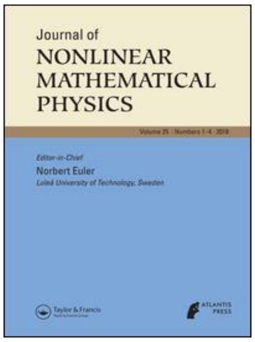

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Time-Dependent Recursion Operators and Symmetries

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Abstract

The recursion operators and symmetries of nonautonomous, $(1+1)$ dimensional integrable evolution equations are considered. It has been previously observed that the symmetries of the integrable evolution equations obtained through their recursion operators do not satisfy the symmetry equations. There have been several attempts to resolve this problem. It is shown that in the case of time-dependent evolution equations or time-dependent recursion operators associativity is lost. Due to this fact such recursion operators need modification. A general formula is given for the missing term of the recursion operators. Apart from the recursion operators a method is introduced to calculate the correct symmetries. For illustrations several examples of scalar and coupled system of equations are considered.

1 Introduction

Time-dependent local and nonlocal symmetries for autonomous and nonautonomous integrable equations have been extensively studied in literature [1]–[24]. In a recent paper [1] by Sanders and Wang it was observed that time-dependent recursion operators associated with some integrable (1+1) dimensional evolution equations do not always generate the higher order symmetries. They explained this fact by the violation of rule $D^{-1}D = 1$ where $D = D_x$. In order to overcome this problem, they presented a method for constructing symmetries of a given integrable evolution equation by a *corrected recursion operator* resulting from the *(weak)* standard one. In this paper we consider the work of [1], but from a new point of view. In fact, we show that an elegant way of understanding this problem is through the action of D^{-1} on arbitrary functions depending on dependent and independent variables and the structure of symmetries of equations. On the other hand we investigate the behavior of recursion operator under a simple Lie point transformation which links evolution equations. For instance the cylindrical Korteweg-de Vries (cKdV) equation is related to the Korteweg-de Vries (KdV) equation by a simple point transformation which allows also a direct construction of symmetries and recursion operators for cKdV from those of KdV. The properties of this transformation or *the principle of covariance* implies that recursion operators must keep their property of *mapping symmetries to symmetries*. The corresponding Lie point transformation maps symmetries of the KdV equation to the correct symmetries of the cKdV equation. On the other hand it maps the recursion operator to the weak one. This fact suggests that under this transformation $D^{-1} \to D^{-1} + h(t)$, where $h(t)$ is a time-dependent function to be determined. If such an integration constant is missed we loose simply the rule of associativity. As a simple example, let $R_0 = D^{-1}$, ${K'}_0 = D^2$, and $\sigma_0 = a_1(t) + a_2(t)x + a_3(t)x^2 + a_4(t)x^3 + \cdots$. Observe that $R_0\,({K'}_0(\sigma_0)) - (R_0\,{K'}_0)\,(\sigma_0) = -a_2 \neq 0$. This integration function can be determined either by using the definition of the recursion operator or the symmetry equation. In this work we use both approaches.

Most of the nonlinear integrable evolution equations, in $(1+1)$ dimensions, admit recursion operators which map symmetries to symmetries. Let A be the space of symmetries of an evolution equation. We assume all symmetries $\sigma \in A$ are differentiable. This space contains two types of functions. Let A_1 be a subset of A containing all functions which vanish in the limit when the jet coordinates go to zero and A_0 be a subset of A the elements of which do not vanish under such a limit. A recursion operator $\mathcal{R}$ is an operator which maps A into itself $\mathcal{R} : A \to A$. This may be implied by the eigenvalue equation $\mathcal{R}\sigma = \lambda\sigma$, where $\sigma \in A$ and λ is the spectral constant. The recursion operators are in general nonlocal operators and the usual characterization of such operators $\mathcal{R}$ of system of evolution equations

$$q^i_t = K[x, t, q^j], \qquad i, j = 1, 2, \ldots, N, \tag{1.1}$$

where K is a locally defined function of q^i and its x-derivatives, is given by the equation [5]

$$\mathcal{R}_t = [K', \mathcal{R}], \tag{1.2}$$

where the operator K' is the Frechét derivative of K. A function σ is called a symmetry of (1.1) if it satisfies the linearized equation

$$\sigma_t = K'\sigma. \tag{1.3}$$

The relation among the symmetries is given by

$$\sigma_{n+1} = \mathcal{R}\sigma_n, \qquad n = 0, 1, 2, \ldots, \tag{1.4}$$

which guarantees the integrability of the equation under study. We note that in [1], the operator $\mathcal{R}$ is called a *weak recursion operator* of (1.1) if it satisfies (1.2) using the rule $D^{-1}D = 1$. In calculating symmetries of an equation with K and recursion operator depending explicitly on x and t, we observed that the problem arises when the coefficient of D^{-1} in the recursion operators contains functions depending only on x and t (no q and its derivatives). As an example, where the function K and the recursion operator depend explicitly on x and t but the symmetries can be calculated as usual, we have

$$u_t = u_{3x} + \frac{1}{2t}uu_x - \frac{1}{2t}xu_x, \tag{1.5}$$

where the corresponding recursion operator is given by

$$\mathcal{R} = tD^2 + \frac{1}{3}u + \frac{1}{6}u_x D^{-1}. \tag{1.6}$$

Keeping the notion of covariance in mind and having the recursion operators found from (1.2) at hand we have to reconsider the action of the operator D^{-1} in the case of function space with elements that have explicit t and x dependencies. In the next section we discuss the principles of covariance by computing the symmetries and recursion operator of cKdV from those of KdV using an invertible Lie point transformation.

2 The link between KdV and cKdV

It is well known that the KdV and cKdV equations are equivalent since their solutions are related by a simple Lie point transformation [12, 13]. This transformation allows also a direct transfer of symmetries in an invariant way. Therefore the invertible point transformation,

$$\tau = -2t^{-1/2}, \qquad \xi = xt^{-1/2}, \qquad v = tu + \frac{1}{2}x, \tag{2.1}$$

takes the cKdV equation

$$u_t = u_{3x} + uu_x - \frac{u}{2t}$$

to the KdV equation

$$v_\tau = v_{\xi\xi\xi} + vv_\xi.$$

Thus we can derive the symmetries of cKdV from those of KdV using the transformation above. The relation between the symmetries of KdV and cKdV, from above transformation, is given as

$$\delta v = t\delta u. \tag{2.2}$$

The first four symmetries of KdV equation, generated by

$$\tau_n = \left(D_\xi^2 + \frac{2}{3}v + \frac{1}{3}v_\xi D_\xi^{-1}\right)^n v_\xi, \tag{2.3}$$

are given as follows:

$$\begin{aligned}
\tau_0 &= v_\xi, \\
\tau_1 &= v_{3\xi} + vv_\xi, \\
\tau_2 &= v_{5\xi} + \frac{10}{3}v_\xi v_{2\xi} + \frac{5}{3}vv_{3\xi} + \frac{5}{6}v^2 v_\xi, \\
\tau_3 &= v_{7\xi} + \frac{21}{3}v_\xi v_{4\xi} + \frac{7}{3}vv_{5\xi} + \frac{35}{3}v_{2\xi}v_{3\xi} \\
&\quad + \frac{70}{9}vv_\xi v_{2\xi} + \frac{35}{18}v^2 v_{3\xi} + \frac{35}{18}v_\xi^3 + \frac{35}{54}v^3 v_\xi.
\end{aligned} \tag{2.4}$$

We see, from (2.1), that differential operators are connected $D_\xi = t^{1/2} D_x$. Now, using (2.1) and (2.2) directly or the transformation [5] $\mathcal{R}_{\rm cKdV} = \chi_v \mathcal{R}_{\rm KdV} \chi_v^{-1}$, where $\chi_v = \frac{\delta u}{\delta v} = 1/t$ we transform the recursion operator of the KdV to the recursion operator of the cKdV

$$\mathcal{R} = tD_x^2 + \frac{2}{3}tu + \frac{1}{3}x + \frac{1}{6}(1 + 2tu_x)D_x^{-1}$$

and the first four symmetries of cKdV as

$$\begin{aligned}
\sigma_0 &= t^{1/2}u_x + \frac{1}{2}t^{-1/2},\\
\sigma_1 &= t^{3/2}u_t + t^{1/2}\left(u + \frac{x}{2}u_x\right) + t^{-1/2}\frac{x}{4},\\
\sigma_2 &= t^{5/2}\left(u_{5x} + \frac{5}{3}uu_{3x} + \frac{10}{3}u_x u_{2x} + \frac{5}{6}u^2 u_x\right)\\
&\quad + t^{3/2}\left(\frac{5}{6}xu_{3x} + \frac{5}{3}u_{2x} + \frac{5}{6}xuu_x + \frac{5}{12}u^2\right) + t^{1/2}\left(\frac{5}{12}xu + \frac{5}{24}x^2 u_x\right) + t^{-1/2}\frac{5x^2}{48},\\
\sigma_3 &= t^{7/2}\left(u_{7x} + \frac{7}{3}uu_{5x} + 7u_x u_{4x} + \frac{35}{18}u^2 u_{3x} + \frac{35}{3}u_{2x}u_{3x} + \frac{70}{9}uu_x u_{2x} + \frac{35}{18}u_x^3\right.\\
&\quad \left. + \frac{35}{54}u^3 u_x\right) + t^{5/2}\left(\frac{7}{6}xu_{5x} + \frac{7}{2}u_{4x} + \frac{35}{18}xuu_{3x} + \frac{35}{9}xu_x u_{2x} + \frac{35}{9}uu_{2x} + \frac{35}{12}u_x^2\right.\\
&\quad \left. + \frac{35}{36}xu^2 u_x + \frac{35}{108}u^3\right) + t^{3/2}\left(\frac{35}{72}x^2 u_{3x} + \frac{35}{18}xu_{2x} + \frac{35}{72}x^2 uu_x + \frac{35}{24}u_x\right.\\
&\quad \left. + \frac{35}{72}xu^2\right) + t^{1/2}\left(\frac{35}{432}x^3 u_x + \frac{35}{144}x^2 u + \frac{35}{144}\right) + t^{-1/2}\frac{35}{864}x^3. \qquad (2.5)
\end{aligned}$$

Here point transformations give the standard (weak) recursion operator and correct symmetries of the cKdV equation. Therefore this point transformation shows that adding a correction term to the recursion operators is necessary. It means that under point transformations the main property *mapping symmetries to symmetries* of the recursion operators should be kept invariant. This is the *covariance*. Recursion operators obtained from (1.2) do not in general obey this covariance principle. Another point is that the action of the operator D^{-1} in (ξ, τ) system (the KdV case) and in (t, x) system (the cKdV case) should not be the same. In the following sections we investigate the corrected recursion operators and the behavior of symmetries and develop a procedure for constructing corrected symmetries generated by weak time-dependent recursion operators.

3 Violation of associativity and the correct recursion operators

The symmetries $\sigma \in A$ of the equation (1.1) obey the evolution equation (1.3) and the recursion operator maps symmetries to symmetries, i.e., $\mathcal{R}\sigma = \lambda\sigma$. We assume that a weak recursion operator satisfying (1.2) takes the following form $\mathcal{R}_w = R_1 + aD^{-1}$, where R_1 is the local part of the recursion operator and a is a function of jet coordinates and x and t. This is a specific example, a recursion operator may take more complicated nonlocal terms

(Example 8). Now we let $\mathcal{R} = \mathcal{R}_w + \frac{a}{g}H$, where H is an operator and the function g is chosen so that a/g is a symmetry. Hence the eigenvalue equation becomes

$$\mathcal{R}_w\sigma + \frac{a}{g}H\sigma = \lambda\sigma. \tag{3.1}$$

Taking the time derivative of the eigenvalue equation and paying attention to the order of the parenthesis we obtain

$$\mathcal{R}_{wt}(\sigma) + \mathcal{R}_w\,(K'(\sigma)) + (a/g)_t H\sigma + (a/g)(H\sigma)_t = K'(\mathcal{R}_w(\sigma) + (a/g)H(\sigma)). \tag{3.2}$$

Since, in general, $\mathcal{R}_w$ contains D^{-1} one should be careful about the parenthesis. For this reason we rewrite the above equation as

$$a\,(H(\sigma))_t + g[\mathrm{As}\,(\mathcal{R}_w, K', \sigma) - \mathrm{As}\,(K', \mathcal{R}_w, \sigma)] = 0, \tag{3.3}$$

where $\mathrm{As}\,(P, Q, \sigma) = P(Q(\sigma)) - (PQ)(\sigma)$ for any operators P, Q and for any $\sigma \in A$. The above equation is the general form of the additional (correction) term.

For local cases the associators $\mathrm{As}\,(A, B, \sigma)$ vanish identically. To correct the symmetries one needs to add only a time-dependent constant, hence the operator H contains a projection operator Π such that $\Pi\sigma = \operatorname*{Lim}_{x,q,q_x,\cdots\to 0} \sigma =$ a time-dependent function. Hence (3.3) reduces to

$$(H\sigma_0)_t + g\left[\mathrm{As}\left(D^{-1}, K_0', \sigma_0\right) - \mathrm{As}\left(K_0', D^{-1}, \sigma_0\right)\right] = 0, \tag{3.4}$$

where σ_0 is the part of the symmetries which depends only on x and t and $K_0' = \operatorname*{Lim}_{q,q_x,\cdots\to 0} K'$.

Example 1 (Burgers equation). $u_t = K[u] = u_{2x} + uu_x$, $K_0' = D^2$. For the well known recursion operator $\mathcal{R} = D + \frac{u}{2} + \frac{u_x}{2}D^{-1}$ there is no problem, $\mathcal{R} = \mathcal{R}_w$. But if we choose the recursion operator $\mathcal{R}_w = tD + \frac{t}{2}u + \frac{1}{2}x + \frac{1}{2}(1 + tu_x)D^{-1}$, there is a problem in the calculation of symmetries. Here $a = \frac{1}{2}(1 + tu_x)$. Since a/g is a symmetry, then g must take the value $g = 1$. Let $\sigma_0 = a_1(t) + a_2(t)x + a_3(t)x^2 + \cdots$ then (3.4) becomes

$$(H\sigma_0)_t = a_2. \tag{3.5}$$

Hence

$$H = D_t^{-1}\,\Pi D. \tag{3.6}$$

Example 2 (cKdV equation). $u_t = K[u] = u_{xxx} + uu_x - \frac{u}{2t}$, $K_0' = D^3 - \frac{1}{2t}$. The recursion operator is $\mathcal{R}_w = tD^2 + \frac{2}{3}tu + \frac{1}{3}x + \frac{1}{6}(1 + 2tu_x)D^{-1}$. Here $a = \frac{1}{6}(1 + 2tu_x)$, and $g = \sqrt{t}$. Using the same ansatz for σ_0 as in the above example we obtain

$$(H\sigma_0)_t = 2g(t)a_3. \tag{3.7}$$

This means that

$$H = D_t^{-1}\sqrt{t}\Pi\, D^2. \tag{3.8}$$

The results are compatible with symmetry calculations in the following sections and also with [1]. One might generalize the formula (3.4) for more complicated evolution equations with $\operatorname*{Lim}_{q\to 0} K' \neq 0$.

4 Construction of symmetries

Firstly we look at the action of D^{-1} on local functions. Let $G \in A_1$. Then we take $D^{-1}G_x = G$ and let $H \in A_0$. Then we take $D^{-1}H_x = H + h(t)$, where h is a function of t. We start with the following definition.

Definition 1. Let $\mathcal{R}_w$ be a recursion operator of the form

$$\mathcal{R}_w = R_1 + R_0, \tag{4.1}$$

where $R_0 = \mathcal{R}_w|_{q\to 0}$. Here and in the sequel $q \to 0$ means all the derivatives of q also go to zero (jet coordinates vanish). Similarly, let σ_n be symmetries of (1.1), generated by the $\mathcal{R}_w$, of the form

$$\sigma_n = \sigma_n^1 + \sigma_n^0, \tag{4.2}$$

where $\sigma_n^0 = \sigma_n|_{q=0}$.

At this point we need the following proposition.

Proposition 1. *Let the function K vanish in the limit when the jet coordinates go to zero, i.e. $\lim\limits_{q\to 0} K = 0$. Then the operator $R_0 = \lim\limits_{q\to 0} \mathcal{R}_w$ satisfies $\sigma_{n+1}^0 = R_0\sigma_n^0$ and $R_{0t} = [K_0', R_0]$ where K_0' is obtained from (1.1) by $K_0' = \lim\limits_{q\to 0} K'$.*

We omit the proof because it is straightforward. The difference between the weak symmetries (the ones obtained from $\mathcal{R}_w$) and the corrected symmetries comes from σ_0 part of the symmetries. For this purpose this proposition will play an important role in the calculation of the missing terms in the symmetries. When we find the correction term $h(t)$ for σ_0 the general corrected symmetry σ takes the form

$$\sigma = \bar{\sigma} + \frac{a}{g}h, \tag{4.3}$$

where $\bar{\sigma}$ is the one obtained by the weak recursion operator. The corresponding corrected recursion operator takes the form

$$\mathcal{R} = \mathcal{R}_w + \frac{a}{g}\,H, \tag{4.4}$$

and $h = H\,\sigma$. Let us illustrate the procedure of how to construct the symmetries of an equation from a time-dependent recursion operator. Firstly we consider the scalar evolution equations of the form $u_t = K[u]$.

Example 3. The Burgers equation

$$u_t = u_{2x} + uu_x, \tag{4.5}$$

possesses a recursion operator of the form

$$\mathcal{R}_w = tD + \frac{1}{2}tu + \frac{1}{2}x + \frac{1}{2}(1 + tu_x)D^{-1}, \tag{4.6}$$

where

$$R_0 = tD + \frac{1}{2}x + \frac{1}{2}D^{-1}. \tag{4.7}$$

Let

$$\sigma_n^0 = a_1 + a_2 x + a_2 x^2 + a_3 x^3 + \cdots, \tag{4.8}$$

where a_i are some functions of t. From the linearized equation $\sigma^0_{(n)t} = \sigma^0_{(n)2x}$

$$a_{1t} = 2a_3, \qquad a_{2t} = 6a_4, \qquad a_{3t} = 12a_5, \quad \ldots \tag{4.9}$$

and by Proposition 1 we obtain

$$\sigma^0_{n+1} = \left(tD + \frac{1}{2}x + \frac{1}{2}D^{-1}\right)\sigma^0_n = t\left(a_2 + 2a_3 x + 3a_4 x^2 + \cdots\right) + \frac{1}{2}x(a_1 + a_2 x + a_3 x^2 + \cdots) + \frac{1}{2}\left(a_1 x + \frac{1}{2}a_2 x^2 + \cdots + h(t)\right), \tag{4.10}$$

or

$$\sigma^0_{n+1} = \left(ta_2 + \frac{1}{2}h\right) + (12ta_3 + a_1)x + \left(3a_4 t + \frac{3}{4}a_2\right)x^2 + \cdots. \tag{4.11}$$

Using $\sigma^0_{(n+1)t} = \sigma^0_{(n+1)2x}$ and equating the coefficients at power of x to zero we obtain the following system of equations for a_i and h

$$\begin{aligned} &\left(ta_2 + \frac{1}{2}h\right)_t = 2\left(3a_4 t + \frac{3}{4}a_2\right), \\ &(2ta_3 + a_1)_t = 6\left(4a_5 t + \frac{2}{3}a_3\right), \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned} \tag{4.12}$$

With (4.9) the first equation gives $h_t = a_2$ and all the others are satisfied identically. Finally we may write h as

$$h = D_t^{-1}\left(\Pi D\sigma^0_n\right), \tag{4.13}$$

where Π is the projection operator defined as $\Pi\, h(x, t, u, u_x, \ldots) = h(t, 0, 0, \ldots)$ for any function h. This calculation allows us to write

$$\sigma^0_{n+1} = \bar{\sigma}^0_{n+1} + \frac{1}{2}D_t^{-1}\left(\Pi D\sigma^0_n\right), \tag{4.14}$$

where $\bar{\sigma}^0_{n+1}$ is the standard part of σ^0_{n+1} without the constant of integration $h(t)$. This means that one should add this constant of integration to D^{-1} in the general symmetry equation (4.3),

$$\sigma_{n+1} = \bar{\sigma}_{n+1} + \frac{1}{2}(1 + tu_x)D_t^{-1}\left(\Pi D\sigma^0_n\right), \tag{4.15}$$

to allow one to generate the whole hierarchy of symmetries. Here $\bar{\sigma}_{n+1}$ is the symmetry obtained by standard application of the operator D^{-1}. The corresponding corrected recursion operator (4.4) for (4.5) is

$$\mathcal{R} = \mathcal{R}_w + \frac{1}{2}(1 + tu_x)D_t^{-1}\Pi D. \tag{4.16}$$

Example 4. The cylindrical Korteweg-de Vries equation (cKdV). The cKdV equation,

$$u_t = u_{3x} + uu_x - \frac{u}{2t}, \tag{4.17}$$

possesses a recursion operator of the form

$$\mathcal{R}_w = tD^2 + \frac{2}{3}tu + \frac{1}{3}x + \frac{1}{6}(1 + 2tu_x)D^{-1}. \tag{4.18}$$

The u-independent part of recursion operator is

$$R_0 = tD^2 + \frac{1}{3}x + \frac{1}{6}D^{-1}. \tag{4.19}$$

Let $\sigma_n^0 = a_1 + a_2x + a_3x^2 + a_4x^3 + \cdots$ with $\sigma_{nt}^0 = \sigma_{n3x}^0 - \frac{1}{2t}\sigma_n^0$. From these we have the following relations among the parameters $a_{1t} = 6a_4 - \frac{a_1}{2t}$, $a_{2t} = 24a_5 - \frac{a_2}{2t}$, $a_{3t} = 60a_6 - \frac{a_3}{2t}$. Then

$$\sigma_{n+1}^0 = R_0\sigma_n^0 = \left(2ta_3 + \frac{1}{6}h\right) + \frac{1}{2}a_1x + \left(12ta_5 + \frac{5}{12}a_2\right)x^2 + \cdots . \tag{4.20}$$

Using the linearized equation satisfied by σ_{n+1}^0 we obtain an equation for h $h_t + \frac{1}{2t}h = 2a_3$ which gives $h = \frac{1}{\sqrt{t}}D_t^{-1}\left(\sqrt{t}\Pi D^2\sigma_n^0\right)$. Hence the symmetry equation (4.3) for cKdV equation is

$$\sigma_{n+1} = \bar{\sigma}_{n+1} + \frac{1}{6}(2tu_x + 1)\frac{1}{\sqrt{t}}D_t^{-1}\left(\sqrt{t}\Pi D^2\sigma_n^0\right). \tag{4.21}$$

The corresponding corrected recursion operator (4.4) for (4.17) is

$$\mathcal{R} = \mathcal{R}_w + \frac{1}{6}(1 + 2tu_x)\frac{1}{\sqrt{t}}D_t^{-1}\sqrt{t}\Pi D^2. \tag{4.22}$$

Now we discuss the symmetries of scalar evolution equation of the following form

$$u_t = F[u] + g(x,t), \tag{4.23}$$

where $\lim\limits_{u\to 0} F = 0$ and $g(x,t)$ is an explicit x and t dependent (differentiable) function. We assume that the above equation admits a recursion operator of the form (4.1) and any symmetry of this equation has the form $\sigma_n = \sigma_n^0 + \sigma_n^1 + \sigma_n^2$, where $\sigma_n^0 = \sigma_n|_{u=0}$ and $\sigma_n^1 = \sum\limits_{i=0} b_iu_i$ and σ_n^2 is the nonlinear part of the symmetry. Here b_i are functions of x and t and $i = 1, 2, \ldots$. Now we give the following Proposition.

Proposition 2. *The operator R_0, such that $\sigma_{n+1}^0 = R_0\sigma_n^0$, can be shown to satisfy $R_{0t} + \mathcal{R}_{wt}|_{u\to 0} = [F_0', R_0]$ and the equation for σ_n^0 is*

$$\sigma_{nt}^0 + \sum_{i=0} b_ig_i = F_0'\sigma_n^0, \tag{4.24}$$

where $F_0' = F'|_{u=0}$.

Example 5. Consider the equation [25]

$$u_t = u_{3x} + 6uu_x - \frac{3u}{t} - \frac{5x}{2t^2} \tag{4.25}$$

which possesses a recursion operator of the form

$$\mathcal{R}_w = t^6 D^2 + 4t^6 u + 2xt^5 + t^5(1 + 2tu_x)D^{-1}. \tag{4.26}$$

Here $R_0 = t^6 D^2 + 2xt^5 + t^5 D^{-1}$ satisfies the relation $R_0 \sigma_n^0 = \sigma_{n+1}^0$. By Proposition 2 the linearized equation for σ_n^0 is

$$\sigma_{nt}^0 - \frac{5xb_0}{2t^2} - \frac{5b_1}{2t^2} = \sigma_{n3x}^0 - \frac{3\sigma_n^0}{t}. \tag{4.27}$$

Now taking

$$b_0 = b_0^0 + b_0^1 x + b_0^2 x^2 + \cdots, \qquad b_1 = b_1^0 + b_1^1 x + b_1^2 x^2 + \cdots, \tag{4.28}$$

where $b_j^i = b_j^i(t)$, we obtain the following equations

$$a_{0t} = 6a_3 - \frac{3a_0}{t} + \frac{5b_1^0}{2t^2}, \qquad a_{1t} = 24a_4 - \frac{3a_1}{t} + \frac{5b_0^0}{2t^2} + \frac{5b_1^1}{2t^2}, \qquad \ldots$$

The next symmetry σ_{n+1}^0 can be generated by the operator R_0 and satisfies the following linearized equation according to Proposition 2

$$\sigma_{(n+1)t}^0 - \frac{5xh_0}{2t^2} - \frac{5h_1}{2t^2} = \sigma_{(n+1)3x}^0 - \frac{3\sigma_{n+1}^0}{t}, \tag{4.29}$$

where

$$h_0 = h_0^0 + h_0^1 x + h_0^2 x^2 + \cdots, \qquad h_1 = h_1^0 + h_1^1 x + h_1^2 x^2 + \cdots, \tag{4.30}$$

and $h_j^i = h_j^i(t)$. The relation between the h_j^i and b_j^i is given by $\sigma_{n+1} = \mathcal{R}_w \sigma_n$. Now using (4.29) we obtain the equation satisfied by the constant of integration f

$$-2t^7 f_t + 4t^7 a_3 - 16t^6 f + 5h_0^0 - 10t^6 \left(h_1^2 - h_0^1\right) = 0, \tag{4.31}$$

where

$$h_0^0 = 2t^6 f + 2t^6 \left(h_1^2 - h_0^1\right) + t^6 \sum_{i=2} (-1)^i \Pi D^{i-2} \frac{\partial \sigma_n}{\partial u_i}. \tag{4.32}$$

Hence the constant of integration f becomes

$$f = \frac{1}{t^3} \left[D_t^{-1} \left(\tau^3 \Pi D^2 \sigma_n\right) + \frac{5}{2} D_t^{-1} \left(\tau \sum_{i=2} \Pi D^{i-2} \frac{\partial \sigma_n}{\partial u_i} \right) \right]. \tag{4.33}$$

Therefore the corrected symmetry equation of (4.25) is of the form

$$\begin{aligned} \sigma_{n+1} = {} & \bar{\sigma}_{n+1} + t^2(1 + 2tu_x) \\ & \times \left[D_t^{-1} \left(\tau^3 \Pi D^2 \sigma_n\right) + \frac{5}{2} D_t^{-1} \left(\tau \sum_{i=2} (-1)^i \, \Pi D^{i-2} \frac{\partial \sigma_n}{\partial u_i} \right) \right]. \end{aligned} \tag{4.34}$$

The first four symmetries of (4.25) are:

$$\begin{aligned}
\sigma_0 &= t^2 + 2t^3 u_x, \\
\sigma_1 &= 2t^9 (u_{3x} + 6uu_x) + 6t^8 (xu_x + u) + 3t^7 x, \\
\sigma_2 &= 2t^{15} \left(u_{5x} + 10uu_{3x} + 20u_x u_{2x} + 30u^2 u_x\right) \\
&\quad + 10t^{14} \left(xu_{3x} + 2u_{2x} + 6xuu_x + 3u^2\right) + 15t^{13} \left(2xu + x^2 u_x\right) + \frac{15}{2} t^{12} x^2, \\
\sigma_3 &= 2t^{21} \left(u_{7x} + 14uu_{5x} + 42u_x u_{4x} + 70u_{2x} u_{3x} + 70u^2 u_{3x} + 280uu_x u_{2x}\right. \\
&\quad \left. + 140u^3 u_x + 70u_x^3\right) + 14t^{20} (xu_{5x} + 3u_{4x} + 10xuu_{3x} + 20xu_x u_{2x} + 20uu_{2x} \\
&\quad + 15u_x^2 + 30xu^2 u_x + 10u^3) + 35t^{19} \left(x^2 u_{3x} + 4xu_{2x} + 6x^2 uu_x + 3u_x + 6xu^2\right) \\
&\quad + \frac{35}{2} t^{18} \left(2x^3 u_x + 6x^2 u + 1\right) + \frac{35}{2} t^{17} x^3.
\end{aligned} \tag{4.35}$$

In the previous examples the nonlocal parts of the recursion operators of evolution equations are of the form aD^{-1}. But there exist some integrable evolution equations admitting recursion operators in which nonlocal parts are of the form $aD^{-1}b$ where a and b are in general functions of both jet coordinates and explicitly x and t. In this case, we may define the nonlocal part of $R_0 = \mathcal{R}_w|_{u\to 0} = ah(t)$ when $b \to 0$ as $u \to 0$.

Example 6. Consider the following extended potential KdV (pKdV) equation [20]

$$u_t = u_{3x} + u_x^2 + c_0 x + c_1, \tag{4.36}$$

where c_0 and c_1 are arbitrary constants. The recursion operator for (4.36) is given by

$$\mathcal{R}_w = D^2 + \frac{4}{3} u_x - \frac{4}{3} c_0 t - \frac{2}{3} D^{-1} u_{2x}. \tag{4.37}$$

As we mentioned above, the form of R_0 will be

$$R_0 = D^2 - \frac{4}{3} c_0 t - \frac{2}{3} h(t). \tag{4.38}$$

We observe that the u-independent part (σ_n^0) of one set of symmetries are functions of t. Therefore from Proposition 2 the linearized equation for σ_n^0 is

$$\sigma_{nt}^0 + b_1^n c_0 = 0, \tag{4.39}$$

where b_1^n depends only on t. The next symmetry σ_{n+1}^0 may be generated by R_0

$$\sigma_{n+1}^0 = -\frac{2}{3} (2c_0 t + h) \sigma_n^0 \tag{4.40}$$

and satisfies

$$\sigma_{(n+1)t}^0 + b_1^{n+1} c_0 = 0, \tag{4.41}$$

where b_1^{n+1} depends only on t. Furthermore the relation between b_1^n and b_1^{n+1} is given by $\sigma_{n+1} = \mathcal{R}_w \sigma_n$ as $b_1^{n+1} = \frac{2}{3}\sigma_n^0 - \frac{4}{3} c_0 t b_1^n$. We can find, together with (4.39), (4.40) and (4.41), an equation for constant of integration h

$$h_t \sigma_n^0 + c_0 D_t^{-1} \sigma_n^0 = 0 \tag{4.42}$$

in terms of which (4.40) becomes

$$\sigma^0_{n+1} = -\frac{2}{3}\left(2c_0 t - c_0 D_t^{-1}\right)\sigma^0_n. \tag{4.43}$$

This leads to the general symmetry equation

$$\sigma_{n+1} = \bar{\sigma}_{n+1} + \frac{2}{3}c_0 D_t^{-1}\Pi\sigma^0_n. \tag{4.44}$$

The first four symmetries (4.36) are:

$$\begin{aligned}
\sigma_0 &= 1,\\
\sigma_1 &= \frac{2}{3}(u_x - c_0 t),\\
\sigma_2 &= \frac{2}{3}\left(u_{3x} + u_x^2 - 2c_0 t u_x + c_0^2 t^2\right),\\
\sigma_3 &= \frac{2}{27}\left(9u_{5x} + 30u_x u_{3x} - 30c_0 t u_{3x} + 15u_{2x}^2\right.\\
&\quad\left. + 10u_x^3 - 30c_0 t u_x^2 + 30c_0^2 t^2 u_x - 10c_0^3 t^3\right).
\end{aligned} \tag{4.45}$$

We finally remark that the corrected recursion operator for (4.36), taking into account the symmetry structure of (4.36), may be written as

$$\mathcal{R} = D^2 + \frac{4}{3}u_x - \frac{4}{3}c_0 t - \frac{2}{3}D^{-1}u_{2x} + \frac{2}{3}c_0 D_t^{-1}\Pi. \tag{4.46}$$

In the following section we will consider the time-dependent symmetries of a system of evolution equations.

5 System of evolution equations

Following the procedure introduced in Section 4, we now discuss the time-dependent symmetries for a system of evolution equations. We present several examples.

Example 7. Consider the following nonautonomous system of equations [24]

$$\begin{aligned}
u_t &= u_{3x},\\
v_t &= v_{3x} + \frac{c_0}{\sqrt{t}}uu_x,
\end{aligned} \tag{5.1}$$

with recursion operator

$$\mathcal{R}_w = \begin{pmatrix} tD^2 + \dfrac{x}{3} + \dfrac{1}{6}D^{-1} & 0 \\ \dfrac{2c_0\sqrt{t}}{3}u + \dfrac{c_0\sqrt{t}}{3}u_x D^{-1} & tD^2 + \dfrac{x}{3} + \dfrac{1}{6}D^{-1} \end{pmatrix}, \tag{5.2}$$

where c_0 is an arbitrary constant. In a similar way as for scalar evolution equations we may take the form of symmetries σ^0_n and ψ^0_n as

$$\begin{aligned}
\sigma^0_n &= a_1 + a_2 x + a_2 x^2 + a_3 x^3 + \cdots,\\
\psi^0_n &= b_1 + b_2 x + b_2 x^2 + b_3 x^3 + \cdots,
\end{aligned} \tag{5.3}$$

where a_i and b_i are some functions of t. The linearized equations, by Proposition 1, for those symmetries, viz.

$$\begin{pmatrix} \sigma_n^0 \\ \psi_n^0 \end{pmatrix}_t = \begin{pmatrix} \sigma_n^0 \\ \psi_n^0 \end{pmatrix}_{3x}, \tag{5.4}$$

with a simple comparison of each power of x, lead to

$$\begin{aligned} &a_{1t} = 6a_4, \qquad a_{2t} = 24a_5, \qquad a_{3t} = 60a_6, \qquad \dots, \\ &b_{1t} = 6b_4, \qquad b_{2t} = 24b_5, \qquad b_{3t} = 60b_6, \qquad \dots. \end{aligned} \tag{5.5}$$

The next symmetry, generated by R_0, is

$$\begin{pmatrix} \sigma_{n+1}^0 \\ \psi_{n+1}^0 \end{pmatrix} = R_0 \begin{pmatrix} \sigma_n^0 \\ \psi_n^0 \end{pmatrix}, \tag{5.6}$$

where

$$R_0 = \begin{pmatrix} tD^2 + \frac{x}{3} + \frac{1}{6}D^{-1} & 0 \\ 0 & tD^2 + \frac{x}{3} + \frac{1}{6}D^{-1} \end{pmatrix} \tag{5.7}$$

is the (u, v) independent part of the recursion operator of $\mathcal{R}_w$ satisfies the Proposition 1. Hence

$$\begin{aligned} \sigma_{n+1}^0 &= \left(2a_3 t + \frac{1}{6}h\right) + \left(\frac{1}{2}a_1 + 6a_4 t\right)x + \left(\frac{5}{12}a_2 + 12a_5 t\right)x^2 + \cdots, \\ \psi_{n+1}^0 &= \left(2b_3 t + \frac{1}{6}g\right) + \left(\frac{1}{2}b_1 + 6b_4 t\right)x + \left(\frac{5}{12}b_2 + 12b_5 t\right)x^2 + \cdots, \end{aligned} \tag{5.8}$$

where $h(t)$ and $g(t)$ are the constants of integrations. Using linearized equations for σ_{n+1}^0 and ψ_{n+1}^0

$$\begin{pmatrix} \sigma_{n+1}^0 \\ \psi_{n+1}^0 \end{pmatrix}_t = \begin{pmatrix} \sigma_{n+1}^0 \\ \psi_{n+1}^0 \end{pmatrix}_{3x} \tag{5.9}$$

together with (5.5) we find the values of the constants of integrations h and g as $h(t) = 2a_3$ and $g(t) = 2b_3$. Finally we may write

$$h(t) = D_t^{-1}\left(\Pi D^2 \sigma_n^0\right), \qquad g(t) = D_t^{-1}\left(\Pi D^2 \psi_n^0\right), \tag{5.10}$$

where the projection Π is defined as $\Pi h(x, t, u, u_x, \dots, v, v_x, \dots) = h(t, 0, 0, \dots)$ for any function h. The general symmetry equations (4.3) for (5.1) are of the form

$$\begin{aligned} \sigma_{n+1} &= \bar{\sigma}_{n+1} + \frac{1}{6}D_t^{-1}\left(\Pi D^2 \sigma_n^0\right), \\ \psi_{n+1} &= \bar{\psi}_{n+1} + \frac{2}{3}\sqrt{t}c_0 u_x D_t^{-1}\left(\Pi D^2 \sigma_n^0\right) + \frac{1}{6}D_t^{-1}\left(\Pi D^2 \psi_n^0\right), \end{aligned} \tag{5.11}$$

where $\bar{\sigma}_{n+1}$ and $\bar{\psi}_{n+1}$ are the symmetries obtained by standard application of the operator D^{-1}. Let

$$\tau_n = \begin{pmatrix} \sigma_n \\ \psi_n \end{pmatrix} \tag{5.12}$$

be the symmetries of (5.1). Then the first four symmetries of (5.1) are:

$$\begin{aligned}
\sigma_0 &= 1, \\
\psi_0 &= 1 + 2c_0 t^{1/2} u_x, \\
\sigma_1 &= \frac{1}{2}x, \\
\psi_1 &= 2t^{3/2}u_{3x} + t^{1/2}c_0(xu_x + u) + \frac{1}{2}x, \\
\sigma_2 &= \frac{5}{24}x^2, \\
\psi_2 &= 2c_0 t^{5/2}u_{5x} + c_0 t^{3/2}\left(\frac{5}{3}xu_{3x} + \frac{10}{3}u_{2x}\right) + \frac{5}{6}c_0 t^{1/2}\left(2x^2u_x + xu\right) + \frac{5}{24}x^2, \\
\sigma_3 &= \frac{35}{72}t + \frac{35}{432}x^3, \\
\psi_3 &= 2c_0 t^{7/2}u_{7x} + c_0 t^{5/2}\left(\frac{7}{3}xu_{5x} + 7u_{4x}\right) + 35c_0 t^{3/2}\left(\frac{1}{36}x^2u_{3x} + \frac{1}{9}xu_{2x} + \frac{1}{12}u_x\right) \\
&\quad + 35c_0 t^{1/2}\left(\frac{1}{216}x^3u_x + \frac{1}{72}u\right) + \frac{35}{72}t + \frac{35}{432}x^3.
\end{aligned} \tag{5.13}$$

Example 8. The system [24]

$$\begin{aligned}
u_t &= u_{3x} + \frac{2c_0}{\sqrt{t}}uu_x, \\
v_t &= v_{3x} + \frac{c_0}{\sqrt{t}}(uv)_x,
\end{aligned} \tag{5.14}$$

is the nonautonomous Jordan Korteweg-de Vries (JKdV), where c_0 is an arbitrary constant. The recursion operator $\mathcal{R}_w$ for this system is

$$\mathcal{R}_w = \begin{pmatrix} \mathcal{R}_0^0 & \mathcal{R}_1^0 \\ \mathcal{R}_0^1 & \mathcal{R}_1^1 \end{pmatrix} \tag{5.15}$$

with

$$\begin{aligned}
\mathcal{R}_0^0 &= tD^2 + \frac{1}{3}x + \frac{4c_0}{3}\sqrt{t}u + \frac{1}{6}\left(4c_0\sqrt{t}u_x + 1\right)D^{-1}, \\
\mathcal{R}_1^0 &= 0, \\
\mathcal{R}_0^1 &= \frac{2c_0}{3}\sqrt{t}v + \frac{c_0}{3}\sqrt{t}v_x D^{-1} - \frac{c_0^2}{9}uD^{-1}vD^{-1}, \\
\mathcal{R}_1^1 &= tD^2 + \frac{1}{3}x + \frac{2c_0}{3}\sqrt{t}u + \frac{1}{6}(2c_0\sqrt{t}u_x + 1)D^{-1} + \frac{c_0^2}{9}uD^{-1}uD^{-1}.
\end{aligned} \tag{5.16}$$

Again we take the form of symmetries σ_n^0 and ψ_n^0 as in (5.3) with the (u, v) independent recursion operator R_0 which is the same as in (5.7). Performing the same steps as in the previous case, we obtain the general symmetry equations (4.3) for (5.14) to be

$$\begin{aligned}
\sigma_{n+1} &= \bar{\sigma}_{n+1} + \frac{1}{6}\left(4c_0\sqrt{t}\,u_x + 1\right) D_t^{-1}\left(\Pi D^2\sigma_n^0\right),\\
\psi_{n+1} &= \bar{\psi}_{n+1} + \frac{1}{3}\sqrt{t}c_0 v_x D_t^{-1}\left(\Pi D^2\sigma_n^0\right) - \frac{1}{9}c_0^2 u D^{-1} v D_t^{-1}\left(\Pi D^2\sigma_n^0\right)\\
&\quad + \left(\frac{1}{6}\left(2c_0\sqrt{t}u_x + 1\right) + \frac{1}{9}c_0^2 u D^{-1} u\right) D_t^{-1}\left(\Pi D^2\psi_n^0\right).
\end{aligned} \tag{5.17}$$

Now we list only the first two symmetries because higher order ones are too long to write down here.

$$\begin{aligned}
\sigma_0 &= \frac{1}{6}\left(4c_0 t^{1/2} u_x + 1\right),\\
\psi_0 &= \frac{1}{18}\left[6c_0 t^{1/2}\left(u_x + v_x\right) + 2c_0^2 uw + 3\right],\\
\sigma_1 &= \frac{1}{12}\left[8c_0 t^{3/2} u_{3x} + 4c_0 t^{1/2}(x u_x + u) + 16c_0^2 t u u_x\right],\\
\psi_1 &= \frac{1}{324}\Big[108c_0 t^{3/2}(u_{3x} + v_{3x}) + c_0 t^{1/2}\left[12c_0^2 h u_x + 54 x u_x + 54 x v_x - 12c_0^2 f u\right.\\
&\quad \left. + 12c_0^2 u n + 24c_0^2 u^2 w + 54u + 54v\right] + 3c_0 t\left[c_0 w u_{2x} + 2c_0 u_x w_x + 3c_0 u u_x\right.\\
&\quad \left. + 5c_0 v u_x + 4c_0 u g + 6c_0^2 h + 6c_0^2 p u - 6c_0^2 r u + 12c_0^2 x u w + 27x\right]\Big],
\end{aligned} \tag{5.18}$$

where $p_x = xu$, $n_x = u^2$, $r_x = xv$, $w_x = u - v$, $g_x = uh$ and $h_x = uw$.

More generally the multicomponent nonautonomous JKdV system [24] is

$$q_t^i = q_{3x}^i + \frac{1}{\sqrt{t}} s_{jk}^i q^j q_x^k, \qquad i, j, k = 1, 2, \dots, N, \tag{5.19}$$

where s_{jk}^i are constants, symmetric in the lower indices and satisfy the Jordan identities

$$s_{pr}^k F^i{}_{ljk} + s_{jr}^k F^i{}_{lpk} + s_{jp}^k F^i{}_{lrk} = 0, \tag{5.20}$$

with $F^i{}_{plj} = s_{jk}^i s_{lp}^k - s_{lk}^i s_{jp}^k$. This system possesses a recursion operator

$$\begin{aligned}
\mathcal{R}_{wj}^i &= t\delta_j^i D^2 + \frac{2}{3}\sqrt{t}\, s_{jk}^i\, q^k + \frac{1}{3}\delta_j^i x + \left(\frac{1}{3}\sqrt{t}\, s_{jk}^i\, q_x^k + \frac{1}{6}\delta_j^i\right) D^{-1}\\
&\quad + \frac{1}{9} F^i{}_{lkj}\, q^l\, D^{-1}\, q^k\, D^{-1}.
\end{aligned} \tag{5.21}$$

The time-dependent symmetries can be computed as

$$\tau_{n+1}^i = \bar{\tau}_{n+1}^i + \Lambda_j^i D_t^{-1}\Pi D^2 \tau_n^j, \tag{5.22}$$

where

$$\Lambda_j^i = \frac{1}{3}\sqrt{t}\, s_{jk}^i\, q_x^k + \frac{1}{6}\,\delta_j^i + \frac{1}{9}\, F_{lkj}^i\, q^l D^{-1}\, q^k. \tag{5.23}$$

The corrected recursion operator (4.4) for (5.19) is given by

$$\mathcal{R} = \mathcal{R}_w + \Lambda D_t^{-1}\Pi D^2. \tag{5.24}$$

Example 9. Consider the following system of equations

$$u_t = v_{3x} + t^{-2/3}(vu_x + vv_x),$$
$$v_t = t^{-2/3} vv_x, \tag{5.25}$$

with recursion operator

$$\mathcal{R}_w = \begin{pmatrix} 3t^{1/3}v + x & 3tD^2 + 3t^{1/3}v + D^{-1} \\ 0 & 3t^{1/3}v + x \end{pmatrix}. \tag{5.26}$$

We take the form of σ_n^0 and ψ_n^0 as in (5.3) and from the linearized equations for σ_n^0 and ψ_n^0

$$\begin{pmatrix} \sigma_n^0 \\ \psi_n^0 \end{pmatrix}_t = \begin{pmatrix} \psi_n^0 \\ 0 \end{pmatrix}_{3x}, \tag{5.27}$$

we obtain

$$a_1 = 6b_4 t, \qquad a_2 = 24 b_5 t, \qquad a_3 = 60 b_6 t, \qquad a_4 = 120 b_7 t, \qquad \ldots, \tag{5.28}$$

where b_i are arbitrary constants. The next symmetries, generated by R_0, are given in (5.6) where, in this case, R_0 has the form

$$R_0 = \begin{pmatrix} x & 3tD^2 + D^{-1} \\ 0 & x \end{pmatrix}. \tag{5.29}$$

They are

$$\sigma_{n+1}^0 = (2b_3 t + h) + (a_1 + b_1 + 18 b_4 t)x + \left(a_2 + \frac{1}{2}b_2 + 36 b_5 t\right)x^2 + \cdots,$$
$$\psi_{n+1}^0 = b_1 x + b_2 x^2 + b_3 x^3 + \cdots, \tag{5.30}$$

where h is the constant of integration. Now using linearized equations for σ_{n+1}^0 and ψ_{n+1}^0 together with (5.28) we obtain the value of $h = 0$. We point out that $\mathcal{R}_w = \mathcal{R}$. The first three classical symmetries of (5.25) are:

$$\tau_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \tau_1 = \begin{pmatrix} 3t^{1/3}v + x \\ 0 \end{pmatrix}, \qquad \tau_2 = \begin{pmatrix} (3t^{1/3}v + x)^2 \\ 0 \end{pmatrix}. \tag{5.31}$$

Example 10. Consider the following system of equations

$$u_t = u_{3x} + e^{-t} vv_x,$$
$$v_t = vv_x, \tag{5.32}$$

with recursion operator

$$\mathcal{R}_w = \begin{pmatrix} D^2 + D^{-1} & e^{-t}v \\ 0 & v \end{pmatrix}. \tag{5.33}$$

We take the form of σ_n^0 and ψ_n^0 as in (5.3) and from the linearized equations (5.27) for σ_n^0 and ψ_n^0 we obtain

$$a_{1t} = 6a_4, \qquad a_{2t} = 24a_5, \qquad a_{3t} = 60a_6, \qquad a_{4t} = 120a_7, \qquad \ldots, \tag{5.34}$$

where b_i are arbitrary constants. The next symmetry, generated by R_0, is given in (5.6), where, in this case, R_0 has the form

$$R_0 = \begin{pmatrix} D^2 + D^{-1} & 0 \\ 0 & 0 \end{pmatrix}. \tag{5.35}$$

They are

$$\begin{aligned} \sigma_{n+1}^0 &= (2a_3t + h) + (a_1 + 6a_4t)x + \left(\frac{1}{2}a_2 + 12a_5\right)x^2 + \cdots, \\ \psi_{n+1}^0 &= 0, \end{aligned} \tag{5.36}$$

where h is the constant of integration. Now using linearized equations for σ_{n+1}^0 and ψ_{n+1}^0 together with (5.34) we obtain the value of $h = D_t^{-1}\left(\Pi D^2 \sigma_n^0\right)$.

Hence the symmetry equations of (5.26) take the following form

$$\begin{aligned} \sigma_{n+1} &= \bar{\sigma}_{n+1} + D_t^{-1}\left(\Pi D^2 \sigma_n^0\right), \\ \psi_{n+1} &= \bar{\psi}_{n+1}. \end{aligned} \tag{5.37}$$

The first four symmetries of (5.32) are:

$$\tau_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \tau_1 = \begin{pmatrix} x \\ 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} x^2 \\ 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} \frac{1}{6}x^3 + t + 1 \\ 0 \end{pmatrix}. \tag{5.38}$$

Example 11. Consider the following system of equations

$$\begin{aligned} u_t &= u_{3x} + c_0 v_{3x} - c_0(u_x v + c_0 v v_x), \\ v_t &= u_x v + c_0 v v_x \end{aligned} \tag{5.39}$$

with recursion operator

$$\mathcal{R}_w = \begin{pmatrix} 3tD^2 + x + 2D^{-1} - 3c_0tv & c_0\left(3tD^2 + x + 2D^{-1} - 3c_0tv\right) \\ 3tv & 3c_0tv + x \end{pmatrix}, \tag{5.40}$$

where c_0 is an arbitrary constant. We take the form of σ_n^0 and ψ_n^0 as in (5.3) and from the linearized equations

$$\begin{pmatrix} \sigma_n^0 \\ \psi_n^0 \end{pmatrix}_t = \begin{pmatrix} 1 & c_0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \sigma_n^0 \\ \psi_n^0 \end{pmatrix}_{3x} \tag{5.41}$$

we obtain

$$a_{1t} = 6a_4 + 6c_0b_4, \qquad a_{2t} = 24a_5 + 24c_0b_5, \qquad a_{3t} = 60a_5 + 60c_0b_6, \qquad \ldots, \tag{5.42}$$

where, in this case, b_i are arbitrary constants. The next symmetry, generated by R_0, is

$$\begin{pmatrix} \sigma^0_{n+1} \\ \psi^0_{n+1} \end{pmatrix} = R_0 \begin{pmatrix} \sigma^0_n \\ \psi^0_n \end{pmatrix}, \tag{5.43}$$

where

$$R_0 = \begin{pmatrix} 3tD^2 + x + 2D^{-1} & c_0\left(3tD^2 + x + 2D^{-1}\right) \\ 0 & x \end{pmatrix}. \tag{5.44}$$

is the (u, v) independent part of the recursion operator of $\mathcal{R}_w$. Using the linearized equations for σ^0_{n+1} and ψ^0_{n+1} together with (5.42) we may determine the value of constant of integration as

$$h = D_t^{-1}\left(\Pi D^2 \sigma^0_n\right) + c_0 t\left(\Pi D^2 \psi^0_n\right). \tag{5.45}$$

Therefore the symmetry equations of (5.39) are:

$$\begin{aligned} &\sigma_{n+1} = \bar{\sigma}_{n+1} + 2(c_0+1)\left(D_t^{-1}\left(\Pi D^2 \sigma^0_n\right) + c_0 t\left(\Pi D^2 \psi^0_n\right)\right), \\ &\psi_{n+1} = \bar{\psi}_{n+1}. \end{aligned} \tag{5.46}$$

The first four symmetries of (5.39) are:

$$\begin{aligned} &\sigma_0 = 2(c_0+1), \qquad \psi_0 = 0, \\ &\sigma_1 = -6c_0(c_0+1)tv + 2(3x(c_0+1)+c_0), \qquad \psi_1 = 6(c_0+1)tv, \\ &\sigma_2 = -6c_0tv(4x(c_0+1)+c_0) + 6x(2x(c_0+1)+c_0)), \\ &\psi_2 = 6vt(4x(c_0+1)+c_0), \\ &\sigma_3 = -12c_0txv(5x(c_0+1)+2c_0) + 4(c_0+1)\left(30t+5x^3\right) + 12c_0x^2 + 120t, \\ &\psi_3 = 12xtv(5x(c_0+1)+2c_0). \end{aligned} \tag{5.47}$$

6 Conclusion

It is well known that one of the effective ways to find symmetries is to use the recursion operator. If the recursion operator or the corresponding evolution equation is time-dependent one is faced with some difficulties in finding the correct symmetries. Here, in this work, we approach this problem in two ways. Firstly we observed that by the standard application of D^{-1} on functions having explicit t and x dependencies the rule of associativity is lost in the application of consecutive two operators on such a function space. Due to this fact the standard equation for the recursion operators is no longer valid. We modified this equation by including the associators. We have given some applications of our formula (3.4) for the modified term of the recursion operators. The second way to calculate the missing parts of the symmetries is to use directly the symmetry equation and the correct application of D^{-1} on functions having explicit x and t dependencies. We have given a general treatment and several examples.

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