \quad \text{where } p_n < \infty \quad n=1,2,\ldots$$

(d_n) be a sequence of non-zero scalars

π " a permutation of N.

Then define: $q_n^k = q^k(t_1^n,\dots,t_{p_n}^n) = q(k,n)$

$$\gamma_n^k = \frac{\alpha_{q(k,n)}}{\beta_{\pi(n)}}$$

$$\mu_n^k = d_n^{1/\beta_{\pi(n)}} \, |t_{q(k,n)}|^{1/\beta_{\pi(n)}} \left(\frac{k}{k+1}\right)^{\gamma_n^k}$$

$$r_n^k = \frac{\mu_n^{k+1}}{\mu_n^k}$$

Y = subspace of $\Lambda_1(\alpha)$ generated by (y_n).

Remark: In the context of preceding notation, we have for all n,k:

$$\left(\frac{k^2+2k+1}{k^2+2k}\right)^{\gamma_n^k} \leq r_n^k \leq \left(\frac{k^2+2k+1}{k^2+2k}\right)^{\gamma_n^{k+1}}$$

Theorem 4: In the context of preceding notation the following are equivalent

(i) $Y \cong \Lambda_\infty(\beta)$

(ii) There exist (d_n) and π such that

a) $\forall j \;\exists k$ and $M>0$ $\; j^{\beta_{\pi(n)}} \leq M \, d_n |t_{q_n^k}^n| \left(\frac{k}{k+1}\right)^{\alpha_{q(k,n)}} \;\forall n$

b) $\forall k \;\exists \ell$ " $N>0$ $\; d_n |t_{q(k,n)}^n| \left(\frac{k}{k+1}\right)^{\alpha_{q(k,n)}} \leq N \, \ell^{\beta_{\pi(n)}} \;\forall n$.

(iii) There exist (d_n) and π such that

a) $\overline{\lim}\, \mu_n^k < \infty \;\forall k$

b) $\sup_k \lim_n \mu_n^k = \infty$

(iv) There exist π such that

a) $\sup \gamma_n^k < \infty$ for all k

b) $\sup_k \lim_n \sum_{j=1}^{\infty} \frac{\gamma_n^j}{j^2} = \infty$

<u>Proof:</u> (i)$\Longleftrightarrow$(ii):

According to theorem of Dragilev (3), $Y \simeq \Lambda_\infty(\beta)$ means, there exist (d_n), π such that the bases (y_n) and $(d_n^{-1} e_{\pi(n)})$ are equivalent where (e_n) is the coordinate basis of $\Lambda_\infty(\beta)$. Thus the sequence spaces determined by these bases are the same.

From absolute basis theorem (10) and from the fact that α is a nuclear exponent sequence of finite, respectively infinite type iff the Fréchet space $\Lambda_1(\alpha)$, pespectively $\Lambda_\infty(\alpha)$, is a nuclear space. In this case the fundamental system of seminorms given by:

$$\tilde{p}_k(\xi) = \sup_n |\xi_n| \left(\frac{k}{k+1}\right)^{\alpha_n}$$

respectively

$$\tilde{p}_k(\xi) = \sup |\xi_n| \; k^{\alpha_n} \quad .$$

Moreover in this case, the coordinate sequence (e_n) forms a basis for the space, called coordinate basis.

Thus the isomorphism is equivalent to the relation

$$\bigcap_k \frac{1}{c^k} \cdot \ell_1 = \bigcap_k \frac{1}{b^k} \ell_1$$

where $b^k = (b_n^k) = |t^n_{q(k,n)}| \left(\frac{k}{k+1}\right)^{\alpha_{q(k,n)}}$

$$c^k = (c_n^k) = \frac{1}{d_n} k^{\beta_{\pi(n)}} .$$

Now the sequence spaces mentioned above are echelon spaces so they are equal iff their corresponding co-echelon spaces are also equal, thus

$$\bigcup_k c^k \ell_\infty = \bigcup_k b^k \ell_\infty$$

is equivalent to (i).

But this equality is equivalent

a) $\forall j \; \exists k$ and $M>0$ $\quad c_n^j < M \, b_n^k \quad \forall n$

b) $\forall k \; \exists \ell$ and $N>0$ $\quad b_n^k \leqq N \, c_n^\ell \quad \forall n$.

so we obtain (ii). (ii) $\Longrightarrow$ (iii):

$\mu_n k = d_n^{j/\beta_{\pi(n)}} |t^n_{q_n^k}|^{1/\beta_{\pi(n)}} \left(\frac{k}{k+1}\right)^{\gamma_n^k}$ by its definition, so raise both sides of the inequalities (ii) to the power $1/\beta_{\pi(n)}$, they become

a) $\forall j \; \exists k$ and $M > 0$ $\quad j \leqq M^{1/\beta_{\pi(n)}} \mu_n^k$ for all n

b) $\forall k \; \exists \ell$ and N 0 $\quad \mu_n^k \leqq N^{1/\beta_{\pi(n)}} \ell$ " .

Then take limit inferior in (a), the limit superior in (b), and apply the fact that if α is a nuclear exponent sequence then $\lim_n \alpha_n = \infty$, to obtain:

a) $\forall j \ \exists k \quad j \leq \underline{\lim}\ \mu_n^k$ equivalent (iii) b

b) $\forall k \ \exists \ell \quad \overline{\lim_n}\ \mu_n^k \leq \ell$ " (iii) a

(iii)$\Longrightarrow$(iv):

suppose there exist (d_n) and π s.t. (iii) (a) and (b) hold and suppose (iv)(a) is not true. Then we would have some k_o and infinite set $N_o \subset N$ s.t. $\lim_{k_o \in N_o} \gamma_n^{k_o} = \infty$ So follows $\lim_{n \in N_o} \gamma_n^k = \infty$

for all $k \geq k_o$ so by our remark it follows that $\lim_{n \in N_o} r_n^k = \infty$

for all $k \geq k_o$ but by (iii) (a) $\overline{\lim_n}\ \mu_n^{k+1} < \infty$ for $k \geq k_o$

so clearly $\lim_{n \in N_o} \mu_n^k = 0$ for $k \geq k_o$ and hence $\underline{\lim}_n \mu_n^k = 0$

$k \geq k_o$ contradiction for (iii) (b). To prove (iv) (b), again using our remark, we have

$$\mu_n^k = \mu_n^1 \ r_n^1 \ \ldots r_n^{k-1} \leq \mu_n^1 \prod_{j=1}^{k-1} \left(\frac{j^2+2j+1}{j^2+2j}\right)^{\gamma_n^{j+1}}$$

so for all k

$$\underline{\lim}\ \mu_n^K \leq (\overline{\lim_n}\ \mu_n^1)\ \underline{\lim_n} \left(\prod_{j=1}^{k-1} \left(\frac{j^2+2j+1}{j^2+2j}\right)^{\gamma_n^{j+1}}\right)$$

by (iii) (a) $\overline{\lim_n}\ \mu_n^1 < \infty$ let $M = \overline{\lim_n}\ \mu_n^1$ then

$$\log\left(\overline{\lim_n}\ \mu_n^k\right) \leq \log M + \overline{\lim_n}\left(\sum_{j=1}^{k-1} \frac{\gamma_n^{j+1}}{(j+1)^2}(j+1)^2 \log\left(\frac{j^2+2j+1}{j^2+2j}\right)\right)$$

This follows from the monotonicity of the logarithm function.

Now consider, $\lim_{j\to\infty} (j+1)^2 \log\left(\frac{j^2+2j+1}{j^2+2j}\right) = \lim_{j\to\infty} \left(\log\left(\frac{j^2+2j+1}{j^2+2j}\right)^{(j+1)^2}\right)$

$$\lim_{j\to\infty} \left(1+\frac{1}{j^2+2j}\right)^{j^2+2j} \cdot \left(1+\frac{1}{j^2+2j}\right) = e$$

so $\lim_{j\to\infty} (j+1)^2 \log\left(\frac{j^2+2j+1}{j^2+2j}\right) = 1$

$$\sup \log\left(\overline{\lim_n}\ \mu_n^k\right) \leq \log M + \sup_k \overline{\lim_n}\left(\sum_{j=1}^{k-1} \frac{\gamma_n^{j+1}}{(j+1)^2}(j+1)^2 \log\left(\frac{j^2+2j+1}{j^2+2j}\right)\right)$$

$$\leq \log M + \sup_{1\leq j<\infty}(j+1)^2 \log\left(\frac{j^2+2j+1}{j^2+2j}\right) \sup_k \overline{\lim_n} \sum_{j=1}^{k-1} \frac{\gamma_n^{j+1}}{(j+1)^2}$$

since $\log M$ and the quantity $\sup (j+1)^2 \log\left(\frac{j^2+2j+1}{j^2+2j}\right)$ are finite

and by (iii) b we have $\sup \log\left(\overline{\lim_n}\ \mu_n^k\right) = \infty$, it follows

$$\sup_k \overline{\lim_n} \sum_{j=1}^{k-1} \frac{\gamma_n^{j+1}}{(j+1)^2} = \infty$$, which is (iv) (b)

(iv) $\Longrightarrow$ (iii)

Suppose π is chosen so that (iv) (a), (b) hold.

Choose $d_n > o$ such that $\mu_n^1 = 1$ for all n, this is possible since $y_n \neq 0$ so $t^n_{q_n^k} \neq 0$ from the definition of q_n^k .

Set $\sup_{j\leq k} \sup_n \gamma_n^j < \infty$ apply remark, to obtain for each n,k

$$\mu_n^k = \mu_n^1 \; r_n^1 \; \ldots r_n^{k-1} \leq \prod_{j=1}^{k-1} \left(\frac{j^2+2j+1}{j^2+2j}\right)^{\gamma_n^{j+1}} \leq \left(\prod_{j=1}^{k-1} \frac{j^2+2j+1}{j^2+2j}\right)^{M^k}$$

which estaplisheas (iii) (a).

Apply remark again to obtain for each n,k:

$$\mu_n^{k+1} = \mu_n^1 \; r_n^1 \; \ldots r_n^k \geq \prod_{j=1}^{k} \left(\frac{j^2+2j+1}{j^2+2j}\right)^{\gamma_n^j} \quad \text{so } \exists \delta>0 \text{ such that}$$

$$\log \mu_n^{k+1} \geq \sum_{j=1}^{k} \frac{\gamma_n^j}{j^2} j^2 \log \left(\frac{j^2+2j+1}{j^2+2j}\right) \geq \delta \sum_{j=1}^{k} \frac{\gamma_n^j}{j^2}$$

$$\sup \underline{\lim}_n \log \mu_n^{k+1} \geq \delta \sup_k \underline{\lim}_n \sum_{j=1}^{k} \frac{\gamma_n^j}{j^2} = \infty \quad \text{Which estaplishes}$$

(iii) (b). The following two theorems concern new classes of finite typ power series space which have infinite type subspaces.

<u>Theorem 5:</u> If (α) is a nuclear exponent sequence of finite type which satisfies the condition $\sup_n \frac{\alpha_{2n}}{\alpha_n} < \infty$. Then $\Lambda_\infty(\alpha)$ is isomorphic to a subspace of $\Lambda_1(\alpha)$, in particular one which is generated by a block basic sequence of a permutation of the basis (e_n).

<u>Proof:</u> We want to apply lemma to (α_n). For this we need that α is satictly increasing. For, if (δ_n) is any sequence of positive numbers with $1\le\delta_n\le 2$, chosen such that $\tilde{\alpha}_n = \alpha_n+\delta_n$ is strictly increasing, then $(\tilde{\alpha}_n)$ is again a nuclear exponent sequence of finite type and we have the isomorphism between $\Lambda_1(\tilde{\alpha}_n)$ and $\Lambda_1(\alpha_n)$ with the identity map.

Moreover $(\tilde{\alpha}_n)$ satisfies our condition, that is:

$$\sup_n \frac{\tilde{\alpha}_{2n}}{\tilde{\alpha}_n} = \sup_n \frac{\alpha_{2n}+\delta_{2n}}{\alpha_n+\delta_n} \leqq \sup \frac{\alpha_{2n}}{\alpha_n} + 2.\ \sup \frac{1}{\alpha_n} < \infty$$

By replacing (α_n) by $(\tilde{\alpha}_n)$ if necessary, we may assume that (α_n) is strictly increasing without any loss of generality. Let $\sigma : NxN \longrightarrow N$

$$\sigma\ (j,n) = 2^{j-1}\ (2n-1)$$

and set $M = \sup_n \frac{\alpha_{2n}}{\alpha_n}$

Let n be fixed and for k=1,2,...,n let j_n^k be the first positive integer such that

$$k^3 < \frac{\alpha_{\sigma(j_n^k,n)}}{\alpha_n} \quad .$$

This is possible since $\sigma\ (.,n)$ and α are increasing. We can select j_n^k in two ways

(1) suppose $j_n^k = 1$, $\sigma(j_n^k,n) = (2n-1)$, so

$$\frac{\alpha_{\sigma(j_n^k,n)}}{\alpha_n} = \frac{\alpha_{2n-1}}{\alpha_n} \leqq \frac{\alpha_{2n}}{\alpha_n} \le M \le Mk^3$$

(2) Suppose $j_n^k > 1$, we have

$$\frac{\alpha_{\sigma(j_n^k - 1,n)}}{\alpha_n} \leq k^3 \quad \text{so that}$$

$$\frac{\alpha_{\sigma(j_n^k,n)}}{\alpha_n} = \frac{\alpha_{\sigma(j_n^k,n)}}{\alpha_{\sigma(j_n^k-1,n)}} \cdot \frac{\alpha_{\sigma(j_n^k-1,n)}}{\alpha_n} \leq M\,.k^3 \;.$$

Thus in either case we obtain

$$k^3 < \frac{\alpha_{\sigma(j_n^k,n)}}{\alpha_n} < M\,k^3 \qquad k = 1,2,\ldots,n$$

So if we set $p_n = \sigma(j_n^n,n)$, since (j_n^k) and hence $\sigma(j_n^k,n)$ are non-decreasing with respect to k. We can apply lemma3, so there exist scalars t_i^n $i = 1,\ldots,p_n$, such that if $y_n = \sum_{i=1}^{p_n} t_i^n e_i$ then $q_n^k = \sigma(j_n^k,n)$ Moreover $t_i^n = 0$ if $i \neq \sigma(j_n^k,n)$ $k = 1,2,\ldots,n$. So

$y_n = \sum_{i=1}^{n} t^n_{\sigma(j_n^i,n)}\, e_{\sigma(j_n^i,n)}$. This shows that (y_n) is a block basic sequence of a permutation of the basis (e_n). Let $\beta = \alpha$ and take π to be the identity map so we get

$$k^3 < \gamma_n^k < M\,k^3 \qquad k = 1,2,\ldots n\;.$$

Hence, for each k,

$$\sup_n \gamma_n^k = \max\{\max_{1\leq n\leq k} \gamma_n^k\,,\, Mk^3\} < \infty$$

and

$$\sup_n \varliminf_n \sum_{j=1}^{k} \frac{\gamma_n^j}{j^2} = \sup_k \varliminf_{n\geq k} \sum_{j=1}^{k} \frac{\gamma_n^j}{j^2} \geq \sup \sum_{j=1}^{k} \frac{j^3}{j^2} = \infty$$

Thus our theorem follows from theorem 4 (iv).

Remark: Following theorem is nicer than the theorem 5. In the sense that, we obtain $\Lambda_\infty(\beta)$ is isomorphic to a subspace of $\Lambda_1(\alpha)$ generated by block basic sequence of (e_n) not a block basic sequence of a permütation of the basis (e_n) .

Theorem 6: Let $I_n = [\phi_n, \lambda_n]$ be a sequence of non-empty closed intervals of positive integers such that

$$\frac{\alpha_{m+1}}{\alpha_m} \leq M \quad \text{for } m \in I_n \ , \quad n=1,2,\ldots$$

and

$$\sup_n \frac{\alpha_{\lambda_n}}{\alpha_\phi} = \infty$$

where α is a nuclear exponent sequence of finite type and M is a constant. Then there exists a nuclear exponent sequence of finite type β such $\Lambda_\infty(\beta)$ is isomorphic to a subspace of $\Lambda_1(\alpha)$ generated by a block basic sequence of (e_n) .

Proof: In this case, we again assume α is strictly increasing and further by disregarding some of I_n's and changing some others if necessary, we get the following properties holds where

$$f(n) = \frac{\alpha_{\lambda_n}}{\alpha_{\phi_n}} : \quad 1 < \phi_1 \quad \lambda_n < \phi_{n+1}$$

$$1 \leq f(n) \leq f(n+1) \quad \text{for } n=1,2,\ldots \text{ and}$$

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^2} = \infty$$

Now fix n, and for k= 1,2,...,n let q(k,n) be the smallest positive integer such that

$$f(k)\ \alpha_{\phi_n} < \alpha_{q(k,n)}$$

since $\phi_n > 1$, $f(k)\ \alpha_{\phi_n} \geq \alpha_{\phi_n}$ and α is increasing, it follows that

$q(k,n) > \phi_n > 1$ for $k=1,2,\dots,n$. and so $\alpha_{q(k,n)-1} \leq f(k)\ \alpha_{\phi_n}$ moreover since $f(k) \leq f(k+1)$ we have

$$q(k,n) \leq q(k+1,n) \quad \text{for } k=1,2,\dots,n-1$$

next we have, for k= 1,2,...,n

$$\alpha_{\lambda_n} = f(n)\alpha_{\phi_n} \geq f(k)\alpha_{\phi_n} \text{ and } f(k)\alpha_{\phi_n} < \alpha_{q(k,n)}$$

so that $q(k,n) \leq \lambda_n$ for $k=1,2,\dots,n$.

so we have

$$\alpha_{q(k,n)} = \frac{\alpha_{q(k,n)}}{\alpha_{q(k,n)-1}} \cdot \alpha_{q(k,n)-1} \leq M\ f(k).\ \alpha_{\phi_n} \quad .$$

Let $P=p_n=q(n,n)$ for $n=1,2,\dots$ then we have for $k=1,2,\dots,n$,

$$p_{n-1} = q(n-1,n-1) \leq \lambda_{n-1} < \phi_n < q(k,n) \leq p_n$$

so replace p_1 by p_n; apply lemma 3, we obtain $y_n \varepsilon \Lambda_1(\alpha)$ with

$$y_n = \sum_{i=p_{n-1}+1}^{pn} t_i e_i$$

and $q(k,n) = q_n^k = q_n^k\ (t_{p_{n-1}+1},\dots,t_{p_1})$ This is our block basic seq.

Set $\beta_n = \alpha_{\phi_n}$ for n=1,2,..., since α is nuclear exponent sequence of finite type so by any subsequence of a nuclear exponent sequence is a nuclear exponent sequence of the same type, we get β is a nuclear exponent sequence of finite type and from the fact that every nuclear exponent sequence of finite type is also a nuclear exponent sequence of infinite type, β is also infinite type.

It remains to prove that subspace generated by (y_n) is isomorphic to $\Lambda_\infty(\beta)$. For take π to be the identity permutation and consider

$$\gamma_n^k = \frac{\alpha_{q(k,n)}}{\beta_{\pi(n)}} = \frac{\alpha_{q(k,n)}}{\alpha_\phi} \quad \text{for } k=1,\dots,n \text{ and } n=1,2,\dots$$

and from above inequalities $f(k) < \gamma_n^k < M.f(k)$ for $n \geq k$, n=1,2,... .
So apply again theorem 4 (iv) .

<u>Remark:</u> It is known that a complemented subspace of power series of finite type can <u>not</u> be isomorphic to power series space of infinite type (2). Thus theorem 5 theorem 6 give examples of block basic sequences which generate uncomplemented subspaces. The following theorem (11) provides us with a limitation on G_∞ - subspaces of power series space of finite type, which actually is an extension result of Dubinsky (4).

<u>Theorem 7:</u> Let $\lambda(P)$ be a G_∞ - space such that $\lambda(P) \cong \lambda(P)xK$. Then $\lambda(P)$ <u>can not</u> be isomorphic to a subspace of $\Lambda_1(\alpha)$ spanned by block basic sequence with respect to canonical basis of $\Lambda_1(\alpha)$.

Proof: Assume contrary, that is, there is a block basic sequence (y_n) in $\Lambda_1(\alpha)$ such that $Y \cong \lambda(P)$. In (11) it is shown that, if a G_∞ -space $\lambda(P)$ is isomorphic to Y, where Y is a subspace of $\Lambda_1(\alpha)$ spanned by basic sequence (y_n). Then there exist a permutation π such that

(a) ... $\forall k \quad \exists j$ with $\exp(k\alpha_{q(k,n)}) = 0\,(p^j_{\pi(n)})$ and

(a) ... $\forall j \quad \exists s$ " $p^j_{\pi(n)} = 0\,(\exp\,(\frac{s-1}{s})\,\alpha_{q(s,n)})$.

So using (a) and nuclearity of $\lambda(P)$ we can determine j with

$$\exp\,(\alpha_{q(1,n)}) = 0\,(p^j_{\pi(n)})$$

From the hypothesis we have $\lambda(P) \times K \cong \lambda\,(P)$, so

$$p^j_{n+1} = 0\,(p^m_n).$$

Let $N_o = \{n : \pi(n+1) \leq \pi\,(n)+1\}$, N_o is an infinite set if not, then there would be an integer n_o such that $\pi\,(n+1) > \pi\,(n)+1 \quad \forall n \geq n_o$, so we have for each $n_1 \geq n_o$ that $\pi(n) \neq \pi(n_1)+1$ for $n \geq n_o$, that is the infinitely many integers $\pi(n_o)+1$, $\pi(n_o+1)+1$,... would be in the set $\{\pi(n) : n \geq n_o\}$ and this contradicts the fact π is onto.

$$\text{So} \quad \lim_{n\in N_o} \; p^j_{\pi(n+1)} \,/\, p^m_{\pi(n)} = 0$$

Using (b) and the nuclearity determine k so that

$p^m_{\pi(n)} = 0\,(\exp(\,(\frac{k-1}{k})\;\alpha_{q(k,n)})$ that is find $n_o \varepsilon N_o$ such that for all $n \geq n_o$ we have

$$\log p^m_{\pi(n)} \leq \frac{k-1}{k} \alpha_{q(k,n)}$$

and

$$\log p^j_{\pi(n+1)} \leq \log p^m_{\pi(n)}$$

on the other hand $q(k,n) \leq q(1,n+1)$ since (y_n) is a block basic sequence. Thus

$$\log p^j_{\pi(n+1)} \leq \frac{k-1}{k} \alpha_{q(k,n)} \leq \frac{k-1}{k} \alpha_{q(1,n+1)} \leq \log p^j_{\pi(n+1)} \cdot \frac{k-1}{k}$$

which is a contradiction.

CHAPTER III CHARACTERIZATION OF SUBSPACES OF (s):

The aim of this chapter is to give an exact characterization of metrizable, nuclear locally convex spaces, which are isomorphic to a subspace of (s). There is two important papers of Dubinsky and Vogt on this subject. Vogt's characterization (18) which we give here is more general.

Definition 1: A sequence (ξ_n) is rapidly decreasing if $\forall k \geq 1$ $\lim_{n\to\infty} \xi_n (n+1)^k = 0$. Let (s) denote the space of all rapidly decreasing sequences and its α-dual is equal to:

$$s' = s^\alpha = \{(\xi_n) : \quad \exists k \geq 1 \quad |\xi_n| = 0\,((n+1)^k)\}$$

Definition 2: Let E be metrizable locally convex space with the topology generating seminorms $\|\ \|_1 \leq \|\ \|_2 \leq \ldots$ Then E has property (DN) in case following holds: There exist a continuous norm $\|\ \|$ on E such that for each $k \in N$ there exist a $p \in N$ and $C>0$ with

$$\|\ \|_k \leq r\,\|\ \| + \frac{C}{r}\,\|\ \|_{k+p} \quad \text{for all } r > 0.$$

The condition (DN) depends on the topology, not on the special choice of semi-norms.

Lemma 3: If E has property (DN), so does each subspace of E and the completion of E.

Remark: s has property (DN). (Any metrizable G_∞-space has property (DN).)

Proof of Remark:

$$\text{Let } \| x \|_k = \sum_{j=1}^{\infty} j^k |x_j| \quad \text{and } \| x \| = \sum_{j=1}^{\infty} |x_j|$$

then we get for $j_o^k < r \leq (j_o+1)^k$:

$$\| x \|_k = \sum_{j=1}^{j_o} j^k \; |x_j| + \sum_{j=j_o+1}^{\infty} j^k \; |x_j| \leq j_o^k \|x\| + (j_o+1)^{-k} \|x\|_{2k}$$

$$\leq r\|x\| + \frac{1}{r} \|x\|_{2k}$$ we have the required inequality

for $r \geq 1$ and $p=k$ if $0<r<1$ the inequality is trivially clear.

Theorem 4: Let E be a metrizable, nuclear, locally convex space, E is isomorphic to a subspace of (s) iff it has property (DN).

Proof: ($\Longrightarrow$:) Since s has property (DN) so does each subspaces of (s) in particular one which is isomorphic to E.

($\Longleftarrow$:) to prove this direction we have to do some extra work. By lemma 3, we may assume E is a complete metrizable locally convex space i.e. E is on (F)-space.

Lemma 5: Let E be an (F)-space, $B_1 \subset B_2 \subset \ldots$ a fundamental system of absolutely convex and bdd (= equicontinuous) sets in E', then we have:

E has property (DN) iff a absolutely convex bounded B in E' exist such that for each $k \in N$ $\exists p \in N$ and $C > 0$ exist with

$$B_k \subset rB + \frac{C}{r} \; B_{k+p} \qquad \text{for all } r>0 .$$

Proof: Proof is immediate using definitions and taking

$$B_k = u_k^o \quad \text{where} \quad u_k = \{ x : \| x \|_k \leq 1 \}$$

Theorem 6: Let $0 \longrightarrow s \longrightarrow \tilde{E} \xrightarrow{\phi} E \longrightarrow 0$ be an exact sequence of Frechêt spaces and E has property (DN). Then there exists a map

$\psi \varepsilon L(E, \tilde{E})$ s.t. $\phi o \psi = 1_E$

Proof: Without loss of generality we may suppose s is a subspace of $\tilde{E}$ and show, s is continuously projected in $\tilde{E}$, i.e. $\tilde{E} = s \oplus H$, $\phi|_H$ bijective so invertible and if ψ is this inverse then $\phi o \psi = 1_E$. Let $f_j \varepsilon s'$ be the j-th coordinate functional, for $x=(x_1,x_2,\ldots)$ $f_j(x)=x_j$. For each k $\{j^k f_j : j=1,2,\ldots\}$ is uniformly continuous. By Hanh-Banach theorem kf_j can be extended to $F_j^k \ \varepsilon \ \tilde{E}'$ so $\{j^k F_j^k : j = 1,2,\ldots\}$ is uniformly continuous and contained U_k^o for a suitable neighbourhod of zero in $\tilde{E}$ assume $U_{k+1} \subset U_k \ \forall k$

Define $G_j^k = F_j^{k+1} - F_j^k$ $\quad G_j^k \ \varepsilon \ s^\perp \subset \tilde{E}'$ and we get

$$\{j^k G_j^k : j= 1,2,\ldots\} \subset 2U_{k+1}^o \cap s^\perp = : B_k$$

Since $S^\perp \cong E'$ and E has property (DN), there exists a bounded set $B \subset s^\perp$, which for a given fundamental system of bounded sets in $s^\perp$ satisfies the condition mentioned in Lemma 5.

Without loss of generality we may assume (in which we make some of U_k smaller) that $B_k \subset rB + \frac{2^{-k-2}}{r} B_{k+1}$ for all $r > o$ $k \varepsilon N$.

In particular for $r=j2^{-k-1}$ and multiplying by $2j^{-k}$ we get

$$2j^{-k}\, B_k \subset j^{k+1}\, 2^{-k}\, B + j^{-k-1}\, B_{k+1} \qquad (*)$$

Now for a fixed j we choose a sequence A_j^k with $A_j^k \in j^{-k} B_k$ we let $A_j^o = 0$ then $G_j^k + A_j^k \in 2j^{-k} B_k$ then by $*$ there exist $A_j^{k+1} \in j^{-k-1} B_{k+1}$ so that

$$G_j^k + A_j^k - A_j^{k+1} \in j^{-k+1}\, 2^{-k}\, B$$

let $\phi_j^k = F_j^k - A_j^k$ so we get for $k \geq 1$

$$\phi_j^{k+1} - \phi_j^k = G_j^k - A_j^{k+1} + A_j^k \in j^{-k+1}\, 2^{-k}\, B \subset 2^{-k} B$$

Also the sequence ϕ_j^k converges in $\tilde{E}'$ (as $k \to \infty$). Let

$$\phi_j = \lim_{k \to \infty} \phi_j^k$$

For $k > n$ since $\phi_j^{n+1} = F_j^{n+1} - A_j^{n+1} \in 3\, j^{-n} U_{n+2}^o$

$$j^n \phi_j^k = j^n \phi_j^{n+1} + \sum_{v=k+1}^{n-1} j^n (\phi_j^{v+1} - \phi_j^v) \in 3U_{n+2}^o + 2^{-n} B$$

So $j^n \phi_j$ is also in $3U_{n+2}^o + 2^{-n} B$ i.e. $\{j^n \phi_j : j=1,2,\ldots\}$ is uniformly continuous in $\tilde{E}'$.

By $x \longrightarrow (\phi_j x)_{j=1,2,\ldots}$ a continuous linear map $\phi : \tilde{E} \longrightarrow s$ defined for $x \in s$ $(\phi x)_j = \phi_j(x) = \lim_{k\to\infty} F_j^k(x) = f_j(x) = x_j$

$\therefore$ ϕ is also a continuous projection of $\tilde{E}$ on s.

Lemma 7: If E is a nuclear (F)-space, then there exist an exact sequence

$$0 \longrightarrow s \longrightarrow \tilde{E} \longrightarrow E \longrightarrow 0$$ where $\tilde{E}$ is a closed subspace of s .

Proof: Let $D[-1,+1]$ be denote the space of all infinitely differentiable functions in $[-1,+1]$ we have a map from $D[-1,+1]$ onto w defined by $D[-1,+1] \longrightarrow w$, $f \quad (f^k(o))$ by Borel's thm. Proof can be found in (17) in detail. So we can construct an exact sequence

$$o \longrightarrow D[-1,0] \times D[o,1] \longrightarrow D[-1,+1] \longrightarrow w \longrightarrow o \quad .$$

now recall that $D[a,b]$ is isomorphic to (s) (10) and s is stable therefore we get

$$0 \longrightarrow s \times s \longrightarrow s \longrightarrow w \longrightarrow o \qquad \text{or}$$

$$0 \longrightarrow s \xrightarrow{\phi_1} s \xrightarrow{\phi_2} w \longrightarrow o$$

using the exactness of the functor $- \hat{\otimes}_\pi S$ we get again on exact sequence; that is:

$$o \longrightarrow s \hat{\otimes}_\pi s \longrightarrow s \hat{\otimes}_\pi s \longrightarrow w \hat{\otimes}_\pi s \longrightarrow o$$

Now again using $s \hat{\otimes}_\pi s \simeq s$ (16) , $w \hat{\otimes}_\pi s = s^N$ we have again an exact sequence:

$$o \longrightarrow s \longrightarrow s \xrightarrow{\phi} s^N \longrightarrow o$$

Now Komura-Komura's thm. (8) states that $(s)^N$ is the largest nuclear Fréchet Space. So we can take E imbeded in $(s)^N$ and let $\tilde{E} = \phi^{-1} E \subset s$, So that

$$o \longrightarrow s \longrightarrow \tilde{E} \longrightarrow E \longrightarrow o \quad \text{exact.}$$

Now we are ready to complete missing part in the proof at theorem 4 that is:

If E is metrizable, nuclear locally convex space having property (DN) then E is isomorphic to a subspace of s.

Proof: Assume E is complete, because if E has property (DN) so does its completion so by lemma 7 there exist an exact sequence

$o \longrightarrow s \longrightarrow \tilde{E} \longrightarrow E \longrightarrow o$ where $\tilde{E} \subset s$.

and by theorem 6, if we have an exact sequence as above then there exists a map $\psi : E \longrightarrow \tilde{E}$ such that $\phi \circ \psi = 1_E$. So $E \xrightarrow{\psi} \tilde{E} \hookrightarrow s$ the map imbeds E isomorphically into s.

Remark: It follows that the nuclear (F)-spaces with property (DN) are just closed subspaces of (s), secondly notice that E has property (DN) iff every exact sequence of (F)-spaces of the form

$o \longrightarrow s \longrightarrow \tilde{E} \longrightarrow E \longrightarrow o$ splits

Theorem 8: Let E be a metrizable, locally convex space, then the following are equivalent:

(1) E has property (DN)

(2) There exist a continuous $\| \ \|$ on E so that $\forall k \in N \ \exists p \in N$ and $C > o$ such that $\| \ \|_k^2 \leq C \ \| \ \| \ \| \ \|_{k+p}$

(3) There exist a topology generating seminorm system $\| \ \|_k$ k=1,2,... so that $\| \ \|_k^2 \leq \| \ \|_{k-1} \ \| \ \|_{k+1}$

Proof: Consider the function $f(r) = r\alpha + \frac{1}{r}\beta \quad \alpha,\beta > o$ holds for $\min_{r>o} f(r) = 2\sqrt{\alpha\beta}$ this implies (1) $\Longleftrightarrow$ (2). If (2) is given, we may assume without loss of generality that $\| \; \| \leq \| \; \|_k \; \forall k$ and replace $\| \; \|_{k+1}$ by $c_k \; \| x \|_{k+p}$ to get (3). If (3) is given, it follows that all $\| \; \|_k$ must be norms and for $x \neq 0$ we get

$$\frac{\|x\|_k}{\|x\|_{k-1}} \leq \frac{\|x\|_{k+1}}{\|x\|_k}$$ for all k the inequality

$$\frac{\|x\|_k}{\|x\|_o} = \prod_{j=1}^{k} \frac{\|x\|_j}{\|x\|_{j-1}} \leq \prod_{j=k+1}^{2k} \frac{\|x\|_j}{\|x\|_{j-1}} = \frac{\|x\|_{2k}}{\|x\|_k}$$

gives with $\|x\|_k^2 \leq \|x\|_o \; \|x\|_{2k}$ for all k.

Definition 9: If E is a nuclear (F)- space with basis $x_1, x_2, \ldots$ so, according to Dubinsky's (6) definition (x_n) is a basis of type (d_3), if there is a fundamental system of norms $(\| \cdot \|_k)$ such that

$$\frac{\|x_n\|_k}{\|x_n\|_{k-1}} \leq \frac{\|x_n\|_{k+1}}{\|x_n\|_k}$$. Notice that there is no assumption of reqularity here. With this notation Vogt has proved a very useful theorem, that is:

Theorem 10: Let E be a nuclear (F)-space with basis then the following are equivalent:

(1) E has propery (DN)

(2) There exists a basis of type (d_3)

(3) Every basis is of type (d_3)

Proof: (3)$\Longrightarrow$(2) clear

(1)$\Longrightarrow$(3) by theorem 8.

(2)$\Longrightarrow$(1): let $x_1, x_2, \ldots$ is a basis of type (d_3) then for

$x = \sum_n \lambda_n x_n$ using schwartz inequality

$$(\sum_n |\lambda_n| \ \|x_n\|_k)^2 \leq (\sum_n |\lambda_n| \sqrt{\|x_n\|_{k-1}} \sqrt{\|x_n\|_{k+1}})^2$$

$$\leq (\sum_n |\lambda_n| \ \|x_n\|_{k-1}) \ (\sum_n |\lambda_n| \ \|x_n\|_{k+1})$$

define $|||\, x \,|||_k = \sum_n |\lambda_n| \ \|x_n\|_k$

Remark: If we combine theorem 4 and theorem 10, we conclude that: A nuclear Fréchet space E with basis is isomorphic to a subspace of (s) iff the basis is of type (d_3) .

Dubinsky has proved above remark as a main theorem in (6), but proof contains an important fact, numely: he showed that if the basis is of type d_3 then E is isomorphic to a subspace of s spanned by block basic sequence with respect to a permutation of the canonical basis of (s).

So we conclude that Dubinsky's characterization is more restrictive in the sense that the subspaces have bases. By theorem 10 Vogt has shown that, if E has basis then (d_3)-condition is equivalent to (DN)-condition.

CHAPTER IV G_∞-SUBSPACES OF G_∞-SPACES

1. G_∞-subspaces of G_∞-spaces:

First we point out some useful fact about G_∞-spaces. It is known that (11) if (y_n) is a block basic sequence w.r.t. canonical basis of a G_∞-space $\lambda(A)$, then (y_n) is semi-equivalent to the canonical basis of $\lambda(A_o)$ where

$A_o = \{(a^k_{q(k,n)}) : k = 1,2,\dots\}$ where $q(k,n) = q^k(y_n)$ as defined before. Note that if (y_n) is a block basis sequence in $\lambda(A)$ then $q(k,n+1) \geq q(k,n)$ and so we obtain that if (y_n) is a block basic sequence w.r.t. the canonical basis in a G_∞-sp. then Y is also a G_∞-space since A_o is a G_∞-set. Let $\lambda(A)$ be a G_∞-space and define

$b^k_n = a^1_n \cdot a^2_n \dots a^k_n$. Then $\lambda(B)$ is also a G_∞-space. Since $a^k_n < b^k_n$ and $(b^k_n) \leq (a^k_n)^k = 0\,(a^j_n)$ we have B is equivalent to A. Moreover, the basis of $\lambda(B)$ is regular. (i.e. $\dfrac{b^k_n}{b^{k+1}_n} \geq \dfrac{b^k_{n+1}}{b^{k+1}_{n+1}}$) In view of this, we can assume that the canonical basis of a G_∞-space is also regular. Now we give a necessary condition below for a G_∞-space to be isomorphic to a subspace of another given G_∞-space.

Proposition 1: Let $\lambda(A)$ be a G_∞-sp. and $A = \{(a^k)\}$ and assume another G_∞-space $\lambda(P)$ is isomorphic to a subspace of $\lambda(A)$. Then there exists a permutation π such that.

$\forall k \quad \exists j \quad$ with $a^k_{q(k,n)} = 0\ (p^j_{\pi(n)})$

and $\quad \forall j \quad \exists s \quad$ " $\quad p^j_{\pi(n)} = 0\ (q^s_{q(s,n)})$.

<u>Proof:</u> We set $\gamma^{k,j}_n = a^k_{q(k,n)} / p^j_{\pi(n)}$ and $\mu^{k,j}_n = d_n \gamma^{k,j}_n$

assume contrary to the first assertion, that $\exists k_o$ such that for each j $\sup_n \gamma^{k_o,j}_n = \infty$ let $N_j = \{n : \gamma^{k,j}_n > j\}$ Then each N_j is an infinite set and $N_{j+1} \subset N_j$ bucause $P^j_{\pi(n)} \leq P^{j+1}_{\pi(n)}$.

Let n_j be a subsquence of integers with $n_j \varepsilon N_j$ and $No = \{n_j : j = 1,2,\ldots\}$. If $j > s$

$\gamma^{k_o,s}_{n_j} \geq \gamma^{k_o,j}_{n_j} > j$ and so $\lim_{n\varepsilon No} \gamma^{k_o,s}_n = \infty$ for every s.

From regularity of the cananical basis of $\lambda(A)$ we have $q(k+1,n) \geq q(k,n)$

$$\frac{\gamma^{k+1,j}_n}{\gamma^{k,j}_n} = \frac{a^{k+1}_{q(k+1,n)}}{a^k_{q(k,n)}} \geq \frac{a^{k+1}_{q(k,n)}}{a^k_{q(k,n)}} \geq 1 .$$

Hence

$\lim_{n\varepsilon N_o} \gamma^{k,j}_n = \infty \quad \forall k \geq k_o$ and $\forall j$. For each k we find $r_k > k$ with $(a^k_n)^2 = 0\ (a^{r_k}_n)$

$$a^k_{q(k,n)} \leq \rho \frac{a^{r_k}_{q(k,n)}}{a^k_{q(k,n)}} \leq \rho \frac{a^{r_k}_{q(r_k,n)}}{a^k_{q(k,n)}} = \rho \frac{\mu^{r_k,j}_n}{\mu^{k,j}_n}$$

$$\text{so } \gamma_n^{k,j} = 0\left(\frac{\mu_n^{r_{k,j}}}{p_{\pi(n)}^j \mu_n^{k,j}}\right)$$

Now by our assumption there is a basic sequence (y_n) in $\lambda(A)$ such that Y is isomorphic to $\lambda(P)$. i.e. (y_n) is quasi-equivalent to the canonical basis of $\lambda(P)$ that is $(p_{\pi(n)}^j)$ is equivalent to $(d_n a_{q(k,n)}^k)$ and hence for each $k \geq k_o$ we find j with $d_n\ a_{q(r_k,n)}^{r_k} = 0(P_{\pi(n)}^j)$ that is $\mu_n^{r_{k,j}} \leq \sigma$ for same σ and for all n.

Thus $\gamma_n^{k,j} = 0\left(\dfrac{1}{p_{\pi(n)}^j\ \mu_n^{k,j}}\right)$ so

$\lim\limits_{n\varepsilon No} p_{\pi(n)}^j\ \mu_n^{k,j} = \lim\limits_{n\varepsilon N_o} d_n\ a_{q(k,n)}^k = 0 \quad \forall k \geq k_o$ but on the other hand we can find s with

$$1 = P_{\pi(n)}^1 = 0\ (d_n\ a_{q(s,n)}^s) \text{ and for } k \geq s, q(k,n) \geq q(s,n)$$

so this sets up a contradiction.

To prove the second assertion, again by assumption $(p_{\pi(n)}^j)$ and $(d_n\ a_{q(k,n)}^k)$ are equivalent for some permutation π and $d_n > o$ find j_o with

$$d_n = a_{q(1,n)}^1\ d_n = 0\ (p_{\pi(n)}^{j_o})$$

For a given j, find k so that $p_n^j \cdot p_n^{j_o} = 0\ (p_n^k)$ then find s with $p_{\pi(n)}^k = 0\ (d_n\ a_{q(s,n)}^s)$

$$d_n\ p_{\pi(n)}^j \leq \rho\ p_{\pi(n)}^{j_o}\ \ p_{\pi(n)}^j = 0\ (d_n\ a_{q(s,n)}^s)$$

Now we give several results about infinite type power series subspaces of G_∞ -spaces.

Proposition 2: Let $\lambda(A)$ be a G_∞ -space such that $\Lambda_\infty(\alpha)$ is isomorphic to a subspace of $\lambda(A)$, then there exists a permutation π, integers i_n and k_o such that $(\alpha_{\pi(n)})$ is asymptotically equivalent to $\log a^k_{i_n}$ for each $k \geq k_o$

Proof: In wiev of proposition 1, find k_o with $2^{\alpha_{\pi(n)}} = 0\ (a^{k_o}_{q(ko,n)})$ set $i_n = q(k_o,n)$. For $k \geq k_o$ $\alpha_{\pi(n)} = 0\ (\log a^k_{i_n})$. we find $R > 1$

$$a^k_{q(k,n)} = 0\ (R^{\alpha_{\pi(n)}}) \text{ since } i_n = q(k_o,n) \leq q(k,n)$$

we get $\log a^k_{i_n} = 0\ (\alpha_{\pi(n)})$

Corollary 3 : Let $\lambda(A)$ be a G_∞- sp. such that for each k $\exists j$ with $j>k$ and $\lim(\log a^j_n / \log a^k_n) = \infty$. Then $\lambda(A)$ has no subspace isomorphic to power series space $\Lambda_\infty(\alpha)$.

There is an application of this corollary in (9). They showed that what is known on $L_f(b,\infty)$-space (that is $L_f(b,\infty)$-space is either isomorphic to power series space or it has no subspace isomorphic to power series space) is not true for G_∞-spaces; namely they set up a G_∞-sp $\lambda(A)$ which is isomorphic to $\Lambda_\infty(\beta) \times \lambda(A_o)$ where $\lambda(A_o)$ is a G_∞-space which has no subspace isomorphic to a power series space.

Proposition 4: Let λ (A) be a G_∞-space such that $\Lambda_\infty(\alpha)$ is isomorphic to a subspace of $\lambda(A)$ and has the property $\Lambda_\infty(\alpha) \cong \Lambda_\infty(\alpha) \times K$. Then $\lambda(A)$ is it self isomorphic to power series space of infinite type.

__Proof:__ By proposition 2, there exist π, (i_n) k_o such that $\log a_i^k \sim \alpha_{\pi(n)}$ for $k \geq k_o$. Define $r_n = \max \{i_{\pi^{-1}(j)} : 1 \leq j \leq n\}$ then $\log a_{r_n}^k \sim \alpha_n$ if $k \geq k_o$ let j_n be the smallest integer with $n \leq r_{j_n}$. Then

$$\frac{\log a_n^{k+1}}{\log a_n^k} \leq \frac{\log a_{r_{jn}}^{k+1}}{\log a_{r_{j_n-1}}^k}$$ let $\rho > o$ and $\sigma > o$ such that

$$\log a_{r_{j_n}}^{k+1} \leq \rho\, \alpha_{j_n} \qquad \log a_{r_{j_n-1}}^k \geq \sigma\, \alpha_{j-1}$$

So $$\frac{\log a_n^{k+1}}{\log a_n^k} = 0\left(\frac{\alpha_{j_n}}{\alpha_{j_n}}\right)$$

by our assumption $\Lambda_\infty(\alpha) \times K \cong \Lambda_\infty(\alpha)$ means $\frac{\alpha_{n+1}}{\alpha_n} = 0\ (1)$

so $\log a_n^{k+1} = 0\ (\log a_n^k)$ for $k \geq k_o$ so let $\beta_n = \log a_n^{k_o}$ then $\lambda(A)$ is isomorphic to $\Lambda_\infty(\beta)$

2. __G_∞ - subspaces of $\Lambda_\infty(\alpha)$:__

First we would like to notice the fact that, the space spanned by a block basic sequence with respect to the canonical basis in a power series space $\Lambda_\infty(\alpha)$ is isomorphic to a G_∞ - space.

Following proposition is a specialized version of the proposition 1.

<u>Proposition 5:</u> If a G_∞- space $\lambda(P)$ is isomorphic to a subspace of $\Lambda_\infty(\alpha)$ then there exist sequences (i_n^k) of positive integers with $i_n^k \leq i_n^{k+1}$ and $i_n^k \leq i_{n+1}^k$ and such that

(i) $\forall k \quad \exists j$ with $k^\alpha i_n^k = 0\ (p_n^j)$

(ii) $\forall k \quad \exists r$ with $p_n^j = 0\ (r^\alpha i_n^r)$

(iii) There exists s_0 with $\alpha_n = 0\ (\alpha_{i_n}^s)$ for every $s \geq s_o$.

<u>Proof:</u> By proposition 1, there exist a permutation π such that $(p_{\pi(n)}^j)$ is equivalent to $(k^\alpha q(k,n))$.

define $i_n^k = \max \{ q(k, \pi^{-1}(i)) \quad 1 \leq i \leq n \}$

then by definition of $q(k,n)$ i_n^k satisfies desired inequalities, namely $i_n^{k+1} \geq i_n^k$ and $i_{n+1}^k \geq i_n^k$.

If $(k^\alpha q(k,n))$ is dominated by $(p_{\pi(n)}^j)$ then

$$k^\alpha i_n^k = \max \ k^\alpha q(k, \pi^{-1}(i)) = 0\ (p_n^j) \quad \text{since } p_i^j \leq p_{i+1}^j$$

so (i) follows. The second assertion follows similirarly. To prove (iii) recall if $\lambda(P)$ is a G_∞-sp. then $\Delta(\lambda(P)) = \lambda(P)'$ (13).

Moreover $\Delta(\Lambda_\infty(\alpha)) \subset \Delta(\lambda(P)) \Rightarrow \Lambda_\infty(\alpha)' \subset \lambda(P)'$. (iii) follows from this and (ii).

3. <u>$\Lambda_\infty(\beta)$ subspaces of $\Lambda_\infty(\alpha)$:</u>

We consider here to find, for a given nuclear exponent sequence α of infinite type, necessary and sufficient conditions on β so that

$\Lambda_\infty(\beta)$ is isomorphic to a subspace of $\Lambda_\infty(\alpha)$.

Following thm is a technical characterization of such β which can be proved analogicaly as thm 4 in chapter II .

Theorem 6: In the context of notation B of chapter II the following are equivalent

(i) Y is isomorphic to $\Lambda_\infty(\beta)$

(ii) There exist d_n and π such that for each k $\overline{\lim_n}\, \mu_n^k < \infty$

$$\sup_k \underline{\lim_n}\, \mu_n^k = \infty$$

(iii) There exist π such that for each k

$$\sup_n \gamma_n^k < \infty \quad \sup_k \underline{\lim} \sum_{j=1}^{k} \left(\frac{\gamma_n^j}{j^2}\right) = \infty$$

Theorem 7: If $\Lambda_\infty(\beta)$ is isomorphic to a subspace of $\Lambda_\infty(\alpha)$ then there exist a non-decreasing unbounded sequence of indices (i_n) such that $\beta \sim (\alpha_{i_n})_n$.

Proof: $\Lambda_\infty(\beta)$ is isomorphic to a subspace of $\Lambda_\infty(\alpha)$ means there is a basic sequence in $\Lambda_\infty(\alpha)$ such that Y is isomorphic to $\Lambda_\infty(\beta)$ choose π according to (iii) of theorem 6. (recall $\gamma_n^k = \frac{\alpha_{q(k,n)}}{\beta_{\pi(n)}}$)

then $\exists\, k_o$; for $k \geq k_o$ we have $\underline{\lim_n}\, \gamma_n^k > 0$ (Dubinsky (4), Proposition 2) so we conculude that $(\alpha_{q(k_o,n)}) \sim (\beta_{\pi(n)})$ and from again same proposition $\lim_n q(k,n) = \infty$ we can find a permutation σ of N such that $i_n = q(k_o,\sigma(n))$ then (i_n) is non-decreasing and unbounded. Thus the map $(\xi_n) \to (\xi_{\pi\sigma(n)})$ defines an isomorphism of $\Lambda_\infty(\beta)$ onto $\Lambda_\infty((\alpha_{i_n}))$. The conculusion follows from elementary properties of power series spaces.

Theorem 7, gives a necessary condition on β to the effect that up to isomorphism β must be obtainable from the sequence α by deleting some terms and repeating others finitely many times.

It is important to know under which conditions set for the subspaces of $\Lambda_\infty(\alpha)$ such that subspace is isomorphic to a power series space of infinite type. M.S. Ramanujan and T.Terzioğlu has given a proposition on this problem. That is:

Proposition 8: Let Y be a subspace of $\Lambda_\infty(\alpha)$ spanned by block basic sequence. If Y x K is isomorphic to Y then Y is isomorphic to power series space of infinite type.

Proof: From the fact that every subspace of $\Lambda_\infty(\alpha)$ spanned by a block basic sequence is isomorphic to a G_∞-space $\lambda(A_o)$ we may assume that Y is equal to the G_∞- space $\lambda(A_o)$ where

$$A_o = \{ (k^{\alpha}q(k,n) : k = 1,2,\ldots\}$$

So $\lambda(A_o)$ is isomorphic to a subspace of $\Lambda_\infty(\alpha)$, from consideration of diametral dimension $\Lambda_\infty(\alpha)' \subset \lambda(A_o)'$ and hence $\alpha_n = 0\ (\alpha_{q(k_o,n)})$ for some integer k_o by our hypothesis $\lambda(A_o) \cong \lambda(A_o) \times K$ this implies the existence of j_o satisfying

$$\alpha_{q(k_o,n+1)} = 0\ (\alpha_{q(j_o,n)})$$

since $q(j,n) \leq q(k_o,n+1)$ For $\forall j$ and $\forall n$. We have

$$\alpha_{q(1,n)} \leq \alpha_{q(k_o,n+1)} = 0\,(\alpha_{q(j_o,n)})$$

$$\beta_n = \alpha_{q(j_o,n)} \quad \text{so we get} \quad \lambda(A_o) = \Lambda_\infty(\beta)$$

Remark: If $\lambda(P)$ is a G_∞-space but not a power series space and if $\lambda(P) \times K \cong \lambda(P)$, then $\lambda(P)$ can not be isomorphic to a subspace Y of any power series space of infinite type spanned by block basic sequence.

There is another result (5) about subspaces of $\Lambda_\infty(\alpha)$ such that the subspace is isomorphic to power series space for the special case $\lim \frac{\alpha_{n+1}}{\alpha_n} = \infty$, that is:

Theorem 9: If $\lim \frac{\alpha_{n+1}}{\alpha_n} = \infty$, then a subspace of $\Lambda_\infty(\alpha)$ is isomorphic to a power series space if and only if it is isomorphic to a subspace of $\Lambda_\infty(\alpha)$ generated by a subsequence of the basis (e_i) .

For the proof we need following results:

Proposition 10: Let Y be a subspace of $\Lambda_\infty(\alpha)$ such that Y is isomorphic to power series space of infinite type and suppose that $\lim \frac{\alpha_{n+1}}{\alpha_n} = \infty$ then any basis for Y has a permutation (y_n) for which there exists a non-decreasing unbounded sequence of indices (i_n) and k_o such that

$\forall k \geq k_o \quad \exists\, n_k \quad \ni q(k,n) = i_n \quad \text{for } n \geq n_k$.

Proof: Take a basic sequence in $\Lambda_\infty(\alpha)$, choose (y_n) so that Y is isomorphic to some $\Lambda_\infty(\beta)$ for some β and π of theorem 6 is the identity.

As in the proof of theorem 7, we obtain (i_n) and k_o such that $(\alpha_{q(k,n)})_n \cong (\beta_{\pi(n)})_n = \beta$ for each $k \geq k_o$. By theorem 7 it follows $(\alpha_{q(k,n)})_n \cong (\alpha_{i_n})$ for each $k \geq k_o$ and the hypothesis that $\lim \frac{\alpha_{n+1}}{\alpha_n} = \infty$ implies that $q(k,n) = i_n$ for n sufficiently large.

NOTATION: In the context of proposition 10, take

(ν_n) = strictly increasing sequence whose range is identical to that of (i_n)

Let $0 = p_o < p_1 < p_2 < \ldots$ then

$i_j = \nu_n$ for $p_{n-1} < j \leq p_n \quad n=1,2,\ldots$

$q(k,j) = i_j = \nu_n \quad \text{for } p_{n-1} < j \leq p_n \quad n \geq n_k$

In the sequel, this notation will be referred to as the "context of proposition 10".

Proposition 11: In the context of proposition 10, there is a m_o such that $i_n \geq n$ and $p_n \leq \nu_n$ for $n > m_o$.

Proof: $\Lambda_\infty(\beta)$ is isomorphic to a subspace of $\Lambda_\infty(\alpha)$ means $\sup_n(\frac{\alpha_n}{\beta_n}) < \infty$ this follows from consideration of diametric dimension. Hence

$\text{Sup}_n \left(\frac{\alpha_n}{\alpha_{i_n}}\right) < \infty$ but from the hypothesis of proposition 10 we know $\sup \frac{\alpha_{n+1}}{\alpha_n} = \infty$ so it follows that $n \leq i_n$ for n large enough. In particular $p_n \leqq i_{p_n} = \nu_n$ for n sufficiently large.

<u>Proposition 12:</u> In the context of proposition 10, we set

$$\tilde{y}_j = \sum_{i=1}^{\nu_n} t_i^j e_i , \qquad p_{n-1} < j \leqq p_n$$

where $y_j = \sum_{i=1}^{\infty} t_i^j e_i \qquad j = 1,2,..$

then there exist a j_1 such that $(\tilde{y}_j)_{j=j_1}^{\infty}$ is a basic sequence in $\Lambda_\infty(\alpha)$ equivalent to $(y_j)_{j=j_1}^{\infty}$.

Now we need an enequality involving ν_n and p_n, that is:

<u>Proposition 13:</u> In the context of proposition 10, it follows that for each $k \geqq k_o$ there is a $\tilde{k}$, c_k and $\tilde{n}_k$

$$k^{\alpha_{\nu_n}} \leqq c_k \; \tilde{k}^{\alpha_{\nu_n} - (p_n - p_{n-1}) + 1} \qquad (p_n - p_{n-1}), n \geqq \tilde{n}_k$$

proof can be find in (5).

<u>Proof of theorem 9:</u> From zahariuta's result (19) we know that it is impossible to embed finite type power series spaces in infinite type power series spaces. So we can assume that the subspace is of infinite type.

In the context of proposition 10, assume $P_n - p_{n-1} \geq 2$ for $n \in N_o$

where N_o is an infinite set of indeces.

Let $k = k_o$ and apply proposition 11 and 13, so we have $\tilde{k}$, C , $\tilde{n}$ such that

$$k_o^{\alpha_{\nu_n}} \leq C_{\nu_n} \tilde{k}^{\alpha_{\nu_n} - (p_n - p_{n-1})^+ 1} \leq C_{\nu_n} \tilde{k}^{\alpha_{\nu_n - 1}} \qquad \text{for } n \geq \tilde{n} \quad n \in N_o$$

this implies $k_o^{\alpha_n} \leq C_1 \, n \, \tilde{k}^{\alpha_{n-1}}$ where $C_1 > 0$ $n \in N_1$, N_1 = infinite set of indices. and $\alpha_n \log k_o \leq \log C_1 + \log n + \alpha_{n-1} \log \tilde{k}$

$$\frac{\alpha_n}{\alpha_{n-1}} \leq \frac{\log C_1}{\alpha_{n-1} \log k_o} + \frac{\log n}{\alpha_{n-1} \log k_o} + \frac{\log \tilde{k}}{\log k_o}$$

$$\operatorname{Sup} \left(\frac{\alpha_n}{\alpha_{n-1}}\right) \leq \operatorname{Sup} \left(\frac{\log C_1}{\alpha_{n-1} \log k_o} + \frac{\log n}{\alpha_{n-1} \log k_o} + \frac{\log \tilde{k}}{\log k_o}\right)$$

since is a nuclear exponent sequence of infinite type we have $\sup \left(\frac{\alpha_n}{\alpha_{n-1}}\right) < \infty$ but this result is a contradiction with our hypothesis

$$\lim \frac{\alpha_{n+1}}{\alpha_n} = \infty$$

Thus $P_n - P_{n-1} = 1$ for n sufficiently large. Therefore (i_n) is strictly increasing for $n \geq n_o$ hence

$$\| y_n \|_k = |t^n_{q_n k}| \, K^{\alpha} q(k,n) = |t^n_{i_n}| \, K^{\alpha_{i_n}} \quad \text{for} \quad n \geq n_k$$

which implies that the map $y_n \longleftrightarrow t^n_{i_n} e_{i_n}$ $n \geq n_o$ defines an isomorphism from the space generated by $(y_n)_{n \geq n_o}$ onto the space generated by the subsequence (e_{i_n}) $n \geq n_o$.

From proposition 11, if we first choose $n_o > m_o$ then $i_n \geq n \geq n_o$ so the space generated by $(y_n)_{n=1}^{\infty}$ is isomorphic to the space generated by the subsequence $(e_1, \ldots, e_{n_o-1}, e_{i_{n_o}}, e_{i_{n_o+1}}, \ldots)$.

<u>Remark 1:</u> By theorem 9 and the fact that, in a nuclear space any subspace generated by a subsequence of a basis is complemented, that if $\lim \frac{\alpha_{n+1}}{\alpha_n} = \infty$ then any subspace of $\Lambda_\infty(\alpha)$ is isomorphic to power series space if and only if it is isomorphic to a complemented subspace.

<u>Remark 2:</u> The following conjecture of C.Bessaga (2) holds for $\Lambda_\infty(\alpha)$, if $\lim_n \frac{\alpha_{n+1}}{\alpha_n} = \infty$ or if $(\alpha_{2n}/\alpha_n) \in \ell_\infty$.

<u>Bessaga's Conjecture:</u> "If X is a nuclear Frêchet space with basis (x_n) then Y is isomorphic to a complemented subspace of X and has basis iff Y is isomorphic to a subspace generated by a subsequence of (x_n)."

Take $X = \Lambda_\infty(\alpha)$ and if F is a complemented subspace of $\Lambda_\infty(\alpha)$ and has a basis then apply thm 2.2 of (2) to conclude that F is isomorphic to power series space, then remark 2 follows from theorem 9.

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