A note on cosmological Levi-Civita spacetimes in higher dimensions

Özgür Sarıoğlu* and Bayram Tekin†
Department of Physics, Faculty of Arts and Sciences,
Middle East Technical University, 06531, Ankara, Turkey
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We find a class of solutions to cosmological Einstein equations that generalizes the four dimensional cylindrically symmetric spacetimes to higher dimensions. The AdS soliton is a special member of this class with a unique singularity structure.

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In this note, we generalize the four dimensional cylindrically symmetric static spacetimes that are solutions to the Einstein equations, with a negative cosmological constant $\Lambda < 0$, to higher dimensions. The flat space ($\Lambda = 0$) analogs of these solutions are known as the Levi-Civita spacetimes [1].

The original four dimensional solutions with a non-vanishing Λ were found in [2] and [3]. Non-singular sheet sources for these spacetimes were recently constructed in [4]. There are subtle issues in the interpretation of the parameters that appear in the four dimensional solution in terms of the physical properties, such as the mass, of material sources. According to [4], "for the $\Lambda < 0$ case the asymptotic forms of the metrics due to material cylinders are more closely related to the asymptotics of bounded sources than for the $\Lambda = 0$ case."

Here we provide a new class of solutions to the cosmological Einstein equations (with a redefined cosmological constant)

$$R_{\mu\nu} = -\frac{n}{\ell^2} g_{\mu\nu}$$

in $D = n + 1 \geq 4$ dimensions. It reads

$$ds^2 = \frac{r^2}{\ell^2}\left[-\left(1-\frac{r_0^n}{r^n}\right)^{p_0} dt^2 + \sum_{k=1}^{n-1}\left(1-\frac{r_0^n}{r^n}\right)^{p_k}(dx^k)^2\right] + \left(1-\frac{r_0^n}{r^n}\right)^{-1}\frac{\ell^2}{r^2}\,dr^2\,, \tag{1}$$

where r_0 is a free parameter, and the constants p_k, with $0 \leq k \leq n-1$, satisfy two Kasner-type conditions:

$$\sum_{k=0}^{n-1} p_k = \sum_{k=0}^{n-1} (p_k)^2 = 1\,. \tag{2}$$

This is a higher dimensional generalization of the metrics in [2, 3]. Specifically, the metric studied by [4] follows from (1) after solving the constraints (2) for $n = 3$ [1].

It immediately follows that as $r \to \infty$, (1) asymptotically approaches the usual maximally symmetric AdS spacetime in horospheric coordinates. Another important observation is that the AdS soliton of [5] is just a special member in this class: It is obtained simply by taking one of the $p_k = 1$, where $1 \leq k \leq n-1$, and setting all the others, including p_0, to zero. In lower dimensions, when $n = 1$ and $n = 2$, (1) becomes the usual two and three dimensional AdS

*Electronic address: sarioglu@metu.edu.tr
†Electronic address: btekin@metu.edu.tr
[1] Their parameter σ can be thought of as the remaining unconstrained p.

spacetimes, respectively. Here we should also note that special forms of (1) with specific choices for the constants p_k have already appeared in the literature for $D = 5$ and $D = 7$ [6, 7, 8].

The singularity structure of the solution (1) is apparent from its Kretschmann scalar, which we give here only for the $n = 3$ and $n = 4$ cases [2]. For $n = 3$, it reads

$$R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} = \frac{24}{\ell^4} + \frac{12}{\ell^4}\frac{r_0^6}{r^6} - \frac{81}{\ell^4}\frac{r_0^9}{r^9}\,\frac{(1+h(r))}{h(r)^2}\,\prod_{k=0}^{2} p_k\,,$$

where $h(r) \equiv 1 - r_0^3/r^3$. As for the $n = 4$ case, it will be convenient to first define $f(r) \equiv 1 - r_0^4/r^4$ and $\Delta \equiv p_0p_1p_2 + p_0p_1p_3 + p_0p_2p_3 + p_1p_2p_3$. Then

$$R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} = \frac{40}{\ell^4} + \frac{72}{\ell^4}\frac{r_0^8}{r^8} - \frac{64}{\ell^4}\frac{r_0^{12}}{r^{12}}\,\frac{1}{f(r)^2}\left(\Delta(4+5f(r)) + 2\,\frac{r_0^4}{r^4}\prod_{k=0}^{3} p_k\right).$$

For the generic choice of the constants p_k, one clearly finds naked singularities at $r = r_0$ and $r = 0$. Remarkably, the AdS soliton is the unique solution with no naked singularities provided that $r \geq r_0$ and the conical singularity at r_0 is avoided by a proper compactification of the corresponding coordinate [5].

It may be of interest to write (1) in different coordinate systems. A somewhat obvious possibility is to consider the coordinate transformation $r = r_0 \cosh^{2/n}(n\rho/(2\ell))$ which takes (1) to

$$ds^2 = d\rho^2 + \frac{r_0^2}{\ell^2}\cosh^{4/n}\left(\frac{n\rho}{2\ell}\right)\left(-\tanh^{2p_0}\left(\frac{n\rho}{2\ell}\right)dt^2 + \sum_{k=1}^{n-1}\tanh^{2p_k}\left(\frac{n\rho}{2\ell}\right)(dx^k)^2\right), \qquad (3)$$

where again the constants p_k are subject to (2), of course.

The interpretation of the constants p_k and r_0 in terms of physical parameters would be of interest. As a step in this direction, we calculate the mass of the solutions (1). For this purpose, we use the procedure described in [9, 10] that requires a choice of a background and a perturbation about it. The correct background in this case is the usual AdS metric obtained from (1) by simply setting r_0 to zero. The background Killing vector that leads to mass/energy is $\bar{\xi}^\mu = -(\partial/\partial t)^\mu$, which in our case yields

$$M = \frac{V_{n-1}}{4\,G_D\,\Omega_{n-2}}\,\frac{r_0^n}{\ell^{n+1}}\,(np_0 - 1)\,, \qquad (4)$$

where V_{n-1} is the volume of the transverse dimensions. The mass of the AdS soliton, for which there is only one non-zero p_k where $k \neq 0$, has already been considered in [5, 11]. It was conjectured in [5] that the AdS soliton, with its negative mass, has the lowest possible energy, (hence the new 'positive' mass conjecture) in its asymptotic class. At first sight, the result (4) seems to contradict this since one can have a solution with $p_0 < 0$ leading to $M < M_{soliton}$. However, one then has naked singularities in which case any kind of 'positive' mass theorem fails. Observing that the AdS soliton is the unique solution with no naked singularities, our result lends further support to the new 'positive' mass conjecture of [5] and the "uniqueness" result of [12]. Amusingly, if one chooses $p_0 = 1/n$, one ends up with massless non-flat $(n-1)$-branes.

To conclude, we have found new Einstein spaces in higher dimensions, generalizing the four dimensional cosmological cylindrically symmetric Levi-Civita spacetimes. We have calculated the mass of these metrics and identified the AdS soliton as a rather unique member.

[2] Its calculation gets rather complicated beyond these dimensions, but is not different in general features.

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