# Maxwell-Chern-Simons Scalar Electrodynamics 

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#### Abstract

The Maxwell-Chern-Simons gauge theory with charged scalar fields is analyzed at two loop level. The effective potential for the scalar fields is derived in the closed form, and studied both analytically and numerically. It is shown that the $U(1)$ symmetry is spontaneously broken in the massless scalar theory. Dimensional transmutation takes place in the Coleman-Weinberg limit in which the Maxwell term vanishes. We point out the subtlety in defining the pure Chern-Simons scalar electrodynamics and show that the Coleman-Weinberg limit must be taken after renormalization. Renormalization group analysis of the effective potential is also given at two loop.


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[^0]
## 1. Introduction

In the previous paper we have evaluated the effective potential of massless scalar fields in three-dimensional $U(1)$ gauge theory to the two loop order and have shown that the $U(1)$ symmetry is spontaneously broken when the Chern-Simons term is present for gauge fields. [1] In this paper we shall give a full account of this theory, including the ColemanWeinberg limit and renormalization group analysis. Subtlety in defining the ColemanWeinberg limit is pointed out. Numerical study of the two loop effective potential is also presented.

There are many reasons for investigating three-dimensional $U(1)$ gauge theory with both Maxwell and Chern-Simons terms. Nonrelativistic Chern-Simons theory serves as an effective theory of the quantum Hall system [2]. Chern-Simons interactions describe the change in statistics, and in general fractional statistics. [3] It was argued that the system of charged anyon gas leads to superconductivity, though experimental support is lacking. [7]

Relativistic three-dimensional gauge theory serves as an effective theory of four dimensional theory at high temperature. In particular, Maxwell-Chern-Simons theory appears as an effective theory of QCD and the standard model of electroweak interactions. [5, 6, 7]

Maxwell-Chern-Simons theory has many unique features. A photon acquires a topological mass without breaking the gauge invariance. [8], 9] When the $U(1)$ symmetry is spontaneously broken, photons appear with two different masses. In self-dual Chern-Simons theories many exact topological and non-topological soliton solutions are available. (10] In the Maxwell-Chern-Simons theory with Dirac fermions a magnetic field can be dynamically generated so that the Lorentz invariance is spontaneously broken. [1] Pure non-Abelian Chern-Simons theory defines a topological field theory, playing an important role in the knot theory. 12]

Quantum aspect of the Maxwell-Chern-Simons gauge theory is under intense investigation in the literature. The Chern-Simons term is induced by Dirac fermions at one loop. [13] In non-Abelian theory the Chern-Simons coefficient is quantized. [8] Non-Abelian gauge fields themselves induce a Chern-Simons term at one loop. 14] Pure non-Abelian

Chern-Simons theory is expected to be ultraviolet finite. [15 The Coleman-Hill theorem assures that corrections to the Chern-Simons coefficient are absent beyond one loop. [17] In the spontaneously broken non-Abelian gauge theory, however, corrections could arise, depending on how the symmetry is broken. In a certain type of scalar field theory it has been argued that symmetry can be broken by radiative corrections even at one loop. In relativistic fermion theories the resummation of ring diagrams leads to spontaneous magnetization. (11] Beta functions have been calculated in pure Chern-Simons gauge theories. [16]

Yet, most arguments in the literature are limited to the one loop approximation or the random phase approximation. One of the main concerns in this paper is the phase structure, namely the symmetry structure, of the scalar gauge theory particularly when the scalar fields are massless. We shall show that one loop result is ambiguious, and one needs to go to two loop to find definitive conclusions.

In this regard there is a subtle difference between the pure Chern-Simons gauge theory and the Maxwell-Chern-Simons gauge theory. Naively defined in three dimensions, these theories have photon propagators which behave, at large momenta, completely differently. In the pure Chern-Simons gauge theory the photon propagator behaves as $1 / p$, whereas in the Maxwell-Chern-Simons gauge theory it behaves as $1 / p^{2}$. The ultraviolet behavior is completely different.

This problem is tied to the renormalizability of the theory. First a regularization method must be specified which works to all orders in perturbation theory. We adopt the dimensional regularization method in the Maxwell-Chern-Simons theory. The pure ChernSimons theory is defined in the limit of the vanishing Maxwell coefficient after renormalization. We show that the limit is well defined and exists only after renormalization.

If the scalar fields self-interact only through $\phi^{6}$ coupling in the pure Chern-Simons theory, the theory at the tree level does not have any dimensional parameter. We define the pure Chern-Simons theory in the manner described above, and show that the dimensional transmutation takes place at two loop.

Section 2 is devoted to the study of pure complex scalar theory in $2+1$ dimensions up to two loop. In section 3 we give an analysis of super renormalizable real scalar $\lambda \phi^{4}$ theory.

Section 4 contains the definition of the gauge theory and the prescription to dimensionally continue it to $n$ dimensions. One and two loop calculations are given in sections 5 and 6, respectively. In Section 6, the renormalized effective potential is given in the analytic form in the limit of small and large scalar fields. In section 7 the Coleman-Weinberg limit of the effective potential is obtained. Section 8 includes an analysis of pure Maxwell theory, namely parity preserving $2+1$ dimensional scalar QED. Renormalization for arbitrary value of the field is carried out numerically in section 9. Divergence structure of the theory by using power counting method is discussed in section 10. We use the renormalization group arguments to find the beta functions in section 11. Summary is given in section 12. Two loop calculations are quite tedious. We have collected relevant integrals in appendices.

## 2. Pure Complex Scalar Theory

In this section we analyze a complex scalar field theory in three dimensions. The most general renormalizable $U(1)$ invariant Lagrangian for $\Phi=\left(\phi_{1}+i \phi_{2}\right) / \sqrt{2}$ is given by

$$
\begin{equation*}
\mathcal{L}=\frac{1}{2}\left(\partial \phi_{1}\right)^{2}+\frac{1}{2}\left(\partial \phi_{2}\right)^{2}-\frac{m^{2}}{2}\left(\phi_{1}^{2}+\phi_{2}^{2}\right)-\frac{\lambda}{4!}\left(\phi_{1}^{2}+\phi_{2}^{2}\right)^{2}-\frac{\nu}{6!}\left(\phi_{1}^{2}+\phi_{2}^{2}\right)^{3} . \tag{2.1}
\end{equation*}
$$

The metric is given by $g^{\mu \nu}=\operatorname{diag}(+,-,-)$. When $m^{2}=\lambda=0$, the theory at the tree level does not contain any dimensional parameter. At the quantum level, however, a dimensional scale enters in the definition of the renormalized coupling constant $\nu$ and a question arises whether or not the $U(1)$ symmetry is spontaneously broken by radiative corrections. We shall show that at two loop the effective potential is minimized at a non-vanishing $\phi$, but the minimum occurs outside the region of the validity of the perturbation theory.

We are going to evaluate the effective potential for (2.1) for arbitrary values of the parameters $m^{2}, \lambda$, and $\nu$ at two loop. Let us recall the general formula for the effective action. For a Lagrangian $\mathcal{L}(\phi)$ the effective action $\Gamma(\varphi)$ is 18]

$$
\begin{align*}
& \Gamma[\varphi]=\int d^{n} x \mathcal{L}[\varphi]+\tilde{\Gamma}[\varphi] \\
& \tilde{\Gamma}[\varphi]=-i \hbar \ln \int \mathcal{D} \phi \exp \frac{i}{\hbar} \int d^{n} x\left[\frac{1}{2} \phi i D_{F}^{-1}(\varphi) \phi+\mathcal{L}_{\mathrm{int}}(\phi ; \varphi)-\frac{\delta \tilde{\Gamma}(\varphi)}{\delta \varphi} \phi\right] \tag{2.2}
\end{align*}
$$

where

$$
\begin{align*}
& \mathcal{L}[\phi+\varphi]=\mathcal{L}[\varphi]+\frac{\delta \mathcal{L}(\varphi)}{\delta \varphi_{a}} \phi_{a}+\frac{1}{2} \phi_{a} i\left[D_{F}^{-1}(\varphi)\right]^{a b} \phi_{b}+\mathcal{L}_{\text {int }}(\phi ; \varphi) \\
& i\left[D_{F}^{-1}(\varphi)\right]^{a b}=\frac{\delta^{2} \mathcal{L}(\varphi)}{\delta \varphi_{a} \delta \varphi_{b}} . \tag{2.3}
\end{align*}
$$

The above matrix equation gives propagators of the theory. $\tilde{\Gamma}[\varphi]$ is the sum of one-particle irreducible diagrams. At one loop

$$
\Gamma[\varphi]^{(1)}= \begin{cases}\frac{i \hbar}{2} \ln \operatorname{det}\left[i D_{F}^{-1}(\varphi)\right] & \text { for bosons }  \tag{2.4}\\ -i \hbar \ln \operatorname{det}\left[i D_{F}^{-1}(\varphi)\right] & \text { for Dirac fields }\end{cases}
$$

For constant $\varphi(x)=\varphi$ the effective potential is

$$
\begin{align*}
V_{\mathrm{eff}}(\varphi)= & V^{(\text {tree })}(\varphi)+\frac{\hbar}{2} \int \frac{d^{n} k}{i(2 \pi)^{n}} \ln \operatorname{det}\left[i D_{F}^{-1}(k ; \varphi)\right] \\
& +i \hbar\left\langle\exp \left(\frac{i}{\hbar} \int d^{n} x \mathcal{L}_{\mathrm{int}}(\phi, \varphi)\right)\right\rangle_{1 \mathrm{PI}} \tag{2.5}
\end{align*}
$$

where the propagator is written in the momentum space.
The Lagrangian (2.1) becomes, after the shifting $\phi_{1} \rightarrow v+\phi_{1}$,

$$
\begin{align*}
\mathcal{L} & =\mathcal{L}_{(0)}+\cdots+\mathcal{L}_{(6)} \\
\mathcal{L}_{(0)} & =-\frac{m^{2}}{2} v^{2}-\frac{\lambda}{4!} v^{4}-\frac{\nu}{6!} v^{6} \\
\mathcal{L}_{(2)} & =\frac{1}{2}\left(\partial \phi_{1}\right)^{2}+\frac{1}{2}\left(\partial \phi_{2}\right)^{2}-\frac{1}{2} m_{1}^{2} \phi_{1}^{2}-\frac{1}{2} m_{2}^{2} \phi_{2}^{2} \\
\mathcal{L}_{(3)} & =-\frac{\lambda}{3!} v \phi_{1}\left(\phi_{1}^{2}+\phi_{2}^{2}\right)-\frac{\nu}{6!} 4 v^{3} \phi_{1}\left(5 \phi_{1}^{2}+3 \phi_{2}^{2}\right) \\
\mathcal{L}_{(4)} & =-\frac{\lambda}{4!}\left(\phi_{1}^{2}+\phi_{2}^{2}\right)^{2}-\frac{\nu}{6!} 3 v^{2}\left(5 \phi_{1}^{2}+\phi_{2}^{2}\right)\left(\phi_{1}^{2}+\phi_{2}^{2}\right) \\
\mathcal{L}_{(5)} & =-\frac{\nu}{5!} v \phi_{1}\left(\phi_{1}^{2}+\phi_{2}^{2}\right)^{2} \\
\mathcal{L}_{(6)} & =-\frac{\nu}{6!}\left(\phi_{1}^{2}+\phi_{2}^{2}\right)^{3} \tag{2.6}
\end{align*}
$$

The linear term $\mathcal{L}_{(1)}$ may absorbed by the redefinition of the source and is irrelevant. The mass parameters are given by :

$$
m_{1}^{2}=m_{1}(v)^{2}=m^{2}+\frac{\lambda}{2} v^{2}+\frac{\nu}{24} v^{4}
$$

$$
\begin{equation*}
m_{2}^{2}=m_{2}(v)^{2}=m^{2}+\frac{\lambda}{6} v^{2}+\frac{\nu}{120} v^{4} \tag{2.7}
\end{equation*}
$$

In $n$ dimensional space-time the dimensions of the coupling constants and fields are

$$
\begin{equation*}
[m]=M \quad, \quad[\lambda]=M^{4-n} \quad, \quad[\nu]=M^{2(3-n)} \quad, \quad\left[v^{2}\right]=M^{n-2} \tag{2.8}
\end{equation*}
$$

The tree level effective potential is

$$
\begin{equation*}
V_{\mathrm{eff}}^{(0)}=\frac{m^{2}}{2} v^{2}+\frac{\lambda}{4!} v^{4}+\frac{\nu}{6!} v^{6}+\Lambda \tag{2.9}
\end{equation*}
$$

The last term in (2.9) is the cosmological constant which is a function of the dimensional parameters. Although it is irrelevant for the discussion of symmetry breaking, it plays an important role in renormalization group analysis. 19]

The one loop effective potential is finite in the dimensional regularization scheme and is given by

$$
\begin{align*}
V_{\mathrm{eff}}^{(1)} & =\frac{\hbar}{2} \int \frac{d^{n} p}{i(2 \pi)^{n}}\left\{\ln \left[p^{2}-m_{1}^{2}\right]+\ln \left[p^{2}-m_{2}^{2}\right]\right\} \\
& =-\frac{\hbar}{2} \frac{\Gamma\left(-\frac{n}{2}\right)}{(4 \pi)^{\frac{n}{2}}}\left[m_{1}^{n}+m_{2}^{n}\right] \\
& =-\frac{\hbar}{12 \pi} \mu^{n-3}\left(m_{1}^{3}+m_{2}^{3}\right)+\mathrm{O}(n-3) \tag{2.10}
\end{align*}
$$

At two loop, we denote

$$
\begin{align*}
& \mathcal{L}_{(3)}=-\beta_{1} \phi_{1}^{3}-\beta_{2} \phi_{1} \phi_{2}^{2} \\
& \beta_{1}=\frac{\lambda}{3!} v+\frac{\nu}{36} v^{3} \quad, \quad \beta_{2}=\frac{\lambda}{3!} v+\frac{\nu}{60} v^{3} \\
& \mathcal{L}_{(4)}=-\alpha_{1} \phi_{1}^{4}-\alpha_{2} \phi_{2}^{4}-\alpha_{3} \phi_{1}^{2} \phi_{2}^{2} \\
& \alpha_{1}=\frac{\lambda}{4!}+\frac{\nu}{2 \cdot 4!} v^{2} \quad, \quad \alpha_{2}=\frac{\lambda}{4!}+\frac{\nu}{2 \cdot 5!} v^{2} \quad, \quad \alpha_{3}=\frac{2 \lambda}{4!}+\frac{3 \nu}{5!} v^{2} \tag{2.11}
\end{align*}
$$

The cubic part of the Lagrangian gives rise to theta shape diagrams which are reduced to the following.


$$
\begin{align*}
I\left(m_{a}, m_{b}, m_{c} ; n\right) & \equiv \int \frac{d^{n} q d^{n} k}{(2 \pi)^{2 n}} \frac{1}{\left[(q+k)^{2}+m_{a}^{2}\right]\left(q^{2}+m_{b}^{2}\right)\left(k^{2}+m_{c}^{2}\right)} \\
& =I\left(m_{b}, m_{a}, m_{c} ; n\right) \text { etc. } \\
& =I^{\mathrm{div}}+\widetilde{I}\left(m_{a}+m_{b}+m_{c}\right) \\
I^{\mathrm{div}} & =\frac{\mu^{2(n-3)}}{32 \pi^{2}}\left\{-\frac{1}{n-3}-\gamma_{E}+1+\ln 4 \pi\right\} \\
\widetilde{I}(m) & =-\frac{\mu^{2(n-3)}}{16 \pi^{2}} \ln \frac{m}{\mu} \tag{2.12}
\end{align*}
$$

The derivation of (2.12) is given in Appendix B. We have split $I$ function into divergent and finite parts for later convenience. The quartic part of the Lagrangian produces two loop diagrams which are reduced to the integral

$$
\begin{align*}
& J\left(m_{a}, m_{b} ; n\right) \equiv \int \frac{d^{n} q d^{n} k}{(2 \pi)^{2 n}} \frac{1}{\left(q^{2}+m_{a}^{2}\right)\left(k^{2}+m_{b}^{2}\right)} \\
&=\frac{\mu^{2(n-3)}}{16 \pi^{2}} m_{a} m_{b}\left[1+(n-3) \psi\left(-\frac{1}{2}\right)+(n-3) \ln \left(\frac{m_{a} m_{b}}{4 \pi \mu^{2}}\right)\right. \\
&\left.\quad+\mathrm{O}(n-3)^{2}\right] \tag{2.13}
\end{align*}
$$

Therefore, two loop contributions are

$$
\begin{align*}
V_{\mathrm{eff}}^{(2)}= & \frac{-\hbar^{2}}{2}\left\{6 \beta_{1}^{2} I\left(m_{1}, m_{1}, m_{1}\right)+2 \beta_{2}^{2} I\left(m_{1}, m_{2}, m_{2}\right)\right\} \\
& +\hbar^{2}\left\{3 \alpha_{1} J\left(m_{1}, m_{1}\right)+3 \alpha_{2} J\left(m_{2}, m_{2}\right)+\alpha_{3} J\left(m_{1}, m_{2}\right)\right\} \\
= & -\hbar^{2}\left[3 \beta_{1}^{2}+\beta_{2}^{2}\right] I^{\mathrm{div}} \\
& +\frac{\mu^{2(n-3)} \hbar^{2}}{32 \pi^{2}} 3 \beta_{1}^{2} \ln \frac{9 m_{1}^{2}}{\mu^{2}}+\frac{\mu^{2(n-3)} \hbar^{2}}{32 \pi^{2}} \beta_{2}^{2} \ln \frac{\left(m_{1}+2 m_{2}\right)^{2}}{\mu^{2}} \\
& +\frac{\mu^{2(n-3)} \hbar^{2}}{16 \pi^{2}}\left\{3 \alpha_{1} m_{1}^{2}+3 \alpha_{2} m_{2}^{2}+\alpha_{3} m_{1} m_{2}\right\} . \tag{2.14}
\end{align*}
$$

Combining (2.9), (2.10), (2.14), one finds the total effective potential to $\mathrm{O}\left(\hbar^{2}\right)$ is, up to counter terms,

$$
V_{\mathrm{eff}}(v ; n)=\frac{1}{2} m^{2} v^{2}+\frac{\lambda}{4!} v^{4}+\frac{\nu}{6!} v^{6}+\Lambda-\frac{\hbar}{12 \pi} \mu^{n-3}\left(m_{1}^{3}+m_{2}^{3}\right)
$$

$$
\begin{align*}
& -\hbar^{2}\left[\left(\frac{\lambda}{6} v+\frac{\nu}{60} v^{3}\right)^{2}+3\left(\frac{\lambda}{6} v+\frac{\nu}{36} v^{3}\right)^{2}\right] I^{\text {div }} \\
& +\frac{\hbar^{2}}{32 \pi^{2}} \mu^{2(n-3)}\left\{\left(\frac{\lambda}{6} v+\frac{\nu}{60} v^{3}\right)^{2} \ln \frac{\left(m_{1}+2 m_{2}\right)^{2}}{\mu^{2}}+3\left(\frac{\lambda}{6} v+\frac{\nu}{36} v^{3}\right)^{2} \ln \frac{9 m_{1}^{2}}{\mu^{2}}\right\} \\
& +\frac{\hbar^{2}}{16 \pi^{2}} \mu^{2(n-3)}\left\{3\left(\frac{\lambda}{4!}+\frac{15 \nu}{6!} v^{2}\right) m_{1}^{2}+3\left(\frac{\lambda}{4!}+\frac{3 \nu}{6!} v^{2}\right) m_{2}^{2}\right. \\
& \left.\quad+2\left(\frac{\lambda}{4!}+\frac{9 \nu}{6!} v^{2}\right) m_{1} m_{2}\right\} \tag{2.15}
\end{align*}
$$

Beta functions for various coupling constants can be found in variety of ways. One way is to evaluate corresponding Feynman diagrams to find divergent parts or counter terms. An alternative way, which is suited in our approach, is to find beta functions from the requirement that the effective potential satisfy the renormalization group equation. Both methods must give the same result. We shall show in the rest of this section that both methods yield the same beta functions in the pure scalar theory.

First we write down the renormalization group equation satisfied by the effective potential in the $\overline{\mathrm{MS}}$ scheme. The $\overline{\mathrm{MS}}$ regularization scheme consists of absorbing terms proportional to $-(n-3)^{-1}-\gamma_{\mathrm{E}}+1+\ln 4 \pi$ by counter terms. The resultant effective potential $V_{\text {eff }}(v)^{\overline{\mathrm{MS}}}$ obtained from (2.15) is finite. As the bare theory is independent of the dimensional parameter $\mu$, it obeys

$$
\begin{equation*}
\left[\mu \frac{\partial}{\partial \mu}+\beta_{m^{2}} \frac{\partial}{\partial m^{2}}+\beta_{\lambda} \frac{\partial}{\partial \lambda}+\beta_{\nu} \frac{\partial}{\partial \nu}+\beta_{\Lambda} \frac{\partial}{\partial \Lambda}-\gamma_{\phi} v \frac{\partial}{\partial v}\right] V_{\mathrm{eff}}(v)^{\overline{\mathrm{MS}}}=0 \tag{2.16}
\end{equation*}
$$

where the beta functions and anomalous dimension are defined by

$$
\begin{array}{ll}
\beta_{\lambda}=\mu \frac{\partial \lambda}{\partial \mu}, & \beta_{m^{2}}=\mu \frac{\partial m^{2}}{\partial \mu} \\
\beta_{\nu}=\mu \frac{\partial \nu}{\partial \mu}, & \beta_{\Lambda}=\mu \frac{\partial \Lambda}{\partial \mu} \\
\gamma_{\phi}=\frac{\mu}{2} \frac{\partial \ln Z_{\phi}}{\partial \mu} . & \tag{2.17}
\end{array}
$$

In the $\overline{\mathrm{MS}}$ scheme, the $\beta$ 's and $\gamma_{\phi}$ are functions of various coupling constants and $\hbar$. Eq. (2.16) is an exact relation, and is valid for arbitrary $v$ and to each order in $\hbar$. As can
be easily shown, the anomalous dimension $\gamma_{\phi}$ vanishes up to two loop, or to $\mathrm{O}\left(\hbar^{2}\right)$. To $\mathrm{O}(\hbar)$

$$
\begin{equation*}
\beta_{m^{2}}^{(1)} \frac{v^{2}}{2}+\beta_{\lambda}^{(1)} \frac{v^{4}}{4!}+\beta_{\nu}^{(1)} \frac{v^{6}}{6!}+\beta_{\Lambda}^{(1)}=0 \tag{2.18}
\end{equation*}
$$

Hence

$$
\begin{equation*}
\beta_{m^{2}}^{(1)}=\beta_{\lambda}^{(1)}=\beta_{\nu}^{(1)}=\beta_{\Lambda}^{(1)}=0 \tag{2.19}
\end{equation*}
$$

To $\mathrm{O}\left(\hbar^{2}\right)$, Eq. (2.16) becomes

$$
\begin{align*}
-\frac{\hbar^{2}}{16 \pi^{2}} & \left\{\left(\frac{\lambda}{6} v+\frac{\nu}{60} v^{3}\right)^{2}+3\left(\frac{\lambda}{6} v+\frac{\nu}{36} v^{3}\right)^{2}\right\} \\
+ & \beta_{m^{2}}^{(2)} \frac{v^{2}}{2}++\beta_{\lambda}^{(2)} \frac{v^{4}}{4!}+\beta_{\nu}^{(2)} \frac{v^{6}}{6!}+\beta_{\Lambda}^{(2)}=0 \tag{2.20}
\end{align*}
$$

Comparing the coefficients term by term, we find

$$
\begin{array}{ll}
\beta_{\Lambda}^{(2)}=0, & \beta_{m^{2}}^{(2)}=\frac{\hbar^{2}}{72 \pi^{2}} \lambda^{2} \\
\beta_{\lambda}^{(2)}=\frac{\hbar^{2}}{20 \pi^{2}} \lambda \nu \quad, & \beta_{\nu}^{(2)}=\frac{7 \hbar^{2}}{60 \pi^{2}} \nu^{2} \tag{2.21}
\end{array}
$$

The same result is obtained by the conventional method of finding beta functions. The superficial degree of divergence $\omega$ for a given Feynman diagram is

$$
\begin{equation*}
\omega=3-\frac{E}{2}-V_{4} \tag{2.22}
\end{equation*}
$$

where $V_{4}$ refers to the number of vertices of quartic coupling while $E$ is the number of the external lines.

For diagrams contributing to $\beta_{m^{2}}, E=2$ and $V_{4}=2$ to $\mathrm{O}\left(\hbar^{2}\right)$. There are two divergent diagrams of the form


The self-energy term for $\phi_{1}\left(\right.$ in $\left.D^{-1}=p^{2}-m_{0}^{2}-\Sigma\right)$ is

$$
\begin{equation*}
\Sigma(p)=-\hbar^{2} \lambda^{2}\left\{\frac{1}{6} I\left(m_{1}, m_{1}, m_{1}\right)+\frac{1}{18} I\left(m_{1}, m_{2}, m_{2}\right)\right\}+\mathrm{O}\left(p^{2}\right) \tag{2.23}
\end{equation*}
$$

The $\mathrm{O}\left(p^{2}\right)$ term is finite. To this order the bare mass is

$$
\begin{equation*}
m_{0}^{2}=m^{2}-\Sigma(0)^{\mathrm{div}}=m^{2}+\frac{2 \hbar^{2} \lambda^{2}}{9} I^{\mathrm{div}} \tag{2.24}
\end{equation*}
$$

where $I^{\text {div }}$ is defined in (2.12). The bare mass does not depend on $\mu$, and $\mu(d / d \mu) \lambda=\mathrm{O}\left(\hbar^{2}\right)$. Hence to $\mathrm{O}\left(\hbar^{2}\right)$

$$
\begin{equation*}
\beta_{m^{2}}=\mu \frac{d}{d \mu} m(\mu)^{2}=-\left.\frac{2 \hbar^{2} \lambda^{2}}{9} \mu \frac{d}{d \mu} I^{\mathrm{div}}\right|_{n=3}=\frac{\hbar^{2} \lambda^{2}}{72 \pi^{2}}, \tag{2.25}
\end{equation*}
$$

which agrees with (2.21).
Two loop diagrams contributing to $\beta_{\lambda}$ must have $E=4, V_{4}=1$, and $V_{6}=1$, taking the form of


The total vertex at zero momentum is

$$
\begin{equation*}
\lambda_{0}-\hbar^{2}\left\{\frac{2 \lambda \nu}{3} I\left(m_{1}, m_{1}, m_{1}\right)+\frac{2 \lambda \nu}{15} I\left(m_{1}, m_{2}, m_{2}\right)\right\} \tag{2.26}
\end{equation*}
$$

The bare coupling constant is

$$
\begin{equation*}
\lambda_{0}=\lambda+\frac{4}{5} \hbar^{2} \lambda \nu I^{\mathrm{div}} \tag{2.27}
\end{equation*}
$$

The same result for $\beta_{\lambda}$ as in (2.21) follows from $\mu(d / d \mu) \lambda_{0}=0$.
Similarly, for $\beta_{\nu}$ there are two diagrams to be considered:


When all external lines are $\phi_{1}$ fields, the six point vertex at zero momenta is

$$
\begin{equation*}
\nu_{0}-\hbar^{2}\left\{\frac{5 \nu^{2}}{3} I\left(m_{1}, m_{1}, m_{1}\right)+\frac{\nu^{2}}{5} I\left(m_{1}, m_{2}, m_{2}\right)\right\} \tag{2.28}
\end{equation*}
$$

so that

$$
\begin{equation*}
\nu_{0}=\nu+\frac{28}{15} \hbar^{2} \nu^{2} I^{\mathrm{div}} \tag{2.29}
\end{equation*}
$$

$\mu(d / d \mu) \nu_{0}=0$ leads to the previous result for $\beta_{\nu}$ in (2.21).
Now, consider the special case $m^{2}=\lambda=0$ i.e. when no dimensionful parameter appears at the tree level. In the $\overline{\mathrm{MS}}$ renormalization scheme, the total effective potential to $\mathrm{O}\left(\hbar^{2}\right)$ takes the form

$$
\begin{equation*}
V_{\text {eff }}(v ; n)=\frac{\bar{A} v^{2}}{2}+\frac{\bar{B} v^{4}}{4!}+\frac{\bar{C} v^{6}}{6!}+\frac{\bar{D} v^{6}}{6!} \ln \frac{\mu^{2(3-n)} \nu v^{4}}{4 \pi \mu^{2}} \tag{2.30}
\end{equation*}
$$

We impose the following renormalization conditions at $n=3$ :

$$
\begin{align*}
\left.V_{\mathrm{eff}}\right|_{v=0} & =0 \\
\left.\frac{\partial^{2} V_{\mathrm{eff}}}{\partial v^{2}}\right|_{v=0} & =m^{2}=0 \\
\left.\frac{\partial^{4} V_{\mathrm{eff}}}{\partial v^{4}}\right|_{v=0} & =\lambda=0 \\
\left.\frac{\partial^{6} V_{\mathrm{eff}}}{\partial v^{6}}\right|_{v=M^{1 / 2}} & =\nu(M)=\nu \tag{2.31}
\end{align*}
$$

Note that $\nu(M)$ has to be defined at $M \neq 0$, as the effective potential has a $\ln v$ singularity at $v=0$. The resultant effective potential is

$$
\begin{equation*}
V_{\mathrm{eff}}(v)^{\text {pure scalar }}=\frac{1}{6!} \nu(M) v^{6}+\frac{1}{6!} \frac{7 \hbar^{2}}{120 \pi^{2}} \nu(M)^{2} v^{6}\left(\ln \frac{v^{4}}{M^{2}}-\frac{49}{5}\right) . \tag{2.32}
\end{equation*}
$$

At first glance, it seems that the potential has a minimum at a nonvanishing $v \equiv v_{\text {min }}$. However,

$$
\begin{equation*}
\nu \ln v_{\min }^{4} / M^{2}=-\frac{120 \pi^{2}}{7 \hbar^{2}}+\frac{137}{15} \nu \tag{2.33}
\end{equation*}
$$

For small $\nu$, the first term dominates and has an absolute value much bigger than one. Since higher order corrections produce higher powers of (2.33), we conclude that the location of the new minimum occurs outside the domain of validity of perturbation theory, as in the Coleman-Weinberg limit of the $3+1$ dimensional pure scalar theory. One cannot draw any definitive conclusion concerning the symmetry breaking from the above perturbative analysis.

## 3. Real Scalar Theory

In this section we analyze the $2+1$ dimensional $\lambda \phi^{4}$ theory with a vanishing $\phi^{6}$ coupling. $\lambda$ has dimension of mass so that the theory is super-renormalizable. By looking at the superficial degree of divergence one can find that $\beta_{\lambda}$ is zero to all orders in perturbation theory. This is seen at two loop by letting $\nu$ be equal to zero in the equation (2.21). Beta functions at three loop are found in [21]. The $Z_{2}$ symmetric version of this theory, namely real scalar theory, has also been studied both at one loop and in the Gaussian approximation which gives an upper bound for the effective potential. Here we would like to extend the analysis to two loop.

In the $Z_{2}$ symmetric case the $\overline{\mathrm{MS}}$ renormalized potential is given by

$$
\begin{align*}
V_{\mathrm{eff}}^{\overline{\mathrm{MS}}}= & \frac{1}{2} m^{2} v^{2}+\frac{\lambda}{24} v^{4}-\frac{\hbar}{12 \pi}\left(m^{2}+\frac{\lambda}{2} v^{2}\right)^{\frac{3}{2}} \\
& +\frac{\hbar^{2}}{128 \pi^{2}} \lambda\left(m^{2}+\frac{\lambda}{2} v^{2}\right)+\frac{\hbar^{2}}{384 \pi^{2}} \lambda^{2} v^{2} \ln \frac{9\left(m^{2}+\frac{\lambda}{2} v^{2}\right)}{\mu^{2}} \tag{3.1}
\end{align*}
$$

The parameters $m^{2}$ and $\lambda$ are finite but otherwise arbitrary. We renormalize by

$$
\begin{equation*}
V_{\mathrm{eff}}(0)=0, \quad V_{\mathrm{eff}}^{(2)}(0)=m^{2}, \quad V_{\mathrm{eff}}^{(4)}(0)=\lambda \tag{3.2}
\end{equation*}
$$

Then one obtains

$$
\begin{align*}
V_{\mathrm{eff}}= & \frac{1}{2}\left(m^{2}+\frac{\hbar \lambda m}{8 \pi}\right) v^{2}+\frac{\lambda}{24} v^{4}-\frac{\hbar}{12 \pi}\left(m^{2}+\frac{\lambda}{2} v^{2}\right)^{\frac{3}{2}} \\
& +\frac{\hbar^{2} \lambda^{2} v^{2}}{384 \pi^{2}} \ln \left(1+\frac{\lambda}{2 m^{2}} v^{2}\right)+\frac{\hbar}{12 \pi} m^{3}+\frac{\hbar \lambda^{2} v^{4}}{128 \pi m}-\frac{\hbar^{2} \lambda^{3} v^{4}}{768 \pi^{2} m^{2}} \tag{3.3}
\end{align*}
$$

In fig. 1 we have plotted one loop and two loop results. In the figure, one can see that for small values of $\lambda / m$, one loop and two loop potential are close to each other. As $\lambda / m$ increases, they start to deviate from each other.

For small $\lambda / m$ the symmetry is unbroken. For $27.811<\lambda / m<29.541$ the two-loop effective potential is minimized at a non-vanishing $v$. It becomes unbounded from below for $\lambda / m>29.541$. However, the perturbation theory breaks down for such a large coupling.

It has been shown by Stevenson and by Olsen et al. that in the Gaussian approximation the symmetry is spontaneously broken if the coupling $\lambda / m$ becomes sufficiently large. 22] Our result in perturbation theory is valid for small $\lambda / m$, and is consistent with the result in the Gaussian approximation. [Note that $\lambda$ in 22] is not normalized by the condition (3.2).]


Figure 1: Effective potential for $\phi^{4}$ real scalar theory using various values of $\lambda / m$. One loop data are represented as points while two loop data are depicted as lines.

## 4. Gauge theory

In the presence of $U(1)$ gauge interactions the most general renormalizable Lagrangian is given by

$$
\begin{align*}
\mathcal{L}= & -\frac{a}{4} F_{\mu \nu} F^{\mu \nu}-\frac{\kappa}{2} \epsilon^{\mu \nu \rho} A_{\mu} \partial_{\nu} A_{\rho}+\mathcal{L}_{\text {g.f. }}+\mathcal{L}_{\text {F.P. }} \\
& +\frac{1}{2}\left(\partial_{\mu} \phi_{1}-e A_{\mu} \phi_{2}\right)^{2}+\frac{1}{2}\left(\partial_{\mu} \phi_{2}+e A_{\mu} \phi_{1}\right)^{2} \\
& -\frac{m^{2}}{2}\left(\phi_{1}^{2}+\phi_{2}^{2}\right)-\frac{\lambda}{4!}\left(\phi_{1}^{2}+\phi_{2}^{2}\right)^{2}-\frac{\nu}{6!}\left(\phi_{1}^{2}+\phi_{2}^{2}\right)^{3} . \tag{4.1}
\end{align*}
$$

In the $R_{\xi}$ gauge

$$
\begin{align*}
& \mathcal{L}_{\text {g.f. }}=-\frac{1}{2 \alpha}\left(\partial_{\mu} A^{\mu}-\alpha e v \phi_{2}\right)^{2} \\
& \mathcal{L}_{\text {F.P. }}=-c^{\dagger}\left(\partial^{2}+\alpha e^{2} v \phi_{1}\right) c \tag{4.2}
\end{align*}
$$

We would like to find the effective potential $V_{\text {eff }}[v]$ for the $\phi$ fields $\left(\operatorname{say}\left\langle\phi_{1}\right\rangle=v,\left\langle\phi_{2}\right\rangle=0\right)$ to the two loop order. In $n$ dimensions

$$
\begin{align*}
& {[\phi]=\left[A_{\mu}\right]=M^{(n-2) / 2} \quad, \quad[a]=[\alpha]=M^{0}} \\
& {[e]=M^{(4-n) / 2} \quad, \quad[\kappa]=M} \\
& {[m]=M^{( } \quad, \quad[\lambda]=M^{4-n} \quad, \quad[\nu]=M^{2(3-n)}} \tag{4.3}
\end{align*}
$$

Not all parameters in the Lagrangian (4.1) are independent. By scaling $A_{\mu}^{\prime}=t A_{\mu}$, one finds the equivalence relation

$$
\begin{equation*}
(a, \kappa, e, \alpha) \sim\left(\frac{a}{t^{2}}, \frac{\kappa}{t^{2}}, \frac{e}{t}, t^{2} \alpha\right) \tag{4.4}
\end{equation*}
$$

or

$$
\begin{equation*}
\left(a, k=\frac{\kappa}{e^{2}}, e, \alpha\right) \sim\left(\frac{a}{t^{2}}, k, \frac{e}{t}, t^{2} \alpha\right) . \tag{4.5}
\end{equation*}
$$

Physics is independent of $t$. If the renormalized $a=0$, physics in the Landau gauge ( $\alpha=0$ ) depends on $k=\kappa / e^{2}, m, \lambda$, and $\nu$. In particular, with $m=\lambda=0$ the classical theory contains no dimensional parameter. As is shown shortly, however, the $a=0$ theory should be defined by the limit $a \rightarrow 0$.

After shifting $\phi_{1} \rightarrow v+\phi_{1}$, the quadratic part of the Lagrangian (4.1) is

$$
\begin{align*}
\mathcal{L}_{(2)} & =\frac{1}{2} A_{\mu} K^{\mu \nu} A_{\nu}-c^{\dagger}\left(\partial^{2}+\alpha e^{2} v^{2}\right) c-\frac{1}{2} \phi_{1}\left(\partial^{2}+m_{1}^{2}\right) \phi_{1}-\frac{1}{2} \phi_{2}\left(\partial^{2}+m_{2}^{2}\right) \phi_{2} \\
K^{\mu \nu} & =\left\{a \partial^{2}+(e v)^{2}\right\} g^{\mu \nu}-\left(a-\frac{1}{\alpha}\right) \partial^{\mu} \partial^{\nu}+\kappa \epsilon^{\mu \nu \rho} \partial_{\rho} \\
m_{1}^{2} & =m_{1}^{2}(v)=m^{2}+\frac{\lambda}{2} v^{2}+\frac{\nu}{24} v^{4} \\
m_{2}^{2} & =m_{2}^{2}(v)=m^{2}+\frac{\lambda}{6} v^{2}+\frac{\nu}{120} v^{4}+\alpha(e v)^{2} \tag{4.6}
\end{align*}
$$

In this paper, we adopt the dimensional regularization method. The definition of the totally antisymmetric tensor, $\epsilon^{\mu \nu \rho}$, depends on the three dimensionality of spacetime. Below we define the $\epsilon^{\mu \nu \rho}$ tensor in $n$ dimensions in a way that it stays essentially in three dimensions. This definition was initially proposed by t'Hooft and Veltman. 233 It has been shown that Slavnov-Taylor identities are satisfied with this definition, 24] and that the Maxwell term improves the ultraviolet behavior of the gauge field propagator. (15]

In $n$ dimensions we define $\epsilon^{\mu \nu \rho}$ and $\hat{g}^{\mu \nu}$ by

$$
\begin{align*}
\epsilon^{\mu \nu \rho} & = \begin{cases} \pm 1 & \text { if }(\mu, \nu, \rho)=\text { permutation of }(0,1,2) \\
0 & \text { otherwise }\end{cases} \\
\hat{g}^{\mu \nu} & = \begin{cases}+1 & \text { for } \mu=\nu=0 \\
-1 & \text { for } \mu=\nu=1,2 \\
0 & \text { otherwise }\end{cases} \tag{4.7}
\end{align*}
$$

Then

$$
\begin{align*}
& \epsilon^{\mu \nu \rho} \epsilon^{\lambda \eta}{ }_{\rho}=\hat{g}^{\mu \lambda} \hat{g}^{\nu \eta}-\hat{g}^{\mu \eta} \hat{g}^{\nu \lambda} \\
& g^{\mu \nu} \hat{g}_{\nu}{ }^{\lambda}=\hat{g}^{\mu \lambda} \tag{4.8}
\end{align*}
$$

We denote $\hat{p}^{\mu}=\hat{g}^{\mu \nu} p_{\nu}$ etc.
The inverse of $K^{\mu \nu}$ in (4.6) is found easily. In general

$$
\begin{align*}
& K^{\mu \nu}=A g^{\mu \nu}+B p^{\mu} p^{\nu}-i \kappa \epsilon^{\mu \nu \rho} p_{\rho} \\
& K_{\nu \lambda}^{-1}=\frac{1}{A}\left(g_{\nu \lambda}-\frac{B}{A+p^{2} B} p_{\nu} p_{\lambda}\right)+\frac{\kappa^{2}}{A\left(A^{2}-\kappa^{2} \hat{p}^{2}\right)}\left(\hat{p}^{2} \hat{g}_{\nu \lambda}-\hat{p}_{\nu} \hat{p}_{\lambda}\right) \\
&+\frac{i \kappa}{A^{2}-\kappa^{2} \hat{p}^{2}} \epsilon_{\nu \lambda \rho} p^{\rho} . \tag{4.9}
\end{align*}
$$

In our case $A=-a p^{2}+(e v)^{2}, B=a-\alpha^{-1}$ so that

$$
\begin{align*}
K_{\nu \lambda}^{-1}= & -\frac{1}{d\left(p^{2}\right)}\left(g_{\nu \lambda}-(1-a \alpha) \frac{p_{\nu} p_{\lambda}}{p^{2}-\alpha(e v)^{2}}\right) \\
& -\frac{\kappa^{2} \hat{p}^{2}}{d\left(p^{2}\right)\left[d\left(p^{2}\right)^{2}-\kappa^{2} \hat{p}^{2}\right]}\left(\hat{g}_{\nu \lambda}-\frac{\hat{p}_{\nu} \hat{p}_{\lambda}}{\hat{p}^{2}}\right)+\frac{i \kappa \epsilon_{\nu \lambda \rho} p^{\rho}}{d\left(p^{2}\right)^{2}-\kappa^{2} \hat{p}^{2}} \\
d\left(p^{2}\right)= & a p^{2}-(e v)^{2} . \tag{4.10}
\end{align*}
$$

In the Landau gauge, $\alpha=0$,

$$
\begin{gather*}
\left.K_{\nu \lambda}^{-1}\right|_{\alpha=0}=-\frac{1}{d\left(p^{2}\right)}\left(g_{\nu \lambda}-\frac{p_{\nu} p_{\lambda}}{p^{2}}\right)-\frac{\kappa^{2} \hat{p}^{2}}{d\left(p^{2}\right)\left[d\left(p^{2}\right)^{2}-\kappa^{2} \hat{p}^{2}\right]}\left(\hat{g}_{\nu \lambda}-\frac{\hat{p}_{\nu} \hat{p}_{\lambda}}{\hat{p}^{2}}\right) \\
+\frac{i \kappa \epsilon_{\nu \lambda \rho} p^{\rho}}{d\left(p^{2}\right)^{2}-\kappa^{2} \hat{p}^{2}} \tag{4.11}
\end{gather*}
$$

In three dimensions, the propagator in the Landau gauge reduces to

$$
\begin{equation*}
\left.K_{\nu \lambda}^{-1}\right|_{3-\operatorname{dim}}=-\frac{1}{d\left(p^{2}\right)^{2}-\kappa^{2} p^{2}}\left\{d\left(p^{2}\right)\left(g_{\nu \lambda}-\frac{p_{\nu} p_{\lambda}}{p^{2}}\right)-i \kappa \epsilon_{\nu \lambda \rho} p^{\rho}\right\} \quad \text { for } \alpha=0 \tag{4.12}
\end{equation*}
$$

The propagator (4.10) can be decomposed into several pieces;

$$
\begin{align*}
& \left.K_{\mu \nu}^{-1}\right|_{\alpha=0}= \\
& -\frac{1}{a}\left\{\frac{1}{m_{+}+m_{-}}\left(\frac{1}{m_{+}} \frac{1}{p^{2}-m_{+}^{2}}+\frac{1}{m_{-}} \frac{1}{p^{2}-m_{-}^{2}}\right)-\frac{1}{m_{3}^{2}} \frac{1}{p^{2}}\right\}\left(\hat{g}_{\mu \nu} \hat{p}^{2}-\hat{p}_{\mu} \hat{p}_{\nu}\right) \\
& +\frac{1}{a} \frac{1}{m_{+}+m_{-}}\left(\frac{1}{p^{2}-m_{+}^{2}}-\frac{1}{p^{2}-m_{-}^{2}}\right) i \frac{\kappa}{|\kappa|} \epsilon_{\mu \nu \rho} p^{\rho} \\
& -\frac{1}{a} \frac{1}{m_{3}^{2}}\left(\frac{1}{p^{2}-m_{3}^{2}}-\frac{1}{p^{2}}\right)\left(\left(g_{\mu \nu} p^{2}-p_{\mu} p_{\nu}\right)-\left(\hat{g}_{\mu \nu} \hat{p}^{2}-\hat{p}_{\mu} \hat{p}_{\nu}\right)\right) \\
& +\frac{\kappa^{2}\left(p^{2}-\hat{p}^{2}\right)}{\left(d^{2}-\kappa^{2} p^{2}\right)\left(d^{2}-\kappa^{2} \hat{p}^{2}\right)}\left\{\frac{\kappa^{2}}{d}\left(\hat{g}_{\mu \nu} \hat{p}^{2}-\hat{p}_{\mu} \hat{p}_{\nu}\right)-i \kappa \epsilon_{\mu \nu \rho} p^{\rho}\right\} \tag{4.13}
\end{align*}
$$

Here

$$
\begin{align*}
m_{ \pm} & =m_{ \pm}(v)=\frac{1}{2}\left\{\sqrt{\frac{\kappa^{2}}{a^{2}}+\frac{4(e v)^{2}}{a}} \pm \frac{|\kappa|}{a}\right\} \\
m_{3}^{2} & =m_{+} m_{-}=\frac{e^{2} v^{2}}{a} \tag{4.14}
\end{align*}
$$

There are several poles. $m_{ \pm}$are the masses of physical gauge bosons in three dimensional spacetime. $m_{3}$ is the mass of photons in the extra-dimensional space. The massless pole corresponds to the gauge degree of freedom. The last term in (4.13) behaves as $1 / p^{5}$ for large $p$. It gives finite contributions which vanish in the $n \rightarrow 3$ limit. It is instructive to write $m_{ \pm}$in terms of Higgs mass $m_{\mathrm{H}}$ and Chern-Simons mass $m_{\mathrm{CS}}$ :

$$
m_{\mathrm{H}}=\frac{e v}{\sqrt{a}} \quad, \quad m_{\mathrm{CS}}=\frac{|\kappa|}{a}
$$

$$
\begin{equation*}
m_{ \pm}=\frac{1}{2}\left\{\sqrt{m_{\mathrm{CS}}^{2}+4 m_{\mathrm{H}}^{2}} \pm m_{\mathrm{CS}}\right\} \tag{4.15}
\end{equation*}
$$

## 5. One loop corrections in gauge theory

The one-loop effective potential can be evaluated easily. For $K^{\mu \nu}$ given in (4.9),

$$
\begin{equation*}
\operatorname{det} K=(-1)^{n-3} A^{n-3}\left(A+p^{2} B\right)\left(A^{2}-\kappa^{2} \hat{p}^{2}\right) \tag{5.1}
\end{equation*}
$$

Hence $V_{\text {eff }}^{(1)}(v)$ is, including the ghost contribution,

$$
\begin{align*}
V_{\text {eff }}(v)^{1-\text { loop }}= & \frac{\hbar}{2} \int \frac{d^{n} p}{i(2 \pi)^{n}}\left\{\ln \left[p^{2}-m_{1}(v)^{2}\right]+\ln \left[p^{2}-m_{2}(v)^{2}\right]\right. \\
& +\ln \left[\left\{a p^{2}-(e v)^{2}\right\}^{2}-\kappa^{2} \hat{p}^{2}\right]+(n-3) \ln \left[a p^{2}-(e v)^{2}\right] \\
& \left.+\ln \frac{1}{\alpha}\left[p^{2}-\alpha(e v)^{2}\right]-2 \ln \left[p^{2}-\alpha(e v)^{2}\right]\right\} \tag{5.2}
\end{align*}
$$

Except for the third term, the integrals can be evaluated by the standard formula (A.1) in Appendix A. The third term contains both $n$-dimensional $p^{2}$ and 3 -dimensional $\hat{p}^{2}$, and needs extra care. To evaluate it we consider

$$
\begin{align*}
F(x)=F(x ; c, n) & =\int \frac{d^{n} p}{i(2 \pi)^{n}} \ln \left[\left(p^{2}-c^{2}\right)^{2}-x \hat{p}^{2}\right] \\
& =\int \frac{d^{n} p}{(2 \pi)^{n}} \ln \left[\left(p^{2}+c^{2}\right)^{2}+x \hat{p}^{2}\right] \tag{5.3}
\end{align*}
$$

We write

$$
\begin{align*}
F(x) & =F(0)+x F^{\prime}(0)+\int_{0}^{x} d x_{1} \int_{0}^{x_{1}} d x_{2} F^{\prime \prime}\left(x_{2}\right) \\
F(0) & =\int \frac{d^{n} p}{(2 \pi)^{n}} \ln \left(p^{2}+c^{2}\right)^{2}=-2 \frac{\Gamma\left(-\frac{1}{2} n\right)}{(4 \pi)^{n / 2}} c^{n} \\
F^{\prime}(0) & =\int \frac{d^{n} p}{(2 \pi)^{n}} \frac{\hat{p}^{2}}{\left(p^{2}+c^{2}\right)^{2}}=\frac{3}{n} \int \frac{d^{n} p}{(2 \pi)^{n}} \frac{p^{2}}{\left(p^{2}+c^{2}\right)^{2}}=\frac{3}{2} \frac{\Gamma\left(1-\frac{1}{2} n\right)}{(4 \pi)^{n / 2}} c^{n-2} \\
F^{\prime \prime}(x) & =-\int \frac{d^{n} p}{(2 \pi)^{n}} \frac{\left(\hat{p}^{2}\right)^{2}}{\left[\left(p^{2}+c^{2}\right)^{2}+x \hat{p}^{2}\right]^{2}} \tag{5.4}
\end{align*}
$$

The integral for $F^{\prime \prime}(x)$ is finite at $n=3$ so that we may set $n=3$ :

$$
\begin{align*}
F^{\prime \prime}(x)_{n=3} & =-\int \frac{d^{3} p}{(2 \pi)^{3}} \frac{\left(p^{2}\right)^{2}}{\left[\left(p^{2}+c^{2}\right)^{2}+x p^{2}\right]^{2}} \\
& =-\frac{1}{2 \pi^{2}} \int_{0}^{\infty} d p \frac{p^{6}}{\left[\left(p^{2}+c^{2}\right)^{2}+x p^{2}\right]^{2}} \\
& =-\frac{1}{8 \pi} \frac{x+5 c^{2}}{\left(x+4 c^{2}\right)^{3 / 2}} \tag{5.5}
\end{align*}
$$

Here we have made use of (A.5). Hence

$$
\begin{align*}
F(x)_{n=3} & =-2 \frac{\Gamma\left(-\frac{3}{2}\right)}{(4 \pi)^{3 / 2}} c^{3}+\frac{3}{2} \frac{\Gamma\left(-\frac{1}{2}\right)}{(4 \pi)^{3 / 2}} c x-\frac{1}{8 \pi} \int_{0}^{x} d x_{1} \int_{0}^{x_{1}} d x_{2} \frac{x+5 c^{2}}{\left(x+4 c^{2}\right)^{3 / 2}} \\
& =-\frac{1}{6 \pi}\left(x+4 c^{2}\right)^{1 / 2}\left(x+c^{2}\right) \tag{5.6}
\end{align*}
$$

In other words, the integral $F(x)$ at $n=3$ is the same as the integral where $\hat{p}^{2}$ is replaced by $p^{2}$ in (5.3).

Returning to (5.2), we find

$$
\begin{equation*}
V_{\mathrm{eff}}(v)_{n=3}^{1-\mathrm{loop}}=-\frac{\hbar}{12 \pi}\left\{m_{1}(v)^{3}+m_{+}(v)^{3}+m_{-}(v)^{3}+m_{2}(v)^{3}-\left[\alpha(e v)^{2}\right]^{3 / 2}\right\} \tag{5.7}
\end{equation*}
$$

where

$$
\begin{equation*}
m_{+}^{3}+m_{-}^{3}=\sqrt{m_{\mathrm{CS}}^{2}+4 m_{\mathrm{H}}^{2}}\left(m_{\mathrm{CS}}^{2}+m_{\mathrm{H}}^{2}\right)=\sqrt{\frac{\kappa^{2}}{a^{2}}+\frac{4 e^{2} v^{2}}{a}}\left(\frac{\kappa^{2}}{a^{2}}+\frac{e^{2} v^{2}}{a}\right) \tag{5.8}
\end{equation*}
$$

Imposing the renormalization conditions (2.31), one finds that the effective potential at one loop is

$$
\begin{align*}
& V_{\mathrm{eff}}(v)^{1 \text { loop }}=\frac{\nu}{6!} v^{6}+\frac{\hbar}{12 \pi} \frac{\kappa^{3}}{a^{3}} G(z) \\
& G(z)=3 z-(1+4 z)^{1 / 2}(1+z)+\frac{2\left(1-62 \tilde{M}+240 \tilde{M}^{2}\right)}{(1+4 \tilde{M})^{11 / 2}} z^{3}+1 \\
& z=\frac{a e^{2} v^{2}}{\kappa^{2}}, \tilde{M}=\frac{a e^{2} M}{\kappa^{2}} . \tag{5.9}
\end{align*}
$$

It was pointed out in the previous paper that one loop calculations do not produce definitive results about symmetry breaking; the minimum occurs at $v=0$ or $v \neq 0$, depending on the choice of $M$. We need to go to two loop.

## 6. Two loop corrections in gauge theory

Relevant vertices for evaluating the two loop effective potential are

$$
\begin{align*}
\mathcal{L}_{\text {cubic }}= & e A^{\mu}\left(\phi_{1} \partial_{\mu} \phi_{2}-\phi_{2} \partial_{\mu} \phi_{1}\right)+e^{2} A_{\mu}^{2} \phi_{1} v \\
& -\frac{\lambda}{3!} v \phi_{1}\left(\phi_{1}^{2}+\phi_{2}^{2}\right)-\frac{\nu}{6!}\left(12 v^{3} \phi_{1} \phi_{2}^{2}+20 v^{3} \phi_{1}^{3}\right) \\
& -\alpha e^{2} v \phi_{1} c^{\dagger} c \tag{6.1}
\end{align*}
$$

and

$$
\begin{align*}
\mathcal{L}_{\text {quartic }}= & \frac{1}{2} e^{2} A_{\mu}^{2}\left(\phi_{1}^{2}+\phi_{2}^{2}\right)-\frac{\lambda}{4!}\left(\phi_{1}^{4}+\phi_{2}^{4}+2 \phi_{1}^{2} \phi_{2}^{2}\right) \\
& -\frac{\nu}{6!}\left(15 v^{2} \phi_{1}^{4}+18 v^{2} \phi_{1}^{2} \phi_{2}^{2}+3 v^{2} \phi_{2}^{4}\right) \tag{6.2}
\end{align*}
$$

The two loop effective potential is found by inserting (6.1) and (6.2) into (2.5). In the Landau gauge there are five types of diagrams to be evaluated.
(1) Two scalar loops


The part of the interaction Lagrangian that produces this diagram is

$$
\begin{equation*}
\tilde{\mathcal{L}}_{q 1}=-\alpha_{1} \phi_{1}^{4}-\alpha_{2} \phi_{2}^{4}-\alpha_{3} \phi_{1}^{2} \phi_{2}^{2} \tag{6.3}
\end{equation*}
$$

where

$$
\begin{align*}
& \alpha_{1}=\frac{\lambda}{4!}+\frac{\nu}{4!} \frac{v^{2}}{2} \\
& \alpha_{2}=\frac{\lambda}{4!}+\frac{\nu}{5!} \frac{v^{2}}{2} \\
& \alpha_{3}=\frac{2 \lambda}{4!}+\frac{\nu}{5!} 3 v^{2} \tag{6.4}
\end{align*}
$$

The effective potential due to this is

$$
V_{\mathrm{eff}(q 1)}=-i^{6} \hbar^{2} \int \frac{d^{n} p d^{n} q}{(2 \pi)^{2 n}}\left\{\frac{3 \alpha_{1}}{\left(p^{2}+m_{1}^{2}\right)\left(q^{2}+m_{1}^{2}\right)}+\frac{3 \alpha_{2}}{\left(p^{2}+m_{2}^{2}\right)\left(q^{2}+m_{2}^{2}\right)}\right.
$$

$$
\begin{gather*}
\left.+\frac{\alpha_{3}}{\left(p^{2}+m_{1}^{2}\right)\left(q^{2}+m_{2}^{2}\right)}\right\} \\
=\frac{\hbar^{2} \mu^{2(n-3)}}{(4 \pi)^{2}}\left\{3\left(\frac{\lambda}{4!}+\frac{15 \nu v^{2}}{6!}\right) m_{1}^{2}+3\left(\frac{\lambda}{4!}+\frac{3 \nu v^{2}}{6!}\right) m_{2}^{2}\right. \\
\left.+2\left(\frac{\lambda}{4!}+\frac{9 \nu v^{2}}{6!}\right) m_{1} m_{2}\right\} \tag{6.5}
\end{gather*}
$$

(2) One scalar and one gauge loop


For this diagram we have

$$
\begin{equation*}
\tilde{\mathcal{L}}_{q 2}=\frac{1}{2} e^{2} A_{\mu}^{2}\left(\phi_{1}^{2}+\phi_{2}^{2}\right) \tag{6.6}
\end{equation*}
$$

The effective potential due to this is

$$
\begin{align*}
& V_{\mathrm{eff}(q 2)}= i^{2} \hbar^{2} \frac{e^{2}}{2} \int \frac{d^{n} p d^{n} q}{(2 \pi)^{2 n}} i K_{\mu}^{\mu}(p)^{-1}\left[\frac{i}{q^{2}-m_{1}^{2}}+\frac{i}{q^{2}-m_{1}^{2}}\right] \\
&-\frac{e^{2} \hbar^{2}}{2 a} \int \frac{d^{n} p d^{n} q}{(2 \pi)^{2 n}}\left[\frac{1}{q^{2}+m_{1}^{2}}+\frac{1}{q^{2}+m_{2}^{2}}\right] \\
& \times\left\{\left[\frac{1}{m_{+}+m_{-}}\left(\frac{1}{m_{+}} \frac{1}{p^{2}+m_{+}^{2}}+\frac{1}{m_{-}} \frac{1}{p^{2}+m_{-}^{2}}\right)-\frac{1}{m_{3}^{2}} \frac{1}{p^{2}}\right]\left(2 \hat{p}^{2}\right)\right. \\
&\left.+\frac{1}{m_{3}^{2}}\left[\frac{1}{p^{2}+m_{3}^{2}}-\frac{1}{p^{2}}\right]\left[(n-1) p^{2}-2 \hat{p}^{2}\right]\right\} . \tag{6.7}
\end{align*}
$$

After the integration

$$
\begin{equation*}
V_{\mathrm{eff}}^{(q 2)}=\frac{e^{2} \hbar^{2} \mu^{2(n-3)}}{16 \pi^{2} a} \frac{\left(m_{1}+m_{2}\right)\left(m_{+}^{2}+m_{-}^{2}\right)}{m_{+}+m_{-}} . \tag{6.8}
\end{equation*}
$$

(3) $\theta$-shape diagram with pure scalar fields


This diagram is due to the following interaction Lagrangian :

$$
\begin{align*}
\tilde{\mathcal{L}}_{\mathrm{c} 1} & =-\frac{\lambda}{3!} v \phi_{1}\left(\phi_{1}^{2}+\phi_{2}^{2}\right)-\frac{\nu}{6!}\left(12 v^{3} \phi_{1} \phi_{2}^{2}+20 v^{3} \phi_{1}^{3}\right) \\
& =-\beta_{1} \phi_{1}^{3}-\beta_{2} \phi_{1} \phi_{2}^{2} \tag{6.9}
\end{align*}
$$

where

$$
\begin{align*}
& \beta_{1}=\frac{\lambda}{3!} v+\frac{\nu}{36} v^{3} \\
& \beta_{2}=\frac{\lambda}{3!} v+\frac{\nu}{60} v^{3} . \tag{6.10}
\end{align*}
$$

The effective potential due to this is the same as the result obtained for the pure scalar theory given in Section 2.

$$
\begin{align*}
V_{\mathrm{eff}(c 1)}= & -\frac{i^{8} \hbar^{2}}{2} \int \frac{d^{n} p d^{n} q}{(2 \pi)^{2 n}} \frac{1}{(p+q)^{2}+m_{1}^{2}}\left\{\frac{6 \beta_{1}^{2}}{\left(p^{2}+m_{1}^{2}\right)\left(q^{2}+m_{1}^{2}\right)}+\frac{2 \beta_{2}^{2}}{\left(p^{2}+m_{2}^{2}\right)\left(q^{2}+m_{2}^{2}\right)}\right\} \\
= & -\hbar^{2}\left[3\left(\frac{\lambda}{3!} v+\frac{\nu}{36} v^{3}\right)^{2}+\left(\frac{\lambda}{3!} v+\frac{\nu}{60} v^{3}\right)^{2}\right] I^{\text {div }} \\
& +\frac{\hbar^{2} \mu^{2(n-3)}}{16 \pi^{2}}\left\{3\left(\frac{\lambda}{3!} v+\frac{\nu}{36} v^{3}\right)^{2} \ln \frac{3 m_{1}}{\mu}+\left(\frac{\lambda}{3!} v+\frac{\nu}{60} v^{3}\right)^{2} \ln \frac{m_{1}+2 m_{2}}{\mu}\right\} . \tag{6.11}
\end{align*}
$$

(4) $\theta$-shape diagram with two scalar and one gauge propagators


For this diagram :

$$
\begin{equation*}
\tilde{\mathcal{L}}_{\mathrm{c} 2}=e A^{\mu}\left(\phi_{1} \partial_{\mu} \phi_{2}-\phi_{2} \partial_{\mu} \phi_{1}\right) \tag{6.12}
\end{equation*}
$$

The effective potential due to this is :

$$
\begin{equation*}
V_{\mathrm{eff}(c 2)}=\frac{i^{6} e^{2} \hbar^{2}}{2} \int \frac{d^{n} p d^{n} q}{(2 \pi)^{2 n}}(p+2 q)^{\mu}(p+2 q)^{\nu} K_{\mu \nu}^{-1}(p) \Delta_{1}(q) \Delta_{2}(-p-q) \tag{6.13}
\end{equation*}
$$

where $K_{\mu \nu}^{-1}, \Delta_{1}$, and $\Delta_{2}$ denote $A_{\mu}, \phi_{1}$ and $\phi_{2}$ propagators, respectively.
Since

$$
(p+2 q)^{\mu}(p+2 q)^{\nu}\left(\hat{p}^{2} \hat{g}_{\mu \nu}-\hat{p}_{\mu} \hat{p}_{\nu}\right)=4\left[\hat{p}^{2} \hat{q}^{2}-(\hat{p} \cdot \hat{q})^{2}\right]
$$

$$
\begin{equation*}
(p+2 q)^{\mu}(p+2 q)^{\nu}\left(p^{2} g_{\mu \nu}-p_{\mu} p_{\nu}\right)=4\left[p^{2} q^{2}-(p \cdot q)^{2}\right] \tag{6.14}
\end{equation*}
$$

(6.13) is reduced to three and $n$-dimensional integrals. The effective potential becomes

$$
\begin{align*}
V_{\text {eff }(c 2)} & =-\frac{4 e^{2} \hbar^{2}}{2} \int \frac{d^{n} p d^{n} q}{(2 \pi)^{2 n}} \frac{1}{q^{2}-m_{1}^{2}} \frac{1}{(p+q)^{2}-m_{2}^{2}} \\
& \times\left\{-\frac{1}{a}\left[\frac{1}{m_{+}+m_{-}}\left(\frac{1}{m_{+}} \frac{1}{p^{2}-m_{+}^{2}}+\frac{1}{m_{-}} \frac{1}{p^{2}-m_{-}^{2}}\right)-\frac{1}{m_{3}^{2}} \frac{1}{p^{2}}\right]\left(\hat{p}^{2} \hat{q}^{2}-(\hat{p} \cdot \hat{q})^{2}\right)\right. \\
& -\frac{1}{a} \frac{1}{m_{3}^{2}}\left[\frac{1}{p^{2}-m_{3}^{2}}-\frac{1}{p^{2}}\right]\left[\left(p^{2} q^{2}-(p \cdot q)^{2}\right)-\left(\hat{p}^{2} \hat{q}^{2}-(\hat{p} \cdot \hat{q})^{2}\right)\right] \\
& \left.+\frac{\kappa^{2}\left(p^{2}-\hat{p}^{2}\right)}{\left(d^{2}-\kappa^{2} p^{2}\right)\left(d^{2}-\kappa^{2} \hat{p}^{2}\right)} \frac{\kappa^{2}}{d}\left(\hat{p}^{2} \hat{q}^{2}-(\hat{p} \cdot \hat{q})^{2}\right)\right\} . \tag{6.15}
\end{align*}
$$

Employing (C.4) and (C.6), one finds

$$
\begin{align*}
& V_{\mathrm{eff}}^{(c 2)}=\frac{e^{2} \hbar^{2}}{2 a}\left[2\left(m_{1}^{2}+m_{2}^{2}\right)-\left(m_{+}+m_{-}\right)^{2}+3 m_{3}^{2}\right] I^{\mathrm{div}} \\
& \quad+\frac{e^{2} \hbar^{2} \mu^{2(n-3)}}{32 \pi^{2} a}\left\{\left[m_{1} m_{2}-\frac{\left(m_{1}+m_{2}\right)\left\{2\left(m_{1}-m_{2}\right)^{2}+m_{+}^{2}+m_{-}^{2}\right\}}{m_{+}+m_{-}}\right]\right. \\
& \quad-\frac{\left(m_{1}^{2}-m_{2}^{2}\right)^{2}}{m_{3}^{2}} \ln \frac{m_{1}+m_{2}}{\mu} \\
& \left.\quad-\sum_{a= \pm} \frac{2 m_{a}^{2}\left(m_{1}^{2}+m_{2}^{2}\right)-m_{a}^{4}-\left(m_{1}^{2}-m_{2}^{2}\right)^{2}}{m_{a}\left(m_{+}+m_{-}\right)} \ln \frac{m_{a}+m_{1}+m_{2}}{\mu}-\frac{5}{12} \frac{\kappa^{2}}{a^{2}}\right\} . \tag{6.16}
\end{align*}
$$

(5) $\theta$-shape diagram with two gauge and one scalar propagators


The interaction Lagrangian is

$$
\begin{equation*}
\tilde{\mathcal{L}}_{c 3}=e^{2} A_{\mu}^{2} \phi_{1} v \tag{6.17}
\end{equation*}
$$

and the corresponding effective potential is

$$
\begin{equation*}
V_{\mathrm{eff}(c 3)}=i^{6} \hbar^{2} e^{4} v^{2} \int \frac{d^{n} p d^{n} q}{(2 \pi)^{2 n}} K^{-1}(p)_{\mu \nu} K^{-1}(q)^{\mu \nu} \frac{1}{(p+q)^{2}-m_{1}^{2}} . \tag{6.18}
\end{equation*}
$$

Upon contracting the tensor indices between the gauge propagators we have

$$
\left(\hat{p}^{2} \hat{g}^{\mu \nu}-\hat{p}^{\mu} \hat{p}^{\nu}\right)\left(\hat{q}^{2} \hat{g}_{\mu \nu}-\hat{q}_{\mu} \hat{q}_{\nu}\right)=\hat{p}^{2} \hat{q}^{2}+(\hat{p} \cdot \hat{q})^{2}
$$

$$
\begin{align*}
& \left(i \kappa \epsilon^{\mu \nu \rho} p_{\rho}\right)\left(i \kappa \epsilon_{\mu \nu \sigma} q^{\sigma}\right)=-2 \kappa^{2}(\hat{p} \cdot \hat{q}) \\
& {\left[\left(p^{2} g^{\mu \nu}-p^{\mu} p^{\nu}\right)-\left(\hat{p}^{2} \hat{g}^{\mu \nu}-\hat{p}^{\mu} \hat{p}^{\nu}\right)\right]\left[\left(q^{2} g_{\mu \nu}-q_{\mu} q_{\nu}\right)-\left(\hat{q}^{2} \hat{g}_{\mu \nu}-\hat{q}_{\mu} \hat{q}_{\nu}\right)\right]} \\
& \quad=(n-2) p^{2} q^{2}+(p \cdot q)^{2}-2\left(p^{2} \hat{q}^{2}+\hat{p}^{2} q^{2}\right)+3 \hat{p}^{2} \hat{q}^{2}-(\hat{p} \cdot \hat{q})^{2} \\
& \left(\hat{p}^{2} \hat{g}^{\mu \nu}-\hat{p}^{\mu} \hat{p}^{\nu}\right)\left[\left(q^{2} g_{\mu \nu}-q_{\mu} q_{\nu}\right)-\left(\hat{q}^{2} \hat{g}_{\mu \nu}-\hat{q}_{\mu} \hat{q}_{\nu}\right)\right] \\
& \quad=2 \hat{p}^{2}\left(q^{2}-\hat{q}^{2}\right) \\
& {\left[\frac{\kappa^{2}}{d\left(p^{2}\right)}\left(\hat{p}^{2} \hat{g}^{\mu \nu}-\hat{p}^{\mu} \hat{p}^{\nu}\right)-i \kappa \epsilon^{\mu \nu \rho} p_{\rho}\right]\left[\frac{\kappa^{2}}{d\left(q^{2}\right)}\left(\hat{q}^{2} \hat{g}_{\mu \nu}-\hat{q}_{\mu} \hat{q}_{\nu}\right)-i \kappa \epsilon_{\mu \nu \sigma} q^{\sigma}\right]} \\
& \quad=\frac{\kappa^{4}}{d\left(p^{2}\right) d\left(q^{2}\right)}\left[\hat{p}^{2} \hat{q}^{2}+(\hat{p} \cdot \hat{q})^{2}\right]-2 \kappa^{2}(\hat{p} \cdot \hat{q}) . \tag{6.19}
\end{align*}
$$

Note that integrals involving the last term in the propagator $K_{\mu \nu}^{-1}$, (4.13), vanish in the limit $n=3$. With (6.19) the effective potential (6.18) reduces to

$$
\begin{equation*}
V_{\mathrm{eff}(c 3)}=\frac{i^{6} \hbar^{2} e^{4} v^{2}}{a^{2}}\left[\tilde{V}_{c 3 a}+\tilde{V}_{c 3 b}+\tilde{V}_{c 3 c}+\tilde{V}_{c 3 d}\right] \tag{6.20}
\end{equation*}
$$

where

$$
\begin{align*}
& \tilde{V}_{c 3 a}= \int \frac{d^{n} p d^{n} q}{(2 \pi)^{2 n}} \frac{\hat{p}^{2} \hat{q}^{2}+(\hat{p} \cdot \hat{q})^{2}}{(p+q)^{2}-m_{1}^{2}}\left[\frac{1}{m_{+}+m_{-}}\left(\frac{1}{m_{+}} \frac{1}{p^{2}-m_{+}^{2}}+\frac{1}{m_{-}} \frac{1}{p^{2}-m_{-}^{2}}\right)-\frac{1}{m_{3}^{2}} \frac{1}{p^{2}}\right] \\
& \times\left[\frac{1}{m_{+}+m_{-}}\left(\frac{1}{m_{+}} \frac{1}{q^{2}-m_{+}^{2}}+\frac{1}{m_{-}} \frac{1}{q^{2}-m_{-}^{2}}\right)-\frac{1}{m_{3}^{2}} \frac{1}{q^{2}}\right] \\
& \tilde{V}_{c 3 b}=-\frac{\kappa^{2}}{a^{2}} \frac{1}{\left(m_{+}^{2}-m_{-}^{2}\right)^{2}} \int \frac{d^{n} p d^{n} q}{(2 \pi)^{2 n}} \frac{2 \hat{p} \cdot \hat{q}}{(p+q)^{2}-m_{1}^{2}}\left[\frac{1}{p^{2}-m_{+}^{2}}-\frac{1}{p^{2}-m_{-}^{2}}\right] \\
& \times\left[\frac{1}{q^{2}-m_{+}^{2}}-\frac{1}{q^{2}-m_{-}^{2}}\right] \\
& \tilde{V}_{c 3 c}= \frac{1}{m_{3}^{4}} \int \frac{d^{n} p d^{n} q}{(2 \pi)^{2 n} \frac{(n-2) p^{2} q^{2}+(p \cdot q)^{2}-2\left(p^{2} \hat{q}^{2}+\hat{p}^{2} q^{2}\right)+3 \hat{p}^{2} \hat{q}^{2}-(\hat{p} \cdot \hat{q})^{2}}{(p+q)^{2}-m_{1}^{2}}} \\
& \quad \times\left[\frac{1}{p^{2}-m_{3}^{2}}-\frac{1}{p^{2}}\right]\left[\frac{1}{q^{2}-m_{3}^{2}}-\frac{1}{q^{2}}\right] \\
& \tilde{V}_{c 3 d}= \frac{4}{m_{3}^{2}} \int \frac{d^{n} p d^{n} q}{(2 \pi)^{2 n} \frac{\hat{p}^{2}\left(q^{2}-\hat{q}^{2}\right)}{(p+q)^{2}-m_{1}^{2}}\left[\frac{1}{q^{2}-m_{3}^{2}}-\frac{1}{q^{2}}\right]} \\
& \quad \times\left[\frac{1}{m_{+}+m_{-}}\left(\frac{1}{m_{+}} \frac{1}{p^{2}-m_{+}^{2}}+\frac{1}{m_{-}} \frac{1}{p^{2}-m_{-}^{2}}\right)-\frac{1}{m_{3}^{2}} \frac{1}{p^{2}}\right] \tag{6.21}
\end{align*}
$$

Once again the integrals above can be evaluated, with the aid of (C.4), (C.6) and (C.7), to be

$$
\begin{align*}
& V_{\mathrm{eff}}^{(c 3)}=-\frac{3 \hbar^{2} e^{4} v^{2}}{2 a^{2}} I^{\mathrm{div}}-\frac{\hbar^{2} e^{4} v^{2} \mu^{2(n-3)}}{32 \pi^{2} a^{2}}\left[-\frac{2 m_{1}}{m_{+}+m_{-}}-\frac{2 m_{1}^{2}+12 m_{3}^{2}}{\left(m_{+}+m_{-}\right)^{2}}+3\right] \\
& +\frac{\hbar^{2} e^{4} v^{2} \mu^{2(n-3)}}{64 \pi^{2} a^{2}}\left\{\frac{2\left[\left(m_{+}-m_{-}\right)^{2}-m_{1}^{2}\right]^{2}}{m_{3}^{2}\left(m_{+}+m_{-}\right)^{2}} \ln \frac{m_{+}+m_{-}+m_{1}}{\mu}+\frac{m_{1}^{4}}{m_{3}^{4}} \ln \frac{m_{1}}{\mu}\right. \\
& \left.\quad+\sum_{a= \pm}\left[\frac{\left(4 m_{a}^{2}-m_{1}^{2}\right)^{2}}{m_{a}^{2}\left(m_{+}+m_{-}\right)^{2}} \ln \frac{2 m_{a}+m_{1}}{\mu}-\frac{2\left(m_{a}^{2}-m_{1}^{2}\right)^{2}}{m_{3}^{2} m_{a}\left(m_{+}+m_{-}\right)} \ln \frac{m_{a}+m_{1}}{\mu}\right]\right\} . \tag{6.22}
\end{align*}
$$

We have obtained the effective potential up to two loop order. Two loop corrections yield divergent contributions. Collecting all divergent terms in (6.11), (6.16), and (6.22), we see

$$
\begin{align*}
V_{\text {eff }}(v)^{\mathrm{div}}= & \hbar^{2} C I^{\text {div }} \\
C= & -3\left(\frac{\lambda}{3!} v+\frac{\nu}{36} v^{3}\right)^{2}-\left(\frac{\lambda}{3!} v+\frac{\nu}{60} v^{3}\right)^{2} \\
& +\frac{2 e^{2}}{a}\left(m^{2}+\frac{\lambda}{3} v^{2}+\frac{\nu}{120} v^{4}\right)-\frac{2 e^{4} v^{2}}{a^{2}}-\frac{e^{2} \kappa^{2}}{2 a^{3}} . \tag{6.23}
\end{align*}
$$

These divergent terms are absorbed by counter terms. The renormalizability guarantees that divergent terms are proportional to $v^{0}, v^{2}, v^{4}$, or $v^{6}$. It is important to recognize that these counter terms are singular in $a$. We shall come back to this point when we discuss the Coleman-Weinberg limit in the next section.

The effective potential in the $\overline{\mathrm{MS}}$ scheme is obtained by simply dropping divergent terms. In the rest of this section we investigate the behavior of the effective potential at small and large $v$ analytically. We investigate the global behavior of the potential numerically in Section 9.

In the $\overline{\mathrm{MS}}$ scheme

$$
\begin{equation*}
V_{\mathrm{eff}}(v)^{\overline{\mathrm{MS}}}=V_{\mathrm{eff}}^{(\text {tree })}+V_{\mathrm{eff}}^{(1-\text { loop })}+V_{\mathrm{eff}}^{(2-\text { loop finite })} . \tag{6.24}
\end{equation*}
$$

We are interested in the behavior of the effective potential in the massless limit defined by $m^{2}=\lambda=0$.

To find the behavior of $V_{\text {eff }}(v)^{\overline{\mathrm{MS}}}$ for small $v$, we note that the expansion parameter $z$ is

$$
\begin{equation*}
z=\frac{a e^{2} v^{2}}{\kappa^{2}} \tag{6.25}
\end{equation*}
$$

The masses of the gauge bosons are expanded as

$$
m_{ \pm}=\left\{\begin{array}{l}
\frac{\kappa}{a}\left(1+z-z^{2}+2 z^{3}+\cdots\right)  \tag{6.26}\\
\frac{\kappa}{a}\left(z-z^{2}+2 z^{3}+\cdots\right)
\end{array}\right.
$$

Up to two loop the small $v$ expansion is

$$
\begin{equation*}
V_{\mathrm{small}}^{\overline{\mathrm{MS}}}=\sum_{n=1}^{\infty} C_{2 n} v^{2 n}+\sum_{n=3}^{\infty} D_{2 n} v^{2 n} \ln v \tag{6.27}
\end{equation*}
$$

The crucially important coefficient is $D_{6}$, which is produced by logarithmic terms originating from $V_{c 1}, V_{c 2}$ and $V_{c 3}$.

$$
\begin{align*}
V_{c 1} & =\frac{\hbar^{2}}{32 \pi^{2}} \frac{7}{675} \nu^{2} v^{6} \ln v+\cdots \\
V_{c 2} & =-\frac{\hbar^{2}}{32 \pi^{2}}\left[\frac{\nu}{5} \frac{e^{4}}{\kappa^{2}}-2 \frac{e^{8}}{\kappa^{4}}\right] v^{6} \ln v+\cdots \\
V_{c 3} & =\frac{\hbar^{2}}{32 \pi^{2}}\left[14 \frac{e^{8}}{\kappa^{4}}-\frac{\nu}{6} \frac{e^{4}}{\kappa^{2}}\right] v^{6} \ln v+\cdots . \tag{6.28}
\end{align*}
$$

Combining all of these, we obtain

$$
\begin{equation*}
D_{6}=\frac{\hbar^{2}}{32 \pi^{2}}\left(16 \frac{e^{8}}{\kappa^{4}}-\frac{11}{30} \frac{\nu e^{4}}{\kappa^{2}}+\frac{7}{675} \nu^{2}\right) \tag{6.29}
\end{equation*}
$$

Since there are no $\ln v, v^{2} \ln v$, or $v^{4} \ln v$ terms in (6.27), we may impose the same renormalization conditions (2.31) as in pure scalar case. With these renormalization conditions, the dominant behavior of the potential at small $v$ is given by

$$
\begin{equation*}
V_{\text {small }}(v) \sim D_{6} v^{6} \ln \frac{v}{\sqrt{M}} . \tag{6.30}
\end{equation*}
$$

Since $D_{6}$ is always positive, we conclude that the tree level minimum at $v=0$ has turned into a maximum.

Next we turn to the behavior at large $v$. Upon using the inverse of the previous expansion parameter, the gauge boson masses are given by

$$
\begin{equation*}
m_{ \pm}=\frac{e v}{\sqrt{a}} \pm \frac{\kappa}{2 a}+\mathrm{O}\left(\frac{1}{v}\right) \tag{6.31}
\end{equation*}
$$

The dominant term for all the gauge boson masses are the same. The potential is parametrized to two loop as

$$
\begin{equation*}
V_{\operatorname{large}}^{\overline{\mathrm{MS}}}=\sum_{n=0}^{\infty} F_{6-n} v^{6-n}+\sum_{n=0}^{\infty} G_{6-n} v^{6-n} \ln v \tag{6.32}
\end{equation*}
$$

Again, terms contributing to $G_{6}$ arise from $V_{c 1}, V_{c 2}$ and $V_{c 3}$. Looking at the logarithmic part term by term, we have

$$
\begin{align*}
V_{c 1} & =\frac{\hbar^{2}}{32 \pi^{2}} \frac{7}{675} \nu^{2} v^{6} \ln v+\cdots \\
V_{c 2} & =O\left(v^{4} \ln v\right)+\cdots \\
V_{c 3} & =O\left(v^{4} \ln v\right)+\cdots \tag{6.33}
\end{align*}
$$

The $v^{6} \ln v$ terms in $V_{c 2}$ and $V_{c 3}$ exactly cancel.
Note that the coefficients of $v^{6} \ln v$ terms in the above are independent of gauge couplings $a, e$ or $\kappa$. $G_{6}$ turns out to be independent of any gauge couplings. $G_{6}$ is determined solely by the $V_{c 1}$ term.

$$
\begin{equation*}
G_{6}=\frac{7 \hbar^{2}}{30 \pi^{2}} \frac{\nu^{2}}{6!} \tag{6.34}
\end{equation*}
$$

Similarly, one can check that $F_{6}$ term comes entirely from $V_{q 1}$ and $V_{c 1}$. The above limit also corresponds to expansion in small $\kappa$ for non-vanishing $a$.

The potential is positive at large $v$, thus establishing the stability of the theory. Combining the result at small $v$, we conclude the symmetry is spontaneously broken in the massless scalar theory.
$D_{6}$ and $G_{6}$ are independent of $a$. This is no coincidence. In general, a Feynman diagram for the effective potential at arbitrary order in the $\overline{\mathrm{MS}}$ scheme is written as a sum of terms of the form

$$
\begin{equation*}
\nu^{n_{1}} \lambda^{n_{2}}\left(\frac{e^{2}}{a}\right)^{n_{3}} v^{n_{4}} f\left[m_{1}, m_{2}, m_{3}, m_{+}, m_{-}\right] \tag{6.35}
\end{equation*}
$$

where $f$ is a finite, well-defined function of various $m_{k}(v, m, \lambda, \nu, a, e, \kappa)$ 's. This follows from the form of various vertices and the gauge field propagator (4.13). As we show in Section 10, the superficial degree of divergence for a diagram involving at least one gauge field propagator with no external legs is at most 2. The last term in (4.13) lowers the divergence degree by 3 , and therefore its contributions to the effective potential are finite and vanish in the $n=3$ limit. This establishes the form (6.35).

The powers $n_{1} \sim n_{4}$ are zero or positive integers. The equivalence relation (4.4) implies that in the Landau gauge $\alpha=0$ the effective potential $V_{\text {eff }}(v)$ can depend on gauge couplings only through $\kappa / e^{2}$ and $\kappa / a$.

When higher loop corrections are included, the dominant part of the effective potential at large $v$ takes the form

$$
\begin{equation*}
V_{\text {large }}^{\overline{\mathrm{MS}}}=\sum_{k=1}^{\infty} G_{6}^{(k)} v^{6}(\ln v)^{k}+\cdots \quad \text { for large } v \tag{6.36}
\end{equation*}
$$

The coefficients $G_{6}^{(p)}$,s are dimensionless. $(\ln v)^{k}$ terms arise from logarithmically divergent integrals. Furthermore for large $v, m_{1}^{2}, m_{2}^{2} \sim \nu v^{4}$ and $m_{ \pm}^{2}, m_{3}^{2} \sim e^{2} v^{2} / a$ so that $(\ln v)^{k}$ terms do not depend on $\kappa$ at all.

The $\lambda$ dependence of $G_{6}^{(p)}$ 's can appear only from vertices, with the power $n_{2} \geq 0$. But available dimensionless combinations $\lambda / m$ and $a \lambda / e^{2}$ are singular in the $m \rightarrow 0$ or $e \rightarrow 0$ limit. Since the $m \rightarrow 0$ limit is well defined for $v \neq 0, G_{6}^{(p)}$ s cannot depend on $\lambda / m$. Similarly the theory with a vanishing gauge coupling $(e=0)$ is well defined. This excludes the dependence on $\lambda$. To summarize, $G_{6}^{(p)}$,s depend on only $\nu$.
[Theorem] In the scalar electrodynamics the coefficients $G_{6}^{(p)}$ 's defined in (6.36) for the effective potential at large $v$ are independent of gauge couplings $a, e$, and $\kappa$ and of $m$ and $\lambda$ to all order in perturbation theory.

For small $v$ we consider the special case $m=\lambda=0$. The dominant part of the effective potential at small $v$ is written as

$$
\begin{equation*}
V_{\mathrm{small}}^{\overline{\mathrm{MS}}}=\sum_{k=1}^{\infty} D_{6}^{(k)} v^{6}(\ln v)^{k}+\cdots \quad \text { for small } v \tag{6.37}
\end{equation*}
$$

The coefficients $D_{6}^{(k)}$ s are dimensionless, and therefore can depend on $\nu$ and $e^{2} / \kappa$ only.
[Theorem] In the massless scalar electrodynamics with $m=\lambda=0$ the coefficients $D_{6}^{(k)}$, $s$ defined in (6.37) for the effective potential at small $v$ are independent of $a$ to all order in perturbation theory.

## 7. Coleman-Weinberg limit

As explained earlier, the Maxwell term is necessary to define the theory in the dimensional regularization scheme. Without it the theory loses renormalizability, as the gauge field propagator in extra-dimensional space behaves badly at high momenta. The Coleman-Weinberg limit is defined as the limit where there is no dimensional parameter to start with. In this case, it corresponds to the $a, m, \lambda \rightarrow 0$ limit.

The subtlety lies in the $a \rightarrow 0$ limit. Loop corrections give rise to terms singular in $a$. We shall see below that all these singular terms are absorbed by counter terms at least to two loop. The $a \rightarrow 0$ limit, the Coleman-Weinberg limit, is well defined after renormalization.

The effective potential is expressed in terms of $m_{1}(v), m_{2}(v), m_{ \pm}(v)$, and $m_{3}(v)$. Only the gauge boson masses depend on $a$. The expansion in $a$ is thus equivalent to the expansion in $z$ defined in 6.25), provided that $\kappa \neq 0$.

Expanding $V_{\mathrm{eff}}(v)$ in $a$, one finds that with $m=\lambda=0$,

$$
\begin{aligned}
V_{\text {eff }}(v) & =V^{\text {tree }}+V^{1-\text { loop }}+V^{2-\text { loop }}+V^{\text {c.t. }} \\
V^{1-\text { loop }} & =-\frac{\hbar \mu^{n-3}}{12 \pi}\left\{\frac{\kappa^{3}}{a^{3}}+\frac{\kappa e^{2}}{a^{2}} v^{2}+\frac{2 e^{6}}{\kappa^{3}} v^{6}+\mathrm{O}(a)\right\}
\end{aligned}
$$

$$
\begin{align*}
V^{2-\mathrm{loop}}= & \left.\hbar^{2} C\right|_{m=\lambda=0} I^{\mathrm{div}}+\sum_{n=0}^{\infty} A_{2 n} a^{n-3} v^{2 n} \\
& +\sum_{n=0}^{\infty} B_{2 n} a^{n-3} v^{2 n} \ln \left(\frac{k}{a}\right)+\sum_{n=3}^{\infty} D_{2 n} v^{2 n} \ln v . \tag{7.1}
\end{align*}
$$

Here $C$ is given in (6.23) and

$$
\begin{aligned}
& A_{0}=-\frac{\hbar}{12 \pi} \kappa^{3}-\frac{5}{6} \frac{\hbar^{2}}{64 \pi^{2}} e^{2} \kappa^{2}-\frac{\hbar^{2}}{64 \pi^{2}} e^{2} \kappa^{2} \ln \mu^{2} \\
& A_{2}=-\frac{\hbar}{4 \pi} e^{2} \kappa+\frac{\hbar^{2}}{16 \pi^{2}} \sqrt{\frac{\nu}{24}} e^{2} \kappa\left(1+\frac{1}{\sqrt{5}}\right)-\frac{\hbar^{2}}{32 \pi^{2}} e^{4}+\frac{\hbar^{2}}{4 \pi^{2}} e^{4} \ln 2-\frac{\hbar^{2}}{16 \pi^{2}} e^{4} \ln \mu^{2} \\
& A_{4}=-\frac{\hbar^{2}}{1280 \pi^{2}} \nu e^{2}+\frac{\hbar^{2}}{640 \pi^{2}} \nu e^{2} \ln \mu^{2}+\frac{\hbar^{2}}{4 \pi^{2}} \frac{e^{6}}{\kappa^{2}}-\frac{\hbar^{2}}{2 \pi^{2}} \frac{e^{6}}{\kappa^{2}} \ln 2 \\
& A_{6}=-\frac{\hbar}{576 \pi} \frac{\nu^{3 / 2}}{\sqrt{6}}\left(1+\frac{1}{5 \sqrt{5}}\right)-\frac{\hbar}{6 \pi} \frac{e^{6}}{\kappa^{3}}+\frac{\hbar^{2}}{16 \pi^{2}} \frac{\nu^{2}}{720}\left[\frac{39}{20}+\frac{3}{4 \sqrt{5}}+\frac{5}{6} \ln \left(\frac{3}{8}\right)\right. \\
& \left.+\frac{1}{5} \ln \left(\frac{1}{\sqrt{24}}+\frac{1}{\sqrt{30}}\right)-\frac{2}{5} \ln \left(1+\frac{1}{\sqrt{5}}\right)-\frac{7}{160} \ln \left(\frac{1}{\sqrt{24}}\right)\right] \\
& +\frac{427 \hbar^{2}}{7680 \pi^{2}} \frac{\nu^{2}}{720} \ln \nu-\frac{\hbar^{2}}{768 \pi^{2}} \frac{\nu^{3 / 2}}{\sqrt{24}} \frac{e^{2}}{\kappa}\left(\frac{52}{15}-\frac{4}{15 \sqrt{5}}\right) \\
& +\frac{\hbar^{2}}{1536 \pi^{2}} \frac{\nu e^{4}}{\kappa^{2}}\left(\frac{7}{5}+\frac{2}{\sqrt{5}}\right)+\frac{\hbar^{2}}{32 \pi^{2}} \sqrt{\frac{\nu}{24}} \frac{e^{6}}{\kappa^{3}}\left(16+\frac{3}{\sqrt{5}}\right)-\frac{13 \hbar^{2}}{24 \pi^{2}} \frac{e^{8}}{\kappa^{4}} \\
& +\left[\frac{7 \hbar^{2}}{4 \pi^{2}} \frac{e^{8}}{\kappa^{4}}-\frac{\hbar^{2}}{192 \pi^{2}} \frac{\nu e^{4}}{\kappa^{2}}\right] \ln 2 \\
& +\left[\frac{\hbar^{2}}{32 \pi^{2}} \frac{e^{8}}{\kappa^{4}}-\frac{\hbar^{2}}{320 \pi^{2}} \frac{\nu e^{4}}{\kappa^{2}}+\frac{\hbar^{2}}{28800 \pi^{2}} \nu^{2}\right] \ln \left(\frac{e^{2}}{\kappa}+\sqrt{\frac{\nu}{24}}\left(1+\frac{1}{\sqrt{5}}\right)\right) \\
& +\left[\frac{\hbar^{2}}{4 \pi^{2}} \frac{e^{8}}{\kappa^{4}}-\frac{\hbar^{2}}{192 \pi^{2}} \frac{\nu e^{4}}{\kappa^{2}}+\frac{\hbar^{2}}{36864 \pi^{2}} \nu^{2}\right] \ln \left(\frac{2 e^{2}}{\kappa}+\sqrt{\frac{\nu}{24}}\right) \\
& +\left[-\frac{\hbar^{2}}{32 \pi^{2}} \frac{e^{8}}{\kappa^{4}}+\frac{\hbar^{2}}{384 \pi^{2}} \frac{\nu e^{4}}{\kappa^{2}}-\frac{\hbar^{2}}{18432 \pi^{2}} \nu^{2}\right] \ln \left(\frac{e^{2}}{\kappa}+\sqrt{\frac{\nu}{24}}\right) \\
& B_{0}=\frac{\hbar^{2}}{32 \pi^{2}} e^{2} \kappa^{2} \\
& B_{2}=\frac{\hbar^{2}}{8 \pi^{2}} e^{4} \\
& B_{4}=-\frac{\hbar^{2}}{320 \pi^{2}} \nu e^{2}
\end{aligned}
$$

$$
\begin{equation*}
B_{6}=-\frac{\hbar^{2}}{4 \pi^{2}} \frac{e^{8}}{\kappa^{4}}+\frac{11 \hbar^{2}}{1920 \pi^{2}} \frac{\nu e^{4}}{\kappa^{2}} \tag{7.2}
\end{equation*}
$$

$D_{6}$ is given in (6.29). Notice that those terms singular in $a$ are of the form $v^{n}$ where $n=0,2,4,6$. They are cancelled by counter terms. The renormalized theory has a welldefined $a \rightarrow 0$ limit. The coefficients of $A_{2 n}$ and $B_{2 n}$ are related to $C_{2 n}$ given in (6.27) by

$$
\begin{equation*}
C_{2 n}=A_{2 n} a^{n-3}+B_{2 n} a^{n-3} \ln \left(\frac{k}{a}\right) \tag{7.3}
\end{equation*}
$$

Let us consider the Coleman-Weinberg limit. We take the $a \rightarrow 0$ limit with a given $v$. Adopting the renormalization condition (2.31), one has

$$
\begin{equation*}
V_{\mathrm{CW}}(v)=\frac{\nu(M)}{6!} v^{6}+D_{6} v^{6}\left(\ln \frac{v}{\sqrt{M}}-\frac{49}{20}\right) . \tag{7.4}
\end{equation*}
$$

Let us choose the renormalization point to be the location of the minimum $\sqrt{M}=v_{\text {min }}$. The condition for minimum value of the effective potential is

$$
\begin{equation*}
V_{\mathrm{CW}}^{\prime}\left(v_{\min }\right)=v_{\min }^{5}\left[\frac{\nu}{5!}-\frac{137}{10} D_{6}\right]=0 \tag{7.5}
\end{equation*}
$$

from which it follows, with the aid of (6.29),

$$
\begin{equation*}
\nu_{\mathrm{CW}}=\nu\left(v_{\mathrm{min}}^{2}\right)=\frac{1}{2}\left[b_{1}-\sqrt{b_{1}^{2}-4 b_{2}}\right] \tag{7.6}
\end{equation*}
$$

where

$$
\begin{align*}
& b_{1}=\frac{495}{14} \frac{e^{4}}{\kappa^{2}}+\frac{1800 \pi^{2}}{959 \hbar^{2}} \\
& b_{2}=\frac{10800}{7} \frac{e^{8}}{\kappa^{4}} \tag{7.7}
\end{align*}
$$

The Coleman-Weinberg limit potential is written as

$$
\begin{equation*}
V_{\mathrm{CW}}=\frac{\nu_{\mathrm{CW}}}{1644} v^{6}\left(\ln \frac{v}{v_{\mathrm{min}}}-\frac{1}{6}\right) \tag{7.8}
\end{equation*}
$$

In the case $\nu=\mathrm{O}\left(e^{8} / \kappa^{4}\right)$, equation (7.6) becomes

$$
\begin{equation*}
\nu_{\mathrm{CW}}=\frac{822 \hbar^{2}}{\pi^{2}} \frac{e^{8}}{\kappa^{4}} \tag{7.9}
\end{equation*}
$$

so that the Coleman-Weinberg potential is

$$
\begin{equation*}
V_{\mathrm{CW}}=\frac{\hbar^{2}}{2 \pi^{2}} \frac{e^{8}}{\kappa^{4}} v^{6}\left(\ln \frac{v}{v_{\min }}-\frac{1}{6}\right) . \tag{7.10}
\end{equation*}
$$

The symmetry is spontaneously broken. Dimensional transmutation takes place. The perturbation theory is reliable as far as $e^{2} / \kappa$ is small.

## 8. Pure Maxwell theory $(\kappa=0)$

Another interesting limit is when the kinetic term for the gauge fields is given by the Maxwell term only. This corresponds to taking $\kappa \rightarrow 0$ in the previous expression for the effective potential, keeping $a$ non-vanishing. Without loss of generality one can set $a=1$. This theory is parity preserving as opposed to Chern-Simons theory. At the tree, level in the limit of vanishing $m^{2}$ and $\lambda$, there is one dimensional parameter $e$. In the Landau gauge, the gauge field propagator reduces to

$$
\begin{equation*}
\left.K_{\nu \lambda}^{-1}\right|_{\alpha=0}=-\frac{1}{\left(p^{2}-e^{2} v^{2}\right)}\left(g_{\nu \lambda}-\frac{p_{\nu} p_{\lambda}}{p^{2}}\right) \tag{8.1}
\end{equation*}
$$

In the rest of this section we set $m^{2}=\lambda=0$. The one loop contributions are simplified to

$$
\begin{equation*}
V_{\mathrm{eff}}^{1-\mathrm{loop}}=-\frac{\hbar}{12 \pi}\left[\left\{\left(\frac{1}{24}\right)^{\frac{3}{2}}+\left(\frac{1}{120}\right)^{\frac{3}{2}}\right\} \nu^{\frac{3}{2}} v^{6}+2 e^{3}|v|^{3}\right] \tag{8.2}
\end{equation*}
$$

There appears a $|v|^{3}$ correction to the tree level effective potential. The loop corrections take the form

$$
\begin{aligned}
V_{\mathrm{eff}}(v)^{\text {loop }}= & \frac{\hbar^{2}}{32 \pi^{2}} v^{2}\left[\left(-e^{2}+\frac{\nu v^{2}}{40}\right)^{2}-\frac{139 \nu^{2} v^{4}}{6!\cdot 60}\right]\left[-\frac{1}{n-3}-\gamma_{E}+1+\ln 4 \pi\right] \\
& +\left[C_{2} v^{2}+C_{3} v^{3}+C_{4} v^{4}+C_{5} v^{5}+C_{6} v^{6}\right] \\
& +\frac{\hbar^{2}}{16 \pi^{2}} v^{2}\left[\left(e^{2}-\frac{\nu v^{2}}{80}\right)^{2}-\frac{149 \nu^{2} v^{4}}{6!\cdot 240}\right] \ln \frac{v^{2}}{\mu} \\
& +\frac{\hbar^{2}}{64 \pi^{2}} v^{2}\left[\left(e^{2}-\frac{\nu v^{2}}{20}\right)^{2}-\frac{\nu^{2} v^{4}}{6!}\right] \ln \frac{1}{\mu}\left(v+\frac{e \sqrt{120 / \nu}}{1+\sqrt{5}}\right)^{2}
\end{aligned}
$$

$$
\begin{align*}
& +\frac{\hbar^{2}}{16 \pi^{2}} v^{2}\left[\left(e^{2}-\frac{\nu v^{2}}{96}\right)^{2}+\frac{\nu^{2} v^{4}}{9216}\right] \ln \frac{(v+2 e \sqrt{24 / \nu})^{2}}{\mu} \\
& -\frac{\hbar^{2}}{64 \pi^{2}} v^{2}\left[\left(e^{2}-\frac{\nu v^{2}}{24}\right)^{2}\right] \ln \frac{(v+e \sqrt{24 / \nu})^{2}}{\mu} \tag{8.3}
\end{align*}
$$

where

$$
\begin{align*}
C_{2}= & \frac{\hbar^{2}}{16 \pi^{2}} e^{4} \ln \frac{\nu}{120}+\frac{\hbar^{2}}{32 \pi^{2}} e^{4} \ln (1+\sqrt{5}) \\
C_{3}= & -\frac{\hbar}{6 \pi} e^{3}+\frac{\hbar^{2}}{32 \pi^{2}} e^{3}(1+2 \sqrt{5}) \sqrt{\frac{\nu}{120}} \\
C_{4}= & \frac{\hbar^{2}}{768 \pi^{2}} e^{2}\left[2 \sqrt{5}+5-24 \ln \sqrt{\frac{\nu}{120}}(1+\sqrt{5})\right] \\
C_{5}= & -\frac{\hbar^{2}}{960 \pi^{2}} e \nu(-1+\sqrt{5}) \sqrt{\frac{\nu}{120}} \\
C_{6}= & -\frac{\hbar}{12 \pi}\left[\left(\frac{1}{24}\right)^{\frac{3}{2}}+\left(\frac{1}{120}\right)^{\frac{3}{2}}\right] \nu^{\frac{3}{2}}+\frac{\hbar^{2}}{16 \pi^{2}} \frac{\nu^{2}}{720}\left(\frac{39}{20}+\frac{3}{4 \sqrt{5}}\right) \\
& +\frac{\hbar^{2}}{32 \pi^{2}} \frac{\nu^{2}}{36}\left[\frac{1}{12} \ln \frac{9 \nu}{24}+\frac{1}{100} \ln \frac{\nu}{120}(9+2 \sqrt{5})\right] \tag{8.4}
\end{align*}
$$

There are terms of the form $v^{2} \ln v$ and $v^{4} \ln v$ so that the renormalization conditions (2.31) cannot be imposed. Both the second and fourth derivative of the effective potential must be evaluated at a non-vanishing value of the field. Both $v^{2} \ln v$ and $v^{4} \ln v$ terms arise from $V_{c 2}$ and $V_{c 3}$ in (6.16) and (6.22), respectively. Their origin is traced back to the logarithmic terms in $m_{ \pm}, m_{1}$ and/or $m_{2}$ appearing in both expressions. The $\ln v^{2}$ terms arise as

$$
\begin{equation*}
\ln \left(\chi_{1} m_{+}+\chi_{2} m_{-}+\chi_{3} m_{1}+\chi_{4} m_{2}\right)^{2}=\ln v^{2}+\ln \left(\left(\chi_{1}+\chi_{2}\right) \tilde{m}_{3}+\left(\chi_{3} \tilde{m}_{1}+\chi_{4} \tilde{m}_{2}\right) v\right)^{2} \tag{8.5}
\end{equation*}
$$

In this equation $\tilde{m}_{3}, \tilde{m}_{1}$ and $\tilde{m}_{2}$ are independent of $v$. In the limit of large $v$, the same expression as in (6.32) is obtained.

As the coefficient of the $v^{2} \ln v$ term in (8.3) is positive, the $U(1)$ symmetry is spontaneously broken at two loop.

## 9. Numerical analysis

In Section 6, we have obtained the analytic expression for the effective potential up to two loop order (see equation (6.24)). To implement the renormalization conditions (2.31), we need to calculate the sixth derivative of the effective potential, which is highly nontrivial. Although we have $V_{\text {eff }}(v)^{\overline{\mathrm{MS}}}$ in the closed form, each term in $V_{\text {eff }}(v)^{\overline{\mathrm{MS}}}$ leads to an extremely lengthy expression when differentiated six times. We have found that standard symbolic manipulation aided by Mathematica or Maple is not of much help.

We adopt numerical evaluation to find the sixth derivative of $V_{\text {eff }}(v)^{\overline{\mathrm{MS}}}$ at finite $v$. We have found that it is best to make use of the Cauchy integral formula. First the effective potential is analytically continued to the complex $v$ plane. We measure all dimensionful quantities in the unit of $e$ to define dimensionless quantities:

$$
\begin{equation*}
\tilde{V}_{\mathrm{eff}}=\frac{V_{\mathrm{eff}}}{e^{6}}, \quad x=\frac{v}{e}, \quad k=\frac{\kappa}{e^{2}}, \quad h^{2}=\frac{M}{e^{2}} \tag{9.1}
\end{equation*}
$$

where M is the renormalization point. Note

$$
\begin{equation*}
\tilde{V}_{\mathrm{eff}}=\tilde{V}_{\mathrm{eff}}(x ; \nu, k, a, h) \tag{9.2}
\end{equation*}
$$

The numerical analysis is further simplified by removing the pole and other terms proportional to $x^{2}, x^{4}$ and $x^{6}$ in $V^{2-l o o p}$ as those terms are completely absorbed in the definition of counter terms. After this procedure the effective potential takes the following form:

$$
\begin{equation*}
\tilde{V}_{\text {eff }}=\frac{\nu}{6!} x^{6}+\tilde{V}_{\text {loop }}+\tilde{V}^{\text {counter-terms }} \tag{9.3}
\end{equation*}
$$

where $\tilde{V}^{\text {counter-terms }}=\alpha_{0}+\frac{1}{2} \alpha_{2} x^{2}+\frac{1}{4!} \alpha_{4} x^{4}+\frac{1}{6!} \alpha_{6} x^{6}$.
The $n$-th $x$-derivative of the potential at $h$ is

$$
\begin{equation*}
\tilde{V}^{(n)}(h)=\frac{n!}{2 \pi i} \int_{C} d z \frac{\tilde{V}(z)}{(z-h)^{n+1}} \tag{9.4}
\end{equation*}
$$

where the contour $C$ should not encircle any singularities of $\tilde{V}$. The imaginary part of the above integral is zero within the numerical precision.

The counter terms are fixed by the renormalization conditions. (9.3) can be rewritten as

$$
\begin{align*}
\tilde{V}_{\text {eff }}(x ; \nu, k, a, h)= & \frac{\nu}{6!} x^{6}+\tilde{V}_{\text {loop }}(x)-\tilde{V}_{\text {loop }}(0)-\frac{1}{2} \tilde{V}_{\text {loop }}^{(2)}(0) x^{2} \\
& -\frac{1}{4!} \tilde{V}_{\text {loop }}^{(4)}(0) x^{4}-\frac{1}{6!} \tilde{V}_{\text {loop }}^{(6)}(h) x^{6} . \tag{9.5}
\end{align*}
$$

With this definition $\nu \equiv \nu(h)=\tilde{V}_{\text {eff }}^{(6)}(h) . \quad \tilde{V}_{\text {loop }}(0), \tilde{V}_{\text {loop }}^{(2)}(0)$, and $\tilde{V}_{\text {loop }}^{(4)}(0)$ are evaluated analytically from the small $v$ expansion in Section $6 . \tilde{V}_{\text {loop }}^{(6)}(h)$ is evaluated numerically by (9.4).

In fig. 2 the tree, 1-loop, and 2-loop effective potentials are plotted for typical values of parameters. The importance of two loop corrections is recognized. We also have depicted


Figure 2: Tree, 1-loop, and 2-loop effective potentials. Plots for $a=1, k=20, \nu=0.0005$, $h=1$
the effective potential for different values of parameters in fig. 3.
Given $\nu, k, a$, and $h$, the potential is fixed. It reaches a minimum at $x=x_{\min } . x_{\min }$ differs in general from $h . \nu$ at the scale $x_{\text {min }}$ is

$$
\begin{equation*}
\nu\left(x_{\min }\right)=\tilde{V}_{\mathrm{eff}}^{(6)}\left(x_{\mathrm{min}}\right) \tag{9.6}
\end{equation*}
$$

which differs from the initial $\nu=\nu(h)$. Hence, the effective potential can be written as $\bar{V}_{\text {eff }}\left(a, k, \nu\left(x_{\min }\right), x_{\mathrm{min}} ; x\right)=\tilde{V}_{\text {eff }}(a, k, \nu, h ; x)$.


Figure 3: The two loop effective potential plot for $a=1$ and $h=1$ using different values of $k$ and $\nu$

A detailed investigation of this yields some interesting properties. For example, a typical plot of $h \equiv h_{\text {in }}$ vs $x_{\min } \equiv h_{\text {out }}$ is shown in fig.4. For particular values of parameters $\nu$ and $k$ the curve unexpectedly blows up in the region between $h_{\text {in }}=10$ and 30 .

The region of small $h_{i n}$ also shows some peculiar behavior which we have not been able to explain. We suspect that it could be due to the limitation of numerical evaluations or some unexplained phenomenon. (See fig. 5.)

We are also interested in the Coleman-Weinberg limit of the potential. From the results of the previous section, the effective potential in the Coleman-Weinberg limit is given by

$$
\begin{equation*}
\tilde{V}_{\mathrm{CW}}=\frac{\hat{\nu}_{\mathrm{CW}}}{1644} x^{6}\left(\ln \frac{x}{x_{\mathrm{min}}}-\frac{1}{6}\right) \tag{9.7}
\end{equation*}
$$

where $\hat{\nu}_{\mathrm{CW}}=V_{\mathrm{CW}}^{(6)}\left(x_{\text {min }}\right)$. The potential is parametrized by two quantities $\hat{\nu}_{\mathrm{CW}}$ and $x_{\text {min }}$. The location of the minimum is not determined by other parameters. Instead it becomes an input parameter. The limit $a \rightarrow 0$ must be taken with due caution.

As explained above, the input $h_{\mathrm{in}}=h$ and output $h_{\text {out }}=x_{\min }$ are different in general. As displayed in fig. 4, there is a fixed point value $h_{\text {out }}=h_{\text {in }}$ for given $\nu, k$, and $a$. Take this value for $h$. Then $x_{\min }=h$ and $\nu=\nu\left(x_{\min }\right)$. In this particular case $x_{\min }=x_{\min }(\nu, k, a)$.


Figure 4: $h_{\text {in }}$ vs $h_{\text {out }}$ plot for $k=20, \nu=0.0005$ using various values of $a$

Now we examine the $a$ dependence of $x_{\min }$. The equivalence relation (4.5) implies that

$$
\begin{equation*}
(a, k, e, \nu) \sim\left(a^{\prime}, k, e^{\prime}=\sqrt{\frac{a^{\prime}}{a}} e, \nu\right) \tag{9.8}
\end{equation*}
$$

The two theories are the same so that the effective potential reach the minimum at the same $v: v_{\text {min }}^{\prime}=v_{\text {min }}$. In terms of the $x$ variable

$$
\begin{equation*}
x_{\min }^{\prime}=\frac{v_{\min }^{\prime}}{e^{\prime}}=\sqrt{\frac{a}{a^{\prime}}} x_{\min } . \tag{9.9}
\end{equation*}
$$

In other words, if the $a \rightarrow 0$ limit is taken with given $\nu$ and $k$, then $x_{\min } \rightarrow \infty$, i.e. the Coleman-Weinberg limit is not obtained. This explains why the fixed point in fig. 4 moves to the right as $a$ gets smaller. The Coleman-Weinberg limit is not attained because the expansion parameter $z$ in (6.25)

$$
\begin{equation*}
z^{\prime}=\frac{a^{\prime} x_{\min }^{\prime}}{k^{\prime 2}}=\frac{a x_{\min }^{2}}{k^{2}}=z \tag{9.10}
\end{equation*}
$$

remains unchanged.


Figure 5: Blown-up region of $h_{\text {in }}$ vs $h_{\text {out }}$ plot for small $h_{\text {in }}$ with $k=20$ and $\nu=0.0005$

To get the Coleman-Weinberg limit one should not pick the fixed point value for $h$. One should choose $\nu, k$, and $h$ such that the expansion parameter at the minimum $z=a x_{\text {min }}^{2} / k^{2}$ becomes small when $a$ becomes small.

Further in the Coleman-Weinberg limit $\nu\left(x_{\min }\right)$ and $k$ are related by (7.6). This guides to the following procedure. Pick a value for $k$ and fix $\nu$ to be $\hat{\nu}_{\mathrm{CW}}(k)$. Next pick values for $h$ and $a$ such that $z<1$ and $h$ and $x_{\min }$ are not terribly far apart. With these $k$ and $h$, we make $a$ smaller to check if the potential approaches the Coleman-Weinberg limit. At a given $a$ we compare the potential $\tilde{V}_{\text {eff }}\left(x ; \hat{\nu}_{\mathrm{CW}}, k, a, h\right)$ with the Coleman-Weinberg potential (9.7) where $x_{\min }=x_{\min }(k, h, a)$ is the location of the minimum of $\tilde{V}_{\text {eff }}(x)$.

In fig. 6 we displayed the result for $k=20$ and $h=1$. For these values $\hat{\nu}_{\mathrm{CW}}=$ $5.18 \times 10^{-3}$. One can see that the two potentials get closer to each other as $a$ becomes smaller.

## 10. Divergence structure

It is helpful to understand the divergence structure of the theory by examining the superficial degree of divergence in perturbation theory. In doing so, one has to distinguish


Figure 6: Plots of $V_{\text {eff }}$ and $V_{\mathrm{CW}}$ as a function of $x$ for different values of $a$. Points correspond to $V_{\text {eff }}$. Lines correspond to $V_{\mathrm{CW}}$ determined by $x_{\min }$ and $V_{\mathrm{CW}}(k)$
the $a=0$ and $a \neq 0$ case. As we have observed in the preceding sections, the theory becomes pathological if the perturbation theory is based on a free gauge field propagator with $a=0$. The Coleman-Weinberg theory has been defined by the limit $a \rightarrow 0$.
(i) The case $a \neq 0$

The gauge field propagator is given by (4.10) - (4.12). Notice that the propagator behaves as $1 / p^{2}$ for large $p^{2}$ :

$$
\begin{equation*}
\text { for } a \neq 0, \quad K_{\mu \nu}^{-1} \sim \frac{1}{p^{2}} \quad \text { as } p^{2} \rightarrow \infty \tag{10.1}
\end{equation*}
$$

It is important that (10.1) is true in arbitrary dimensions $n$ and irrespective of whether $v=0$ or $v \neq 0$. Hence it is sufficient to examine the superficial degree of divergence in the unbroken theory $v=0$. The ultra-violet behavior does not depend on whether $v=0$ or $v \neq 0$.

The Lagrangian (4.1) yields various vertices. Let $V_{4}, V_{6}, V_{3 A}, V_{4 A}$, and $V_{3 c}$ be the numbers of vertices $\phi^{4}, \phi^{6}, A \phi \partial \phi, A^{2} \phi^{2}$, and $\phi c^{\dagger} c$ in a given Feynman diagram $F$, respectively. We denote by $E$ and $I$ the number of external and internal lines contained in $F$,
respectively. Then the number of loop momenta $L$ is

$$
\begin{equation*}
L=I-V+1 \tag{10.2}
\end{equation*}
$$

where $V=V_{4}+V_{6}+V_{3 A}+V_{4 A}+V_{3 c}$.
Since all propagators behave as $1 / p^{2}$ and the vertex $A \phi \partial \phi$ carries a derivative, the superficial degree of divergence in $n$ dimensions is

$$
\begin{equation*}
\omega=n L-2 I+V_{3 A} . \tag{10.3}
\end{equation*}
$$

The topological identity $3\left(V_{3 c}+V_{3 A}\right)+4\left(V_{4 A}+V_{4}\right)+6 V_{6}+E=2(E+I)$ gives

$$
\begin{equation*}
I=\frac{1}{2}\left\{3 V_{3 c}+3 V_{3 A}+4 V_{4 A}+4 V_{4}+6 V_{6}-E\right\} \tag{10.4}
\end{equation*}
$$

Combining (10.2) - (10.4), one finds

$$
\begin{equation*}
\omega=2(n-3) V_{6}+(n-4)\left(V_{4}+V_{4 A}\right)+\frac{1}{2}(n-4) V_{3 A}+\frac{1}{2}(n-6) V_{3 c}-\frac{1}{2}(n-2) E+n . \tag{10.5}
\end{equation*}
$$

In three dimensions $n=3$

$$
\begin{align*}
\omega & =-V_{4}-V_{4 A}-\frac{1}{2} V_{3 A}-\frac{3}{2} V_{3 c}-\frac{1}{2} E+3 \\
L & =2 V_{6}+V_{4}+V_{4 A}+\frac{1}{2} V_{3 A}+\frac{1}{2} V_{3 c}-\frac{1}{2} E+1 . \tag{10.6}
\end{align*}
$$

For propagators $(E=2), \omega=2-V_{4}-V_{4 A}-\frac{1}{2} V_{3 A}-\frac{3}{2} V_{3 c}$. Divergent contributions to the wave function renormalization for the scalar field, $Z_{\phi}$, come from only $\nu^{n}$ terms. The anomalous dimension is

$$
\begin{equation*}
\gamma_{\phi}=\gamma_{\phi}(\nu) \text { to all orders }, \tag{10.7}
\end{equation*}
$$

i.e. it does not depend on gauge couplings. Since a diagram of a single loop is finite in the dimensional regularization scheme, $\delta Z_{\phi}=\mathrm{O}\left(\nu^{2}\right)$. In other words, the anomalous dimension vanishes, $\gamma_{\phi}=0$, to the two loop order. The mass counter term for scalar fields is $\mathrm{O}\left(\lambda, e^{2}, \lambda^{2}, \lambda e^{2}, e^{4}\right) \times \mathrm{O}\left(\nu^{n}\right)$. To two loop $\delta m^{2}=\mathrm{O}\left(\lambda^{2}, \lambda e^{2}, e^{4}\right)$.

Contributions to the gauge field propagator must satisfy $V_{4 A} \geq 1$ or $V_{3 A} \geq 2$. There is no divergent contribution proportional to $A_{\mu}^{2}$ from the gauge invariance. This implies that
the coefficient of $F_{\mu \nu} F^{\mu \nu}$ remains finite so that the wave function renormalization factor $Z_{A}=1$. Hence the anomalous dimension vanishes.

$$
\begin{equation*}
\gamma_{A}=0 \quad \text { to all orders } \tag{10.8}
\end{equation*}
$$

There could appear divergent contributions to the Chern-Simons coefficient $\epsilon^{\mu \nu \rho} A_{\mu} F_{\nu \rho}$. The superficial degree of divergence for the Chern-Simons coefficient is $1-V_{4}-V_{4 A}-\frac{1}{2} V_{3 A}-\frac{3}{2} V_{3 c}$. And hence $\delta \kappa=\mathrm{O}\left(e^{2} \nu^{n}\right)(n \geq 2)$. Since $V_{4 A} \geq 1$ or $V_{3 A} \geq 2$, it vanishes at two loop.

Contributions to the coefficient of the vertex $A \phi \partial \phi$ have $\omega=\frac{1}{2}-V_{4}-V_{4 A}-\frac{1}{2} V_{3 A}-\frac{3}{2} V_{3 c}$. A diagram must have at least one $e, V_{3 A} \geq 1$. Hence $\delta e=\mathrm{O}\left(e \nu^{n}\right)(n \geq 2) . \delta \kappa$ and $\delta e$ are not independent. The Coleman-Hill theorem [17] ensures that $\delta\left(\kappa / e^{2}\right)=0$.

Contributions to the vertex $\lambda \phi^{4}$ have $\omega=1-V_{4}-V_{4 A}-\frac{1}{2} V_{3 A}-\frac{3}{2} V_{3 c}$. Hence $\delta \lambda=\mathrm{O}\left(\nu^{n}\right)$ $(n \geq 2)$ or $\mathrm{O}\left(\lambda \nu^{n}, e^{2} \nu^{n}\right)(n \geq 1)$. Contributions to the vertex $\nu \phi^{6}$ have $\omega=-V_{4}-V_{4 A}-$ $\frac{1}{2} V_{3 A}-\frac{3}{2} V_{3 c}$. Since $\gamma_{\phi}=\gamma_{\phi}(\nu)$, the beta function depends on only $\nu$ :

$$
\begin{equation*}
\beta_{\nu}=\beta_{\nu}(\nu) \quad \text { to all orders. } \tag{10.9}
\end{equation*}
$$

(ii) The case $a=0$

We have to stress that the perturbation theory based on $a=0$ is inconsistent in the dimensional regularization supplemented with $\epsilon^{\mu \nu \rho}$ in (4.7). This is due to the behavior of the gauge field propagator at large momenta. The propagator (4.10) behaves at large $p^{2}$ and $a=0$

$$
\begin{equation*}
K_{\nu \lambda}^{-1} \sim-\frac{1}{(e v)^{2}}\left[\left(g_{\nu \lambda}-\frac{p_{\nu} p_{\lambda}}{p^{2}}\right)-\left(\hat{g}_{\nu \lambda}-\frac{\hat{p}_{\nu} \hat{p}_{\lambda}}{\hat{p}^{2}}\right)\right]-\frac{i \epsilon_{\nu \lambda \rho} p^{\rho}}{\kappa \hat{p}^{2}} . \tag{10.10}
\end{equation*}
$$

The first term does not vanish. In particular, extra-dimensional components of $K_{\nu \lambda}^{-1}$ behaves as $\mathrm{O}\left(p^{0}\right)$. In other words, higher loop diagrams with many gauge field propagators behave very badly. The theory in the dimensional regularization scheme loses the renormalizability if $a$ is set to be zero in defining the perturbation theory. One consistent way to define the $a=0$ theory (the Coleman-Weinberg theory) is to take the limit $a \rightarrow 0$ after renormalization, which we have adopted in this paper.

Yet this does not entirely exclude the possibility of defining a theory with $a=0$. One possibility is to stay in three dimensions, adopting the Pauli-Villars regularization method.

We have not checked the feasibility of the Pauli-Villars regularization method beyond one loop, particularly when the symmetry breaking takes place. There is ambiguity in defining regulator fields.

Here we add an argument concerning the divergence structure, assuming that there exists a regularization method defined entirely in three dimensions, consistent to all orders at $a=0$. Should such a regularization scheme exist, the gauge field propagator in the Landau gauge would be, as inferred from (4.13),

$$
\begin{equation*}
\left.K_{\nu \lambda}^{-1}\right|_{3-\operatorname{dim}}=\frac{-1}{\kappa^{2} p^{2}-(e v)^{4}}\left\{-(e v)^{2}\left(g_{\nu \lambda}-\frac{p_{\nu} p_{\lambda}}{p^{2}}\right)-i \kappa \epsilon_{\nu \lambda \rho} p^{\rho}\right\} . \tag{10.11}
\end{equation*}
$$

The asymptotic behavior is

$$
\begin{equation*}
K_{\nu \lambda}^{-1} \sim \frac{-i \epsilon_{\nu \lambda \rho} p^{\rho}}{\kappa p^{2}}=\mathrm{O}\left(\frac{1}{p}\right) . \tag{10.12}
\end{equation*}
$$

We suppose that regulator fields have the same behavior.
Accepting (10.12), we derive the formula for the superficial degree of divergence. To distinguish gauge field propagators we introduce the following notation. The number of external gauge, Fadeev-Popov ghost, or scalar fields is denoted by $E_{A}, E_{c}$, or $E_{\phi}$, respectively. Similarly the number of internal gauge, Faddeev-Popov ghost, or scalar fields is denoted by $I_{A}, I_{c}$, or $I_{\phi}$. We have $E=E_{A}+E_{c}+E_{\phi}$ and $I=I_{A}+I_{c}+I_{\phi}$.

The identities (10.2) and (10.4) are still valid. Because of (10.12), (10.3) is modified to

$$
\begin{equation*}
\omega=n L-2\left(I_{\phi}+I_{c}\right)-I_{A}+V_{3 A} . \tag{10.13}
\end{equation*}
$$

The topological identity associated with gauge couplings is $2 V_{4 A}+V_{3 A}+E_{A}=2\left(E_{A}+I_{A}\right)$, from which it follows that

$$
\begin{equation*}
I_{A}=\frac{1}{2}\left(V_{3 A}+2 V_{4 A}-E_{A}\right) . \tag{10.14}
\end{equation*}
$$

Combining these, we have

$$
\begin{equation*}
\omega=3-V_{4}-\frac{3}{2} V_{3 c}-\frac{1}{2} E_{\phi}-E_{A}-\frac{1}{2} E_{c} . \tag{10.15}
\end{equation*}
$$

The formula for $L$ remains the same as in (10.6). Notice that gauge couplings become marginal; the superficial degree of divergence does not depend on $\nu$ or $e$. (Recall that in the $a=0$ theory $e$ appears only in the combination $\kappa / e^{2}$ which is dimensionless.)

The divergence structure is quite different. This time one would conclude that $\gamma_{\phi}, \beta_{\nu}$, $\beta_{e}$ and $\beta_{\kappa}$ are all functions of $\nu$ and $e^{2} / \kappa . \gamma_{A}=0$ still holds. We stress that this conclusion is drawn on the assumption of the existence of a consistent regularization method to all orders, which needs to be established.

## 11. Renormalization Group Analysis

The RG equation for the effective potential in the $\overline{M S}$ scheme is

$$
\begin{align*}
{\left[\mu \frac{\partial}{\partial \mu}+\beta_{\nu} \frac{\partial}{\partial \nu}+\beta_{\kappa} \frac{\partial}{\partial \kappa}+\frac{\beta_{e^{2}}}{2} \frac{\partial}{\partial e^{2}}+\beta_{a} \frac{\partial}{\partial a}+\right.} & \left.\beta_{m^{2}} \frac{\partial}{\partial m^{2}}+\beta_{\lambda} \frac{\partial}{\partial \lambda}+\beta_{\Lambda} \frac{\partial}{\partial \Lambda}-\gamma_{\phi} v \frac{\partial}{\partial v}\right] \\
& \times V\left(v ; \nu, m^{2}, \lambda, \kappa, e^{2}, a, \Lambda, \mu\right)^{\overline{\mathrm{MS}}}=0 \tag{11.1}
\end{align*}
$$

where various $\beta$ functions are given by (2.17) and

$$
\begin{equation*}
\beta_{\kappa}=\mu \frac{\partial}{\partial \mu} \kappa \quad, \quad \beta_{e^{2}}=\mu \frac{\partial}{\partial \mu} e^{2}, \beta_{a}=\mu \frac{\partial}{\partial \mu} a \tag{11.2}
\end{equation*}
$$

The renormalization for $a$ is the same as the wave function renormalization for $A_{\mu}$. Although the result (10.8) implies $\beta_{a}=0$, we have kept the $\beta_{a}$ term in (11.1) to show a useful relation below. Note that up to $\mathrm{O}\left(\hbar^{2}\right), \gamma_{\phi}=0$.

In Section 2 we have shown that beta functions in pure scalar theory can be determined from the renormalization group equation for the effective potential. We employ the same technique to find beta functions in the gauge theory.

At $\mathrm{O}(\hbar)$, (11.1) yields

$$
\begin{equation*}
\beta_{m^{2}}^{(1)} \frac{v^{2}}{2}+\beta_{\lambda}^{(1)} \frac{v^{4}}{4!}+\beta_{\nu}^{(1)} \frac{v^{6}}{6!}+\beta_{\Lambda}^{(1)}=0 \tag{11.3}
\end{equation*}
$$

so that

$$
\begin{equation*}
\beta_{m^{2}}^{(1)}=\beta_{\lambda}^{(1)}=\beta_{\nu}^{(1)}=\beta_{\Lambda}^{(1)}=0 \tag{11.4}
\end{equation*}
$$

At $\mathrm{O}\left(\hbar^{2}\right)$, Eq. (11.1) becomes :

$$
\begin{align*}
& -\frac{\hbar^{2}}{16 \pi^{2}}\left[\left(\frac{\lambda}{6} v+\frac{\nu}{60} v^{3}\right)^{2}+3\left(\frac{\lambda}{6} v+\frac{\nu}{36} v^{3}\right)^{2}\right] \\
& +\frac{e^{2} \hbar^{2}}{32 \pi^{2} a}\left[4 m^{2}+\frac{4}{3} \lambda v^{2}+\frac{\nu}{10} v^{4}-\frac{\kappa^{2}}{a^{2}}-\frac{e^{2} v^{2}}{a}\right] \\
& -\frac{3 e^{4} \hbar^{2}}{32 \pi^{2} a^{2}} v^{2}+\beta_{m^{2}}^{(2)} \frac{v^{2}}{2}+\beta_{\lambda}^{(2)} \frac{v^{4}}{4!}+\beta_{\nu}^{(2)} \frac{v^{6}}{6!}+\beta_{\Lambda}^{(2)} \\
& -\frac{\hbar}{12 \pi}\left(\frac{\kappa^{2}}{a^{2}}+\frac{4 e^{2} v^{2}}{a}\right)^{-1 / 2}\left[\frac{\beta_{\kappa}^{(1)}}{\kappa} \frac{3 \kappa^{2}}{a^{2}}\left(\frac{\kappa^{2}}{a^{2}}+\frac{3 e^{2} v^{2}}{a}\right)+\frac{\beta_{e^{2}}^{(1)}}{e^{2}} \frac{3 e^{2} v^{2}}{a}\left(\frac{\kappa^{2}}{a^{2}}+\frac{2 e^{2} v^{2}}{a}\right)\right. \\
& \left.\quad-\frac{\beta_{a}^{(1)}}{a}\left(\frac{3 \kappa^{4}}{a^{4}}+\frac{12 \kappa^{2} e^{2} v^{2}}{a^{3}}+\frac{6 e^{4} v^{4}}{a^{2}}\right)\right]=0 . \tag{11.5}
\end{align*}
$$

The above equation is quite complicated but must be satisfied for arbitrary $v$. Since the last term contains a square root, it must vanish identically. It follows immediately that

$$
\begin{equation*}
\frac{\beta_{\kappa}^{(1)}}{\kappa}=\frac{\beta_{e^{2}}^{(1)}}{e^{2}}=\frac{\beta_{a}^{(1)}}{a} . \tag{11.6}
\end{equation*}
$$

Upon making use of $\beta_{a}=0$, one concludes that $\beta_{\kappa}^{(1)}=\beta_{e^{2}}^{(1)}=0$.
Then the rest of the equation becomes

$$
\begin{array}{ll}
\mathrm{O}\left(v^{0}\right): & \frac{e^{2} \hbar^{2}}{32 \pi^{2} a}\left[4 m^{2}-\frac{\kappa^{2}}{a^{2}}\right]+\beta_{\Lambda}^{(2)}=0 \\
\mathrm{O}\left(v^{2}\right): & -\frac{\hbar^{2}}{144 \pi^{2}} \lambda^{2}+\frac{\hbar^{2}}{32 \pi^{2}}\left(\frac{4}{3} \lambda \frac{e^{2}}{a}-\frac{e^{4}}{a^{2}}\right)-\frac{3 \hbar^{2}}{32 \pi^{2}} \frac{e^{4}}{a^{2}}+\frac{1}{2} \beta_{m^{2}}^{(2)}=0 \\
\mathrm{O}\left(v^{4}\right): \quad-\frac{\hbar^{2}}{480 \pi^{2}} \lambda \nu+\frac{\hbar^{2}}{320 \pi^{2}} \frac{e^{2}}{a} \nu+\frac{\beta_{\lambda}^{(2)}}{24}=0 \\
\mathrm{O}\left(v^{6}\right): \quad-\frac{\hbar^{2}}{16 \pi^{2}} \frac{7}{2700} \nu^{2}+\frac{\beta_{\nu}^{(2)}}{720}=0 \tag{11.7}
\end{array}
$$

It follows that

$$
\begin{aligned}
& \beta_{\Lambda}^{(2)}=\frac{e^{2} \hbar^{2}}{32 \pi^{2} a}\left(4 m^{2}-\frac{\kappa^{2}}{a^{2}}\right) \\
& \beta_{m^{2}}^{(2)}=\frac{\hbar^{2}}{72 \pi^{2}} \lambda^{2}-\frac{\hbar^{2}}{12 \pi^{2}} \lambda \frac{e^{2}}{a}+\frac{\hbar^{2}}{4 \pi^{2}} \frac{e^{4}}{a^{2}}
\end{aligned}
$$

$$
\begin{align*}
& \beta_{\lambda}^{(2)}=\frac{\hbar^{2}}{20 \pi^{2}} \lambda \nu-\frac{3 \hbar^{2}}{40 \pi^{2}} \frac{e^{2}}{a} \nu \\
& \beta_{\nu}^{(2)}=\frac{7 \hbar^{2}}{60 \pi^{2}} \nu^{2} \tag{11.8}
\end{align*}
$$

The last relation for $\beta_{\nu}$ confirms the result (10.9) at two loop.
The beta functions in the $\overline{\mathrm{MS}}$ scheme are singular the $a \rightarrow 0$ limit. The the renormalization group equation for the Coleman-Weinberg potential (7.4) is more involved than naively expected. Eq. (11.1) is for the effective potential in the $\overline{\mathrm{MS}}$ scheme before renormalization. These two are related by

$$
\begin{align*}
& V_{\mathrm{CW}}\left(v ; \nu, \kappa / e^{2}, M\right) \\
& \quad=\lim _{a \rightarrow 0}\left\{V_{\overline{\mathrm{MS}}}(v)-V_{\overline{\mathrm{MS}}}(0)-\frac{v^{2}}{2} V_{\overline{\mathrm{MS}}}^{(2)}(0)-\frac{v^{4}}{4!} V_{\overline{\mathrm{MS}}}^{(4)}(0)-\frac{v^{6}}{6!} V_{\overline{\mathrm{MS}}}^{(6)}\left(M^{1 / 2}\right)\right\} \\
& V_{\overline{\mathrm{MS}}}(v) \equiv V\left(v ; \nu, m^{2}=0, \lambda=0, \kappa, e^{2}, a, \Lambda, \mu\right)^{\overline{\mathrm{MS}}} . \tag{11.9}
\end{align*}
$$

The subtraction terms give additional contributions to the renormalization group equation.

## 12. Conclusion

We have examined the Maxwell-Chern-Simons gauge theory with complex scalar fields with the most general renormalizable interactions at two loop. The effective potential for the scalar fields was obtained in the closed form in dimensional regularization scheme. In the massless scalar theory the $\phi^{6}$ coupling constant $\nu$ cannot be renormalized at $\phi=0$ as two loop corrections yield terms of the form $\phi^{6} \ln \phi$. Evaluation of the sixth derivative of the effective potential at finite $\phi$ is a formidable task, which we have done by numerical method. The renormalized effective potential for general couplings was evaluated numerically.

We have found that two loop corrections are decisive to determine the phase. The $U(1)$ symmetry is spontaneously broken in the massless theory ( $m=\lambda=0$ ) by radiative corrections. In particular, in the Coleman-Weinberg limit in which the Maxwell term is absent for gauge fields, the dimensional transmutation takes place at two loop.

From the effective potential we have also determined beta functions for various couplings. Two loop results confirm the general theorem that the beta function $\beta_{\nu}$ is independent of gauge couplings and a function of $\nu$ only.

Here we would like to stress again that the regularization of the theory is a delicate matter. The Maxwell term (with the coefficient $a$ ) must be introduced to have improved ultraviolet behavior of the gauge field propagator in the dimensional regularization. We have demonstrated that only after renormalization one can take the limit $a \rightarrow 0$. Counter terms are singular in $a$. The perturbation theory defined with $a=0$ is inconsistent in the dimensional regularization scheme. It is not renormalizable.

Avdeev, Grigoryev and Kazakov have studied the pure Chern-Simons theory coupled to scalar matter to find beta functions differing from ours. 16] They evaluated diagrams in three dimensional space to eliminate all $\epsilon^{\mu \nu \rho}$ tensors, and then extend and perform momentum integrals in $n$ dimensions to define "dimensional regularization". This is incorrect. Everything must be defined in $n$ dimensions first. This is the source of the discrepancy.

In the absence of the Maxwell term one of the gauge degrees of freedom becomes infinitely massive. However, it cannot be completely discarded. It gives nontrivial cancellations and the beta function for the scalar field becomes independent of the gauge couplings.

The $U(1)$ symmetry is spontaneously broken at the two loop level. Our results can be extended to supersymmetric self-dual Chern-Simons theory. As it was pointed out by [28], in the $N=2$ and 3 supersymmetric models the scaling symmetry broken at two loop may be restored quantum mechanically.

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## Appendix A. Some Useful Formulas

In this appendix we collect $n$ dimensional integrals which we have made use of in the paper. See also 29.

In Minkowski space we have

$$
\begin{align*}
\int \frac{d^{n} k}{i(2 \pi)^{n}} \ln \left(m^{2}-k^{2}\right) & =-\frac{\Gamma\left(-\frac{1}{2} n\right)}{(4 \pi)^{n / 2}} m^{n} \\
\int \frac{d^{n} k}{i(2 \pi)^{n}} \frac{1}{\left(m^{2}-k^{2}\right)^{\alpha}} & =+\frac{\Gamma\left(\alpha-\frac{1}{2} n\right)}{(4 \pi)^{n / 2} \Gamma(\alpha)}\left(m^{2}\right)^{(n / 2)-\alpha} \\
\int \frac{d^{n} k}{i(2 \pi)^{n}} \frac{k^{\mu} k^{\nu}}{\left(m^{2}-k^{2}\right)^{\alpha}} & =-\frac{\Gamma\left(\alpha-1-\frac{1}{2} n\right)}{(4 \pi)^{n / 2} \Gamma(\alpha)} \frac{g^{\mu \nu}}{2}\left(m^{2}\right)^{(n / 2)+1-\alpha} \tag{A.1}
\end{align*}
$$

Also we have

$$
\begin{equation*}
\int \frac{d^{n} k}{i(2 \pi)^{n}} \frac{1}{\left(-k^{2}\right)^{\alpha}}=0 \tag{A.2}
\end{equation*}
$$

Similarly in Euclidean space

$$
\begin{align*}
J_{1}[p, \alpha]= & \int \frac{d^{n} k}{(2 \pi)^{n}} \frac{1}{\left[k^{2}+m^{2}+x\left(2 p k+p^{2}\right)\right]^{\alpha}} \\
= & +\frac{\Gamma\left(\alpha-\frac{1}{2} n\right)}{(4 \pi)^{n / 2} \Gamma(\alpha)}\left[p^{2} x(1-x)+m^{2}\right]^{(n / 2)-\alpha} \\
J_{2}^{\mu}[p, \alpha]= & \int \frac{d^{n} k}{(2 \pi)^{n}} \frac{k^{\mu}}{\left[k^{2}+m^{2}+x\left(2 p k+p^{2}\right)\right]^{\alpha}}=-x p^{\mu} J_{1}[p, \alpha] \\
J_{3}^{\mu \nu}[p, \alpha]= & \int \frac{d^{n} k}{(2 \pi)^{n}} \frac{k^{\mu} k^{\nu}}{\left[k^{2}+m^{2}+x\left(2 p k+p^{2}\right)\right]^{\alpha}} \\
= & x^{2} p^{\mu} p^{\nu} J_{1}[p, \alpha]+\frac{\delta^{\mu \nu}}{2(\alpha-1)} J_{1}[p, \alpha-1] \\
= & x^{2} p^{\mu} p^{\nu} \frac{\Gamma\left(\alpha-\frac{1}{2} n\right)}{(4 \pi)^{n / 2} \Gamma(\alpha)}\left[p^{2} x(1-x)+m^{2}\right]^{(n / 2)-\alpha} \\
& \quad+\frac{1}{2} \delta^{\mu \nu} \frac{\Gamma\left(\alpha-1-\frac{1}{2} n\right)}{(4 \pi)^{n / 2} \Gamma(\alpha)}\left[p^{2} x(1-x)+m^{2}\right]^{(n / 2)-\alpha+1} . \tag{A.3}
\end{align*}
$$

Below we give the integrals in the regularized form

$$
\int \frac{d^{3} p}{(2 \pi)^{3}} \frac{1}{p^{2}+a^{2}}=-\frac{a}{4 \pi}
$$

$$
\begin{gather*}
\int \frac{d^{3} p}{(2 \pi)^{3}} \frac{1}{\left(p^{2}+a^{2}\right)\left(p^{2}+b^{2}\right)}=\frac{1}{4 \pi} \frac{1}{a+b} \\
\int \frac{d^{3} p}{(2 \pi)^{3}} \frac{1}{\left(p^{2}+a^{2}\right)\left(p^{2}+b^{2}\right)\left(p^{2}+c^{2}\right)}=-\frac{1}{4 \pi}\{a f(a ; b, c)+b f(b ; c, a)+c f(c ; a, b)\} \\
\int \frac{d^{3} p}{(2 \pi)^{3}} \frac{p^{2}}{\left(p^{2}+a^{2}\right)\left(p^{2}+b^{2}\right)\left(p^{2}+c^{2}\right)}=\frac{1}{4 \pi}\left\{a^{3} f(a ; b, c)+b^{3} f(b ; c, a)+c^{3} f(c ; a, b)\right\} \\
\quad f(a ; b, c)=\frac{1}{\left(a^{2}-b^{2}\right)\left(a^{2}-c^{2}\right)} . \tag{A.4}
\end{gather*}
$$

As an application we have

$$
\begin{equation*}
\int_{-\infty}^{\infty} d p \frac{p^{6}}{\left(p^{2}+a^{2}\right)^{2}\left(p^{2}+b^{2}\right)^{2}}=\frac{\pi}{2} \frac{a^{2}+3 a b+b^{2}}{(a+b)^{3}} \tag{A.5}
\end{equation*}
$$

## Appendix B. Two loop integrals

A basic two loop diagram yields

$$
\begin{align*}
I\left(m_{1}, m_{2}, m_{3} ; n\right) & \equiv \int \frac{d^{n} q d^{n} k}{(2 \pi)^{2 n}} \frac{1}{\left[(q+k)^{2}+m_{1}^{2}\right]\left(q^{2}+m_{2}^{2}\right)\left(k^{2}+m_{3}^{2}\right)} \\
& =I\left(m_{2}, m_{1}, m_{3} ; n\right) \text { etc. } \\
& =\frac{\mu^{2(n-3)}}{32 \pi^{2}}\left\{-\frac{1}{n-3}-\gamma_{E}+1-\ln \frac{\left(m_{1}+m_{2}+m_{3}\right)^{2}}{4 \pi \mu^{2}}\right\} \tag{B.1}
\end{align*}
$$

This result was first derived in [6].
To show this, we note

$$
\begin{align*}
I= & \mu^{2(n-3)} \frac{\Gamma(3-n)}{(4 \pi)^{3}} \int_{0}^{1} d x d y \frac{1}{\sqrt{x(1-x) y}}\left[\frac{y}{x(1-x)}\right]^{(3-n) / 2} \\
& \times\left[\frac{1}{4 \pi \mu^{2}}\left\{y\left(\frac{m_{1}^{2}}{1-x}+\frac{m_{2}^{2}}{x}\right)+(1-y) m_{3}^{2}\right\}\right]^{n-3} \\
= & \frac{\mu^{2(n-3)}}{(4 \pi)^{3}} \int_{0}^{1} \frac{d x d y}{\sqrt{x(1-x) y}}\left\{-\frac{1}{n-3}-\gamma_{E}+\frac{1}{2} \ln \frac{y}{x(1-x)}\right. \\
& \left.-\ln \left[\frac{1}{4 \pi \mu^{2}}\left\{y\left(\frac{m_{1}^{2}}{1-x}+\frac{m_{2}^{2}}{x}\right)+(1-y) m_{3}^{2}\right\}\right]\right\} \tag{B.2}
\end{align*}
$$

In the $y$ integral, we have

$$
\begin{equation*}
\ln \{p(x) y+q\} \quad, \quad p(x)=\frac{m_{1}^{2}}{1-x}+\frac{m_{2}^{2}}{x}-m_{3}^{2} \quad, \quad q=m_{3}^{2}>0 \tag{B.3}
\end{equation*}
$$

$p(x)(0<x<1)$ reaches a minimum at $x=m_{2} /\left(m_{1}+m_{2}\right)$. Its value is $p_{\min }=\left(m_{1}+\right.$ $\left.m_{2}\right)^{2}-m_{3}^{2}$. Hence, making use of

$$
\int_{0}^{1} d y \frac{\ln (p y+q)}{\sqrt{y}}=2 \ln (p+q)-4+4 \sqrt{\frac{q}{p}} \tan ^{-1} \sqrt{\frac{p}{q}}
$$

we find, for $m_{1}+m_{2} \geq m_{3}$,

$$
\begin{align*}
I= & \frac{\mu^{2(n-3)}}{32 \pi^{2}}\left(-\frac{1}{n-3}-\gamma_{E}+2 \ln 2+1\right) \\
& -\frac{\mu^{2(n-3)}}{32 \pi^{3}} \int_{0}^{1} \frac{d x}{\sqrt{x(1-x)}} \ln \left[\frac{1}{4 \pi \mu^{2}}\left(\frac{m_{1}^{2}}{1-x}+\frac{m_{2}^{2}}{x}\right)\right] \\
& -\frac{\mu^{2(n-3)}}{16 \pi^{3}} \int_{0}^{1} \frac{d x}{\sqrt{x(1-x)}} \frac{\tan ^{-1}\left(\frac{1}{1-x} \frac{m_{1}^{2}}{m_{3}^{2}}+\frac{1}{x} \frac{m_{2}^{2}}{m_{3}^{2}}-1\right)^{1 / 2}}{\left(\frac{1}{1-x} \frac{m_{1}^{2}}{m_{3}^{2}}+\frac{1}{x} \frac{m_{2}^{2}}{m_{3}^{2}}-1\right)^{1 / 2}} \tag{B.4}
\end{align*}
$$

The second term is evaluated as

$$
\begin{equation*}
\int_{0}^{1} \frac{d x}{\sqrt{x(1-x)}} \ln \left[\frac{1}{4 \pi \mu^{2}}\left(\frac{m_{1}^{2}}{1-x}+\frac{m_{2}^{2}}{x}\right)\right]=\pi \ln \frac{\left(m_{1}+m_{2}\right)^{2}}{\pi \mu^{2}} \tag{B.5}
\end{equation*}
$$

Hence

$$
\begin{align*}
I_{m_{1}+m_{2} \geq m_{3}} & =\frac{\mu^{2(n-3)}}{32 \pi^{2}}\left\{-\frac{1}{n-3}-\ln \frac{\left(m_{1}+m_{2}\right)^{2}}{4 \pi \mu^{2}}-\gamma_{E}+1+f\left(\frac{m_{1}^{2}}{m_{3}^{2}}, \frac{m_{2}^{2}}{m_{3}^{2}}\right)\right\} \\
f(a, b) & =-\frac{2}{\pi} \int_{0}^{1} \frac{d x}{\sqrt{x(1-x)}} \frac{\tan ^{-1}\left(\frac{a}{1-x}+\frac{b}{x}-1\right)^{1 / 2}}{\left(\frac{a}{1-x}+\frac{b}{x}-1\right)^{1 / 2}} . \tag{B.6}
\end{align*}
$$

As a special case we have

$$
\begin{equation*}
I\left(m_{1}, m_{2}, 0 ; n\right)=\frac{\mu^{2(n-3)}}{32 \pi^{2}}\left\{-\frac{1}{n-3}-\ln \frac{\left(m_{1}+m_{2}\right)^{2}}{4 \pi \mu^{2}}-\gamma_{E}+1\right\} \tag{B.7}
\end{equation*}
$$

which can be obtained directly from (B.2), too. Since the divergent term in (B.7) is independent of $m_{j}$, one can write

$$
\begin{equation*}
I\left(m_{1}, m_{2}, m_{3} ; n\right)=I\left(m_{1}, m_{2}, 0 ; n\right)+\left.\int_{0}^{m_{3}} d m_{3} \frac{\partial}{\partial m_{3}} I\left(m_{1}, m_{2}, m_{3} ; n\right)\right|_{n=3}+\mathrm{O}(n-3) \tag{B.8}
\end{equation*}
$$

Now we evaluate

$$
\begin{align*}
& \left.\frac{\partial}{\partial m_{3}} I\left(m_{1}, m_{2}, m_{3} ; n\right)\right|_{n=3} \\
& =-2 m_{3} \int \frac{d^{3} q d^{3} k}{(2 \pi)^{6}} \frac{1}{\left[(q+k)^{2}+m_{1}^{2}\right]^{2}\left(q^{2}+m_{2}^{2}\right)\left(k^{2}+m_{3}^{2}\right)} \\
& =-\frac{m_{3}}{4 \pi^{4}} \int_{0}^{\infty} d q d k \int_{-1}^{1} d(\cos \theta) \frac{q^{2} k^{2}}{\left[q^{2}+k^{2}+2 q k \cos \theta+m_{3}^{2}\right]^{2}\left(q^{2}+m_{1}^{2}\right)\left(k^{2}+m_{2}^{2}\right)} \\
& =\frac{m_{3}}{8 \pi^{4}} \int_{0}^{\infty} d q d k \frac{q k}{\left(q^{2}+m_{1}^{2}\right)\left(k^{2}+m_{2}^{2}\right)}\left\{\frac{1}{(q+k)^{2}+m_{3}}-\frac{1}{(q-k)^{2}+m_{3}^{2}}\right\} \\
& =\frac{m_{3}}{32 \pi^{4}} \int_{-\infty}^{\infty} d q d k \frac{q k}{\left(q^{2}+m_{1}^{2}\right)\left(k^{2}+m_{2}^{2}\right)}\left\{\frac{1}{(q+k)^{2}+m_{3}^{2}}-\frac{1}{(q-k)^{2}+m_{3}^{2}}\right\} . \tag{B.9}
\end{align*}
$$

Making use of the residue theorem, one finds

$$
\begin{aligned}
& \left.\frac{\partial}{\partial m_{3}} I\left(m_{1}, m_{2}, m_{3} ; n\right)\right|_{n=3} \\
& =\frac{m_{3}}{16 \pi^{3}} \int_{-\infty}^{\infty} d k \frac{i k}{k^{2}+m_{2}^{2}}\left\{\frac{1}{2}\left(\frac{1}{\left(k+i m_{1}\right)^{2}+m_{3}^{2}}-\frac{1}{\left(k-i m_{1}\right)^{2}+m_{3}^{2}}\right)\right. \\
& \left.\quad+\frac{-k+i m_{3}}{\left(k-i m_{3}\right)^{2}+m_{1}^{2}} \frac{1}{2 i m_{3}}-\frac{k+i m_{3}}{\left(k+i m_{3}\right)^{2}+m_{1}^{2}} \frac{1}{2 i m_{3}}\right\} \\
& = \\
& \left\{\begin{array}{l}
64 \pi^{3} \int d k \frac{k}{k^{2}+m_{2}^{2}} \\
\\
\left\{\frac{1}{k+i m_{1}-i m_{3}}-\frac{1}{k+i m_{1}+i m_{3}}-\frac{1}{k-i m_{1}-i m_{3}}+\frac{1}{k-i m_{1}+i m_{3}}\right. \\
\left.=\frac{1}{k-i m_{3}-i m_{1}}-\frac{1}{k-i m_{3}+i m_{1}}-\frac{1}{k+i m_{3}-i m_{1}}-\frac{1}{k+i m_{3}+i m_{1}}\right\} \\
= \\
16 \pi^{3} \int d k \frac{k^{2}}{\left(k^{2}+m_{2}^{2}\right)\left[k^{2}+\left(m_{1}+m_{3}\right)^{2}\right]}
\end{array}\right.
\end{aligned}
$$

$$
\begin{equation*}
=-\frac{1}{16 \pi^{2}} \frac{1}{m_{1}+m_{2}+m_{3}} . \tag{B.10}
\end{equation*}
$$

Hence eq. (B.8) becomes

$$
\begin{equation*}
I\left(m_{1}, m_{2}, m_{3} ; n\right)=-\frac{\mu^{2(n-3)}}{16 \pi^{2}} \ln \frac{m_{1}+m_{2}+m_{3}}{m_{1}+m_{2}}+I\left(m_{1}, m_{2}, 0 ; n\right)+\mathrm{O}(n-3) \tag{B.11}
\end{equation*}
$$

which leads, with (B.7), to (B.1).

## Appendix C. More loop integrals

In the course of evaluating loop corrections, we would typically encounter integrals involving both three and $n$-dimensional momenta. These integrals can be reduced into a basic Euclidean integral of the following form :

$$
\begin{align*}
J^{\mu \nu \rho \sigma} & =\int \frac{d^{n} p d^{n} q}{(2 \pi)^{2 n}} \frac{p^{\mu} p^{\nu} q^{\rho} q^{\sigma}}{\left(p^{2}+m_{1}^{2}\right)\left(q^{2}+m_{2}^{2}\right)\left[(p+q)^{2}+m_{3}^{2}\right]} \\
& =A g^{\mu \nu} g^{\rho \sigma}+B\left(g^{\mu \rho} g^{\nu \sigma}+g^{\mu \sigma} g^{\nu \rho}\right) \tag{C.1}
\end{align*}
$$

Upon contracting (C.1) with $g_{\mu \nu} g_{\rho \sigma}$ and $g_{\mu \rho} g_{\nu \sigma}$ independently, we obtain

$$
\begin{align*}
& n^{2} A+2 n B=K_{1} \\
& n A+\left(n^{2}+n\right) B=K_{2} \tag{C.2}
\end{align*}
$$

where

$$
\begin{align*}
K_{1} & =\int \frac{d^{n} p d^{n} q}{(2 \pi)^{2 n}} \frac{p^{2} q^{2}}{\left(p^{2}+m_{1}^{2}\right)\left(q^{2}+m_{2}^{2}\right)\left[(p+q)^{2}+m_{3}^{2}\right]} \\
K_{2} & =\int \frac{d^{n} p d^{n} q}{(2 \pi)^{2 n}} \frac{(p \cdot q)^{2}}{\left(p^{2}+m_{1}^{2}\right)\left(q^{2}+m_{2}^{2}\right)\left[(p+q)^{2}+m_{3}^{2}\right]} \tag{C.3}
\end{align*}
$$

Both $K_{1}$ and $K_{2}$ are expressed in terms of the $I$ integral given in the previous section.

$$
\begin{aligned}
& K_{1}=-\frac{\mu^{2(n-3)}}{16 \pi^{2}}\left(m_{1}^{3}+m_{2}^{3}\right) m_{3}+m_{1}^{2}+m_{2}^{2} I\left(m_{1}, m_{2}, m_{3}\right) \\
& K_{2}=\frac{\mu^{2(n-3)}}{64 \pi^{2}}\left\{m_{3}^{2}\left[m_{1} m_{3}+m_{2} m_{3}-m_{1} m_{2}\right]-m_{3}\left[3 m_{1}^{3}+3 m_{2}^{3}+m_{1}^{2} m_{2}+m_{2}^{2} m_{1}\right]\right.
\end{aligned}
$$

$$
\begin{equation*}
\left.+\left(m_{1}^{2}+m_{2}^{2}\right) m_{1} m_{2}\right\}+\frac{1}{4}\left[4 m_{1}^{2} m_{2}^{2}-\left(m_{1}^{2}+m_{2}^{2}-m_{3}^{2}\right)^{2}\right] I\left(m_{1}, m_{2}, m_{3}\right) \tag{C.4}
\end{equation*}
$$

Solving (C.2), we may write (C.1) in terms of $K_{1}$ and $K_{2}$.

$$
\begin{equation*}
J^{\mu \nu \rho \sigma}=\frac{(n+1) K_{1}-2 K_{2}}{n(n-1)(n+2)} g^{\mu \nu} g^{\rho \sigma}+\frac{n K_{2}-K_{1}}{n(n-1)(n+2)}\left(g^{\mu \rho} g^{\nu \sigma}+g^{\mu \sigma} g^{\nu \rho}\right) \tag{C.5}
\end{equation*}
$$

Making use of (C.5), one finds

$$
\begin{align*}
& \int \frac{d^{n} p d^{n} q}{(2 \pi)^{2 n}} \frac{\hat{p}^{2} \hat{q}^{2}}{\left(p^{2}+m_{1}^{2}\right)\left(q^{2}+m_{2}^{2}\right)\left[(p+q)^{2}+m_{3}^{2}\right]}=\frac{(9 n+3) K_{1}+(6 n-18) K_{2}}{n(n-1)(n+2)} \\
& \int \frac{d^{n} p d^{n} q}{(2 \pi)^{2 n}} \frac{(\hat{p} \cdot \hat{q})^{2}}{\left(p^{2}+m_{1}^{2}\right)\left(q^{2}+m_{2}^{2}\right)\left[(p+q)^{2}+m_{3}^{2}\right]}=\frac{(3 n-9) K_{1}+(12 n-6) K_{2}}{n(n-1)(n+2)} \\
& \int \frac{d^{n} p d^{n} q}{(2 \pi)^{2 n}} \frac{\hat{p}^{2} \hat{q}^{2}-(\hat{p} \cdot \hat{q})^{2}}{\left(p^{2}+m_{1}^{2}\right)\left(q^{2}+m_{2}^{2}\right)\left[(p+q)^{2}+m_{3}^{2}\right]}=\frac{6}{n(n-1)}\left(K_{1}-K_{2}\right) \\
& \int \frac{d^{n} p d^{n} q}{(2 \pi)^{2 n}} \frac{\hat{p}^{2} \hat{q}^{2}+(\hat{p} \cdot \hat{q})^{2}}{\left(p^{2}+m_{1}^{2}\right)\left(q^{2}+m_{2}^{2}\right)\left[(p+q)^{2}+m_{3}^{2}\right]}=\frac{6(2 n-1) K_{1}+6(3 n-4) K_{2}}{n(n-1)(n+2)} \\
& \int \frac{d^{n} p d^{n} q}{(2 \pi)^{2 n}} \frac{p^{2} \hat{q}^{2}}{\left(p^{2}+m_{1}^{2}\right)\left(q^{2}+m_{2}^{2}\right)\left[(p+q)^{2}+m_{3}^{2}\right]}=\frac{3}{n} K_{1} \tag{C.6}
\end{align*}
$$

One must be careful when solving the above integrals for $n=3$. The pole terms in $K_{1}$ and $K_{2}$ give finite contributions upon being multiplied by $(n-3)$.

Other useful 2-loop integrals are :

$$
\begin{align*}
& K_{3}=\int \frac{d^{n} p d^{n} q}{(2 \pi)^{2 n}} \frac{2 p \cdot q}{\left(p^{2}+m_{1}^{2}\right)\left(q^{2}+m_{2}^{2}\right)\left[(p+q)^{2}+m_{3}^{2}\right]} \\
&=\frac{\mu^{2(n-3)}}{16 \pi^{2}}\left(m_{1} m_{2}-m_{2} m_{3}-m_{1} m_{3}\right)+\left(m_{1}^{2}+m_{2}^{2}-m_{3}^{2}\right) I\left(m_{1}, m_{2}, m_{3}\right) \\
& \int \frac{d^{n} p d^{n} q}{(2 \pi)^{2 n}} \frac{2 \hat{p} \cdot \hat{q}}{\left(p^{2}+m_{1}^{2}\right)\left(q^{2}+m_{2}^{2}\right)\left[(p+q)^{2}+m_{3}^{2}\right]}=\frac{3}{n} K_{3} \\
&=\frac{\mu^{2(n-3)}}{16 \pi^{2}}\left(m_{1} m_{2}-m_{2} m_{3}-m_{1} m_{3}\right)+\frac{\mu^{2(n-3)}}{96 \pi^{2}}\left(m_{1}^{2}+m_{2}^{2}-m_{3}^{2}\right) \\
& \quad \quad+\left(m_{1}^{2}+m_{2}^{2}-m_{3}^{2}\right) I\left(m_{1}, m_{2}, m_{3}\right)+O(n-3) . \tag{C.7}
\end{align*}
$$

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