Entanglement assisted state discrimination and entanglement preservation

Özenç Güngör[1, 2, *] and Sadi Turgut[2]

[1]*Department of Physics, Case Western Reserve University,*
10900 Euclid Avenue, Cleveland, Ohio 44106, USA
[2]*Department of Physics, Middle East Technical University, 06800, Ankara, Turkey*
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In this paper, the following scenario is considered: there are two qubits possessed by two parties at different locations. Qubits have been prepared in one of a maximum of four, mutually-orthogonal, entangled states and the parties wish to distinguish between the states by using local operations and classical communication. Although in general it is not possible to distinguish between four arbitrary states, the parties can spend some pre-shared entanglement to achieve perfect discrimination between four qubit states and can also preserve the entanglement of the states after discrimination. This is shown by employing the theory of majorization and the connections between entanglement transformations and state discrimination protocols.

I. INTRODUCTION

One of the consequences of quantum mechanics is that it is impossible to perfectly distinguish between non-orthogonal pure quantum states. Much work has been done in this field and several procedures have been identified for minimizing the probability of error when discriminating between pure, monopartite quantum states [1–4]. While not as general, partial results and necessary conditions on the separability of monopartite, mixed quantum states exist in the literature [5, 6].

While any number of monopartite, orthogonal states can be perfectly distinguished, this is not true in general for multipartite, entangled states under local operations and classical communication (LOCC). Walgate[7] has shown that two, orthogonal, bipartite states can be perfectly distinguished using LOCC alone and Virmani and his collaborators[8] have extended this result into non orthogonal states. Bandyopadhyay and his collaborators have shown that the number of states that are perfectly distinguishable under LOCC is closely related to the entanglement content of the states [9, 10] and they have found a general bound on the number

$$N \le \frac{D}{1 + R(|\psi_i\rangle)} \le \frac{D}{2^{E_R(|\psi_i\rangle)}} \le \frac{D}{2^{E_g(|\psi_i\rangle)}} \qquad (1)$$

where $R(|\psi_i\rangle)$ [11] is the global robustness of entanglement, $E_R(|\psi_i\rangle)$ is the relative entanglement entropy [12], $E_g(|\psi_i\rangle)$ is the geometric measure of entanglement [13] for the state $|\psi_i\rangle$ and D is the dimension. An explicit example of 4 Bell type states being indistinguishable can be found in the work of Ghosh et. al. [14] .

The results listed above by various authors have one aspect in common: the entanglement in the states is lost after discrimination, the parties are left with a product state in their hands. Cohen has considered the possibility of preserving entanglement after discrimination in his

paper [15] by considering the Schmidt ranks of the states to be discriminated as a measure of the entanglement without explicitly referring to the Schmidt coefficients themselves. He has also shown that entanglement can be used as a resource in state discrimination procedures [16] enabling the parties to achieve tasks that are otherwise impossible.

In this paper, our main point is to argue that the following set of mutually orthogonal Bell type states

$$
\begin{aligned}
|\psi_1\rangle &= a\,|00\rangle + b\,|11\rangle \\
|\psi_2\rangle &= b^*\,|00\rangle - a^*\,|11\rangle \\
|\psi_3\rangle &= c\,|01\rangle + d\,|10\rangle \\
|\psi_4\rangle &= d^*\,|01\rangle - c^*\,|10\rangle
\end{aligned}
\qquad (2)
$$

are distinguishable under LOCC if the parties agree to spend some pre-shared entanglement and the parties can achieve entanglement preservation after discrimination by also spending pre-shared entanglement. To prove these claims, we first view multipartite state discrimination as a local entanglement transformation and use the theory of majorization [17, 18] to classify these transformations. These ideas also serve to show the usefulness of majorization relations for characterizing local state discrimination procedures.

Majorization defines a partial order between vectors whose components sum up to the same value. It is defined as follows. Let x and y be d dimensional vectors

$$
x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{pmatrix}, \; y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_d \end{pmatrix}.
\qquad (3)
$$

Then, y majorizes x, or $x \prec y$, if the following inequality holds for every $k = 1, 2, \ldots, d$ and equality holds for $k = d$

$$\sum_{i=1}^{k} x_i^\downarrow \le \sum_{i=1}^{k} y_i^\downarrow, \qquad (4)$$

where $x^\downarrow$ denotes the vector obtained by organizing the components of x in descending order. Specifically, $x_i^\downarrow$

* oxg34@case.edu

is the ith largest component of x. If the dimensions of the two vectors are not equal, the lower dimensional one can be "padded" with enough zeros. Majorization relations can also be extended to matrices of equal trace by constructing a vector $\lambda(A)$ where the elements are the eigenvalues of A *sorted in a decreasing order*. Nielsen has shown that for two entangled states in Schmidt forms

$$\begin{aligned}|\psi\rangle &= \sum_i \sqrt{x_i}\,|ii\rangle \\ |\phi\rangle &= \sum_i \sqrt{y_i}\,|i'i'\rangle \end{aligned} \qquad (5)$$

where x_i and y_i are the respective Schmidt coefficients, which both sum up to 1, the state $|\psi\rangle$ can be converted into $|\phi\rangle$ under LOCC with unit probability if and only if $\lambda(\psi) \prec \lambda(\phi)$ (or $x \prec y$) is satisfied where $\lambda(\psi) = x$ is defined to be the eigenvalue vector of the reduced density matrix $\rho_A = \mathrm{tr}_B |\psi\rangle\langle\psi|$. Subsequently, Jonathan and Plenio [19] have shown that the state $|\psi\rangle$ can be converted into a set of different states where $|\phi_i\rangle$ is obtained with probability p_i, if and only if

$$\lambda(\psi) \prec \sum_i p_i \lambda(\phi_i). \qquad (6)$$

The above equation is of central importance to our discussion on the distinguishability of entangled quantum states as it establishes the connection between entangled state transformations and state discrimination.

The majorization relations described above don't explicitly describe how to construct the local operators to realize the transformation but the construction of the majorization relation guarantees the existence of a LOCC measurement scheme. In the case of a deterministic transformation, the operation is equivalent to one of the parties making a general measurement and the other party using a unitary transformation conditional on the outcome of the general measurement. In the case of transformations for mixed bipartite or pure multipartite states, a one-way transformation is not enough.

The above relation can be used in the following way to investigate the conditions for the distinguishability of the states in Eq. (2). Suppose Alice, Bob, Charlie and Devin share the pure state

$$|\Psi\rangle = \frac{1}{2} \sum_{i=1}^{4} |\psi_i\rangle_{AB} \, |\phi_i\rangle_{CD} \qquad (7)$$

where $|\phi_i\rangle$ are the Bell states and we have chosen that the initial probabilities p_i of the states $|\psi_i\rangle$ to be equal. If Alice and Bob are able to distinguish between the states $|\psi_i\rangle$, Charlie and Devin will share the Bell state $|\phi_i\rangle$ with probability $p_i = 1/4$. Distinguishing between the states $|\psi_i\rangle$ in this case is equal to transforming the state $|\Psi\rangle$ into $|\phi_i\rangle$ with $p_i = 1/4$ with i identified meaning that Alice and Bob have succeeded in discriminating the states $|\psi_i\rangle$. If $|\Psi\rangle$ is reinterpreted as a bipartite state where Alice and Charlie together is considered as one of the

parties (and Bob and Devin is the other party), the following majorization relation can be constructed for this entanglement transformation, after padding the eigenvalue matrix of $\frac{1}{2}\mathbb{1}_2$ with enough zeros

$$\lambda(\Psi) \prec \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \\ 0 \end{pmatrix}, \qquad (8)$$

leading to the inequality

$$\frac{1}{8}(a+b+c+d)^2 \le \frac{1}{2}\,, \qquad (9)$$

where we have assumed, without losing any generality, that a, b, c, d are real and $a \ge b$ and $c \ge d$. Since $x + \sqrt{1-x^2} \ge 1$ for $1 \ge x \ge 0$, the inequality in Eq. (9) can only be satisfied if both $a, c = 1$ which is the case that the states $|\psi_i\rangle$ are product states. This serves to show that the 4 orthogonal states $|\psi_i\rangle$ cannot be perfectly discriminated if they are entangled.

The same procedure can be applied to investigate the distinguishability of 3 of the four states in Eq. (2) in the following way. Consider the state shared between Alice, Bob, Charlie and Devin

$$|\Psi\rangle = \frac{1}{\sqrt{3}} \sum_{i=1}^{3} |\psi_i\rangle_{AB} \, |\phi_i\rangle_{CD}\,. \qquad (10)$$

Using the majorization relation in Eq. (8), we find that if two of the three states are maximally entangled, discrimination is not possible but numerical calculations show that discrimination is possible for a range of values for the Schmidt parameters a and c. The conditions for discrimination found by majorization relations are in complete agreement with the results in Ref. 14.

II. ENTANGLEMENT ASSISTED DISCRIMINATON

Majorization relations are also helpful in entanglement assisted discrimination. The situation that is considered is as follows. Alice and Bob have qubits in an unknown state among $|\psi_i\rangle$, but they also have other qubits in a known entangled state $|\phi\rangle$. Can they successfully achieve discrimination with an incurred cost of consuming the entanglement in $|\phi\rangle$? In other words, can they discriminate between the states $|\phi\rangle \otimes |\psi_i\rangle$? The entangled state to be used up will be denoted by

$$|\phi\rangle = \alpha\,|00\rangle + \beta\,|11\rangle \qquad (11)$$

where the important parameters are the Schmidt parameters α and β.

The same idea used before to characterize entangled state discrimination will be applied here; if Alice and Bob are able to distinguish between the states $|\phi\rangle \otimes |\psi_i\rangle$,

then when they apply the same protocol to $|\phi\rangle \otimes |\Psi\rangle$, the parties Charlie and Devin will be left with a Bell state $|\phi_i\rangle$, with probability $p_i = 1/4$ for each. This is equivalent to a probabilistic entanglement transformation of the state $|\phi\rangle \otimes |\Psi\rangle$ to $|\phi_i\rangle$ with probability distribution p_i. Using this idea, the following majorization relation can be constructed

$$\lambda(\phi) \otimes \lambda(\Psi) \prec \lambda(\tfrac{1}{2}\mathbb{1}) \ . \tag{12}$$

The following bound for the Schmidt coefficients of the state $|\phi\rangle$ can be obtained from this relation

$$|\alpha|^2 \leq \frac{4}{(a+b+c+d)^2} \ , \tag{13}$$

where it is assumed that $|\alpha|^2 \geq |\beta|^2$. The entanglement entropy for the state $|\phi\rangle$ can be calculated for various entanglement of the states $|\psi_i\rangle$ using the regular expression for entanglement entropy

$$\mathcal{E}(\phi) = -\sum_{i=1}^{2} \lambda_i \log_2 \lambda_i \tag{14}$$

where λ_i are the Schmidt coefficients of the state $|\phi\rangle$ and the entanglement needed to discriminate the states $|\psi_i\rangle$ versus their average entanglement is shown in Fig. 1.
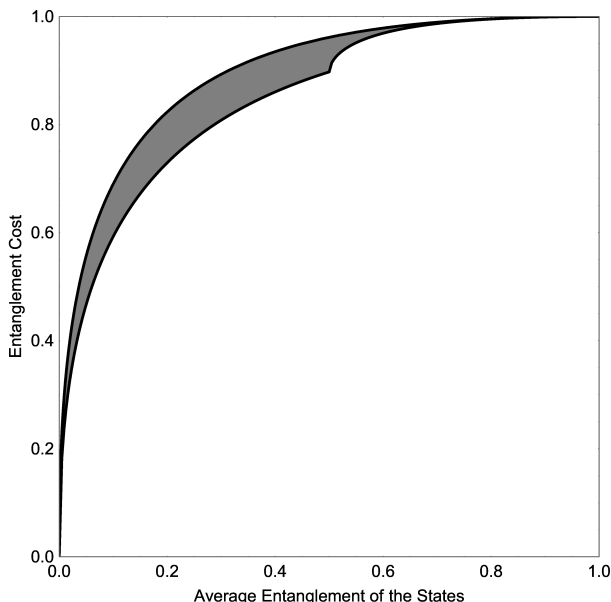

FIG. 1. Plot of the entanglement cost of discrimination versus the average entanglement of the states $|\psi_i\rangle$ for $p_i = 1/4$. The entanglement cost is not a simple function of the average entanglement as the average entanglement is not a one-to-one function of the parameters a and c.

The parties can also succeed in discriminating between the states $|\psi_i\rangle$ in the following way: one of the parties can teleport her part of the states to the other party using an e-bit of entanglement and the states can be locally

discriminated in a perfect way afterwards. The cost calculated using majorization, however, shows that there are protocols for which the entanglement cost can be lower than the teleportation cost provided that all the states are not maximally entangled.

III. DISCRIMINATION WITH REMAINING ENTANGLEMENT

In this section we turn our attention to discrimination of entangled states with preservation of their entanglement. We use entanglement entropy as before to quantify entanglement and use the fact that a bipartite pure state of qubits is convertible to another one with the same or smaller entanglement entropy under LOCC. This enables us to characterize the discrimination protocol by considering only the Schmidt coefficients of the states since two states with the same Schmidt coefficients are same up to local unitaries.

We consider two observers, Alice and Bob, in possession of qubits AB whose state is one of the states $|\psi_i\rangle$ as in Eq. (2).They also possess quantum systems A′B′ with a known but arbitrary entangled state $|\phi\rangle$ whose entanglement is usable by the parties. An absolute upper bound on the entanglement cost of this protocol can be found by considering the cost of using quantum teleportation. Alice can teleport her part of the states to Bob using up one e-bit of entanglement after which Bob can perfectly discriminate between 4 orthogonal quantum states. Bob then teleports to Alice what was originally her part of the state using up another e-bit of entanglement bringing up the total cost to 2 e-bits of entanglement. In other words, with 2 e-bits of entanglement of A′B′, the task can always be accomplished.

To obtain a better bound on the amount of entanglement necessary to accomplish the task, the discrimination procedure will be used to design an entanglement transformation protocol. Suppose that the parties are able to discriminate between the states $|\psi_i\rangle$ of qubits AB and preserve the states, with the aid of the other pair A′B′ in state $|\phi\rangle$. In other words, there is a local procedure implemented by the parties such that if the pair AB is initially in state $|\psi_i\rangle_{AB}$, then the procedure causes the transformation

$$|\phi\rangle_{A'B'} \otimes |\psi_i\rangle_{AB} \longrightarrow |00\rangle_{A'B'} \otimes |\psi_i\rangle_{AB} \tag{15}$$

and the parties identify the index i. (In here and the following, the symbols A, A′, $\overline{A}$, etc. represent particles possessed by Alice and, similarly, B, B′, $\overline{B}$ belong to Bob.) Now, we consider a situation where four particles AB$\overline{AB}$ are in state

$$|\Psi\rangle_{AB\overline{AB}} = \sum_i |\psi_i\rangle_{AB} \otimes |\varphi_i\rangle_{\overline{A}\,\overline{B}} \tag{16}$$

and the same procedure is applied. This then causes a probabilistic transformation

$$|\phi\rangle_{A'B'} \otimes |\Psi\rangle_{AB\overline{AB}} \longrightarrow |00\rangle_{A'B'} \otimes |\psi_i\rangle_{AB} \otimes |\varphi_i\rangle_{\overline{AB}} \tag{17}$$

where the ith outcome is identified and obtained with probability $p_i = \|\varphi_i\|^2$. The majorization relation will then put on bounds on the possibility of the original transformation in (15). Note that there are infinitely many such bounds because $|\varphi_i\rangle$ can be chosen arbitrarily. However, the best bounds are obtained when $|\Psi\rangle$ is a product state in the $A\overline{A}$:$B\overline{B}$ cut because

$$\lambda(\Psi) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} . \tag{18}$$

If $|\Psi\rangle$ is an entangled state, largest entry in the Schmidt coefficients will be smaller than 1; this will give us looser bounds for $|\phi\rangle$.

When we choose $|\Psi\rangle$ to be a product state, the majorization relation that characterizes the transformation and therefore gives us bounds on $|\phi\rangle$ can be expressed as

$$\lambda(\phi) \prec \sum_i \|\varphi_i\|^2 \lambda(\psi_i) \otimes \lambda\left(\frac{\varphi_i}{\|\varphi_i\|}\right). \tag{19}$$

The state $|\Psi\rangle$, which is unentangled in the $A\overline{A}$:$B\overline{B}$ cut, can be expressed as $|\Psi\rangle = |u\rangle_{A\overline{A}} \otimes |v\rangle_{B\overline{B}}$. We choose $|u\rangle$ and $|v\rangle$ as Bell states

$$|u\rangle = |v\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \tag{20}$$

and re-express $|\Psi\rangle$ as

$$|\Psi\rangle = \frac{1}{2} \sum_{i,j,k,l=0}^{1} \delta_{ij}\delta_{kl} |ik\rangle_{AB} \otimes |jl\rangle_{\overline{A}\overline{B}} . \tag{21}$$

We define, for the ease of manipulation, what we call a partial inner product between bras and kets in the following way

$$|\varphi_m\rangle = \langle\psi_m|\Psi\rangle\rangle \tag{22}$$

and express $|\psi_m\rangle$ as $|\psi_m\rangle = \sum_{i,k}(\psi_m)_{ik} |ik\rangle_{AB}$. The states $|\varphi_m\rangle$ can then be found as

$$|\varphi_m\rangle = \langle\psi_m|\Psi\rangle\rangle = \frac{1}{2} \sum_{i,j,k,l} (\psi_m)_{ik}^\star \delta_{ij}\delta_{kl} |jl\rangle_{\overline{A}\overline{B}}$$
$$|\varphi_m\rangle = \frac{1}{2} |\psi_m^\star\rangle . \tag{23}$$

where $|\psi_m^\star\rangle$ is the state obtained by complex conjugating the coefficients of $|\psi_m\rangle$ in the "computational basis". The majorization relation can now be expressed as

$$\lambda(\phi) \prec \frac{1}{4} \sum_i \lambda(\psi_i) \otimes \lambda(\psi_i),$$

$$\lambda(\phi) \prec \frac{1}{2} \begin{pmatrix} a^4 + c^4 \\ a^2b^2 + c^2d^2 \\ a^2b^2 + c^2d^2 \\ b^4 + d^4 \end{pmatrix} \tag{24}$$

where we assumed $a \geq b$, $c \geq d$ and a, b, c, d are real without losing any generality. The majorization relation in Eq. (24) shows that the state $|\phi\rangle$ has at least 4 Schmidt coefficients, and it has 2 e-bits of entanglement if all the $|\psi_i\rangle$ are maximally entangled, and no entanglement if all the $|\psi_i\rangle$ are product states. This result is in agreement and is expected since the upper bound on the entanglement cost of such a protocol is 2 e-bits if teleportation is used. The entanglement cost versus the average entanglement of the states is plotten in Fig. 2
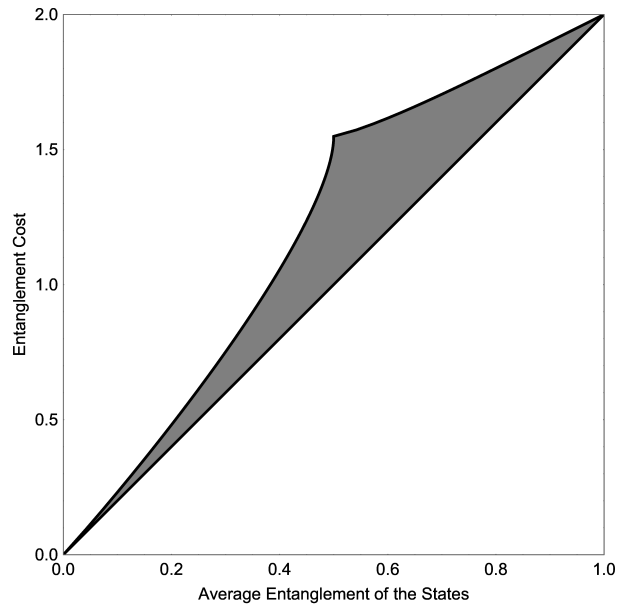

FIG. 2. Plot of the entanglement cost versus the average entanglement of the states. The diagonal line is when all the states have equal entanglement and the cusp is when two of the states are maximally entangled and the others are product states.

IV. CONCLUSION

In this work, we have considered the problem of local discrimination of multipartite states and shown that majorization relations prove very useful in characterizing various types of state discrimination problems. Using majorization relation, it is possible to say something about the minimum amount of entanglement that must be used to discriminate between 4 orthogonal, bipartite, entangled states. Bounds have been provided on the entanglement that needs to be spent. These results are in agreement with the teleportation bound and the cost that we have calculated is always lower than or equal to the teleportation bound.

The question of preserving entanglement after discrimination is also considered. This problem has also been discussed by Cohen [15], and the protocols have been examined. Using majorization relations for probabilistic

entanglement transformations, we have found bounds on the Schmidt coefficients of the resource state and calculated the entanglement cost of entanglement preserving discrimination problems. The upper bound on the cost is in agreement with the teleportation cost of 2 e-bits and is always smaller than 2 e-bits for non maximally entangled states.

We haven't constructed explicit protocols that would achieve discrimination for the given problems but as it is stated above, the majorization relations, by construction, guarantee the existence of a LOCC protocol. There is however, a caveat, the protocols constructed from the majorization relations are local protocols in the sense that Alice can act on the Hilbert spaces $A, \overline{A}$. In the case of a protocol achieving discrimination with remaining entanglement, Alice can't act on the Hilbert space $\overline{A}$ at all. Thus, an entanglement transformation protocol on 4 particles $A, \overline{A}, B, \overline{B}$ can be constructed for the discrimination problem of 2 particles A, B but not the other way around. In the light of this identification, the bounds calculated in this work are necessary conditions for parties to achieve discrimination with remaining entanglement. Sufficient conditions for the existence of such protocols and their explicit construction is still an open question.

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