Transformations of W-Type Entangled States

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The transformations of W-type entangled states by using local operations assisted with classical communication are investigated. For this purpose, a parametrization of the W-type states which remains invariant under local unitary transformations is proposed and a complete characterization of the local operations carried out by a single party is given. These are used for deriving the necessary and sufficient conditions for deterministic transformations. A convenient upper bound for the maximum probability of distillation of arbitrary target states is also found.

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I. INTRODUCTION

Entanglement can be regarded as a nonlocal resource which enables us to achieve some classically impossible tasks such as dense coding[1] and teleportation[2]. However, such tasks can be accomplished only when certain special entangled states have already been established between particles distributed to remote parties. As the transfer of such particles over noisy quantum channels results in the decoherence of the state, the transformation of entangled states using only local operations assisted with classical communication (LOCC) has arisen as an important problem in quantum information theory. Such transformations also form an operational basis for quantifying[3, 4] and distinguishing the different types[5, 6] of entanglement. Alternatively, the deterministic transformations can be used for defining a natural partial order between the states[7], which provides another approach for assessing the entanglement content of the states.

Nearly all aspects of the transformations of pure bipartite states have been understood[3, 7–10]. The existence of the Schmidt decomposition of such states appears to be immensely useful in the analysis of the associated transformations. A particularly convenient aspect of the decomposition is the existence of a one-to-one correspondence between the equivalence classes of states under local unitary (LU) transformations and the unique Schmidt coefficients of the state. As a result of this, all statements about the transformations of bipartite pure states can be expressed entirely in terms of the Schmidt coefficients.

The absence of a simple and powerful representation like the Schmidt decomposition has been a great impediment in the analysis of the transformations of multipartite entangled states. For this reason, studies on such transformations are focused on specific types of states. For transformations between multipartite states that have a generalized Schmidt decomposition, it is found that the results obtained for bipartite states can be directly applied without changes[11]. The two special types of genuine multipartite states of three qubits generally attracts a special interest, and for this reason various aspects of the transformations of Greenberger-Horne-Zeilinger (GHZ) type states[12–15] and W-type states[16–19] are investigated. Most of these studies are concerned with special source or target states, however, and lack a systematic analysis of the transformations. Such a systematic investigation not only enables us to better assess the order between the states in terms of their entanglement content, but also provides a framework upon which future studies can be built. A systematic treatment of the transformations of GHZ-type states has recently been given [15] and the purpose of the current article is to do the same for the transformations of the W-type states.

The organization of the article is as follows. In Sec. II, the W-type states are defined and a convenient parametrization of these states is given. The LU equivalence relation between states is also described in this section. In Sec. III, a complete characterization of the local quantum operations carried out by a single party is given. Sec. IV contains two particular applications: complete characterization of the deterministic transformations and finding an upper bound for the maximum distillation probability for arbitrary states.

II. STATES AND THEIR PARAMETRIZATION

Standard W state of p qubits ($p \geq 3$) distributed to p different parties is given by

$$|W\rangle = \frac{1}{\sqrt{p}} \left(|10\cdots 0\rangle + |01\cdots 0\rangle + \cdots + |00\cdots 1\rangle \right) , \quad (1)$$

where the labels in kets denote states of the particles $1, 2, \ldots, p$, in that order. A state $|\Psi\rangle$ of p particles will be called a *W-type state*, if it is stochastically reducible[20] from the state $|W\rangle$ by LOCC. Note that, by the term W-type, we also embrace states where some parties are unentangled from the rest. Especially, the product states and bipartite entangled states with Schmidt rank 2 are covered by this definition. Since this set of states is closed under stochastic reducibility relation by definition, any state in this set can only be transformed to states in the same set. Hence, all possible local manipulations of these states are within the scope of the current work.

If $|\Psi\rangle$ is a W-type state, then there are local operators A_k such that $|\Psi\rangle = (A_1 \otimes \cdots \otimes A_p)|W\rangle$ and therefore it can be expressed as

$$|\Psi\rangle = \sum_{k=1}^{p} |u_1 \otimes \cdots \otimes u_{k-1} \otimes v_k \otimes u_{k+1} \otimes \cdots u_p\rangle \quad (2)$$

where $|u_k\rangle$ and $|v_k\rangle$ are vectors (which need to be neither normalized nor orthogonal) in the Hilbert space $\mathcal{H}_k$ of the kth particle. Since only two vectors are involved for each party, it can always be assumed that each particle is qubit and $\dim \mathcal{H}_k = 2$.

For the purpose of obtaining a unique representation, orthonormal sets $\{|\alpha_k\rangle, |\beta_k\rangle\}$ can be defined in $\mathcal{H}_k$ in such a way that $|\alpha_k\rangle$ is parallel to $|u_k\rangle$ and hence

$$|u_k\rangle = c_k |\alpha_k\rangle \quad , \quad (3)$$

$$|v_k\rangle = c'_k |\alpha_k\rangle + c''_k |\beta_k\rangle \quad , \quad (4)$$

for some expansion constants. Expanding $|\Psi\rangle$ in these bases we get

$$|\Psi\rangle = z_0 |\alpha_1 \cdots \alpha_p\rangle + \sum_{k=1}^{p} z_k |\alpha_1 \cdots \alpha_{k-1}\beta_k\alpha_{k+1} \cdots \alpha_p\rangle \quad (5)$$

for some complex coefficients $z_0, z_1, \ldots, z_p$. Finally, the phases of these basis vectors can be redefined so that all expansion coefficients are nonnegative real numbers. With this redefinition, the state becomes

$$\begin{aligned} |\Psi\rangle &= \sqrt{x_0}\,|\alpha_1 \cdots \alpha_p\rangle \quad (6) \\ &+ \sum_{k=1}^{p} \sqrt{x_k}\,|\alpha_1 \cdots \alpha_{k-1}\beta_k\alpha_{k+1} \cdots \alpha_p\rangle \quad , \end{aligned}$$

where $x_k = |z_k|^2$. In short, it is shown that for any W-type state there are nonnegative real numbers, $x_0, x_1, \cdots, x_p$ and local orthonormal bases $\{|\alpha_k\rangle, |\beta_k\rangle\}$ in $\mathcal{H}_k$ such that the expansion above is valid.

Note that we have

$$x_0 + \sum_{k=1}^{p} x_k = 1 \quad (7)$$

by normalization. The vector $\mathbf{x} = (x_1, x_2, \ldots, x_p)$ is the basic mathematical object that will be used in describing the W-type states. The zeroth component x_0 is not considered as an independent component; it will rather be considered as a function of the vector $\mathbf{x}$ given by Eq. (7). For distinguishing from the zeroth, all other numbers $x_1, \ldots, x_p$ will be called *party components*. All possible allowed vectors $\mathbf{x}$ form a subset $\mathcal{S}$ of $\mathbb{R}^p$ which is actually a simplex defined by $x_k \geq 0$ for $k = 1, 2, \ldots, p$ and $x_1 + \cdots + x_p \leq 1$.

An LU transformation of the state $|\Psi\rangle$ changes only the local orthonormal bases $\{|\alpha_k\rangle, |\beta_k\rangle\}$. Therefore the vector parameter $\mathbf{x}$ is invariant under LU transformations, which makes it a good candidate for parametrizing LU equivalence classes. The state $|\Psi\rangle$ given in Eq. (6) is LU equivalent to the following representative state

$$\begin{aligned} |\Phi(\mathbf{x})\rangle &= \sqrt{x_0}\,|000\cdots0\rangle + \sqrt{x_1}\,|100\cdots0\rangle \quad (8) \\ &\quad + \sqrt{x_2}\,|010\cdots0\rangle + \cdots + \sqrt{x_p}\,|000\cdots1\rangle \\ &= \sqrt{x_0}\,|\mathbf{0}\rangle + \sum_{k=1}^{p} \sqrt{x_k}\,|\mathbf{1}_k\rangle \quad (9) \end{aligned}$$

where $|\mathbf{0}\rangle$ is shorthand for the state where all qubits are in 0 state and $|\mathbf{1}_k\rangle$ represents the state where the kth qubit is in 1 state and all the others are in 0 state.

For some states $|\Psi\rangle$, there might be different representations of the form (6), i.e., the state might be associated with two different parameter vectors $\mathbf{x}$ and $\mathbf{x}'$. This is equivalent to saying that the vectors $|\Phi(\mathbf{x})\rangle$ and $|\Phi(\mathbf{x}')\rangle$ are LU equivalent. In such a case, we will say that the vectors $\mathbf{x}$ and $\mathbf{x}'$ are equivalent and show this relation by $\mathbf{x} \sim \mathbf{x}'$. With the complete characterization of this equivalence relation, the simplex $\mathcal{S}$ becomes a natural working ground in the investigation of the transformation of W-type states. Consequently, we will usually talk about the transformations of points in $\mathcal{S}$. For example, we say that $\mathbf{x}$ can be transformed deterministically to $\mathbf{y}$, when it is actually meant that a state associated with the vector $\mathbf{x}$ can be deterministically transformed to another state that has parameter vector $\mathbf{y}$.

For investigating the equivalence relation in the parameter space $\mathcal{S}$, it is useful to utilize quantities that do not depend on the particular representation used for the state. One such property is the concurrence[21] corresponding to the bipartite entanglement between a subset of the parties and the rest. Let $\mathcal{C}_k$ be the concurrence of the entanglement of the kth party with the rest. It is given by

$$\mathcal{C}_k = 2\sqrt{\det \rho^{(k)}} = 2\sqrt{x_k(1 - x_0 - x_k)} \quad (10)$$

where $\rho^{(k)}$ is the reduced density matrix of the kth particle. The corresponding quantity for a subset $R = \{k_1, k_2, \ldots, k_r\}$ of parties can be computed easily with a simple trick. Note that the state $|\Psi\rangle$ will still be classified as a W-type state when all parties in R is reinterpreted as a single party. In that case, the expansion in Eq. (6) is altered by taking the x-parameter corresponding to R as

$$x_R = x_{k_1} + x_{k_2} + \cdots + x_{k_r} \quad , \quad (11)$$

and by taking the associated local orthonormal basis as $\{|A_R\rangle, |B_R\rangle\}$ where

$$|A_R\rangle = |\alpha_{k_1} \otimes \alpha_{k_2} \otimes \cdots \otimes \alpha_{k_r}\rangle \quad , \quad (12)$$

$$|B_R\rangle = \sum_{j=1}^{r} \sqrt{\frac{x_{k_j}}{x_R}}\,|\alpha_{k_1} \otimes \cdots \otimes \beta_{k_j} \otimes \cdots \otimes \alpha_{k_r}\rangle \quad (13)$$

As a result, the concurrence of the entanglement between R and the rest of the parties is given by $\mathcal{C}_R = 2\sqrt{x_R(1 - x_0 - x_R)}$.

For any pair of distinct parties k and ℓ, define the following quantity

$$\mathcal{D}_{k\ell} = \frac{1}{8}(\mathcal{C}_k^2 + \mathcal{C}_\ell^2 - \mathcal{C}_{k\ell}^2) \quad . \tag{14}$$

Note that $\mathcal{D}_{k\ell}$ does not depend on the way $|\Psi\rangle$ is represented. Moreover, it is related to the parameter vector $\mathbf{x}$ by $\mathcal{D}_{k\ell} = x_k x_\ell$. This shows that if $\mathbf{x} \sim \mathbf{x}'$ then $x_k x_\ell = x'_k x'_\ell$ for all distinct pairs of parties k and ℓ.

The last relation implies that when $|\Psi\rangle$ has a representation (6) for which at least two components of $\mathbf{x}$ is nonzero, then all other party-components are unique. In particular, if x_r and x_s are nonzero, then for any party $k \neq r, s$, the coefficient x_k is given as

$$x_k = \sqrt{\frac{\mathcal{D}_{kr}\mathcal{D}_{ks}}{\mathcal{D}_{rs}}} \tag{15}$$

and hence such x_k are unique. Consequently, when three components of $\mathbf{x}$ are nonzero, then every component of $\mathbf{x}$ is unique.

The equivalence relation in $\mathcal{S}$ and the type of entanglement that these vectors represent can be completely described as follows. Let $|\Psi\rangle$ be a W-type state having a parameter vector $\mathbf{x}$. There are three possibilities depending on the number of nonzero party components of $\mathbf{x}$.

(i) At least three party-components of $\mathbf{x}$ are nonzero: in this case, the representation in Eq. (6) is unique; in other words $\mathbf{x} \sim \mathbf{x}'$ if and only if $\mathbf{x} = \mathbf{x}'$. Moreover, $|\Psi\rangle$ is a *truly multipartite* state (i.e., at least three parties are entangled with each other) and a given party k is entangled with the rest ($\mathcal{C}_k \neq 0$) if and only if $x_k \neq 0$.

(ii) Only two party-components of $\mathbf{x}$ are nonzero, say $x_r, x_s \neq 0$: in that case the vector parameter is not unique in general[22]. The state $|\Psi\rangle$ is a bipartite entangled state between rth and sth particles having the concurrence $\mathcal{C} = 2\sqrt{x_r x_s}$. We have $\mathbf{x} \sim \mathbf{x}'$ if and only if $x_r x_s = x'_r x'_s$ and $x'_k = 0$ for all $k \neq 0, r, s$.

(iii) Finally, in all the other cases, i.e., when all components of $\mathbf{x}$ are zero or if only one of them is nonzero, then $|\Psi\rangle$ is a product state. Obviously, $\mathbf{x} \sim \mathbf{x}'$ if and only if $\mathbf{x}'$ satisfies the same property.

When a parameter vector $\mathbf{x}$ is unique, then the basis vectors used in the representation in Eq. (6) are also unique. This can be seen as follows. First, note that if $x_k = 0$ for some k, then the kth particle is unentangled from the rest and therefore only $|\alpha_k\rangle$ is uniquely defined (up to a phase) and the vector $|\beta_k\rangle$ is irrelevant. For two distinct parties k and ℓ for which $\mathcal{C}_{k\ell} \neq 0$ (i.e., these two parties are entangled with the rest), consider the reduced density matrix $\rho^{(k\ell)}$ on the Hilbert space $\mathcal{H}_k \otimes \mathcal{H}_\ell$ of the corresponding particles. This is an operator on a 4-dimensional space having rank 2. Hence, its eigensubspace corresponding to zero eigenvalue is 2-dimensional and spanned by the following two vectors

$$\left|\Theta_1^{(k,\ell)}\right\rangle = |\beta_k \otimes \beta_\ell\rangle \quad , \tag{16}$$

$$\left|\Theta_2^{(k,\ell)}\right\rangle = \sqrt{x_\ell}\,|\beta_k \otimes \alpha_\ell\rangle - \sqrt{x_k}\,|\alpha_k \otimes \beta_\ell\rangle \quad . \tag{17}$$

It is clear that, if both x_k and x_ℓ are nonzero, then this eigensubspace contains only one vector (direction) in product form, namely $\left|\Theta_1^{(k,\ell)}\right\rangle$. This enables us to define $|\beta_k\rangle$ and $|\beta_\ell\rangle$ uniquely up to a phase. By orthogonality, the vectors $|\alpha_k\rangle$ and $|\alpha_\ell\rangle$ are also unique. In summary, when the W-type state is truly multipartite, then the basis vectors are also uniquely defined.

III. LOCAL OPERATIONS BY ONE PARTY

In order to be able to analyze the transformations of W-type states, one should first describe the local quantum operations carried out by a single party and its effect on the parameters. This section is concerned with this description and its immediate implications. The main result is given by the following theorem.

Theorem 1 *Suppose that the kth party carries out a local operation on a W-type state with parameter vector $\mathbf{x}$. The set of final states with vectors $\mathbf{x}_\lambda$ can be produced with probabilities P_λ if and only if for each possible outcome λ there are vectors $\mathbf{x}'_\lambda \sim \mathbf{x}_\lambda$ and there are non-negative scale factors s_λ such that*

(i) $x'_{\lambda,\ell} = s_\lambda x_\ell$ for all $\ell \neq k, 0$;

(ii) $\sum_\lambda P_\lambda s_\lambda = 1$ and

(iii)

$$\sum_\lambda P_\lambda \sqrt{s_\lambda x'_{\lambda,0}} \geq \sqrt{x_0} \quad . \tag{18}$$

Note that the requirement of finding a new vector $\mathbf{x}'_\lambda$ may become necessary only if the final state $\mathbf{x}_\lambda$ corresponds to a bipartite or a product state. Otherwise, if $\mathbf{x}_\lambda$ corresponds to a truly multipartite final state, we necessarily have $\mathbf{x}'_\lambda = \mathbf{x}_\lambda$. Note also that, it is not necessary that the initial state $\mathbf{x}$ is truly multipartite; the theorem is valid for all the other cases as well.

We first start proving the necessity of the conditions listed. Suppose that the kth party carries out a local operation on the state $|\Phi(\mathbf{x})\rangle$ and let $\{M_\lambda\}$ be the set of measurement operators that describe this operation. They satisfy the normalization relation $\sum_\lambda M_\lambda^\dagger M_\lambda = \mathbb{1}_k$ where $\mathbb{1}_k$ denotes the identity operator on $\mathcal{H}_k$. For each outcome λ, an orthonormal basis $\{|a_\lambda\rangle, |b_\lambda\rangle\}$ can be found in $\mathcal{H}_k$ such that

$$M_\lambda |0\rangle = A_\lambda |a_\lambda\rangle \quad , \tag{19}$$

$$M_\lambda |1\rangle = B_\lambda |a_\lambda\rangle + C_\lambda |b_\lambda\rangle \quad , \tag{20}$$

$$\tag{21}$$

where A_λ and C_λ are nonnegative real numbers and B_λ is some complex number. Note that the basis $\{|a_\lambda\rangle, |b_\lambda\rangle\}$ depends only on a local unitary transformation that may be applied by the kth party after the outcome λ is obtained and hence it is irrelevant. This basis will be chosen to be $\{|0\rangle, |1\rangle\}$ for simplifying the notation below. The normalization relation of the measurement operators can be expressed as

$$\sum_\lambda A_\lambda^2 = \sum_\lambda |B_\lambda|^2 + C_\lambda^2 = 1 \quad , \quad \sum_\lambda A_\lambda B_\lambda = 0 \quad . \tag{22}$$

When the outcome λ is obtained, the final state is

$$\begin{aligned}(M_\lambda \otimes \mathbb{1}'_k)\,|\Phi(\mathbf{x})\rangle &= (A_\lambda\sqrt{x_0} + B_\lambda\sqrt{x_k})\,|\mathbf{0}\rangle \\ &+ \sum_{\ell\neq k} A_\lambda \sqrt{x_\ell}\,|\mathbf{1}_\ell\rangle + C_\lambda\sqrt{x_k}\,|\mathbf{1}_k\rangle\end{aligned} \tag{23}$$

where $\mathbb{1}'_k$ represents the identity operator on all particles except the kth. The probability of the outcome is given by the square of the norm of the vector above

$$P_\lambda = |A_\lambda\sqrt{x_0} + B_\lambda\sqrt{x_k}|^2 + A_\lambda^2(1-x_0-x_k) + C_\lambda^2 x_k \ , \tag{24}$$

and after an appropriate phase redefinition of the local basis vectors of the kth qubit, the vector parameter $\mathbf{x}_\lambda$ of the final state can be found as

$$x_{\lambda,k} = \frac{C_\lambda^2 x_k}{P_\lambda} \quad , \tag{25}$$

$$x_{\lambda,\ell} = \frac{A_\lambda^2 x_\ell}{P_\lambda} \quad (\ell \neq 0, k) \quad , \tag{26}$$

$$x_{\lambda,0} = \frac{\left|A_\lambda\sqrt{x_0} + B_\lambda\sqrt{x_k}\right|^2}{P_\lambda} \quad . \tag{27}$$

Obviously, $s_\lambda = A_\lambda^2/P_\lambda$ and it can be immediately seen that the conditions (i) and (ii) in the theorem is satisfied. For proving condition (iii), note that

$$\begin{aligned}\sum_\lambda P_\lambda\sqrt{s_\lambda x_{\lambda,0}} &= \sum_\lambda A_\lambda\,|A_\lambda\sqrt{x_0} + B_\lambda\sqrt{x_k}| \\ &\geq \sum_\lambda A_\lambda\,(A_\lambda\sqrt{x_0} + \mathrm{Re}(B_\lambda)\sqrt{x_k}) \\ &= \sqrt{x_0} \quad ,\end{aligned} \tag{28}$$

where we have used the fact that the modulus of a complex number is greater than its real part in order to introduce the inequality and then invoked Eq. (22). This completes the proof of the necessity of the conditions.

In order to prove the sufficiency, first suppose that the conditions (i), (ii) and (iii) are satisfied. Let λ_0 be the outcome which contributes the largest term in the summation in Eq. (18). If it happens that there is only one possible outcome (i.e., $P_{\lambda_0} = 1$) or if

$$P_{\lambda_0}\sqrt{s_{\lambda_0} x'_{\lambda_0,0}} > \sum_{\lambda\neq\lambda_0} P_\lambda\sqrt{s_\lambda x'_{\lambda,0}} \tag{29}$$

then redefine the outcomes so that $\mathbf{x}_{\lambda_0}$ appears twice in the result set with probabilities $P_{\lambda_0}/2$ each. In that case, Eq. (18) can be interpreted as the polygonal inequality, i.e., the generalization of the triangle inequality to polygons. Imagining the corresponding polygon on the complex plane, we can find angles ϕ_λ such that the equality

$$\sum_\lambda P_\lambda\sqrt{s_\lambda x'_{\lambda,0}}\,e^{i\phi_\lambda} = \sqrt{x_0} \tag{30}$$

is satisfied. In that case, it is possible to go backwards in the derivation given above by defining

$$A_\lambda = \sqrt{P_\lambda s_\lambda} \quad , \tag{31}$$

$$B_\lambda = \sqrt{\frac{P_\lambda}{x_k}}\left(\sqrt{x'_{\lambda,0}}\,e^{i\phi_\lambda} - \sqrt{s_\lambda x_0}\right) \quad , \tag{32}$$

$$C_\lambda = \sqrt{\frac{P_\lambda x'_{\lambda,k}}{x_k}} \quad , \tag{33}$$

$$M_\lambda = \begin{bmatrix} A_\lambda & B_\lambda \\ 0 & C_\lambda \end{bmatrix} \quad . \tag{34}$$

It is straightforward to see that the relations (22) are satisfied and hence $\{M_\lambda\}$ satisfy the normalization condition of measurement operators. It is also straightforward to check that the final state $\mathbf{x}'_\lambda$ is produced with probability λ. This completes the proof of the theorem.□

Let us investigate some implications of the conditions in the theorem. First, applying the Schwarz inequality to condition (iii) of the theorem gives

$$\sqrt{x_0} \leq \sum_\lambda P_\lambda\sqrt{s_\lambda x'_{\lambda,0}} \leq \sqrt{\sum_\lambda P_\lambda s_\lambda}\sqrt{\sum_\lambda P_\lambda x'_{\lambda,0}} \tag{35}$$

which shows that

$$x_0 \leq \sum_\lambda P_\lambda x'_{\lambda,0} \quad , \tag{36}$$

with equality holding if and only if $x'_{\lambda,0}$ is proportional to s_λ. In short, the zeroth component of the parameter vector is nondecreasing on the average. Together with the condition (i), the following inequalities can be written for the party components of the vectors.

$$x_\ell = \sum_\lambda P_\lambda x'_{\lambda,\ell} \quad (\ell \neq 0, k), \tag{37}$$

$$x_k \geq \sum_\lambda P_\lambda x'_{\lambda,k} \quad , \tag{38}$$

i.e., the kth component of the parameter vector is nonincreasing on the average, while the averages of all the other components do not change. These inequalities imply that, for any to distinct parties k and ℓ, the function $\sqrt{x_k x_\ell}$, which depends only on the LU equivalence class of the state, is an entanglement monotone[4].

An important corollary that can be deduced from the theorem is concerned with the deterministic transformation induced only by the local operations of the kth party.

Corollary 1 *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be two vectors in* $\mathcal{S}$ *that differ only in their kth component such that* $x_k > y_k$. *Then, the state* $|\Phi(\mathbf{x})\rangle$ *can be transformed to* $|\Phi(\mathbf{y})\rangle$ *by the local operations of the kth party only.*

The corollary follows straightforwardly from Theorem 1 as there is only one outcome λ; necessarily $s_\lambda = 1$ and finally $y_0 > x_0$. Hence, all three conditions of the theorem are satisfied. A two-outcome general measurement that does the transformation can be constructed easily by using the reasoning given in the proof of the Theorem 1. Note that, in order to bring the state to the final form $|\Phi(\mathbf{y})\rangle$, additional LU transformations by all the other parties are also needed.

IV. GENERAL LOCC TRANSFORMATIONS

The relations in Eq. (36-38) can be easily extended to a general protocol where a sequence of local operations is carried out by different parties. Let the variable Λ be used for labeling the sequence of outcomes obtained in these operations and let P_Λ be the probability for this outcome. Let $\mathbf{x}$ be the parameter vector corresponding to the initial state and let $\mathbf{x}'_\Lambda$ denote the vector, whose existence is guaranteed by Theorem 1, that corresponds to the final state when the outcome Λ is obtained. Using Eq. (37,38) the following inequality can be seen to be valid

$$x_\ell \geq \sum_\Lambda P_\Lambda x'_{\Lambda,\ell} \quad (\ell = 1, 2, \ldots, p) \ . \tag{39}$$

Obviously, if the ℓth party never does a local operation, then equality sign holds in Eq. (39).

At this point, it is very convenient to introduce a partial order between the vectors in $\mathcal{S}$. For two such vectors $\mathbf{x}$ and $\mathbf{y}$, we will say that $\mathbf{x} \geq \mathbf{y}$ if and only if $x_\ell \geq y_\ell$ for all parties $\ell = 1, 2, \ldots, p$. Using this relation, Eq. (39) can be expressed more compactly as

$$\mathbf{x} \geq \sum_\Lambda P_\Lambda \mathbf{x}'_\Lambda \quad . \tag{40}$$

Therefore, during all the intermediate moments of the whole transformation process, the average of the state vectors follows a monotonically decreasing path in $\mathcal{S}$. It turns out that this partial order completely describes the deterministic transformations.

Theorem 2 *A W-type state with parameter vector* $\mathbf{x}$ *can be transformed by LOCC to another state with vector* $\mathbf{y}$ *if and only if there is* $\mathbf{y}' \sim \mathbf{y}$ *such that* $\mathbf{x} \geq \mathbf{y}'$.

Again, the need for using another final state vector $\mathbf{y}'$ may become necessary only when $\mathbf{y}$ corresponds to a bipartite state. The proof of the theorem is rather straightforward. The sufficiency part is already partially covered by the Corollary 1. Hence, if $\mathbf{x} \geq \mathbf{y}'$, then every party carries out a local operation whose sole effect is the decrease of the corresponding component of the parameter vector. In other words, for all k, kth party changes the kth component of the vector from x_k to y'_k.

For the sufficiency part of the proof, suppose that $\mathbf{x}$ to $\mathbf{y}$ conversion is possible. If $\mathbf{y}$ is a truly multipartite state, then Eq. (40) already gives us $\mathbf{x} \geq \mathbf{y}$ and there is nothing further to do. On the other hand, if $\mathbf{y}$ corresponds to a bipartite state, then different final state vectors $\mathbf{x}'_\Lambda$ may appear in Eq. (40) in which case the following analysis has to be made. Let the final state be a bipartite entangled state between the parties r and s. Let $\bar{x}'_r$ and $\bar{x}'_s$ represent the average of the rth and sth components of the final vectors $\mathbf{x}'_\Lambda$. By Eq. (40) we have

$$x_r \ \geq \ \bar{x}'_r = \sum_\Lambda P_\Lambda x'_{\Lambda,r} \quad , \tag{41}$$

$$x_s \ \geq \ \bar{x}'_s = \sum_\Lambda P_\Lambda x'_{\Lambda,s} \quad . \tag{42}$$

On the other hand $\mathbf{x}'_\Lambda \sim \mathbf{y}$ and hence we have $x'_{\Lambda,r} x'_{\Lambda,s} = \mathcal{C}^2/4$ where $\mathcal{C} = 2\sqrt{y_r y_s}$ is the concurrence of the final state. We then use the Schwarz inequality as follows

$$\frac{\mathcal{C}}{2} \ = \ \sum_\Lambda P_\Lambda \sqrt{x'_{\Lambda,r} x'_{\Lambda,s}} \tag{43}$$

$$\leq \ \sqrt{\sum_\Lambda P_\Lambda x'_{\Lambda,r}} \sqrt{\sum_\Lambda P_\Lambda x'_{\Lambda,s}} \tag{44}$$

$$= \ \sqrt{\bar{x}'_r \bar{x}'_s} \ . \tag{45}$$

We can now define a suitable $\mathbf{y}'$ vector by

$$y'_r = \bar{x}'_r \quad , \quad y'_s = \frac{\mathcal{C}^2}{4\bar{x}'_r} \quad , \quad y'_\ell = 0 \quad (\ell \neq 0, r, s) \ . \tag{46}$$

It can now be easily verified that $\mathbf{y}' \sim \mathbf{y}$, $y'_r \leq x_r$ and finally $y'_s \leq \bar{x}'_s \leq x_s$. Therefore we have $\mathbf{y}' \leq \mathbf{x}$. This completes the proof of the theorem. □

As a result, the order $\geq$ defined on the points of the simplex $\mathcal{S}$ is closely related to the partial order defined by the deterministic LOCC convertibility relation. Obviously, there are pairs of states which cannot be converted into each other, hence the order is partial. A few words can also be said about the maximal states. Note that any state having a parameter vector $\mathbf{x}$ with nonzero zeroth component ($x_0 \neq 0$) is not maximally entangled; such a state can always be deterministically reducible from a different state. The truly multipartite states that have a vanishing zeroth component ($x_0 = 0$) are always maximally entangled. The standard W state with

$$\mathbf{x}_W = \frac{1}{p}(1, 1, \ldots, 1) \quad , \tag{47}$$

is in this set but there are many more states that are also maximally entangled.

Next, we turn our attention to the probabilistic transformations. It appears that the inequality (40) is not sufficient for a complete description of possible probabilistic transformations. Still, it can be used for finding suitable bounds on some problems of interest. For example, it can be utilized for finding a good upper bound for the maximum probability of distilling a truly multipartite target state $\mathbf{y}$ from an initial state $\mathbf{x}$. To do this, consider a transformation where $\mathbf{x}$ is converted into various states $\mathbf{x}'_\Lambda$, where the parametrization is such that Eq. (40) holds. Let P be the total probability of outcomes Λ where $\mathbf{x}'_\Lambda = \mathbf{y}$. By Eq. (40), we have

$$P\mathbf{y} + \sum_{\Lambda(\mathbf{x}'_\Lambda \nsim \mathbf{y})} P_\Lambda \mathbf{x}'_\Lambda \leq \mathbf{x} \ . \tag{48}$$

Since the sum term gives a vector with nonnegative components, we have $P\mathbf{y} \leq \mathbf{x}$. Therefore, the probability P is bounded from above by

$$P \leq \mathcal{P}_{\rm bnd}(\mathbf{x} \to \mathbf{y}) = \min\left(\frac{x_1}{y_1}, \frac{x_2}{y_2}, \ldots, \frac{x_p}{y_p}, 1\right) \ . \tag{49}$$

Although this bound is tight for some transitions, e.g., deterministic transformations, there are some transitions where the maximum probability and the bound above differ. For example, consider the case where $\mathbf{x} = t\mathbf{y}$ and $0 < t < 1$ for which the bound is $\mathcal{P}_{\rm bnd}(t\mathbf{y} \to \mathbf{y}) = t$. If it were possible that $P = t$, then it would be necessary that all of the other outcomes are equal to the zero vector, $\mathbf{x}'_\Lambda = \mathbf{0}$, and Eq. (48) is an equality. Therefore, Eq. (36) must be an equality as well for all intermediate local operations. Since the Schwarz inequality has been used for obtaining Eq. (36), the equality conditions of that inequality implies that $x'_{\lambda,0} \propto s_\lambda$. However, this is in contradiction with the existence of outcomes $\mathbf{x}'_\lambda = \mathbf{0}$ for which we have $s_\lambda = 0$, but $x'_{\lambda,0} = 1$. This shows that the distillation probability $P = t$ cannot be reached.

It appears that for cases where the zeroth component of both the initial and final vectors are zero ($x_0 = y_0 = 0$), the maximum probability of distillation reaches to the bound in Eq. (49). The special case where the final state is the standard W state (i.e., $\mathbf{y} = \mathbf{x}_W$) has already been investigated before[16, 17] for which a protocol is proposed which achieves the distillation probability of $P = p\min(x_1, \ldots, x_p)$. Since this is also equal to $\mathcal{P}_{\rm bnd}(\mathbf{x} \to \mathbf{y})$, their protocol is optimal.

It appears that their protocol can be directly adapted to all other transformations on this particular face of the simplex $\mathcal{S}$, i.e., the face formed by the vectors that have a vanishing zeroth component. First, note that when all produced states are on that face, the relation (iii) of Theorem 1 is automatically satisfied. This simplifies the analysis of such transformations considerably. The protocol can be constructed as follows: Suppose that the ratio $r_k = x_k/y_k$ becomes the minimum for the first m parties, i.e., let $r_1 = \cdots = r_m < r_{m+1}, \ldots, r_p$. The local operations are done by the last $p - m$ parties, in any order they wish. For simplicity, it will be assumed that the operations are carried out in the order of increasing index, i.e., first $(m+1)$th party applies an operation, then $(m+2)$th, etc. Let $s_\lambda^{(k)}$ and $P_\lambda^{(k)}$ denote the scale factors and probabilities of kth party's operation. Each local operation is a two outcome general measurement where $\lambda = 0$ corresponds to the failure result with $s_0^{(k)} = 0$ and $\lambda = 1$ corresponds to the success result. In order to satisfy the condition (i) of Theorem 1, we should have $s_1^{(k)} \geq 1$ and $P_1^{(k)} = 1/s_1^{(k)}$. Let $\mathbf{x}^{(k)}$ denote the state after the successful operation of the kth party ($k = m + 1, \cdots, p$). It is given as

$$\mathbf{x}^{(k)} = (r_1 y_1, \ldots, r_1 y_k, x_{k+1}, \ldots, x_p) s_1^{(m+1)} \cdots s_1^{(k)} \ .$$

As the components of all such vectors add up to 1, the scale factors are

$$s_1^{(k)} = \frac{r_1(y_1 + \cdots + y_{k-1}) + x_k + x_{k+1} + \cdots + x_p}{r_1(y_1 + \cdots + y_k) + x_{k+1} + \cdots + x_p} \ . \tag{50}$$

Since $r_k > r_1$, we have $s_1^{(k)} > 1$ and therefore the kth operation can be carried out. The final state is obviously $\mathbf{x}^{(p)} = \mathbf{y}$, and the probability of distillation is given by

$$P = P_1^{(m+1)} \cdots P_1^{(p)} = \frac{1}{s_1^{(m+1)} \cdots s_1^{(p)}} = r_1 \quad , \tag{51}$$

i.e., the upper bound $\mathcal{P}_{\rm bnd}(\mathbf{x} \to \mathbf{y})$ has been attained.

Finally, if $\mathbf{y}$ corresponds to a bipartite state between parties r and s with concurrence $\mathcal{C}$, it can be shown that $P \leq \mathcal{P}_{\rm bnd}(\mathbf{x} \to \mathbf{y}) = 2\sqrt{x_r x_s}/\mathcal{C}$. Considering the special case where $\mathbf{x}$ is also bipartite entangled, it can be seen that this bound is also not tight.

V. CONCLUSION

The essential mathematical tools for the systematic investigation of all possible transformations of W-type states have been obtained. These tools include the simplex $\mathcal{S}$, which is extremely useful in the identification of LU equivalence classes of the states. It appears that the LU equivalence classes of truly multipartite states are represented by a single point in $\mathcal{S}$, which considerably simplifies the analysis of the transformations of such states. A complete characterization of transformations that can be carried out by a single party is also given. Finally, these tools are used for identifying all necessary and sufficient conditions for deterministic transformations. An upper bound for maximum distillation probability of arbitrary multipartite states is also given. A complete characterization of the probabilistic transformations of W-type states is still an open problem. It is hoped that this article lays a good background from which such problems and related questions about the multipartite entanglement can be studied.

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