

UNIVERSAL GROUPS OF INTERMEDIATE GROWTH AND THEIR INVARIANT RANDOM SUBGROUPS

MUSTAFA GÖKHAN BENLİ, ROSTISLAV GRIGORCHUK AND TATIANA NAGNIBEDA

To A.M. Vershik on the occasion of his 80-th birthday, with admiration and respect

ABSTRACT. We exhibit examples of groups of intermediate growth with 2^{\aleph_0} ergodic continuous invariant random subgroups. The examples are the universal groups associated with a family of groups of intermediate growth.

1. INTRODUCTION

The goal of this paper is to show the existence of groups of intermediate growth with 2^{\aleph_0} distinct ergodic continuous invariant random subgroups.

Invariant random subgroup (abbreviated *IRS*) is a convenient term that stands for a probability measure on the space of subgroups in a locally compact group, invariant under the action of the group by conjugation. In the case of a countable group G (only such groups will be considered here), the space $S(G)$ of subgroups of G is supplied with the topology induced from the Tychonoff topology on $\{0, 1\}^G$ where a subgroup $H \leq G$ is identified with its characteristic function $\chi_H(g) = 1$ if $g \in H$ and 0 otherwise.

The delta mass corresponding to a normal subgroup is a trivial example of an *IRS*, as well as the average over a finite orbit of delta masses associated with groups in a finite conjugacy class. Hence, we are rather interested in continuous invariant probability measures on $S(G)$. Clearly, such a measure does not necessarily exist, for example if the group only has countably many subgroups.

Given a countable group G , a basic question is whether a continuous *IRS* exists. Ultimately one wants to describe the structure of the simplex of invariant probability measures of the topological dynamical system $(\text{Inn}(G), S(G))$ where $\text{Inn}(G)$ is the group of inner automorphisms of G acting on $S(G)$. Of particular interest are ergodic measures, i.e., the extremal points in the simplex.

A more general problem is the identification of the simplex of invariant probability measures of the system $(\Phi, S(G))$ where Φ is a subgroup of the group $\text{Aut}(G)$ of automorphisms of G (see [AGV12, Bow12, Ver12]). A closely related problem is the study of invariant measures on the space of rooted Schreier graphs of G , with G acting by change of the root. This point of view is presented in [Gri11, Vor12].

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A very fruitful idea in the subject belongs to Anatoly Vershik who introduced the notion of a totally non-free action of a locally compact group G on a space X with invariant measure μ , i.e., an action with the property that different points $x \in X$ have different stabilizers $St_G(x)$ μ -almost surely. Then the map $St : X \rightarrow S(G)$ defined by $x \mapsto St_G(x)$ is injective μ -almost surely and the image of μ under this map is the law of an *IRS* on G which is continuous and ergodic whenever μ is. In [Ver12], Vershik showed that a totally non-free action of a group G provides us not only with an *IRS* but also with a factor representation of G . He also realized the plan outlined above and described all the ergodic $Aut(G)$ -invariant measures on $S(G)$ in the case when G is the infinite symmetric group, see [Ver10, Ver12].

Bowen showed in [Bow12] that non-abelian free groups of finite rank possess a whole “zoo” of ergodic continuous *IRS*, and that a big part of the simplex of *IRS* on a free group F_r , $r \geq 2$, is a Poulsen simplex. (A simplex is called a Poulsen if its extremal points are dense. It is unique up to affine isomorphism by [LOS78]). As shown in [BGK12], already the so-called lamplighter group $\mathcal{L} = \mathbb{Z}_2 \wr \mathbb{Z}$ (the “simplest” finitely generated group that has 2^{\aleph_0} subgroups) has a Poulsen simplex of *IRS*. Given a surjection $\phi : G \twoheadrightarrow H$, there is a natural homeomorphism $\tilde{\phi} : S(H) \rightarrow S(G, Ker(\phi))$ where $S(G, Ker(\phi))$ denotes the subspace of $S(G)$ consisting of subgroups of G containing $Ker(\phi)$. This allows to lift any *IRS* on H to G thus providing a large spectrum of *IRS* on G from that on H . This applies in particular to the free group F_2 that covers \mathcal{L} .

A finitely generated virtually nilpotent group has only countably many subgroups and therefore does not possess any continuous *IRS*. By Gromov’s theorem, the class of finitely generated virtually nilpotent groups coincide with the class of groups of polynomial growth. Recall that, given a finitely generated group G with a system of generators S , one can consider its growth function $\gamma(n) = \gamma_{(G,S)}(n)$ which counts the number of elements of length at most n . The growth type of this function when $n \rightarrow \infty$ does not depend on the generating set S and can be polynomial, exponential or intermediate. The question of existence of groups of intermediate growth was raised by Milnor [Mil68] and was answered by the second author in [Gri84b]. The main construction associates with every sequence $\omega \in \Omega = \{0, 1, 2\}^{\mathbb{N}}$ a group G_ω generated by four involutions $a_\omega, b_\omega, c_\omega, d_\omega$ and if ω is not an eventually constant sequence, then G_ω has intermediate growth. Moreover, it was also observed in [Gri84b] that the groups G_ω fall into the class of just-infinite branch groups. A group is just infinite if it is infinite but every proper quotient is finite. A group is branch if it has a faithful level transitive action on a spherically homogeneous rooted tree with the property that rigid stabilizers of the levels of the tree are of finite index, see Section 4.2 for precise definitions. Just-infinite branch groups constitute one of three classes in which the class of just-infinite groups naturally splits [Gri00].

Since the groups G_ω are just-infinite, they only have countably many quotients. This raised the question of existence of groups of intermediate growth having 2^{\aleph_0}

quotients, answered in [Gri84a]. The main idea was to take a suitable subset $\Lambda \subset \Omega$ of cardinality 2^{\aleph_0} and consider the group U_Λ defined as the quotient of the free group F_4 by a normal subgroup N which is the intersection of normal subgroups $N_\omega, \omega \in \Lambda$ where $G_\omega = F_4/N_\omega$. In this paper we explore this idea further by using *IRS* on G_ω and lift them to U_Λ deducing the main result.

Branch groups give us the most transparent examples of totally non-free actions and thus of *IRS*. Indeed, as shown in [BG02], the natural extension of a branch action on a spherically homogeneous tree T to its boundary ∂T is totally non free with respect to the uniform probability measure on ∂T . It is even completely non free, i.e., different points have different stabilizers. The uniform probability measure on ∂T is ergodic and invariant. The groups G_ω act on the binary rooted tree in a branch way. By lifting the uniform measure to $S(G_\omega)$ and then to $S(U_\Lambda)$, one obtains a host of *IRS* on U_Λ . We then proceed to showing that the *IRS* obtained in this way are distinct. These considerations allow us to prove our main theorem:

Main Theorem. *There exists a finitely generated group of intermediate growth with 2^{\aleph_0} distinct continuous ergodic invariant random subgroups.*

We also investigate some additional properties of groups of the form $U_\Lambda, \Lambda \subset \Omega$, including finite presentability, branching property and self-similarity.

2. SPACE OF MARKED GROUPS AND UNIVERSAL GROUPS

Definition 1. A *k*-marked group is a pair (G, S) , where G is a group and $S = (s_1, \dots, s_k)$ is an ordered set of (not necessarily distinct) elements such that the set $\{s_1, \dots, s_k\}$ generates the group G . The *canonical map* between two *k*-marked groups (G, S) and (H, T) is the map sending $s_i \mapsto t_i, i = 1, 2, \dots, k$. If this map defines an epimorphism, it will be called the marked epimorphism and (H, T) will be called a marked image of (G, S) . Two *k*-marked groups (G, S) and (H, T) are equivalent if the canonical map defines an isomorphism between G and H .

The space of (equivalence classes of) *k*-marked groups will be denoted by \mathcal{M}_k . This space has a natural topology, which for instance can be defined by the following metric: Two *k*-marked groups (G, S) and (H, K) are of distance 2^{-m} , where m is the largest natural number such that the balls of radius m of the Cayley graphs of (G, S) and (H, K) are isomorphic (as directed labeled graphs). In [Gri84b] it was observed that this makes \mathcal{M}_k into a compact totally disconnected space.

Alternatively, this space can be defined in the following way: Let F_k be a free group of rank k with a basis $\{x_1, \dots, x_k\}$. Let \mathcal{N}_k denote the set of all normal subgroups of F_k , together with the topology inherited from the power set $\mathcal{P}(F_k) \cong \{0, 1\}^{F_k}$ supplied with the Tychonoff topology. This topology has basis consisting of sets of the form $\mathcal{O}_{A,B} = \{N \triangleleft F_k \mid A \subset N, B \cap N = \emptyset\}$ where A and B are finite subsets of F_k . Given $(G, S) \in \mathcal{M}_k$, let $N_{(G,S)} \in \mathcal{N}_k$ be the kernel of the natural map $\pi_{(G,S)} : F_k \rightarrow G$ sending $x_i \mapsto s_i$. This gives a homeomorphism

between \mathcal{M}_k and \mathcal{N}_k (depending on the basis of F_k) (See [Cha00]). We will interchangeably use these two spaces.

Definition 2. Let $\mathcal{C} = \{(G_i, S_i) \mid i \in I\}$ be a subset of \mathcal{M}_k . Let $N_{\mathcal{C}} = \bigcap_{i \in I} N_{(G_i, S_i)}$. The *universal group of the family \mathcal{C}* is the k -marked group $(U_{\mathcal{C}}, S_{\mathcal{C}})$ where $U_{\mathcal{C}} = F_k/N_{\mathcal{C}}$ and $S_{\mathcal{C}}$ is the image of the basis $\{x_1, \dots, x_k\}$.

$U_{\mathcal{C}}$ has the following universal property: If (H, T) is a marked group such that for all $i \in I$ the canonical map from (H, T) to (G_i, S_i) defines a group homomorphism, then the canonical map from (H, T) to $(U_{\mathcal{C}}, S_{\mathcal{C}})$ defines a group homomorphism.

An alternative way to define the universal group is the following:

Definition 3. Given $\mathcal{C} = \{(G_i, S_i) \mid i \in I\} \subset \mathcal{M}_k$, write $S_i = (s_1^i, \dots, s_k^i)$. Let $U_{\mathcal{C}}^{diag}$ be the subgroup of the (unrestricted) direct product $\prod_{i \in I} G_i$ generated by the elements $s_j = (s_j^i)_{i \in I}$ $j = 1, \dots, k$. The k -marked group $(U_{\mathcal{C}}^{diag}, S_{\mathcal{C}}^{diag})$ is called the *diagonal group* of the family \mathcal{C} .

It is straightforward to check that $(U_{\mathcal{C}}^{diag}, S_{\mathcal{C}}^{diag})$ equivalent (as a marked group) to the universal group $(U_{\mathcal{C}}, S_{\mathcal{C}})$ of Definition 2.

Proposition 1. *Let $\mathcal{C} \subset \mathcal{M}_k$. Then the marked groups $(U_{\mathcal{C}}, S_{\mathcal{C}})$ and $(U_{\bar{\mathcal{C}}}, S_{\bar{\mathcal{C}}})$ are equivalent, where $\bar{\mathcal{C}}$ denotes the closure of \mathcal{C} in \mathcal{M}_k .*

Proof. We need to show that

$$\bigcap_{(G, S) \in \mathcal{C}} N_{(G, S)} = \bigcap_{(G, S) \in \bar{\mathcal{C}}} N_{(G, S)}.$$

Clearly, the right hand side is contained in the left. Suppose that some $g \in F_k$ belongs to the left hand side but not to the right. Then there exists $(G, S) \in \mathcal{C}$ such that $g \notin N_{(G, S)}$. Let $\{(G_n, S_n)\}_{n \geq 0}$ be a sequence in \mathcal{C} converging to (G, S) . Since g belongs to the left hand side, g belongs to each $N_{(G_n, S_n)}$ and by definition of the topology in \mathcal{N}_k , to $N_{(G, S)}$ which gives a contradiction. \square

For an element $w \in F_k, w \neq 1$, denote $\mathcal{O}_w = \{N \triangleleft F_k \mid w \in N\}$.

Lemma 1. *Let $H \leq F_k$ be a subgroup and $w_1, \dots, w_m \in H, w_i \neq 1$ for each i . Then there exists $w \in H, w \neq 1$ such that $\bigcup_{i=1}^m \mathcal{O}_{w_i} \subset \mathcal{O}_w$.*

Proof. By induction on m . The case $m = 1$ is clear, one can take $w = w_1$. So, assume $m > 1$.

Case 1: $[w_1, w_2] = 1$ in F_k . In this case there exists $w \in F_k$ and $s, t \in \mathbb{Z}$ such that $w_1^s = w_2^t = w$ (all non-trivial abelian subgroups in a free group are cyclic). Therefore, $\mathcal{O}_{w_1} \cup \mathcal{O}_{w_2} \subset \mathcal{O}_w$ and hence we can apply the induction hypothesis by replacing \mathcal{O}_{w_1} and \mathcal{O}_{w_2} by \mathcal{O}_w .

Case 2: $[w_1, w_2] \neq 1$ in F_k . In this case we can replace \mathcal{O}_{w_1} and \mathcal{O}_{w_2} by $\mathcal{O}_{[w_1, w_2]}$ and apply the induction hypothesis. \square

Proposition 2. *Let $\mathcal{C} \subset \mathcal{M}_k$ be a closed subset and assume that no group in \mathcal{C} contains a nonabelian free subgroup. Then the universal group $U_{\mathcal{C}}$ also has no nonabelian free subgroups.*

Proof. Let $\mathcal{C} = \{(G_i, S_i) \mid i \in I\}$. Let $a, b \in U_{\mathcal{C}}$ be two distinct elements, given as words in the generators $S_{\mathcal{C}}$. Let $w_a, w_b \in F_k$ such that $\pi_{(G,S)}(w_a) = a$ and $\pi_{(G,S)}(w_b) = b$. For each $i \in I$, since G_i has no (non-abelian) free subgroups, there is nontrivial $w_i \in \langle w_a, w_b \rangle \leq F_k$ such that $\pi_{(G_i, S_i)}(w_i) = 1$, i.e., $w_i \in N_{(G_i, S_i)}$. Hence $\{\mathcal{O}_{w_i}\}_{i \in I}$ is an open cover of \mathcal{C} . Since \mathcal{C} is compact, there is a finite subcover $\mathcal{O}_{w_1}, \dots, \mathcal{O}_{w_n}$. By Lemma 1, there exists non-trivial $w \in \langle w_a, w_b \rangle$ such that $\mathcal{C} \subset \mathcal{O}_w$. This shows that $w = 1$ in $U_{\mathcal{C}}$. \square

3. GRIGORCHUK 2-GROUPS¹

We recall here the construction of [Gri84b]. Note that in the original construction in [Gri84b] the groups are defined as measure preserving transformations of the unit interval. Here we will define them as groups of automorphisms of the binary rooted tree.

Let $\Omega = \{0, 1, 2\}^{\mathbb{N}}$ be the space of infinite sequences $\omega = \omega_1\omega_2\dots\omega_n\dots$ where $\omega_i \in \{0, 1, 2\}$, considered with its natural product topology. Let τ be the shift transformation, i.e., if $\omega = \omega_1\omega_2\dots \in \Omega$ then $\tau\omega = \omega_2\omega_3\dots$. Let T be the binary rooted tree whose vertices are identified with the set of all finite binary words $\{0, 1\}^*$ and edges defined in standard way: $E = \{(w, wx) \mid w \in \{0, 1\}^*, x \in \{0, 1\}\}$. For each $\omega \in \Omega$, consider the automorphisms $\{a, b_{\omega}, c_{\omega}, d_{\omega}\}$ of T defined recursively as follows:

For $v \in \{0, 1\}^*$

$$a(0v) = 1v \text{ and } a(1v) = 0v$$

$$\begin{aligned} b_{\omega}(0v) &= 0\beta(\omega_1)(v) & c_{\omega}(0v) &= 0\zeta(\omega_1)(v) & d_{\omega}(0v) &= 0\delta(\omega_1)(v) \\ b_{\omega}(1v) &= 1b_{\tau\omega}(v) & c_{\omega}(1v) &= 1c_{\tau\omega}(v) & d_{\omega}(1v) &= 1d_{\tau\omega}(v), \end{aligned}$$

where

$$\begin{aligned} \beta(0) &= a & \zeta(0) &= a & \delta(0) &= e \\ \beta(1) &= a & \zeta(1) &= e & \delta(1) &= a \\ \beta(2) &= e & \zeta(2) &= a & \delta(2) &= a \end{aligned}$$

and e denotes the identity automorphism of T .

For each $\omega \in \Omega$, let G_{ω} be the subgroup of $\text{Aut}(T)$ generated by the set $S_{\omega} = \{a, b_{\omega}, c_{\omega}, d_{\omega}\}$ so that $\mathcal{G} = \{(G_{\omega}, S_{\omega}) \mid \omega \in \Omega\}$ is a subset of \mathcal{M}_4 . In [Gri84b] it was observed that if two sequences $\omega, \eta \in \Omega$ which are not eventually constant, have long common beginning, then the 4-marked groups (G_{ω}, S_{ω}) and (G_{η}, S_{η}) are close to each other in \mathcal{M}_4 . It was also observed that the groups (G_{ω}, S_{ω}) for eventually constant sequences ω are isolated in $\{(G_{\omega}, S_{\omega}) \mid \omega \in \Omega\}$. Hence, removing these isolated points from this set and taking its closure in \mathcal{M}_4 , one obtains a compact

¹ The first and the third authors insist on using this standard terminology.

subset $\tilde{\mathcal{G}} = \{(\tilde{G}_\omega, \tilde{S}_\omega) \mid \omega \in \Omega\} \subset \mathcal{M}_4$ which is homeomorphic to Ω (and hence to a Cantor set) via $\omega \mapsto (\tilde{G}_\omega, \tilde{S}_\omega)$. Note that $(\tilde{G}_\omega, \tilde{S}_\omega) = (G_\omega, S_\omega)$ if and only if ω is not eventually constant and $(\tilde{G}_\omega, \tilde{S}_\omega) = \lim_{n \rightarrow \infty} (G_{\omega^{(n)}}, S_{\omega^{(n)}})$ when ω is eventually constant and where $\{\omega^{(n)}\}_{n \geq 0}$ is a sequence of not eventually constant elements in Ω converging to ω (the limit does not depend on the choice of the sequence $\{\omega^{(n)}\}_{n \geq 0}$). In other words, the families \mathcal{G} and $\tilde{\mathcal{G}}$ differ only on countably many points.

Note that we have the following: $N_{(\tilde{G}_\omega, \tilde{S}_\omega)} = N_{(G_\omega, S_\omega)}$ if ω is not eventually constant and $N_{(\tilde{G}_\omega, \tilde{S}_\omega)} \subset N_{(G_\omega, S_\omega)}$ for eventually constant $\omega \in \Omega$.

Let Ω_∞ be the set of sequences in Ω in which all three letters $\{0, 1, 2\}$ occur infinitely often and Ω_0 be the set of eventually constant sequences. Regarding the groups in \mathcal{G} and $\tilde{\mathcal{G}}$ the following are known:

Theorem 1 ([Gri84b]).

- (1) All groups G_ω , $\omega \in \Omega$ are infinite residually finite groups.
- (2) G_ω is virtually \mathbb{Z}^{2^n} if ω becomes constant starting with n -th coordinate.
- (3) If $\omega \notin \Omega_0$ then G_ω has intermediate growth between polynomial and exponential.
- (4) If $\omega \in \Omega_0$ then \tilde{G}_ω is virtually metabelian, infinitely presented and has exponential growth.
- (5) If $\omega \in \Omega_\infty$ then G_ω is a torsion 2-group.
- (6) If $\omega \in \Omega_\infty$ then G_ω is just-infinite, i.e., all its nontrivial quotients are finite.
- (7) For $\omega_1, \omega_2 \in \Omega_\infty$ we have $G_{\omega_1} \cong G_{\omega_2}$ if and only if ω_1 can be obtained from ω_2 by applying a permutation from $\text{Sym}(\{0, 1, 2\})$ letter by letter.

Proof. For proofs of (1),(2),(3) and (5) see [Gri84b, Theorem 2.1]. (4) is proven in [Gri84b, Theorem 6.1,6.2] and (6) in [Gri84b, Theorem 8.1]. (7) is proven in [Nek05, Theorem 2.10.13]. \square

4. SOME PROPERTIES OF THE FULL UNIVERSAL GROUP U

Regarding the universal groups corresponding to the families \mathcal{G} and $\tilde{\mathcal{G}}$ we have the following:

Proposition 3. $U_{\mathcal{G}} = U_{\tilde{\mathcal{G}}}$.

Proof. Referring to the notation of Definition 2, we need to show the following equality:

$$N_{\mathcal{G}} := \bigcap_{\omega \in \Omega} N_{(G_\omega, S_\omega)} = N_{\tilde{\mathcal{G}}} := \bigcap_{\omega \in \Omega} N_{(\tilde{G}_\omega, \tilde{S}_\omega)}.$$

Since $N_{(\tilde{G}_\omega, \tilde{S}_\omega)} \subset N_{(G_\omega, S_\omega)}$ for each $\omega \in \Omega$, the right-hand side of the above equation is contained in the left-hand side. Since $\{(\tilde{G}_\omega, \tilde{S}_\omega) \mid \omega \in \Omega \setminus \Omega_0\}$ is dense in $\tilde{\mathcal{G}}$, by

Proposition 1 we have

$$N_{\tilde{\mathcal{G}}} = \bigcap_{\omega \in \Omega \setminus \Omega_0} N_{(\tilde{G}_\omega, \tilde{S}_\omega)}.$$

Therefore,

$$N_{\mathcal{G}} \subset \bigcap_{\omega \in \Omega \setminus \Omega_0} N_{(G_\omega, S_\omega)} = \bigcap_{\omega \in \Omega \setminus \Omega_0} N_{(\tilde{G}_\omega, \tilde{S}_\omega)} = N_{\tilde{\mathcal{G}}}.$$

□

We will use the notation $U = U_{\mathcal{G}}$ for the full universal group and denote by $S = \{a, b, c, d\}$ its canonical generators. Note that the basic relations $a^2 = b^2 = c^2 = d^2 = bcd = 1$ hold in U .

Theorem 2. *U contains no nonabelian free subgroups, has uniformly exponential growth and is not finitely presented.*

Proof. Since all groups in $\tilde{\mathcal{G}}$ are amenable (and hence cannot contain nonabelian free subgroups), the first assertion follows from Proposition 2. By Theorem 1 part (4), the group \tilde{G}_η for $\eta = 000\dots$ is an elementary amenable group of exponential growth, and hence of uniformly exponential growth by [Osi04]. Therefore U has uniformly exponential growth. By [BGdlH13, Theorem 1.10], any finitely presented group mapping onto the groups $G_\omega, \omega \in \Omega$ must be large, i.e., has a finite index subgroup mapping onto a nonabelian free group. In particular, such group contains a nonabelian free subgroup. Therefore U cannot be finitely presented. □

4.1. U as an automaton group.

In this section we will realize U as an automaton group and explore further properties. Firstly, we will recall some basics.

Let T_d denote the d -ary rooted tree with vertex set $\{0, 1, 2, \dots, d-1\}^*$. For an automorphism $g \in \text{Aut}(T_d)$ and $x \in \{0, 1, \dots, d-1\}$, the *section* of g at x (denoted by g_x) is the automorphism of T_d defined uniquely by

$$g(xv) = g(x)g_x(v) \text{ for all } v \in \{0, 1, \dots, d-1\}.$$

This gives an isomorphism

$$\begin{aligned} \text{Aut}(T_d) &\rightarrow S_d \rtimes (\text{Aut}(T_d) \times \dots \times \text{Aut}(T_d)) \\ g &\mapsto \sigma_g(g_0, \dots, g_{d-1}) \end{aligned}$$

where σ_g describes how g permutes the first level subtrees and g_i describe its action within each subtree. (Here S_d is the symmetric group on d letters).

Definition 4. A subgroup $G \leq \text{Aut}(T_d)$ is called self-similar if for all $g \in G$ and $x \in \{0, 1, \dots, d-1\}$, $g_x \in G$.

For an overview of self-similar groups and related topics we refer to [GŠ07].

A standard way to construct self-similar groups is to start with a list of symbols $S = \{s^1, \dots, s^m\}$ and permutations $\sigma_1, \dots, \sigma_m \in S_d$ and consider the system

$$\begin{aligned} s^1 &= \sigma_1(s_0^1, \dots, s_{d-1}^1) \\ &\vdots \\ s^m &= \sigma_m(s_0^m, \dots, s_{d-1}^m) \end{aligned}$$

where $s_j^i \in S$. If $\sigma_i = id$, we will omit writing it. Such a system defines a unique set of m automorphisms of T_d . Clearly the group $G = \langle S \rangle$ will be self-similar. Since in this case the generating set S is closed under taking sections, the action of the group can be modeled by a Mealy type automaton where each generator will correspond to a state of the automaton (see the figure below for an example). Such groups, i.e., groups generated by the states of a Mealy type automaton are called *automata groups*. We refer to [GNS00] for a detailed account on automata groups.

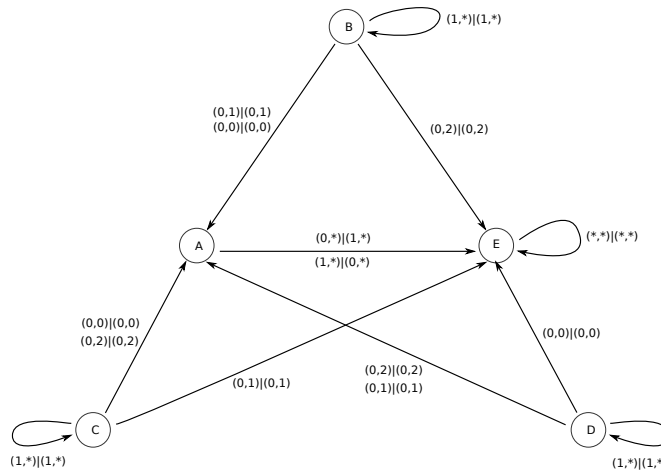
Consider the tree T_6 determined by the alphabet

$$\mathcal{A} = \{0, 1\} \times \{0, 1, 2\} = \{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2)\}$$

whose elements are enumerated as $0, 1, \dots, 5$. Let $V \leq Aut(T_6)$ be generated by the following elements:

$$(1) \quad \begin{aligned} A &= (14)(25)(36) \quad (E, E, E, E, E, E) \\ B &= \quad \quad \quad (A, A, E, B, B, B) \\ C &= \quad \quad \quad (A, E, A, C, C, C) \\ D &= \quad \quad \quad (E, A, A, D, D, D) \end{aligned}$$

where, $(14)(25)(36)$ is an element of the symmetric group S_6 and E corresponds to the identity automorphism. Observe that $A^2 = B^2 = C^2 = D^2 = BCD = 1$. The corresponding automaton is as follows:



We will show that the group V is isomorphic to U (as a marked group).

Given $\omega \in \Omega$ and $u \in \{0, 1\}^*$ let $\omega^u \in \{0, 1, 2\}^*$ be the beginning of ω of length $|u|$. Note that

$$\omega^{uv} = \omega^u (\tau^{|u|}\omega)^v$$

for all $u, v \in \{0, 1\}^*$

For any $\omega \in \Omega$ let $T_\omega = \{(u, v) \in T_6 \mid u \in \{0, 1\}^*, v = \omega^u\}$. Clearly T_ω is a binary subtree of T_6 . Denote $\{0, 1\}^*$ by T_2 and let $\phi_\omega : T_\omega \rightarrow T_2$ be defined as

$$\phi_\omega(u, v) = u$$

which clearly is a bijection. For $(u, v) \in T_\omega$ and $(u', v') \in T_{\tau^{|u|}\omega}$ we have

$$(2) \quad \phi_\omega(uu', vv') = \phi_\omega(u, v)\phi_{\tau^{|u|}\omega}(u', v')$$

Given $g \in V$ and $\omega \in \Omega$ define a group homomorphism $\psi_\omega : V \rightarrow \text{Aut}(T_2)$ by

$$\psi_\omega(g)(u) = \phi_\omega(g(u, \omega^u)) \text{ for all } u \in T_2$$

It is straightforward to verify that $\psi_\omega(g) \in \text{Aut}(T_2)$ and that ψ_ω defines a group homomorphism.

Lemma 2. *For all $u \in T_2$ with $|u| = n$ have*

$$\psi_\omega(g)_u = \psi_{\tau^n \omega}(g_{(u, \omega^u)})$$

Proof. Let $u, z \in \{0, 1\}^*$, $|u| = n$ and denote $\omega^u = v$, $(\tau^n \omega)^z = v'$

$$g(uz, \omega^{uz}) = g(uz, vv') = g((u, v)(z, v')) = g(u, v)g_{(u, v)}(z, v')$$

Hence, by Equation 2,

$$\begin{aligned} \psi_\omega(g)(uz) &= \phi_\omega(g(uz, \omega^{uz})) = \phi_\omega(g(u, v)g_{(u, v)}(z, v')) = \phi_\omega(g(u, v))\phi_{\tau^n \omega}(g_{(u, v)}(z, v')) \\ &= \psi_\omega(g)(u)\psi_{\tau^n \omega}(g_{(u, v)})(z) \end{aligned}$$

The result follows. \square

Lemma 3. *For any $\omega \in \Omega$, ψ_ω defines a marked surjective homomorphism $\psi_\omega : V \rightarrow G_\omega$.*

Proof. It is enough to show that ψ_ω maps generators of V to the generators of G_ω . Firstly, by definition of A we have

$$\psi_\omega(A)(u) = \phi_\omega(A(u, \omega^u)) = \phi_\omega((a(u), \omega^u)) = a(u) \text{ for all } u.$$

We will show by induction on $|u|$ that B, C, D are mapped to $b_\omega, c_\omega, d_\omega$ respectively. If $|u| = 1$ it is straightforward to check this. Using Lemma 2 and induction assumption we have for $u \in \{0, 1\}^*$

$$\psi_\omega(B)(0u) = 0\psi_\omega(B)_0(u) = 0\psi_\omega(B_{(0, \omega^0)})(u) = \begin{cases} 0a(u) & \text{if } \omega^0 = 0, 1 \\ 0u & \text{if } \omega^0 = 2 \end{cases} = b_\omega(0u)$$

Similarly one can check that $\psi_\omega(B)(1u) = b_\omega(1u)$ for all $u \in \{0, 1\}^*$ and hence $\psi_\omega(B) = b_\omega$. Repeating the argument shows that $\psi_\omega(C) = c_\omega, \psi_\omega(D) = d_\omega$. \square

Theorem 3. *The group V is isomorphic to the universal group U (as a marked group).*

Proof. By Lemma 3, for each $\omega \in \Omega$ there exists a marked surjection $\psi_\omega : V \rightarrow G_\omega$, and hence there exists a marked surjection $\psi : V \rightarrow U$. If $g \in V$ is a nontrivial, let $v \in T_6$ such that $gv \neq v$. Let $\omega \in \Omega$ be such that $v \in T_\omega$. This shows that $\psi_\omega(g) \neq 1$ and hence $\psi(g) \neq 1$. This shows that ψ is a marked isomorphism. \square

From now on we will identify U with V .

Sidki in [Sid00] classified automata groups according to their "activity growth" and conjectured that automata groups of polynomial growth are amenable. Note that the automaton defining U has exponential activity growth in Sidki's classification. The question of amenability of the group U remains open. The note [Muc05] claiming amenability of this group unfortunately contains a mistake.

4.2. Branch Structure of U .

Let G be a group acting on a rooted d -ary tree T_d . For a vertex v of T_d , let T_v denote the subtree hanging down at vertex v and for an element $g \in G$ let $\text{supp}(g)$ be the support of g i.e., the set of vertices not fixed by g . The stabilizer of a vertex v is the subgroup $\text{St}_G(v) = \{g \in G \mid g(v) = v\}$. The rigid stabilizer of a vertex v is the subgroup $\text{Rist}_G(v) = \{g \in G \mid \text{supp}(g) \subset T_v\}$. The rigid stabilizer of level n is the subgroup $\text{Rist}_G(n) = \langle \text{Rist}_G(v) \mid |v| = n \rangle$. Since rigid stabilizer of distinct vertices of the same level commute, we have $\text{Rist}_G(n) = \prod_{|v|=n} \text{Rist}_G(v)$.

Definition 5. Let G be group of automorphisms of a rooted tree T . G is said to be a near branch group (resp. weakly near branch group) if for all $n \geq 1$, the subgroup $\text{Rist}_G(n)$ has finite index in G (resp. is nontrivial). If in addition G acts level transitively (i.e., transitively on each level of the tree) then G is called a branch group (weakly branch group) respectively.

The class of (weakly) branch groups is interesting from various points of view and plays an important role in the classification of just-infinite groups, i.e., infinite groups whose proper homomorphic images are all finite (see [Gri00] for a detailed account on branch groups and just-infinite groups).

Let us mention the following fact which will be used in the forthcoming sections. We will also give an alternative proof of this fact later.

Theorem 4. [Gri84b] *For $\omega \in \Omega_\infty$, the group G_ω is a branch group.*

Note that at the terminology "branch group" was not used in [Gri84b].

If G is a self-similar group, a standard way to show near branch property (resp. weakly near branch property) is to find a finite index subgroup K (resp. nontrivial subgroup) of G such that the image $\phi(K)$ contains the subgroup $K \times \cdots \times K$ where

$\phi : \text{Aut}(T_d) \rightarrow S_d \ltimes (\text{Aut}(T_d) \times \cdots \times \text{Aut}(T_d))$ is as defined in the previous section. This inclusion is denoted by $K \succcurlyeq K \times \cdots \times K$. In this case the group is said to be a regular ((weakly) near) branch group over the subgroup K .

Definition 6. Let G be a self-similar group of automorphisms of a d -ary rooted tree d . G is said to be *self-replicating* if for all $g \in G$ and all $x \in \{0, 1, 2, \dots, d-1\}$, there exists an element $h \in \text{St}_G(1)$ such that $h_x = g$.

Regarding the action of U on T_6 we have the following:

Theorem 5. U is a self-replicating weakly near branch group, regular branching over the third commutator subgroup U''' .

Proof. Note that $\text{St}_U(1)$ is generated by the elements $\{b, c, d, aba, aca, ada\}$. Since we have

$$\begin{aligned} b &= (a, a, 1, b, b, b) \\ c &= (a, 1, a, c, c, c) \\ d &= (1, a, a, d, d, d) \\ aba &= (b, b, b, a, a, 1) \\ aca &= (c, c, c, a, 1, a) \\ ada &= (d, d, d, 1, a, a) \end{aligned}$$

it follows that U is self-replicating.

We claim that the derived subgroup U' is generated by $(ab)^2, (ac)^2, (ad)^2$. From the basic relations we have that a, b, c, d are of order 2 and b, c, d commute with each other. Hence U' is generated as a normal subgroup by

$$[a, b] = (ab)^2, [a, c] = (ac)^2, [a, d] = (ad)^2$$

Therefore it is enough to show that the subgroup generated by $(ab)^2, (ac)^2, (ad)^2$ is normal in U . Clearly conjugation by a inverts the elements $(ab)^2, (ac)^2, (ad)^2$. For other conjugations we have (using the relation $bcd = 1$):

$$x(ax)^2x = (xa)^2 = ((ax)^2)^{-1}$$

and

$$y(ax)^2y = (ya)^2(az)^2 = ((ay)^2)^{-1}(az)^2$$

where $x, y, z \in \{b, c, d\}$ are distinct. Therefore U' is generated by $(ab)^2, (ac)^2, (ad)^2$.

Next we claim that U is near weakly branch over the third derived subgroup U''' , that is: $U''' \succcurlyeq U''' \times U''' \times U''' \times U''' \times U''' \times U'''$. Let

$$t = [(ab)^2, (ac)^2], \quad v = [(ab)^2, (ad)^2] \quad w = [(ac)^2, (ad)^2]$$

U'' is generated as a normal subgroup by t, v and w . Hence U'' is generated by the set

$$\{t^{g_1}, v^{g_2}, w^{g_3} \mid g_i \in U\}$$

It follows that U''' is generated as a normal subgroup by the set

$$S = \{[t^{g_1}, v^{g_2}], [t^{g_3}, w^{g_4}], [v^{g_5}, w^{g_6}] \mid g_i \in U\}$$

We have the following equalities (this can best be checked with the GAP package

<http://www.gap-system.org/Packages/automgrp.html>):

$$\begin{aligned}
h_1 &= [[(ab)^2, b], [b, (ca)^2]] = (t, *, 1, 1, 1, 1) \\
h_2 &= [[(ab)^2, b], [c, (da)^2]] = (v, 1, 1, 1, *, 1) \\
h_3 &= [[c, (ca)^2], [b, (da)^2]] = (w, 1, 1, 1, 1, *) \\
h_4 &= [[b, (ba)^2], [d, (ca)^2]] = (1, t, 1, *, 1, 1) \\
h_5 &= [[d, (ad)^2], [b, (ba)^2]] = (1, v, 1, 1, *, 1) \\
h_6 &= [[d, (ca)^2], [b, (da)^2]] = (1, w, 1, 1, 1, *) \\
h_7 &= [[c, (ba)^2], [d, (ca)^2]] = (1, 1, t, *, 1, 1) \\
h_8 &= [[d, (ba)^2], [c, (da)^2]] = (1, 1, v, 1, *, 1) \\
h_9 &= [[c, (ca)^2], [d, (da)^2]] = (1, 1, w, 1, 1, *)
\end{aligned}$$

where $*$ are elements of U not of importance. Clearly $h_i \in U'''$ for $i = 1, 2, 3, 4, 5, 6$. Given $g_1, g_2, g_3, g_4, g_5, g_6 \in U$, due the fact that U is self-replicating, there are elements $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6 \in U$ such that

$$\begin{aligned}
\gamma_1 &= (g_1, *, *, *, *, *) \\
\gamma_2 &= (g_2, *, *, *, *, *) \\
\gamma_3 &= (g_3, *, *, *, *, *) \\
\gamma_4 &= (g_4, *, *, *, *, *) \\
\gamma_5 &= (g_5, *, *, *, *, *) \\
\gamma_6 &= (g_6, *, *, *, *, *)
\end{aligned}$$

So,

$$\begin{aligned}
[h_1^{\gamma_1}, h_2^{\gamma_2}] &= ([t^{g_1}, v^{g_2}], 1, 1, 1, 1, 1) \\
[h_1^{\gamma_3}, h_3^{\gamma_4}] &= ([t^{g_3}, w^{g_4}], 1, 1, 1, 1, 1) \\
[h_2^{\gamma_5}, h_3^{\gamma_6}] &= ([v^{g_5}, w^{g_6}], 1, 1, 1, 1, 1)
\end{aligned}$$

and clearly left hand sides are elements of U''' . Using the fact that U is self-replicating we see that

$$U''' \succcurlyeq U''' \times 1 \times 1 \times 1 \times 1 \times 1.$$

Doing same thing in second and third coordinates and using other h_i we see that

$$U''' \succcurlyeq 1 \times U''' \times 1 \times 1 \times 1 \times 1$$

and

$$U''' \succcurlyeq 1 \times 1 \times U''' \times 1 \times 1 \times 1$$

and finally conjugating with a we also have

$$\begin{aligned}
U''' &\succcurlyeq 1 \times 1 \times 1 \times U''' \times 1 \times 1 \\
U''' &\succcurlyeq 1 \times 1 \times 1 \times 1 \times U''' \times 1 \\
U''' &\succcurlyeq 1 \times 1 \times 1 \times 1 \times 1 \times U'''
\end{aligned}$$

which shows that

$$U''' \succcurlyeq U''' \times U''' \times U''' \times U''' \times U''' \times U'''.$$

Clearly U''' is non-trivial since U has non-solvable quotients. \square

Note that U/U''' maps onto the group $\tilde{G}_{000\dots}$ and hence is infinite. Also, U cannot have a branch type action (on any rooted tree) since all non-trivial quotients of branch groups are virtually abelian, a fact proven in [Gri00].

4.3. Branch structure of general universal groups.

In this subsection we will investigate the branch structure of arbitrary universal groups.

For $\omega \in \Omega$ we have an injection

$$\begin{aligned} \phi_\omega : G_\omega &\rightarrow S_2 \times (G_{\tau\omega} \times G_{\tau\omega}) \\ a &\mapsto (01)(1, 1) \\ b_\omega &\mapsto (\beta(\omega_0), b_{\tau\omega}) \\ c_\omega &\mapsto (\zeta(\omega_0), c_{\tau\omega}) \\ d_\omega &\mapsto (\delta(\omega_0), d_{\tau\omega}) \end{aligned}$$

For subgroups $H \leq G_\omega$ and $H \leq G_{\tau\omega}$ let us write $K \times K \preceq H$ if $K \times K \leq \phi_\omega(H)$. Note that this means H contains a subgroup isomorphic to $K \times K$.

Proposition 4. *For $\omega \in \Omega$ we have $G_{\tau\omega}''' \times G_{\tau\omega}''' \preceq G_\omega'''$*

Proof. Let us assume that $\omega_0 = 0$. Define

$$\pi : U \times U \times U \times U \times U \times U \rightarrow G_\omega \times G_\omega$$

by $\pi(u_1, u_2, u_3, u_4, u_5, u_6) = (\psi_\omega(u_1), \psi_\omega(u_4))$ where ψ_ω is as defined in section 4.1. Let $\phi : U \rightarrow S_6 \times U \times U \times U \times U \times U$ be the canonical map.

Then the following diagram commutes.

$$\begin{array}{ccc} St_U(1) & \xrightarrow{\phi} & U \times U \times U \times U \times U \times U \\ \downarrow \psi_\omega & & \downarrow \pi \\ St_{G_\omega}(1) & \xrightarrow{\phi_\omega} & G_{\tau\omega} \times G_{\tau\omega} \end{array}$$

By Theorem 5, we have $U''' \times U''' \times U''' \times U''' \times U''' \times U''' \preceq U'''$. Since $\psi_\omega(U''') = G_\omega'''$ we see that $G_{\tau\omega}''' \times G_{\tau\omega}''' \preceq G_\omega'''$.

The case when $\omega_0 = 1$ or $\omega_0 = 2$ can be proven similarly by modifying π . □

Corollary 1. *For $\omega \in \Omega_\infty$, G_ω is a branch group.*

Proof. It follows by Proposition 4 and an induction argument that for any $n \geq 1$ we have

$$\prod_1^{2^n} G_{\tau^n \omega}''' \preceq G_\omega'''$$

It follows that for any $n \geq 1$, $\prod_1^{2^n} G_{\tau^n \omega}''' \preceq Rist_{G_\omega}(n)$. Note that for any $\omega \in \Omega \setminus \Omega_0$, G_ω''' is nontrivial (since G_ω is not solvable) and also have finite index (since G_ω are just-infinite.) It follows that $Rist_{G_\omega}(n)$ has finite index for all $n \geq 1$. □

For a non-empty subset $\Lambda \subset \Omega$, let us denote the universal group corresponding to the family $\{(G_\omega, S_\omega) \mid \omega \in \Lambda\}$ by U_Λ . Given $\Lambda \subset \Omega$ let $T_\Lambda = \bigcup_{\omega \in \Lambda} T_\omega$ and note that T_Λ is a (not necessarily regular) subtree of T_6 . Also note that T_Λ is U invariant (since each T_ω is so) and the restriction of U onto T_Λ gives the universal group U_Λ .

Proposition 5. *If $\Lambda \subset \Omega \setminus \Omega_0$ then with the action onto T_Λ , U_Λ is a weakly near branch group.*

Proof. Let $v \in T_\Lambda$ and let $v \in T_\omega$ for some $\omega \in \Lambda$. Let g be a non-trivial element of $\text{Rist}_{G_\omega}(v)$. Then by the proof of Proposition 4, there exists $h \in \text{Rist}_U(v)$ such that $\psi_\omega(h) = g$. The restriction of h onto T_Λ gives a non-trivial element in $\text{Rist}_{U_\Lambda}(v)$. \square

5. UNIVERSAL GROUPS OF INTERMEDIATE GROWTH

The aim of this section is to show that there exists a subset $\Lambda \subset \Omega$ of cardinality 2^{\aleph_0} such that U_Λ has intermediate growth. This fact was first established in [Gri84a], we fix some inaccuracies in the proof of this fact.

First, let us briefly recall basic notions related to the growth of groups. We refer to [dlH00, Man12, Gri13] for a detailed account on growth and related topics.

Let G be a finitely generated group and S a finite generating set. The length of an element (with respect to S) is given by $\ell_S(g) = \min\{n \mid g = s_1 s_2 \dots s_n, i \in S^\pm\}$. The growth function of G (with respect to S) is $\gamma_{G,S}(n) = \#B(G, S, n)$ where $B(G, S, n) = \{g \in G \mid \ell_S(g) \leq n\}$ is the ball of radius n . For two increasing functions f_1, f_2 defined on the set of natural numbers, let us write $f_1 \preceq f_2$ if there exists $C > 0$ such that $f_1(n) \leq f_2(Cn)$ for all n . Let us also write $f_1 \sim f_2$ if $f_1 \preceq f_2$ and $f_2 \preceq f_1$, which defines an equivalence relation. It can be observed that the growth functions of a group with respect to different finite generating sets are \sim equivalent and hence the asymptotic behavior of the growth functions of a group is an invariant of the group.

There are three types of growth for groups: If $\gamma_G \preceq n^d$ for some $d \geq 0$ then G is said to be of polynomial growth, if $\gamma_G \sim e^n$ then it is said to have exponential growth. If neither of this happens then the group is said to have *intermediate growth*.

If we are talking about the growth of a marked group (G, S) , we will simply write γ_G for the growth function of G with respect to S .

Lemma 4. *Let $F = \{(G_i, S_i) \mid i \in I\} \subset \mathcal{M}_k$ be a non-empty subset. Denote by γ_F the growth function of the diagonal group $(U_F^{\text{diag}}, S_F^{\text{diag}})$ of Definition 3. Then*

- (1) *For all $i \in I, \gamma_F(n) \geq \gamma_i(n)$ for all n ,*
- (2) *If I is finite then, $\gamma_F(n) \leq \prod_{i \in I} \gamma_i(n)$ for all n .*

Proof. In general, if (H, K) is a marked image of (G, S) , then $\gamma_G(n) \geq \gamma_H(n)$ for every n . Since all (G_i, S_i) are marked images of the diagonal group, we obtain the first assertion. For the second assertion, observe that $B(U_F^{diag}, S_F^{diag}, n) \subset \prod_{i \in I} B(G_i, S_i, n)$. \square

For a positive integer M let $\Omega_M \subset \Omega_\infty$ be the set of all sequences for which every subword of length M contains all symbols $0, 1, 2$.

Theorem 6. [Gri84b, Theorem 3.3] *There exist constants C and $\alpha < 1$ depending only on M , such that if $\omega \in \Omega_M$ then*

$$\gamma_{G_\omega}(n) \leq C^{n^\alpha} \text{ for all } n.$$

Given natural numbers r_1, \dots, r_k let

$$\Lambda_{r_1, \dots, r_k} = \{(012)^{r_1} \eta_1 (012)^{r_2} \eta_2 \dots (012)^{r_k} \eta_k (012)^\infty \mid \eta_i \in \{0, 1, 2\}\} \subset \Omega.$$

where $(012)^\infty$ stands for the periodic sequence $012012012\dots$

For a sequence of natural numbers $\mathbf{r} = \{r_k\}$, let

$$\Lambda_{\mathbf{r}} = \{(012)^{r_1} \eta_1 (012)^{r_2} \eta_2 \dots (012)^{r_k} \eta_k \dots \mid \eta_i \in \{0, 1, 2\}\} \subset \Omega.$$

Note that both $\Lambda_{r_1, \dots, r_k}$ and $\Lambda_{\mathbf{r}}$ are subsets of Ω_4 . Let us denote the universal groups $U_{\Lambda_{r_1, \dots, r_k}}$ and $U_{\Lambda_{\mathbf{r}}}$ by U_{r_1, \dots, r_k} and $U_{\mathbf{r}}$ respectively. Let γ_{r_1, \dots, r_k} and $\gamma_{\mathbf{r}}$ denote the growth functions (with respect to the canonical generating sets) of U_{r_1, \dots, r_k} and $U_{\mathbf{r}}$ respectively.

Lemma 5. *Given natural numbers r_1, \dots, r_k , there exists a natural number m such that*

$$\gamma_{r_1, \dots, r_k, x}(m) \leq \left(1 + \frac{1}{k}\right)^m \text{ for any } x \in \mathbb{N}.$$

Proof. Since $\Lambda_{r_1, \dots, r_k, x} \subset \Omega_4$, by Theorem 6 there exists C and $\alpha < 1$ (not depending on x) such that for all $\omega \in \Lambda_{r_1, \dots, r_k, x}$ we have

$$\gamma_\omega(n) \leq C^{n^\alpha} \text{ for all } n.$$

Therefore, by Lemma 4 (using the fact that $|\Lambda_{r_1, \dots, r_k, x}| = 3^{k+1}$) we have

$$\gamma_{r_1, \dots, r_k, x}(n) \leq (C^{n^\alpha})^{3^{k+1}} = D^{n^\alpha} \text{ for all } n$$

where $D = C^{3^{k+1}}$ does not depend on x . Therefore there exists a natural number m such that

$$\gamma_{r_1, \dots, r_k, x}(m) \leq \left(1 + \frac{1}{k}\right)^m \text{ for any } x \in \mathbb{N}.$$

\square

Lemma 6. [Gri84a, Lemma 3] *Let $\mathbf{r} = \{r_k\}$ be a sequence of natural numbers. If for some k*

$$k + r_1 + r_2 + \dots + r_k \geq \log_2 2n$$

then $\gamma_{r_1, \dots, r_k}(n) = \gamma_{\mathbf{r}}(n)$.

Theorem 7. [Gri84a, Theorem 1] *There exists a sequence $\mathbf{r} = \{r_k\}$ such that $U_{\mathbf{r}}$ has intermediate growth.*

Proof. Let $r_1 = 1$. By Lemma 5, there exists a natural number n_1 such that

$$\gamma_{r_1, x}(n_1) \leq \left(1 + \frac{1}{1}\right)^{n_1} \text{ for any } x.$$

Choose r_2 such that $2 + r_1 + r_2 \geq \log_2 2n_1$. Again by Lemma 5 there exists $n_2 > n_1$ such that

$$\gamma_{r_1, r_2, x}(n_2) \leq \left(1 + \frac{1}{2}\right)^{n_2} \text{ for any } x.$$

Assume r_1, \dots, r_k has been already chosen. By Lemma 5, there exists $n_k > n_{k-1}$ such that

$$(3) \quad \gamma_{r_1, \dots, r_k, x}(n_k) \leq \left(1 + \frac{1}{k}\right)^{n_k} \text{ for any } x.$$

Choose r_{k+1} such that

$$(4) \quad k + 1 + r_1 + \dots + r_{k+1} \geq \log_2 2n_k.$$

Continuing in this manner we construct sequences $\mathbf{r} = \{r_k\}$ and $\{n_k\}$ for which Equations 3 and 4 are satisfied. Lemma 6 and Equation 4 shows that for all k we have

$$\gamma_{r_1, \dots, r_{k+1}}(n_k) = \gamma_{\mathbf{r}}(n_k).$$

Using this and Equation 3 we have,

$$\lim_{n \rightarrow \infty} \gamma_{\mathbf{r}}(n)^{\frac{1}{n}} = \lim_{k \rightarrow \infty} \gamma_{\mathbf{r}}(n_k)^{\frac{1}{n_k}} = \lim_{k \rightarrow \infty} \gamma_{r_1, \dots, r_{k+1}}(n_k)^{\frac{1}{n_k}} \leq \lim_{k \rightarrow \infty} \left(1 + \frac{1}{k}\right) = 1$$

□

Corollary 2. *There exists a finitely generated group of intermediate growth with 2^{\aleph_0} non-isomorphic homomorphic images.*

As mentioned in the beginning of this section, this fact was established in [Gri84a] with a small inaccuracy. Our proof mainly follows the lines of [Gri84a] only difference being that one needs Lemma 5.

6. INVARIANT RANDOM SUBGROUPS OF UNIVERSAL GROUPS

The aim of this section is to show that there are universal groups with many invariant random subgroups.

6.1. Preliminaries About Invariant Random Subgroups.

Let G be a countable group and let $S(G)$ be the space of subgroups of G endowed with the topology having sets of the form $\mathcal{O}_{A,B} = \{N \leq G \mid A \subset N, B \cap N = \emptyset\}$ where A, B are finite subsets of G , as basis.. $S(G)$ can be identified with a closed subspace of $\{0, 1\}^G$ supplied with by the topology induced from the Tychonoff topology. The group G acts on $S(G)$ by conjugation and hence forming a topological dynamical system $(G, S(G))$. We are interested in dynamical system of the form $(G, S(G), \mu)$ where μ is a conjugation invariant probability measure on $S(G)$.

Definition 7. A conjugation invariant Borel probability measure on $S(G)$ is called an invariant random subgroup (IRS in short).

The space $S(G)$ is a compact, metrizable, totally disconnected space which (applying the Cantor-Bendixon procedure [Kec95, I.6]) consists of a perfect kernel $\kappa(G)$ and its complement $S(G) \setminus \kappa(G)$ which is countable. The perfect kernel $\kappa(G)$ is either empty or is homeomorphic to a Cantor set, and it is empty if and only if $S(G)$ is countable, that is G has only countably many subgroups. This is the case, for instance, for finitely generated virtually nilpotent groups, virtually polycyclic groups, some metabelian groups like Baumslag-Solitar groups $B(1, n)$, or Tarski monsters [Ol'80].

As $\kappa(G)$ is an invariant subset of $S(G)$ with respect to the action of $Aut(G)$ and as the complement $S(G) \setminus \kappa(G)$ is countable, it is clear that a continuous IRS has law μ supported on $\kappa(G)$.

Given a subgroup $L \leq G$, let $S(G, L) \subset S(G)$ be the set of subgroups containing L , which clearly is closed. Note that, if L is a normal subgroup of G , then $S(G, L)$ is invariant under the action of G .

Let $\varphi : G \longrightarrow H$ be a homomorphism. It induces two maps

$$\begin{aligned} \bar{\varphi} : S(G) &\longrightarrow S(H) \\ N &\longmapsto \varphi(N) \end{aligned}$$

and

$$\begin{aligned} \tilde{\varphi} : S(H) &\longrightarrow S(G, Ker(\varphi)) \\ K &\longmapsto \varphi^{-1}(K) \end{aligned}$$

Lemma 7.

- (1) $\bar{\varphi}$ is Borel.
- (2) $\tilde{\varphi}$ is continuous.
- (3) $\tilde{\varphi}(K^{\varphi(g)}) = \tilde{\varphi}(K)^g$ for all $g \in G$ and $K \leq H$.
- (4) $\tilde{\varphi}^{-1}(C^g) = \tilde{\varphi}^{-1}(C)^{\varphi(g)}$ for all $g \in G$ and $C \subset S(G, Ker(\varphi))$.
- (5) If φ is surjective, then $\tilde{\varphi}$ is a homeomorphism.

Proof.

(1) We claim that

$$\bar{\varphi}^{-1}(\mathcal{O}_{A,B}) = \bigcap_{a \in A} \bigcup_{x \in \varphi^{-1}(a)} \bigcap_{y \in \varphi^{-1}(B)} \mathcal{O}_{\{x\},\{y\}}$$

where A, B are finite subsets of H .

If $\varphi(N) \in \mathcal{O}_{A,B}$ then $A \subset \varphi(N)$ and $B \cap \varphi(N) = \emptyset$. This shows that for any $a \in A$ there exists $n_a \in N$ such that $\varphi(n_a) = a$. Also, for all $y \in \varphi^{-1}(B)$ we have $y \notin N$. Hence N belongs to the right hand side.

Conversely, let $N \leq G$ belong to the right hand side. This means that for all $a \in A$ there exists $n_a \in \varphi^{-1}(a)$ such that for all $y \in \varphi^{-1}(B)$ we have $N \in \mathcal{O}_{\{n_a\},\{y\}}$. For any $a \in A$, we have $\varphi(n_a) = a$ and hence $A \subset \varphi(N)$. Also, if $B \cap \varphi(N)$ is nonempty, then the set $N \cap \varphi^{-1}(B)$ is nonempty which is not true. Hence $\varphi(N) \in \mathcal{O}_{A,B}$.

Note that in general $\bar{\varphi}$ is not continuous. For example, the sequence of subgroups $(2n+1)\mathbb{Z}$, $n \geq 1$ of \mathbb{Z} converge to the trivial subgroup, but their images in \mathbb{Z}_2 converge to the whole group.

- (2) We claim that $\tilde{\varphi}^{-1}(\mathcal{O}_{C,D}) = \mathcal{O}_{\varphi(C),\varphi(D)}$ where C, D are finite subsets of G . In fact, if $\tilde{\varphi}(K) \in \mathcal{O}_{C,D}$ for some $K \leq H$, then $C \subset \varphi^{-1}(K)$ and $D \cap \varphi^{-1}(K) = \emptyset$. It follows that $\varphi(C) \subset K$ and $\varphi(D) \cap K = \emptyset$. This shows that $K \in \mathcal{O}_{\varphi(C),\varphi(D)}$. Conversely, if $K \in \mathcal{O}_{\varphi(C),\varphi(D)}$ for some $K \leq H$, then $\varphi(C) \subset K$ and $D \cap \varphi(K) = \emptyset$. It follows that $C \subset \varphi^{-1}(K)$ and $D \cap \varphi^{-1}(K) = \emptyset$ and hence $\tilde{\varphi}(K) = \varphi^{-1}(K) \in \mathcal{O}_{C,D}$.
- (3) This can be verified directly.
- (4) This follows from part (2).
- (5) If φ is surjective, then clearly $\tilde{\varphi}$ is bijective. Since $S(H)$ is compact, it follows that $\tilde{\varphi}$ is a homeomorphism, □

Corollary 3. *If μ is an IRS of H then the measure $\nu = \tilde{\varphi}_*(\mu)$ is an IRS of G supported on the set $\{\varphi^{-1}(K) \mid K \in \text{supp}(\mu)\}$. If moreover μ is continuous, ergodic with respect to the action of H and φ is surjective, then ν is continuous and ergodic with respect to the action of G .*

Proof. The first part is immediate consequence of Lemma 7 parts (1) and (3). Note that the measure $\tilde{\varphi}_*(\mu)$ is defined on the closed subset $S(G, \text{Ker}(\varphi))$ of $S(G)$, and hence can be considered as a measure on $S(G)$ with support in $S(G, \text{Ker}(\varphi))$. Suppose that μ is continuous, ergodic and φ is surjective. Since $\tilde{\varphi}$ is a homeomorphism the measure ν is continuous. Let $C \subset S(G, \text{Ker}(\varphi))$ be G -invariant. Given $h \in H$, pick $g \in G$ such that $\varphi(g) = h$. By Lemma 7 part (3), $\tilde{\varphi}^{-1}(C)^h = \tilde{\varphi}^{-1}(C)^{\varphi(g)} = \tilde{\varphi}^{-1}(C^g) = \tilde{\varphi}^{-1}(C)$. Therefore $\tilde{\varphi}^{-1}(C)$ is H invariant, from which it follows that $\nu(C) = \mu(\tilde{\varphi}^{-1}(C)) \in \{0, 1\}$. □

Proposition 6. *Let X be a metrizable Hausdorff topological space and let μ be a Borel measure on X . Suppose also that a group G acts on the Borel space (X, μ) by measure preserving transformations. Then the map $St : X \rightarrow S(G)$ given by $x \mapsto St_G(x)$ is Borel. Moreover, the measure $\nu = St_*(\mu)$ is an IRS supported on $\{St_G(x) \mid x \in X\}$.*

Proof. Observe that the Borel σ -algebra on $S(G)$ is generated by sets of the form $\mathcal{O}_g = \{N \leq G \mid g \in N\}$. Also observe that $St^{-1}(\mathcal{O}_g) = Fix(\varphi_g)$ where $\varphi_g : X \rightarrow X$ given by $\varphi_g(x) = g.x$. Therefore $St^{-1}(\mathcal{O}_g)$ is a Borel set (see e.g. [Par05, Chapter 1] on sections of Borel maps). This shows that the measure $\nu = St_*(\mu)$ is a Borel measure on $S(G)$ with support $\{St_G(x) \mid x \in X\}$. The relation $St_G(g.x) = St_G(x)^{g^{-1}}$ and the G invariance of μ show that ν is conjugation invariant. \square

It is known (see [AGV12]) that every IRS of a finitely generated group arises from a measure preserving action on a Borel probability space (X, μ) .

If T_d is the rooted d -ary tree, its boundary ∂T_d is the set of all infinite rays emanating from the root vertex. ∂T_d is in bijection with infinite sequences over the alphabet $\{0, 1, \dots, d-1\}$ and hence homeomorphic to a Cantor Set. If G is a group of automorphisms of a rooted tree T_d , its action on T_d extends to an action onto the boundary ∂T_d and this action is by homeomorphisms. Let μ be the uniform Bernoulli measure on ∂T_d , (i.e., the product of uniform measures on the set $\{0, 1, \dots, d-1\}$). Observe that μ is continuous and invariant under the action of $Aut(T_d)$ and hence invariant under the action of any subgroup $G \leq Aut(T_d)$. Regarding the the dynamics of such actions the following is known:

Proposition 7. [Gri11] *Let G be a countable group of automorphisms of a regular rooted tree T_d . Then, the following are equivalent:*

- (1) *the group G acts transitively on the levels of T_d ,*
- (2) *the action of G on ∂T_d is minimal (i.e., orbits are dense),*
- (3) *the action of G on ∂T_d is ergodic with respect to the uniform Bernoulli measure on ∂T_d .*
- (4) *the action is uniquely ergodic.*

An action of weakly branch type on T gives a totally non-free action on the boundary ∂T .

Proposition 8. [BG02, Gri11] *Let $G \leq Aut(T)$ be weakly branch. Then the map $St : \partial T \rightarrow S(G)$ given by $\xi \mapsto St_G(\xi)$ is injective.*

Proof. Let $\xi, \eta \in \partial T$ be distinct elements and let u, v be distinct prefixes of ξ and η respectively, of same length, say, n . Infinite sequences starting with u (respectively with v) form a neighbourhood of ξ (respectively of η) in ∂T . We will show that the stabilizer of the neighbourhood of η (which is a subgroup in $St_G(\eta)$) is not contained in $St_G(\xi)$.

Let u and v be distinct prefixes of length n of ξ and η respectively. Let $g \in \text{Rist}_G(u)$ be nontrivial. Since v is not contained in the subtree T_u , g fixes every infinite sequence starting with v . Since g is nontrivial it moves some vertex in $uu_1 \in T_u$, say $g(uu_1) = uu_2$ for some $u_1 \neq u_2$ of lengths m . Let uu' be the prefix of ξ of length $n + m$.

If $u' = u$ or $u' = u_2$, then $g(uu') \neq uu'$ and hence $g \notin \text{St}_G(\xi)$. If both $u' \neq u_1$ and $u' \neq u_2$, by level transitivity let $h \in G$ such that $h(uu_1) = uu'$. Then

$$(hgh^{-1})(uu') = (hg)(uu_1) = h(uu_2) \neq uu'$$

because $u_1 \neq u_2$. Therefore $hgh^{-1} \notin \text{St}_G(\xi)$. Since $h(uu_1) = uu_2$, we have $h \in \text{St}_G(u)$ and hence $hgh^{-1} \in \text{Rist}_G(u)$. It follows that $hgh^{-1} \in \text{St}_G^\circ(\eta)$. \square

As explained in Introduction, this readily provides us with a continuous ergodic *IRS* on G . See for example [DDMN10] for a detailed study of this and related measures on the space of Schreier graphs of the Basilica group.

Regarding the action of the Grigorchuk groups $G_\omega, \omega \in \Omega$ on the boundary ∂T_2 of the binary tree we obtain the following.

Proposition 9. *For $\omega \in \Omega$ the action of G_ω on T_2 is level transitive and hence the action of G_ω on $(\partial T_2, \mu)$ is ergodic. Therefore, the induced *IRS* on G_ω is continuous and ergodic.*

Proof. By Proposition 6 the action of G_ω on $(\partial T_2, \mu)$ induces an *IRS* on G_ω . This *IRS* will be continuous by Proposition 8 and ergodic by Proposition 7. \square

6.2. *IRS* on universal groups.

Given $\omega_1, \omega_2 \in \Omega$, let us write $\omega_1 \sim \omega_2$ if there exists $\sigma \in \text{Sym}(\{0, 1, 2\})$ such that ω_2 is obtained from ω_1 by application of σ to each letter of ω_1 . Recall that by Theorem 1 part (7) we have that for $\omega_1, \omega_2 \in \Omega_\infty$, $G_{\omega_1} \cong G_{\omega_2}$ if and only if $\omega_1 \sim \omega_2$.

For a subset $\Lambda \subset \Omega$ let $|\Lambda|_\sim$ denote the cardinality of the set of \sim equivalence classes in Λ .

Proposition 10. *For $\Lambda \subset \Omega_\infty$, U_Λ has at least $|\Lambda|_\sim$ distinct continuous, ergodic invariant random subgroups.*

Proof. Fix $\Lambda \subset \Omega_\infty$. Let $\varphi_\omega : U_\Lambda \rightarrow G_\omega$ be the canonical surjection and let $N_\omega = \text{Ker}(\varphi_\omega)$. Note that if $\omega \approx \eta$, then by Theorem 1 part (7) and the fact that G_η is just infinite, we have $N_\eta \not\subseteq N_\omega$. For $\omega \in \Omega$ and $\xi \in \partial T_2$ let $W_{\omega, \xi} = \text{St}_{G_\omega}(\xi)$. By Proposition 9, the canonical action of G_ω onto $(\partial T_2, \mu)$ induces a continuous, ergodic *IRS* μ_ω on G_ω . Moreover, μ_ω is supported on $\{W_{\omega, \xi} \mid \xi \in \partial T_2\}$.

Let ν_ω denote the induced *IRS* on U_Λ obtained as described in Corollary 3 (i.e., $\nu_\omega = (\tilde{\varphi}_\omega)_*(\mu_\omega)$). Again by Corollary 3, ν_ω is continuous and ergodic. Let $L_{\omega, \xi} = \varphi_\omega^{-1}(W_{\omega, \xi})$ and note that ν_ω is supported on $Y_\omega = \{L_{\omega, \xi} \mid \xi \in \partial T_2\}$. Observe that for all $\xi \in \partial T_2$, $L_{\omega, \xi}$ contains N_ω .

Suppose that for some $\omega \approx \eta \in \Lambda$ and $\xi, \rho \in \partial T_2$ we have $L_{\omega, \xi} = L_{\eta, \rho}$. Then $N_\omega, N_\eta \leq L_{\omega, \xi}$ and hence $L_{\omega, \xi}$ contains the subgroup $N = N_\omega N_\eta$. Since $N_\eta \not\leq N_\omega$, N contains N_ω as a proper subgroup. It follows that the group U_Λ/N is a proper quotient of the group $U_\Lambda/N_\omega \cong G_\omega$. Since G_ω is a just infinite group it follows that N and hence $L_{\omega, \xi}$ has finite index in U_Λ . This, in turn shows that $St_{G_\omega}(\xi)$ has finite index in G_ω which is a contradiction. Therefore if $\omega \approx \eta$ we see that the measures ν_ω and ν_η have disjoint supports and are in particular distinct. \square

Combining this with results from Section 5 we obtain the main theorem:

Main Theorem. *There is a subset $\Lambda \subset \Omega$ such that the corresponding universal group U_Λ has intermediate growth and has 2^{\aleph_0} distinct continuous ergodic invariant random subgroups.*

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MUSTAFA GÖKHAN BENLİ, MIDDLE EAST TECHNICAL UNIVERSITY, ANKARA, TURKEY
E-mail address: benli@metu.edu.tr

ROSTISLAV GRIGORCHUK, TEXAS A&M UNIVERSITY, COLLEGE STATION, TX, USA
E-mail address: grigorch@math.tamu.edu

TATIANA NAGNIBEDA, SECTION DE MATHÉMATIQUES, UNIVERSITÉ DE GENÈVE, C.P. 64,
 CH-1211 GENÈVE 4, SUISSE.
E-mail address: tatiana.smirnova-nagnibeda@unige.ch