On the reduction principle for the hybrid equation

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Proposed running head: The Reduction Principle

Abstract

In this paper we introduce a new type of differential equations with piecewise constant argument (EPCAG), more general than EPCA [11, 41]. The Reduction Principle [35] is proved for EPCAG. The continuation of solutions is investigated. We establish the existence of global integral manifolds of quasilinear EPCAG and investigate the stability of the zero solution. Since the method of reduction to discrete equations [11] is difficult to apply to EPCAG, a new technique of investigation of equations with piecewise argument, based on an integral representation formula, is proposed. The approach can be fruitfully applied for investigation stability, oscillations, controllability and many other problems of EPCAG.

1 Introduction and Preliminaries

1.1 Definitions and the description of the system

The theory of integral manifolds was founded by H. Poincaré and A. M. Lyapunov [37, 29], and it became a very powerful instrument for investigating various problems of the qualitative theory of differential equations. For the last several decades, many researchers have been studying the methods of reducing high dimensional problems to low dimensional ones. When discussing this problem for long-time dynamics of differential equations, we should consider the Reduction Principle [35, 36]. One can read about the history of the principle in [27, 30, 35] and papers cited there. The principle was utilized in the center manifold theory, as well as in the theory of inertial manifolds [7, 18, 20]. It is natural that the exploration of the properties and neighborhoods of manifolds is one of the most interesting problems of the theory of differential equations [6, 7, 9, 15, 16, 23, 26, 33, 38]. One should not be surprised that manifolds and the reduction principle are one of the major subjects of investigation for specific types of differential and difference equations [2, 5, 8, 10, 14, 15, 16, 17, 18, 20, 28, 34, 38, 39, 43]. Our main goal in this paper is to extend the principle to the differential equations with piecewise constant

argument of generalized type. For this purpose, we have developed another approach to the investigation, different from what was proposed by the founders of the EPCA theory [11, 41].

Let $\mathbb{Z}, \mathbb{N}$ and $\mathbb{R}$ be the sets of all integers, natural and real numbers, respectively. Denote by $|| \cdot ||$ the Euclidean norm in $\mathbb{R}^n$, $n \in \mathbb{N}$. Fix two real-valued sequences $\theta_i, \zeta_i, i \in \mathbb{Z}$, such that $\theta_i < \theta_{i+1}, \theta_i \leq \zeta_i \leq \theta_{i+1}$ for all $i \in \mathbb{Z}$, $|\theta_i| \to \infty$ as $|i| \to \infty$, and there exists a number $\theta > 0$ such that $\theta_{i+1} - \theta_i \leq \theta, i \in \mathbb{Z}$. In this paper we are concerned with the quasilinear system

$$z' = Az + f(t, z(t), z(\beta(t))), \tag{1}$$

where $z \in \mathbb{R}^n, t \in \mathbb{R}, \beta(t) = \zeta_i$, if $t \in [\theta_i, \theta_{i+1}), i \in \mathbb{Z}$. One can easily see that equation (1) has the form

$$z' = Az + f(t, z(t), \bar{z}), \tag{2}$$

if $t \in [\theta_i, \theta_{i+1}), \bar{z} = z(\zeta_i), i \in \mathbb{Z}$.

The theory of differential equations with piecewise constant argument (EPCA) of the type

$$\frac{dx(t)}{dt} = f(t, x(t), x([t])), \tag{3}$$

where $[\cdot]$ signifies the greatest integer function, was initiated in [11] and has been developed by many authors [4, 12, 22, 24, 34, 41, 42]. They are hybrid equations, in that they combine the properties of both continuous systems and discrete equations. For example, even a scalar logistic equation may produce chaos [19, 21, 25], when the solutions are continuous functions.

The novel idea of our paper is that system (1) is EPCA of general type (EPCAG) for equation (3). Indeed if we take $\zeta_i = \theta_i = i, i \in \mathbb{Z}$, then (1) takes the form of (3). Another EPCA which can be easily written as EPCAG is the equation alternately of retarded and advanced type [12, 41]

$$\frac{dx(t)}{dt} = f(t, x(t), x(2[(t+1)/2])). \tag{4}$$

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One can check that (1) takes the form of (4) if $\theta_i = 2i - 1, \zeta_i = 2i, i \in \mathbb{Z}$. Moreover, the system considered in [3] is a particular case of (1), too.

The following assumptions will be needed throughout the paper:

(C1) A is a constant $n \times n$ real valued matrix;

(C2) $f(t, x, z)$ is continuous in the first argument, $f(t, 0, 0) = 0, t \in \mathbb{R}$, and f is Lipschitzian in the second and third arguments with a positive Lipschitz constant l such that

$$||f(t, z_1, w_1) - f(t, z_2, w_2)|| \leq l(||z_1 - z_2|| + ||w_1 - w_2||)$$

for all $t \in \mathbb{R}$ and $z_1, z_2, w_1, w_2 \in \mathbb{R}^n$.

(C3) If we denote by $\lambda_j, j = \overline{1, n}$, the eigenvalues of matrix A, then there exists a positive integer k such that $\mu = \max_{j=\overline{1,k}} \Re\lambda_j < 0$, and $\min_{j=\overline{k+1,n}} \Re\lambda_j = 0$, where $\Re\lambda_j$ denotes the real part of the eigenvalue λ_j of matrix A.

The previous condition implies that, without loss of generality, we can assume that

(C4)

$$A = \begin{pmatrix} B_+ & 0 \\ 0 & B_- \end{pmatrix},$$

where square matrices B_+ and B_- are of dimension k and $n-k$ respectively, $\lambda_j, j = \overline{1, k}$, are eigenvalues of the matrix B_- and $\lambda_j, j = \overline{k+1, n}$, are the eigenvalues of matrix B_+.

The existing method of investigation of EPCA, as proposed by founders, is based on the reduction of EPCA to discrete equations. It is obvious that this method is not applicable to the present problem. A new approach is based on the construction of an equivalent integral equation. Consequently, we prove a corresponding equivalence lemma for every result of our paper. Thus, when

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investigating EPCAG, we need not impose any conditions on the reduced discrete equations, and hence require more easily verifiable conditions, similar to those for ordinary differential equations. It becomes less cumbersome to solve the problems of EPCAG theory (as well as of EPCA theory).

The theory of EPCAG (EPCA) necessitates a more careful discussion of the continuation problem. The subject of backward continuation for functional differential equations was considered in [16]. In our paper it is necessary to analyze the forward continuation, too, as we also deal with equations alternately of retarded and advanced type. The backward continuation of the solutions of EPCA was investigated in [11] through the solvability of certain difference equations. For our needs, we shall introduce less formal definitions than those in [11], since we consider integral manifolds, and it is natural to discuss the global continuation of a solution of (1) as well as its uniqueness on these manifolds.

DEFINITION 1.1 *A solution* $z(t, t_0, z_0), \zeta_i < t_0 \le \theta_{i+1}$, *of (1) is said to be backward continued to* $t = \zeta_i$ *if there exists a solution* $z(t, \zeta_i, \bar{z})$ *of (2) such that* $z(t_0, \zeta_i, \bar{z}) = z_0$. *The solution* $z(t, t_0, z_0)$ *is uniquely backward continued to* $t = \zeta_i$ *if the continuation is unique.*

DEFINITION 1.2 *A solution* $z(t, t_0, z_0), \theta_i \le t_0 < \zeta_i$, *of (1) is said to be forward continued to* $t = \zeta_i$, *if there exists a solution* $z(t, \zeta_i, \bar{z})$ *of (2) such that* $z(t_0, \zeta_i, \bar{z}) = z_0$. *The solution* $z(t, t_0, z_0)$ *is uniquely continued to* $t = \zeta_i$ *if the continuation is unique.*

The following example shows that even for simple EPCAG the continuation of some solutions can fail.

EXAMPLE 1.1 *Consider the following EPCA*

$$z' = 3z - z^2(2[(t+1)/2]), \tag{5}$$

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where $z \in \mathbb{R}, t \in \mathbb{R}$. Let us show that not all solutions of (5) can be forward continued. Consider the interval $[-1, 0]$. Fix $z_0 \in \mathbb{R}$, and let $z(t, 0, z_0)$ be a solution of (5). It is clear that for a solution $z(t, -1, x_0)$ to be forward continued to $t = 0$, in the sense of Definition 1.1, the equation $[e^3 - 1]z_0^2 - e^3 z_0 - x_0 = 0$ must be solvable with respect to z_0. Since this equation can not be solved for all $z_0 \in \mathbb{R}$, the assertion is proved. Further, we shall consider the uniqueness of the continuation. Now let us focus on the backward continuation. Consider the interval $[0, 1]$. Fix numbers $z_0, z_1 \in \mathbb{R}$ such that $(z_0 + z_1)(1 - e^3) = e^3$. Denote $z_0(t) = z(t, 0, z_0)$ and $z_1(t) = z(t, 0, z_1)$, which are solutions of (5). They are the solutions of the equations $z' = 3z - z_0^2$ and $z' = 3z - z_1^2$, respectively, and they are defined on $[0, 1)$. Since $z_j(t) = e^{3t}z_j + \int_0^t e^{3(t-s)}z_j^2 ds, j = 0, 1$, one can obtain, using the continuity of solutions, that $z_0(1) = z_1(1)$. That is, the solution $z(t, 1, z_1(1))$ of (5) can not be continued back to $t = 0$ uniquely.

Next we consider the construction procedure for a solution of an initial value problem. We define the solution only for decreasing t, but one can easily see that the definition is similar for increasing t.

Let us assume that $\theta_i \leq \zeta_i < t_0 \leq \theta_{i+1}$ for some $i \in \mathbb{Z}$. Suppose that $z(t) = z(t, t_0, z_0)$ is back continued from t_0 to $t = \zeta_i$ in the sense of Definition 1.1. Then conditions $(C1)$ and $(C2)$ imply that $z(t)$ can be continued to $t = \theta_i$, as it is a solution of the following system of ordinary differential equations $z' = Az + f(t, z(t), z(\zeta_i))$ on $[\theta_i, \theta_{i+1})$.

Next, we suppose that $z(t, \theta_i, z(\theta_i, t_0, z_0))$ is back continued from $t = \theta_i$ to $t = \zeta_{i-1}$ in the sense of Definition 1.1. Then again we can conclude that $z(t)$ can be continued to $t = \theta_{i-1}$. If $z(t, \theta_{i-1}, z(\theta_{i-1}, t_0, z_0))$ is back continued from θ_{i-1} to $t = \zeta_{i-2}$ in the sense of Definition 1.1, then $z(t)$ can be continued to $t = \theta_{i-2}$. Proceeding in this way and assuming that $z(t, \theta_j, z(\theta_j, t_0, z_0))$ is back continued from θ_j to $t = \zeta_{j-1}$ in the sense of Definition 1.1 for all $j \leq i$, we can find that

$z(t, t_0, z_0)$ is back continuable to $-\infty$. *We call this continuation of $z(t, t_0, z_0)$ to $-\infty$ as a solution of (1) on $(-\infty, t_0]$.* Similarly, one can define a solution as the continuation on an interval $[t_0, \infty)$ as time is increasing. On the basis of the above discussion we can conclude that the following theorem is valid.

THEOREM 1.1 *Assume that conditions (C1) and (C2) hold, ζ_i is the maximal among ζ_j which are smaller than t_0. Then $z(t, t_0, z_0)$ exists on $(-\infty, t_0]$ if and only if $z(t, \theta_j, z(\theta_j, t_0, z_0))$ is back continuable to ζ_{j-1} in the sense of Definition 1.1 for all $j \leq i$.*

A similar theorem can be proved for forward continuation.

In what follows we shall say that a solution $z(t)$ *is continued* if it *is continued backward* or/and *forward*.

DEFINITION 1.3 *A function $z(t) = z(t, t_0, z_0), z(t_0) = z_0, \theta_i \leq t_0 < \theta_{i+1}, i \in \mathbb{Z}$, is a solution of (1) on the interval $[\theta_i, \infty)$ if the following conditions are fulfilled:*

(i) *$z(t)$ is continuable to $t = \zeta_i$;*

(ii) *the derivative $z'(t)$ exists at each point $t \in [\theta_i, \infty)$ with the possible exception of the points $\theta_j \in [\theta_i, \infty)$ where one-sided derivatives exist;*

(iii) *equation (1) is satisfied for $z(t)$ at each point $t \in [\theta_i, \infty) \backslash \{\theta_j\}$, and it holds for the right derivative of $z(t)$ at the points $\theta_j \in [\theta_i, \infty), j \in \mathbb{Z}$.*

Remark 1.1 One can see that Definition 1.3 is a slightly changed version of a definition from [11], adapted for our general case.

DEFINITION 1.4 *A function $z(t) = z(t, t_0, z_0), z(t_0) = z_0, \theta_i < t_0 \leq \theta_{i+1}, i \in \mathbb{Z}$, is a solution of (1) on the interval $(-\infty, \theta_{i+1}]$ if the following conditions are fulfilled:*

(i) *$z(t)$ is continuable to $t = \zeta_i$;*

(ii) the derivative $z'(t)$ exists at each point $t \in (-\infty, \theta_{i+1})$ with the possible exception of the points $\theta_j \in (-\infty, \theta_{i+1})$, where the one-sided derivatives exist;

(iii) equation (1) is satisfied with $z(t)$ at each point $t \in (-\infty, \theta_{i+1}) \backslash \{\theta_i\}$, and at the points $\theta_j \in (-\infty, \theta_{i+1})$ it holds for the right derivative of $z(t)$.

We shall also use the following definition, which is a version of a definition from [34], modified for our general case.

DEFINITION 1.5 *A function $z(t)$ is a solution of (1) on $\mathbb{R}$ if:*

(i) $z(t)$ is continuous on $\mathbb{R}$;

(ii) the derivative $z'(t)$ exists at each point $t \in \mathbb{R}$ with the possible exception of the points $\theta_i, i \in \mathbb{Z}$, where the one-sided derivatives exist;

(iii) equation (1) is satisfied for $z(t)$ on each interval $(\theta_i, \theta_{i+1}), i \in \mathbb{Z}$, and it holds for the right derivative of $z(t)$ at the points $\theta_i, i \in \mathbb{Z}$.

DEFINITION 1.6 *A set Σ in the $(t, z)-$ space is said to be an integral set of system (1) if any solution $z(t) = z(t, t_0, z_0), z(t_0) = z_0$, with $(t_0, z_0) \in \Sigma$, has the property that $(t, z(t)) \in \Sigma, t \in \mathbb{R}$. In other words, for every $(t_0, z_0) \in \Sigma$ the solution $z(t) = z(t, t_0, z_0), z(t_0) = z_0$, is continuable on $\mathbb{R}$ and $(t, z(t)) \in \Sigma, t \in \mathbb{R}$.*

DEFINITION 1.7 *A set Σ in the $(t, z)-$ space is said to be a local integral set of system (1) if for every $(t_0, z_0) \in \Sigma$ there exists $\epsilon > 0, \epsilon = \epsilon(t_0, z_0)$, such that if $z(t) = z(t, t_0, z_0)$ is a solution of (1) and $|t - t_0| < \epsilon$ then $(t, z(t)) \in \Sigma$.*

1.2 The existence and uniqueness of solutions on $\mathbb{R}$

In what follows we use the uniform norm $||T|| = \sup\{||Tz||||z|| \le 1\}$ for matrices.

It is known that there exists a constant $\Omega > 0$ such that $||e^{A(t-s)}|| \le e^{\Omega|t-s|}$, $t, s \in \mathbb{R}$. Hence, one can show that

$$||e^{A(t-s)}|| \ge e^{-\Omega|t-s|}, t, s \in \mathbb{R}.$$

The last two inequalities imply the following, very simple but useful in what follows, estimates

$$||e^{A(t-s)}|| \le M, \quad ||e^{A(t-s)}|| \ge m,$$

if $|t - s| \le \theta$, where $M = e^{\Omega\theta}, m = e^{-\Omega\theta}$.

From now on we make the assumption:

(C5) $Ml\theta e^{Ml\theta} < 1$,

$\quad\quad 2Ml\theta < 1$,

$\quad\quad M^2 l\theta \left\{ \frac{Ml\theta e^{Ml\theta}+1}{1-Ml\theta e^{Ml\theta}} + Ml\theta e^{Ml\theta} \right\} < m.$

THEOREM 1.2 *Assume that conditions $(C1)-(C3)$, and $(C5)$ are fulfilled. Then for every $(t_0, z_0) \in \mathbb{R} \times \mathbb{R}^n$ there exists a solution $z(t) = z(t, t_0, z_0)$ of (1) which is defined on $\mathbb{R}$ and is unique.*

Proof. The existence of the solution. Let us consider only backward continuation, since forward continuation can be investigated in a similar manner. Theorem 1.1 implies that it is sufficient to consider the continuation of a solution $z(t) = z(t, \theta_i, z(\theta_i, t_0, z_0))$ from θ_i to ζ_{i-1}, for all $i \in \mathbb{Z}$. We have that

$$z(t) = e^{A(t-\theta_i)} z(\theta_i) + \int_{\theta_i}^{t} e^{A(t-s)} f(s, z(s), z(\zeta_{i-1})) ds$$

on $[\zeta_{i-1}, \theta_i]$.

Define a norm $||z(t)||_0 = \max_{[\zeta_{i-1}, \theta_i]} ||z(t)||$, and take $z_0(t) = e^{A(t-\theta_i)} z(\theta_i)$. Define a sequence

$$z_{m+1}(t) = e^{A(t-\theta_i)} z(\theta_i) + \int_{\theta_i}^{t} e^{A(t-s)} f(s, z_m(s), z_m(\zeta_{i-1})) ds, \ m \geq 0.$$

The last expression implies that

$$||z_{m+1}(t) - z_m(t)||_0 \leq [2Ml\theta]^{m+1} M ||z(\theta_i)||.$$

The existence is proved.

The uniqueness of the solution. Denote $z_j(t) = z(t, t_0, z_0^j), z_j(t_0) = z_0^j, j = 1, 2$, solutions of (1), where $\theta_i \leq t_0 \leq \theta_{i+1}$. It is sufficient to check that for every $t \in [\theta_i, \theta_{i+1}], z_0^1 \neq z_0^2$ implies $z_1(t) \neq z_2(t)$. We have that

$$z_1(t) - z_2(t) = e^{A(t-\theta_i)} (z_0^2 - z_0^1) - \int_{t_0}^{t} e^{A(t-s)} [f(s, z_1(s), z_1(\zeta_i) - f(s, z_2(s), z_2(\zeta_i))] ds.$$

Hence,

$$||z_1(t) - z_2(t)|| \leq M ||z_0^2 - z_0^1|| + Ml\theta ||z_1(\zeta_i) - z_2(\zeta_i)|| + Ml \left| \int_{t_0}^{t} ||z_1(s) - z_2(s)|| ds \right|.$$

The Gronwall-Bellman Lemma yields that

$$||z_1(t) - z_2(t)|| \leq M(||z_0^2 - z_0^1|| + l\theta ||z_1(\zeta_i) - z_2(\zeta_i)||) e^{Ml\theta}.$$

Particularly,

$$||z_1(\zeta_i) - z_2(\zeta_i)|| \leq M(||z_0^2 - z_0^1|| + l\theta ||z_1(\zeta_i) - z_2(\zeta_i)||) e^{Ml\theta}.$$

Then,

$$||z_1(\zeta_i) - z_2(\zeta_i)|| \leq \frac{M}{1 - Ml\theta e^{Ml\theta}} ||z_0^2 - z_0^1||.$$

Hence,

$$||z_1(t) - z_2(t)|| \leq M e^{Ml\theta} [1 + \frac{Ml\theta}{1 - Ml\theta e^{Ml\theta}}] ||z_0^2 - z_0^1||. \tag{6}$$

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Assume on the contrary that there exists $t \in [\theta_i, \theta_{i+1}]$ such that $z_1(t) = z_2(t)$. Then

$$e^{A(t-t_0)}(z_0^1 - z_0^2) =$$
$$\int_{t_0}^t e^{A(t-s)}[f(s, z_2(s), z_2(\zeta_i)) - f(s, z_1(s), z_1(\zeta_i))]ds. \tag{7}$$

We have that

$$||e^{A(t-t_0)}(z_0^2 - z_0^1)|| \geq m||z_0^2 - z_0^1||. \tag{8}$$

Moreover, (6) implies that

$$||\int_{t_0}^t e^{A(t-s)}[f(s, z_1(s), z_1(\zeta_i)) - f(s, z_2(s), z_2(\zeta_i)]ds|| \leq$$
$$M^2 l\theta \left\{ \frac{Ml\theta e^{Ml\theta} + 1}{1 - Ml\theta e^{Ml\theta}} + Ml\theta e^{Ml\theta} \right\} ||z_0^2 - z_0^1||. \tag{9}$$

Finally, one can see that $(C5)$, (8) and (9) contradict (7). The theorem is proved.

Remark 1.2 Inequality (6) implies continuous dependence of solutions of (1) on the initial value.

2 The existence of integral surfaces

Fix a number $\sigma \in \mathbb{R}$ such that $\mu < -\sigma < 0$. Clearly, there exist constants $K \geq 1$ and $m \in \mathbb{N}, m < n - k$, such that

$$||e^{B_+ t}|| \leq Ke^{-\sigma t}, \text{ and } ||e^{-B_- t}|| \leq K(1 + t^m),$$

for all $t \in R_+ = [0, \infty)$.

Using condition $(C4)$ one can write equation (1) as the following system

$$\frac{du}{dt} = B_+ u + f_+(t, z(t), z(\beta(t))),$$
$$\frac{dv}{dt} = B_- v + f_-(t, z(t), z(\beta(t))), \tag{10}$$

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where $z = (u,v), u \in R^k, v \in R^{n-k}, (f_+, f_-) = f(t, z(t), z(\beta(t)))$.

Fix a number $\alpha, 0 < \alpha < \sigma$, and denote

$$\gamma = \int_0^\infty (1 + t^m) e^{-\alpha t} dt.$$

We shall establish the validity of the following lemma.

LEMMA 2.1 *Fix $N \in \mathbb{R}, N > 0$, and assume that conditions $(C1) - (C3)$ are valid. A continuous function $z(t) = (u, v), ||z(t)|| \le N e^{-\alpha(t-t_0)}, t \ge t_0$, is a solution of (1) on $\mathbb{R}$ if and only if $z(t)$ is a solution on $\mathbb{R}$ of the following system of integral equations*

$$u(t) = e^{B_+(t-t_0)} u(t_0) + \int_{t_0}^t e^{B_+(t-s)} f_+(s, z(s), z(\beta(s))) ds,$$

$$v(t) = -\int_t^\infty e^{B_-(t-s)} f_-(s, z(s), z(\beta(s))) ds. \tag{11}$$

Proof. Necessity. Assume that $z(t) = (u,v), ||z(t)|| \le N e^{-\alpha(t-t_0)}, t \in [t_0, \infty)$, is a solution of (1). Denote

$$\phi(t) = e^{B_+(t-t_0)} u(t_0) + \int_{t_0}^t e^{B_+(t-s)} f_+(s, z(s), z(\beta(s))) ds,$$

$$\psi(t) = -\int_t^\infty e^{B_-(t-s)} f_-(s, z(s), z(\beta(s))) ds. \tag{12}$$

By straightforward evaluation we can see that the integrals converge, are bounded on $[t_0, \infty)$, and, moreover,

$$||\phi(t)|| \le K e^{-\sigma(t-t_0)} ||u(t_0)|| +$$
$$Kl \left[\frac{N(1 + e^{\alpha\theta})}{\sigma - \alpha} + 2\frac{1}{\sigma} \max_{[\theta_i, \theta_{i+1}]} ||z(s)|| e^{\sigma(|t_0| + \theta)} \right] e^{-\alpha(t-t_0)},$$
$$||\psi(t)|| \le Kl\gamma N(1 + e^{\alpha\theta}) e^{-\alpha(t-t_0)}. \tag{13}$$

If $t \ne \theta_i, i \in \mathbb{Z}$, then

$$\phi'(t) = B_+\phi(t) + f_+(t, z(t), z(\beta(t))),$$

$$\psi'(t) = B_-\psi(t) + f_-(t, z(t), z(\beta(t))),$$

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and

$$u'(t) = B_+ u(t) + f_+(t, z(t), z(\beta(t))),$$

$$v'(t) = B_- v(t) + f_-(t, z(t), z(\beta(t))).$$

Hence,

$$[\phi(t) - u(t)]' = B_+[\phi(t) - u(t)],$$

$$[\psi(t) - v(t)]' = B_-[\psi(t) - v(t)].$$

Calculating the limit values at $\theta_j \in \mathbb{Z}$ we can find that

$$\phi'(\theta_j \pm 0) = B_+ \phi(\theta_j \pm 0) + f_+(\theta_j \pm 0, z(\theta_j \pm 0), z(\beta(\theta_j \pm 0))),$$

$$u'(\theta_j \pm 0) = B_+ u(\theta_j \pm 0) + f_+(\theta_j \pm 0, z(\theta_j \pm 0), z(\beta(\theta_j \pm 0))),$$

$$\psi'(\theta_j \pm 0) = B_+ \psi(\theta_j \pm 0) + f_(\theta_j \pm 0, z(\theta_j \pm 0), z(\beta(\theta_j \pm 0))),$$

$$v'(\theta_j \pm 0) = B_+ v(\theta_j \pm 0) + f_-(\theta_j \pm 0, z(\theta_j \pm 0), z(\beta(\theta_j \pm 0))).$$

Consequently,

$$[\phi(t) - u(t)]'|_{t=\theta_j+0} = [\phi(t) - u(t)]'|_{t=\theta_j-0}, [\psi(t) - v(t)]'|_{t=\theta_j+0} = [\psi(t) - v(t)]'|_{t=\theta_j-0}.$$

Thus, $(\phi(t) - u(t), \psi(t) - v(t))$ is a continuously differentiable on $\mathbb{R}$ function satisfying $u'(t) = B_+ u(t), v'(t) = B_- v(t)$ with the initial condition $\phi(t_0) - u(t_0) = 0$. Assume that $\psi(t_0) - v(t_0) \neq 0$. Then $\psi(t) - v(t)$ is not a decay solution, which contradicts (13). Hence, $\phi(t) - u(t) = 0, \psi(t) - v(t) = 0$ on $\mathbb{R}$.

Sufficiency. Suppose that $z(t)$ is a solution of (11). Differentiating $z(t)$ in $t \in (\theta_i, \theta_{i+1})$, $i \in \mathbb{Z}$, one can see that the function satisfies (1). Moreover, letting $t \to \theta_i+$, and remembering that $z(\beta(t))$ is a right-continuous function, we find that $z(t)$ satisfies (1) on $[\theta_i, \theta_{i+1})$. The Lemma is proved.

Denote

$$p = K(1 + e^{\alpha\theta})[\frac{1}{\sigma - \alpha} + \gamma].$$

In what follows we mainly use the technique of [36]. See also [7, 33].

THEOREM 2.1 *Suppose conditions* $(C1) - (C5)$ *are fulfilled and, moreover,*

$$2pl < 1. \tag{14}$$

Then for arbitrary $\alpha \in (0, \sigma)$ *there exists a function* $F(\zeta_i, u), i \in \mathbb{Z}$, *satisfying*

$$F(\zeta_i, 0) = 0, \tag{15}$$

$$||F(\zeta_i, u_1) - F(\zeta_i, u_2)|| \leq pKl||u_1 - u_2||, \tag{16}$$

for all i, u_1, u_2, *such that a solution* $z(t)$ *of (1) with* $z(\zeta_i) = (c, F(\zeta_i, c), c \in \mathbb{R}^k$, *is defined on* $\mathbb{R}$ *and satisfies*

$$||z(t)|| \leq 2K||c||e^{-\alpha(t-\zeta_i)}, t \geq \zeta_i. \tag{17}$$

Proof. Let us consider system (11) and apply the method of successive approximations. Denote $z_0(t) = (0, 0)^T, z_m = (u_m, v_m)^T, m \in \mathbb{N}$, where for $m \geq 0$

$$u_{m+1}(t) = e^{B_+(t-\zeta_i)}c + \int_{\zeta_i}^t e^{B_+(t-s)} f_+(s, z_m(s), z_m(\beta(s)))ds,$$

$$v_{m+1}(t) = - \int_t^\infty e^{B_-(t-s)} f_-(s, z_m(s), z_m(\beta(s)))ds.$$

Let us show that

$$||z_m(t)|| \leq 2K||c||e^{-\alpha(t-\zeta_i)}, t \leq \zeta_i. \tag{18}$$

Indeed, z_0 satisfies the relation. Assume that z_{m-1} satisfies (18). Then

$$||u_m(t)|| \leq Ke^{-\sigma(t-\zeta_i)}||c|| + \frac{2K^2l(1 + e^{\alpha\theta})}{\sigma - \alpha}e^{-\alpha(t-\zeta_i)}||c||,$$

$$||v_m(t)|| \leq 2\gamma K^2l(1 + e^{\alpha\theta})e^{-\alpha(t-\zeta_i)}||c||, \tag{19}$$

and (18) is valid provided (14) is correct. Similarly, one can establish the following inequality

$$||z_{m+1}(t) - z_m(t)|| \leq K||c||(2pl)^m e^{-\alpha(t-\zeta_i)}. \tag{20}$$

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The last inequality and assumption (14) imply that the sequence z_m converges uniformly for all c and $t \geq \zeta_i$. Let $z(t, \zeta_i, c) = (u(t, \zeta_i, c), v(t, \zeta_i, c))$ be the limit function. It is obvious that the function is a solution of (11). By Lemma 2.1 $z(t, \zeta_i, c)$ is a solution of (1), too. Taking $t_0 = \zeta_i$ in (11) we have that

$$u(\zeta_i, \zeta_i, c) = c,$$

$$v(\zeta_i, \zeta_i, c) = -\int_{\zeta_i}^{\infty} e^{B_-(t-s)} f_-(s, z(s, \zeta_i, c), z(\beta(s), \zeta_i, c))) ds.$$

Denote $F(\zeta_i, c) = v(\zeta_i, \zeta_i, c)$. Since

$$||v_m(t, \zeta_i, c_1) - v_m(t, \zeta_i, c_2)|| \leq pKl||c_1 - c_2||, \ m \geq 1, \tag{21}$$

inequality (16) is valid. The theorem is proved.

For every $i \in \mathbb{Z}$ consider a set Ψ_i of continuous on $\mathbb{R}$ functions such that if $\psi \in \Psi_i$ then there exists a positive constant K_ψ, satisfying $||\psi(t)|| \leq K_\psi e^{-\alpha(t-\zeta_i)}, \zeta_i \leq t$, where constant α is defined for Theorem 2.1.

LEMMA 2.2 *For every* $\zeta_i, i \in \mathbb{Z}, c \in \mathbb{R}^k$, *the system*

$$u(t) = e^{B_+(t-t_0)} c + \int_{\zeta_i}^{t} e^{B_+(t-s)} f_+(s, z(s), z(\beta(s))) ds,$$

$$v(t) = -\int_{t}^{\infty} e^{B_-(t-s)} f_-(s, z(s), z(\beta(s))) ds.$$

has only one solution from Ψ_i.

Proof. If z_1 and z_2 are two solutions of the system bounded on $[\zeta_i, \infty)$, then by straightforward evaluation one can show that

$$\sup_{[\zeta_i, \infty)} ||z_1 - z_2|| \leq 2pl \sup_{[\zeta_i, \infty)} ||z_1 - z_2||.$$

Hence, in view of (14) the lemma is proved.

Let us denote by S_i^+ the set of all points from the $(t, z)-$ space $(z = (u, v))$ such that $t = \zeta_i$, $v = F(\zeta_i, u)$.

LEMMA 2.3 *If $(\zeta_i, z_0) \notin S_i^+$, then the solution $z(t, \zeta_i, z_0)$ of (1) is not from Ψ_i.*

Proof. Assume on the contrary that $z(t) \in \Psi_i, z(t) = z(t, \zeta_i, z_0) = (u, v)$, is a solution of (1) and $(\zeta_i, z_0) \notin S_i^+$. It is obvious that

$$u(t) = U(t, \zeta_i)u(\zeta_i) + \int_{\zeta_i}^t U(t, s)f_+(s, z(s), z(\beta(s)))ds,$$

$$v(t) = V(t, \zeta_i)\hat{\kappa} - \int_t^\infty V(t, s)f_-(s, z(s), z(\beta(s)))ds, \qquad (22)$$

where

$$\hat{\kappa} = v(\zeta_i) + \int_{\zeta_i}^\infty V(t, s)g_-(s, z(s), z(\beta(s)))ds,$$

and the improper integral converges and is bounded on $[\zeta_i, \infty)$. But condition $(C3)$ on eigenvalues of matrix B_- imply that $||V(t, \zeta_i)\hat{\kappa}|| \to 0$ as $t \to \infty$ if only $\hat{\kappa} = 0$. By Lemma 2.1 $z(t)$ satisfies (11) with $t_0 = \zeta_i$. The contradiction proves the lemma.

Let S^+ be the set of all points from the $(t, z)-$ space $(z = (u, v))$ such that either $(t, z) \in S_i^+$ for some $i \in \mathbb{Z}$, or there exist $\zeta_i, \zeta_i < t, c \in \mathbb{R}^k$, such that $(\zeta_i, c) \in S_i^+$ and $z = z(t, \zeta_i, c)$.

THEOREM 2.2 *S^+ is an invariant set.*

Proof. Assume that $(\zeta_i, u_0, v_0) \in S_i^+, z_0 = (u_0, v_0)$. We show that if $z(t) = z(t, \zeta_i, z_0)$, then $(\theta_j, z(\theta_j)) \in S_j^+$ for all $j \geq i$. Indeed, if $t \geq \theta_j$ then $||z(t)|| \leq (K + \epsilon)||u_0||e^{-\alpha(t-\theta_j)}e^{-\alpha(\theta_j-\zeta_i)}$. Lemma 2.1 implies that the point $(\theta_j, z(\theta_j))$ satisfies the equation $v = F(t, u)$. If $(\theta, \xi) \in S^+ \backslash \cup_{i \in \mathbb{Z}} S_i^+$, then by the definition and the previous part of the proof $z(t, \theta, \xi) \in S^+$ for all $t \geq \theta$. Assume that $(\zeta_i, z_0) \in S_i^+$, and denote $z(t) = z(t, \zeta_i, z_0)$. Lemma 2.3 implies that $(z(\theta_{i-1}), \theta_{i-1}) \in S_j^+$. The theorem is proved.

On the basis of Theorem 2.2, Lemmas 2.1 and 2.3 we can conclude that there exists an invariant surface S^+ of equation (1), such that every solution starting at S^+ tends to zero as $t \to \infty$.

Denote $z(t, r, c) = (u(t, r, c), v(t, r, c)), t, r \in \mathbb{R}, c \in \mathbb{R}^k$, a solution of (1) such that $u(r, r, c) = c$. From the discussion above it can be seen that surface S^+ contains solutions which satisfy the equation $v = F(t, u), (t, u) \in \mathbb{R} \times \mathbb{R}^k$, where

$$F(t, u) = -\int_t^\infty e^{B_-(t-s)} f_-(s, z(s, t, u), z(\beta(s), t, u)) ds. \tag{23}$$

It is obvious that $F(t, u)$ is a function continuous in both arguments.

THEOREM 2.3 *Suppose conditions* $(C1) - (C5)$ *are fulfilled. Then for an arbitrarily small positive* $\tilde{\alpha}$ *and a sufficiently small Lipschitz constant* l *there exists a function* $G(\zeta_i, v), i \in \mathbb{Z}$, *from* $\mathbb{R}^{n-k}$ *to* $\mathbb{R}^k$, *satisfying*

$$G(\zeta_i, 0) = 0, \tag{24}$$

$$||G(\zeta_i, d_1) - G(\zeta_i, d_2)|| \le Pl||d_1 - d_2|| \tag{25}$$

for all d_1, d_2, *such that a solution* $z(t)$ *of (1) with* $z(\zeta_i) = (G(\zeta_i, v_0), v_0), v_0 \in \mathbb{R}^{n-k}$, *is defined on* $\mathbb{R}$ *and satisfies*

$$||z(t)|| \le D||v_0|| e^{-\tilde{\alpha}(t-\zeta_i)}, \ t \le \zeta_i, \tag{26}$$

where $P, D > 0$ *are constant.*

Proof. Let us denote $\kappa = \frac{\sigma}{2}$ and $\eta(t) = z(t) e^{\kappa t}$. Then system (1) is transformed into the equation

$$\begin{aligned} \frac{d\xi}{dt} &= (B_+ + \kappa I)\xi + g_+(t, \eta(t), \eta(\beta(t))), \\ \frac{d\omega}{dt} &= (B_- + \kappa I)\zeta + g_-(t, \eta(t), \eta(\beta(t))), \end{aligned} \tag{27}$$

where $\eta = (\xi, \omega), I$ is an identity matrix, $\eta(\beta(t)) = z(\beta(t)) e^{-\kappa\beta(t)}$, and $g(t, z, y) = (g_+, g_-) = e^{\kappa t} f(t, z e^{-\kappa t}, y e^{-\kappa\beta(t)})$. It is easy to see that the function $g(t, z, y)$ satisfies the Lipschitz condition in z, y with a constant $l e^{\kappa\theta}$, and the eigenvalues of the matrices $B_+ + \kappa I$ and $B_- + \kappa I$ have negative and positive real parts, respectively, such that $\mu + \kappa = \max_{j=\overline{1,k}} \Re\lambda_j(B_+ + \kappa I) < -\sigma + \kappa < 0$, and

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$\min_{j=\overline{k+1,n}} \Re\lambda_j(B_- + \kappa I) = \kappa > 0$. Fix a positive number $\bar{\kappa} < \min\{\sigma - \kappa, \kappa\} = \kappa$. There exists a positive number $\bar{K}$ such that

$$||e^{(B_+ + \kappa I)(t-s)}|| \le \bar{K}e^{-\bar{\kappa}(t-s)}, t \ge s$$

$$||e^{(B_- + \kappa I)(t-s)}|| \le \bar{K}e^{\bar{\kappa}(t-s)}, t \le s.$$

To continue the proof we need the following two assertions which can be proved similarly to Lemma 2.1 and Theorem 2.1.

LEMMA 2.4 *Fix* $N \in \mathbb{R}, N > 0$, *and assume that conditions* $(C1) - (C3)$ *are valid. A continuous function* $\eta(t) = (\xi, \omega), ||\eta(t)|| \le Ne^{\tilde{\alpha}(t-t_0)}, 0 < \tilde{\alpha} < \bar{\kappa}, t \le t_0, \theta_j < t_0 \le \theta_{j+1}$, *is a solution of (27) on* $(-\infty, t_0]$ *if and only if* $\eta(t)$ *is a solution of the following system of integral equations*

$$\xi(t) = \int_{-\infty}^{t} e^{(B_+ + \kappa I)(t-s)} g_+(s, \eta(s), \eta(\beta(s)))ds,$$

$$\omega(t) = e^{(B_- + \kappa I)(t-t_0)}\omega(t_0) + \int_{t_0}^{t} e^{(B_- + \kappa I)(t-s)} g_-(s, \eta(s), \eta(\beta(s)))ds. \quad (28)$$

LEMMA 2.5 *Suppose conditions* $(C1) - (C5)$ *are fulfilled. Then for an arbitrary* $\alpha_1 \in (0, \bar{\kappa})$ *and a sufficiently small Lipschitz constant* l *there exists a function* $\bar{G}(\zeta_i, u), i \in \mathbb{Z}$, *satisfying*

$$\bar{G}(\zeta_i, 0) = 0, \quad (29)$$

$$||\bar{G}(\zeta_i, d_1) - \bar{G}(\zeta_i, d_2)|| \le Pl||d_1 - d_2||, \quad (30)$$

where P *is a positive constant, and such that* $\xi_0 = \bar{G}(\zeta_i, \omega_0)$ *defines a solution* $\eta(t)$ *of (1) with* $\eta(\zeta_i) = (\bar{G}(\zeta_i, \omega_0), \omega_0)$ *and*

$$||\eta(t)|| \le 2\bar{K}||\omega_0||e^{\alpha_1(t-\zeta_i)}, t \le \zeta_i. \quad (31)$$

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Similarly to (23) one can show that if $\eta(t, r, c) = (\xi, \omega)$ is a solution of (1) such that $\omega(r) = c$, then

$$\bar{G}(t, \omega) = \int_{-\infty}^{t} \mathrm{e}^{(B_+ + \kappa I)(t-s)} g_+(s, \eta(s, t, \omega), \eta(\beta(s), t, \omega)) ds. \tag{32}$$

Let us now finish the proof of Theorem 2.3. Applying the inverse transformation $z(t) = \eta(t)\mathrm{e}^{-\kappa t}$ we can define a new function $G(\zeta_i, v) = \mathrm{e}^{-\kappa t}\bar{G}(\zeta_i, v\mathrm{e}^{\kappa t})$ and check that

$$||G(\zeta_i, v_1) - G(\zeta_i, v_2)|| \leq Pl||v_1 - v_2||, \tag{33}$$

and

$$||z(t)|| \leq 2\bar{K}||v_0||\mathrm{e}^{(\alpha_1 - \kappa)(t - \zeta_i)}, t \leq \zeta_i,$$

if $u_0 = G(\zeta_i, v_0)$. If we denote now $D = 2\bar{K}$ and choose $\bar{\kappa}$ sufficiently close to κ then we can take $\alpha_1 = \kappa - \tilde{\alpha} > 0$ such that the last inequality implies (26). The theorem is proved.

Using the equation $\xi = \bar{G}(t, \omega)$ we can define, similarly to S^+, an integral surface $\bar{S}_0$ such that every solution of (27) starting on $\bar{S}_0$ tends to the origin as $t \to -\infty$. Then an integral set S_0 for (1) can be defined by the equation $u = G(t, v)$.

3   The stability of the zero solution

We shall need the following definitions.

DEFINITION 3.1 *The trivial solution of (1) is stable, if for any $\epsilon > 0$ and any $t_0 \in \mathbb{R}$, there exists a $\delta(t_0, \epsilon) > 0$ such that if $||z_0|| < \delta(t_0, \epsilon)$, then $||z(t, t_0, z_0)|| < \epsilon$ for all $t \geq t_0$. If the δ above is independent of t_0 then the zero solution is uniformly stable.*

DEFINITION 3.2 *The zero solution of (1) is asymptotically stable, if it is stable and if there exists a $\delta_0(t_0) > 0$ such that if $||z_0|| < \delta_0(t_0)$, then $z(t, t_0, z_0) \to 0$, as $t \to \infty$.*

DEFINITION 3.3 *The zero solution of (1) is uniformly asymptotically stable, if it is uniformly stable and there is a $\kappa_0 > 0$ such that for any $t_0 \in \mathbb{R}$, there exists a $T(\epsilon) > 0$, independent of t_0, such that $||z(t, t_0, z_0)|| < \epsilon$ for all $t \geq t_0 + T(\epsilon)$ whenever $||z_0|| < \kappa_0$.*

DEFINITION 3.4 *The zero solution of (1) is exponentially stable if there exists an $\alpha > 0$, and for every $\epsilon > 0$ and t_0 there exists a $\delta(\epsilon, t_0) > 0$, such that*

$$||z(t, t_0, z_0)|| \leq \epsilon e^{-\alpha(t-t_0)}$$

for all $t \geq t_0$, whenever $||z_0|| < \delta$. If the δ above is independent of t_0 then the zero solution is uniformly exponentially stable.

System (1) is an equation with a deviating argument, but one can easily see that Definitions 3.1 - 3.4 coincide with the definitions of stability in the Lyapunov sense for ordinary differential equations [17, 31]. They do not involve the concept of initial interval for an initial value problem. This phenomenon must not surprise us, as the right side of (1) depends only on one "delayed" value of a solution at $t = \zeta_i$ if $\theta_i \leq t < \theta_{i+1}, i \in \mathbb{Z}$. For EPCA where argument is delayed [12, 41] the stability is investigated with $t_0 = 0$. Continuous dependence on the initial value provided by (6) helps us to investigate stability assuming that the initial moment t_0 can be an arbitrary real number.

Theorem 2.1 considered with $k = n$ and inequality (6) imply that the following assertion is valid.

THEOREM 3.1 *Suppose that conditions $(C1), (C2)$ and $(C5)$ are fulfilled, and all eigenvalues of matrix A have negative real parts. Then the zero solution of (1) is uniformly exponentially stable if the Lipschitz constant l is sufficiently small.*

Comparing Definitions 3.3 and 3.4 we can conclude that if the zero solution is uniformly exponentially stable then it is uniformly asymptotically stable.

4 The stability of the integral surface S_0

THEOREM 4.1 *If the Lipschitz constant l is sufficiently small then for every solution $z(t) = (u, v)$ of (1) there exists a solution $\mu(t) = (\phi, \psi)$ on S_0 such that*

$$||u(t) - \phi(t)|| \leq 2K||u(\zeta_i) - \phi(\zeta_i)||e^{-\alpha(t-\zeta_i)},$$

$$||v(t) - \psi(t)|| \leq K||v(\zeta_i) - \psi(\zeta_i)||e^{-\alpha(t-\zeta_i)}, \zeta_i \leq t, \qquad (34)$$

where α is the coefficient defined for Theorem 2.1.

Proof. Fix a solution $z(t, \zeta_i, z_0)$ of (1). Denote by $\mu(t, \zeta_i, d) = (\phi, \psi)$ a solution of (1) such that $\psi(\zeta_i, \zeta_i, d) = d, \phi(\zeta_i, \zeta_i, d) = G(\zeta_i, d)$. Let us carry out the transformation

$$X(t) = u - \phi(t), Y(t) = v - \psi(t), \qquad (35)$$

in system (1) and denote $Z = (X, Y)$. The transformed equation has the form

$$\frac{dX}{dt} = B_+ X + Q_+(t, Z(t), Z(\beta(t)), d),$$
$$\frac{dY}{dt} = B_- Y + Q_-(t, Z(t), Z(\beta(t)), d), \qquad (36)$$

where $Q(t, X, Y, d) = (Q_+, Q_-) = f(t, z(t), z(\beta(t))) - f(t, \mu(t), \mu(\beta(t)))$. One can see that Q satisfies the Lipschitz condition with the same constant l. By Theorem 2.1 there exists a function $\tilde{F}$ such that the equation $Y = \tilde{F}(\zeta_i, X, d)$ defines a set for (36) which satisfies, according to (15) and (16), the following properties

$$\tilde{F}(\zeta_i, 0, d) = 0,$$

$$||\tilde{F}(\zeta_i, X_1, d) - \tilde{F}(\zeta_i, X_2, d)|| \leq pKl||X_1 - X_2||. \qquad (37)$$

Using (14) and formulas similar to (19) one can see that every solution $Z(t), Z(\zeta_i) = (X_0, \tilde{F}(\zeta_i, X_0))$, satisfies

$$||X(t)|| \leq Ke^{-\sigma(t-\zeta_i)}||X_0|| + \frac{2K^2 l(1 + e^{\alpha\theta})}{\sigma - \alpha}e^{-\alpha(t-\zeta_i)}||X_0||,$$

$$||Y(t)|| \leq 2\gamma K^2 l(1 + e^{\alpha\theta})e^{-\alpha(t-\zeta_i)}||X_0||. \qquad (38)$$

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Let us show that there exist X_0 and d such that for solutions $z(t)$ and $(X(t), Y(t))$ of systems (1) and (36), respectively,

$$X(t) = u(t, \zeta_i, z_0) - \phi(t), Y(t) = v(t, \zeta_i, z_0) - \psi(t).$$

The last equalities for $t = \zeta_i$ have the form

$$X_0 = u_0 - G(\zeta_i, d), \tilde{F}(\zeta_i, X_0, d) = v_0 - d. \tag{39}$$

Let us consider the system as an equation with respect to X_0 and d. We shall show that it has a solution for every pair (u_0, v_0). Equation (39) implies that

$$d = v_0 - \tilde{F}(\zeta_i, u_0 - G(\zeta_i, d), d). \tag{40}$$

Applying properties (37) of the function $\tilde{F}$ and equality (40) we can write that

$$||d - v_0|| \le pKl||u_0 - G(\zeta_i, d)||.$$

Since the function G satisfies the Lipschitz condition, using

$$||d - v_0|| \le pKl||u_0 - G(\zeta_i, v_0)|| + pKl||G(\zeta_i, d) - G(\zeta_i, v_0)||,$$

one can show that

$$||d - v_0|| \le \frac{pKl}{1 - pPKl^2}||u_0 - G(\zeta_i, v_0)||. \tag{41}$$

We assume that $1 - pPKl^2 > 0, pKl(1 + Pl) \le 1$, and will consider the ball $\hat{B} = \{d : ||d - v_0|| \le ||u_0 - G(\zeta_i, v_0)||\}$. Inequality (41) implies that (40) transforms $\hat{B}$ into itself, and by Brauer's theorem there exists a fixed point of the transformation. Denote the point by $\bar{d}$. Substituting $\bar{d}$ into the first equation of (39) one can obtain the value $\bar{X}_0$. The pair $(\bar{X}_0, \bar{d})$ satisfies system (39). Now,

applying (38), (14) and the theorem of existence and uniqueness we can complete the proof of the theorem.

We shall introduce a notion of stability for an integral set [7, 35]. Denote by $M \subset \mathbb{R} \times \mathbb{R}^n$ an integral surface of (1) and by $d(z, M)$ the distance between a point $z \in \mathbb{R}^n$ and the set M.

DEFINITION 4.1 *M is a stable integral surface of (1), if for any $\epsilon > 0$, there exists a number $\delta > 0, \delta = \delta(\epsilon, t_0)$, such that if $d(z_0, M(t_0)) < \delta$, then $d(z(t, t_0, z_0), M(t)) < \epsilon$ for all $t \geq t_0$.*

DEFINITION 4.2 *A stable integral surface M is stable in large, if every solution of (1) approaches M as $t \to \infty$.*

Theorem 4.1 implies that the surface S_0 is stable, and, moreover, it is stable in large.

5   The reduction principle

The following conditions are needed in this part of the paper.

(C6) The function $f(t, z, w)$ is uniformly continuously differentiable in z, w for all t, z, w, and
$$\frac{\partial f(t, 0, 0)}{\partial z} = 0, \frac{\partial f(t, 0, 0)}{\partial w} = 0.$$

(C7) If we denote by $\lambda_j, j = \overline{1, n}$, the eigenvalues of matrix A, then there exists a positive integer k such that $\mu = \max_{j=\overline{1,k}} \Re\lambda_j < 0$, and $\Re\lambda_j = 0, j = \overline{k+1, n}$, where $\Re\lambda_j$ denotes the real part of the eigenvalue λ_j of matrix A.

Denote
$$T(h) = \{(t, z) \in \mathbb{R} \times \mathbb{R}^n : ||z|| < h\}$$

for a fixed number $h > 0$. Assume that $\epsilon_0 > 0$ is sufficiently small for the Lipschitz constant l, provided by (C6), to satisfy all conditions of Theorem 2.3 in $T(\epsilon_0)$.

Denote $\epsilon_1 = \frac{\epsilon_0}{2\bar{K}}$, where $\bar{K}$ is the constant from (31).

By Lemma 2.5 there exists a local integral manifold of (27) in $T(\epsilon_1)$ such that a solution starting on the manifold is continuable to $-\infty$, and is exponentially decaying.

Using the inverse transformation $z = \eta e^{-\kappa t}$ one can obtain a local integral manifold of (1) in $T(\epsilon_1)$ given by equation $u = G(t, v)$. Solutions of (1) on the manifold are not necessarily continuable to $-\infty$ in $T(\epsilon_1)$. For the function G condition (33) is true and $G(t, 0) = 0, t \in \mathbb{R}$. On the local manifold solutions of (1) satisfy the following system

$$\frac{dv}{dt} = B_- v + f_-(t, (G(t, v(t)), v(t)), (G(t, v(\beta(t))), v(\beta(t)))). \qquad (42)$$

We can see that the function $f_-(t, (G(t, v), v), (G(t, \bar{v}), \bar{v}))$ satisfies the Lipschitz condition in $v, \bar{v}$ with the constant $l(1 + Pl)$.

THEOREM 5.1 *Assume that conditions $(C1) - (C2), (C4) - (C7)$ are fulfilled. The trivial solution of (1) is stable, asymptotically stable or unstable in Lyapunov sense, if the trivial solution of (42) is stable, asymptotically stable or unstable, respectively.*

Proof. Consider system (1) in $T(\epsilon_1)$. We assume, additionally, that ϵ_0 is sufficiently small such that conditions of Theorem 4.1 are valid in $T(\epsilon_0)$, and, moreover,

$$1 + Pl \le 2. \qquad (43)$$

Suppose that the zero solution of (42) is stable in the sense of Definition 3.1. Fix an $\epsilon > 0$. Without loss of generality we assume that $\epsilon < \epsilon_1$.

In view of Remark 1.2 we can assume that $t_0 = \zeta_i$ for some fixed $i \in \mathbb{Z}$. Fix a positive number ν such that the inequality

$$2\nu(1 + Pl) < 1 \qquad (44)$$

is true. The stability implies the existence of $\delta > 0, 0 < 2\delta < \epsilon$, such that if $d \in \mathbb{R}^{n-k}, ||d|| < 2\delta$, then the solution $v = \psi(t, \zeta_i, d)$ of (42) satisfies the inequality

$$||\psi(t, \zeta_i, d)|| < \nu\epsilon, \ \zeta_i \leq t. \tag{45}$$

Let u_0 and v_0 be arbitrary vectors satisfying $||u_0|| + ||v_0|| < \delta$. Denote $z(t) = z(t, \zeta_i, z_0), z(\zeta_i) = z_0, z_0 = (u_0, v_0)$, a solution of (1). Further we shall follow the proof of Theorem 4.1 specifying it for the local case. Let $\mu(t) = \mu(t, \zeta_i, d) = (\phi, \psi)$ be a solution of (1) such that $\psi(\zeta_i, \zeta_i, d) = d, \phi(\zeta_i, \zeta_i, d) = G(\zeta_i, d)$ and $\psi(\zeta_i, \zeta_i, d)$ satisfies (45). Applying (45) and the Lipschitz condition on G we have that $||\phi(t, \zeta_i, d)|| \leq Pl\nu\epsilon, \zeta_i \leq t$. Then $||\mu((t, \zeta_i, d)|| \leq (1 + Pl)\nu\epsilon, \zeta_i \geq t$. Finally using (44) we can write that

$$||\mu(t)|| < \frac{1}{2}\epsilon. \tag{46}$$

Applying transformation (35) we obtain equation (36). From (46) it follows that (35) transforms neighborhood $T(\frac{\epsilon}{2})$ for (36) into neighborhood $T(\epsilon)$ for (1). So, the conditions set by Theorem 4.1 for the coefficient l are valid if (36) is considered in $\frac{\epsilon}{2}$−neighborhood of $X = 0, Y = 0, t \in \mathbb{R}$.

Now, if we assume that

$$||X(\zeta_i)|| < \frac{\epsilon}{2K(1 + 2pl)}, \tag{47}$$

then similarly to the sequence (u_m, v_m) in Theorem 2.1 we can construct a sequence $Z_m = (X_m, Y_m), m \geq 0$, such that $(X_0, Y_0) = (0, 0)^T$,

$$X_{m+1}(t) = e^{B_+(t-\zeta_i)}X(\zeta_i) + \int_{\zeta_i}^t e^{B_+(t-s)}Q_+(s, Z_m(s), Z_m(\beta(s)))ds,$$

$$Y_{m+1}(t) = -\int_t^\infty e^{B_-(t-s)}Q_-(s, Z_m(s), Z_m(\beta(s)))ds,$$

$$||X_m(t)|| \leq Ke^{-\sigma(t-\zeta_i)}||X(\zeta_i)|| + \frac{2K^2l(1+e^{\alpha\theta})}{\sigma - \alpha}e^{-\alpha(t-\zeta_i)}||X(\zeta_i)||,$$

$$||Y_m(t)|| \leq 2\gamma K^2l(1+e^{\alpha\theta})e^{-\alpha(t-\zeta_i)}||X(\zeta_i)||, \tag{48}$$

and, hence,

$$||Z_m(t)|| \leq K(1+2pl)||X(\zeta_i)||e^{-\alpha(t-\zeta_i)} < \frac{\epsilon}{2}, \; \zeta_i \leq t,$$

The limit function $Z(t) = (X(t), Y(t))$ of the sequence is a solution of (36) and satisfies

$$||Z(t)|| \leq K(1+2pl)||X(\zeta_i)||e^{-\alpha(t-\zeta_i)} < \frac{\epsilon}{2}, \; \zeta_i \leq t, \tag{49}$$

Hence, we can define a function $\tilde{F}(\zeta_i, X, d)$ such that $Y(\zeta_i) = \tilde{F}(\zeta_i, X(\zeta_i), d)$, which satisfies (37). Next, we can prove using (43) and (48) the existence of a pair $(\bar{X}_0, \bar{d})$ such that

$$\bar{X}_0 = u_0 - G(\zeta_i, \bar{d}), v_0 - \bar{d} = \tilde{F}(\zeta_i, \bar{X}_0, \bar{d}),$$

$$||\bar{X}_0|| < \frac{\epsilon}{2K(1+2pl)}, \;\; ||\bar{d}|| < 2\delta.$$

Now, transformation (35) and (49) imply that

$$||z(t, \zeta_i, z_0) - \mu((t, \zeta_i, \bar{d})|| \leq K(1+2pl)||\bar{X}_0||e^{-\alpha(t-\zeta_i)}, \; \zeta_i \leq t. \tag{50}$$

From (46) and (50) it follows that

$$||z(t, \zeta_i, z_0)|| < \epsilon, \; \zeta_i \leq t. \tag{51}$$

Now, we can conclude in view of (51) that the zero solution of (1) is stable. Assume that the zero solution of (42) is asymptotically stable, then (50) implies that the zero solution of (1) is also asymptotically stable. Finally, it is obvious that if the zero solution of (42) is unstable, then the trivial solution of (1) is unstable as well. The theorem is proved.

References

[1] A. R. Aftabizadeh, J. Wiener and J.- M. Xu, *Oscillatory and periodic solutions of delay differential equations with piecewise constant argument,* Proc. Amer. Math. Soc. 99 (1987), 673-679.

[2] M. U. Akhmetov and N. A. Perestyuk, *Differential properties of solutions and integral surfaces of nonlinear impulse systems,* Differential Equations, 28 (1992), 445-453.

[3] M. U. Akhmet, *Almost periodic solutions of differential equations with piecewise constant argument of generalized type.* (submitted)

[4] A. Alonso, J. Hong and R. Obaya, *Almost periodic type solutions of differential equations with piecewise constant argument via almost periodic type sequences,* Appl. Math. Lett. 13 (2000), 131-137.

[5] B. Aulbach and T. Wanner, *Integral manifolds for Caratheodory type differential equations in Banach spaces,* in "Six lectures on dynamical systems," (eds. B. Aulbach and F. Colonius), World Scientific, Singapore (1966), 45-119.

[6] N.N. Bogolyubov, "On some statistical methods in mathematical physics," (Russian), Acad. Nauk U.R.S.R., 1945.

[7] J. Carr, "Applications of center manifold theory," Springer-Verlag, New York, 1981.

[8] C. Chicone and Y.Latushkin, *Center manifolds for infinite-dimensional non -autonomous differential equations,* J. Differential Equations. 141 (1997), 356-399.

[9] S.-N. Chow and K. Lu, C^k *centre unstable manifold,* Proc. Royal. Soc Edinburgh, 108A (1988), 285-317.

[10] I. Chueshov, *A reduction principle for coupled nonlinear parabolic-hyperbolic PDE,* J. Evol. Equ. 4 (2004),591-612.

[11] K. L. Cooke and J. Wiener, *Retarded differential equations with piecewise constant delays,* J. Math. Anal. Appl. 99 (1984), 265-297.

[12] K. L. Cooke and J. Wiener, *An equation alternately of retarded and advanced type,* Proceedings of the American Mathematical Society. 99 (1987), 726-732.

[13] W.A. Coppel, "Dichotomies in stability theory, " Lecture notes in mathematics, Springer-Verlag, Berlin, Heidelberg, New York, 1978.

[14] O. Diekmann, S.A. van Gils, S.M. Verduyn Lunel, and H.-O. Walther, "Delay equations. Functional-, Complex,- and Nonlinear Analysis,"Springer Verlag, New York, 1995.

[15] J. Hale and S.M.V. Lunel, "Introduction to functional differential equations," Springer-Verlag, New York, 1993.

[16] J. Hale, "Theory of Functional Differential Equations," Springer-Verlag, New York, Heidelberg, Berlin, 1977.

[17] P. Hartman, "Ordinary Differential Equations," Wiley, New York, 1964.

[18] D.B. Henry, "Geometric theory of semi-linear parabolic equations,"Lecture notes in mathematics, **840**, Springer-Verlag, New York, 1981.

[19] F.C. Hoppensteadt and C.S. Peskin, "Mathematics in Medicine and the Life Science," Springer-Verlag, New-York, Berlin, Heidelberg, 1992.

[20] C. Foias, G.R. Sell and R. Temam, *Inertial manifolds for nonlinear evolutionary equations,* J. Differential Equations. 73 (988), 309-353.

[21] K. Gopalsamy, "Stability and oscillations in delay differential equations," Kluwer Academic Publishers Group, Dordrecht, 1992.

[22] I. Györi and G. Ladas, "Oscillation theory of delay differential equations. With applications."Oxford University Press, New York, 1991.

[23] A. Kelley, *The stable, center-stable, center, center-unstable, unstable manifolds,* An appendix in "Transversal mappings and flows," R. Abraham and J. Robbin, Benjamin, New York (1967).

[24] T. Küpper and R. Yuan, *On quasi-periodic solutions of differential equations with piecewise constant argument,* J. Math. Anal. Appl. 267 (2002), 173-193.

[25] Yu. A. Kuznetsov, "Elements of Applied Bifurcation Theory, " Springer-Verlag, New York, Berlin, Heidelberg, 1995.

[26] Y. Latushkin and B. Layton, *The optimal gap condition for invariant manifolds,* Discrete and continuous dynamical systems, 5 (1999), 233-268.

[27] S. Lefschetz, "Differential equations, Geometric theory," Wiley, New York, 1957.

[28] K. H. Lee and E. H.Ong, *A reduction principle for singular perturbation problems,* Appl. Math. Comput. 101 (1999), 45-62.

[29] A.M. Lyapunov, "Probléme général de la stabilité du mouvement," Princeton Univ. Press, Princeton, N.J.,1949.

[30] I.G. Malkin, "Theory of stability of motion," M.-L., Moskow, 1952. Translated by Atomic Energy Comission, AEC-TR-3352.

[31] R.K. Miller and A.N. Michel, "Ordinary differential equations, " Academic Press, New York, 1982

[32] Muroya, Yoshiaki, *Persistence, contractivity and global stability in logistic equations with piecewise constant delays,* J. Math. Anal. Appl. 270 (2002), 602-635.

[33] K.J. Palmer, *Linearization near an integral manifold,* J. Math. Anal. Appl. 51 (1975), 243-255.

[34] G. Papaschinopoulos, *Linearization near the integral manifold for a system of differential equations with piecewise constant argument,* J. of Math. Anal. and Appl. 215 (1997), 317-333.

[35] V. A. Pliss, *The reduction principle in the theory of the stability of motion,* Izv. Akad. Nauk SSSR, Ser Mat. 27 (1964), 1297-1324.(Russian).

[36] V. A. Pliss, "Integral sets of periodic systems of differential equations," Nauka, Moskow, 1977. (Russian).

[37] Poincaré H., "Les méthodes nouvelles de la mécanique céleste. Vol.1,2," Gauthier-Villars, Paris, 1892.

[38] C. Pugh and M. Shub, *Linearization of normally hyperbolic diffeomorphisms and flows,* Invent. Math. 10 (1970), 187-190.

[39] A. M. Samoilenko and N. A. Perestyuk, "Impulsive Differential Equations," World Scientific, Singapore, 1995.

[40] G. Seifert, *Almost periodic solutions of certain differential equations with piecewise constant delays and almost periodic time dependence,* J. Differential equations. 164 (2000), 451-458.

[41] J. Wiener, "Generalized solutions of functional differential equations," World Scientific, Singapore, 1993.

[42] J. Wiener and V. Lakshmikantham, *A damped oscillator with piecewise constant time delay,* Nonlinear Stud. 7 (2000), 78–84.

[43] A. Stokes, *Local coordinates around a limit cycle of a functional differential equation with applications,* J. Differential Equations. 24 (1977) 153-172.