Integrability of a Generalized Ito System: the Painlevé Test

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Abstract

It is shown that a generalized Ito system of four coupled nonlinear evolution equations passes the Painlevé test for integrability in five distinct cases, of which two were introduced recently by Tam, Hu and Wang. A conjecture is formulated on integrability of a vector generalization of the Ito system.

KEYWORDS: integrable systems, Painlevé analysis

Recently, Tam, Hu and Wang [1] introduced the following two systems of coupled nonlinear evolution equations:

$$\begin{aligned} &u_t = v_x, \\ &v_t = -2v_{xxx} - 6\,(uv)_x - 12ww_x + 6p_x, \\ &w_t = w_{xxx} + 3uw_x, \\ &p_t = p_{xxx} + 3up_x, \end{aligned} \tag{1}$$

and

$$\begin{aligned} &u_t = v_x, \\ &v_t = -2v_{xxx} - 6\,(uv)_x - 6\,(wp)_x\,, \\ &w_t = w_{xxx} + 3uw_x, \\ &p_t = p_{xxx} + 3up_x, \end{aligned} \tag{2}$$

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which are generalizations of the well-known integrable Ito system [2]

$$
\begin{aligned}
&u_t = v_x, \\
&v_t = -2v_{xxx} - 6\,(uv)_x\,.
\end{aligned}
\tag{3}
$$

Hirota bilinear representations of the systems (1) and (2), 4-soliton solutions of the system (1) with $p = 0$, and 3-soliton solutions of the system (2) were found in ref. [1]. More recently, a new type of 3-soliton solutions with constant boundary conditions at infinity was found for the system (1) with $p = 0$ in ref. [3]. The question on integrability of the systems (1) and (2), in the sense of existence of Lax pairs, infinitely many conservation laws, and N-soliton solutions, was posed in ref. [1].

In this Letter, we show that the Tam-Hu-Wang systems (1) and (2) must be integrable according to positive results of the Painlevé test. We apply the Painlevé test for integrability to the following generalized Ito system:

$$
\begin{aligned}
&u_t = v_x, \\
&v_t = -2v_{xxx} - 6\,(uv)_x + aww_x + bpw_x + cwp_x + dpp_x + fw_x + gp_x, \\
&w_t = w_{xxx} + 3uw_x, \\
&p_t = p_{xxx} + 3up_x,
\end{aligned}
\tag{4}
$$

where a, b, c, d, f, g are arbitrary constants. The system (4) turns out to possess the Painlevé property under the only constraint imposed on its coefficients: $c = b$. Then, using affine transformations of w and p, we reduce the system (4) with $c = b$ to five distinct cases, of which two being eqs. (1) and (2). Finally, we propose a conjecture on integrability of a multi-component generalization of the Tam-Hu-Wang systems (1) and (2).

Let us apply the Painlevé test for integrability to the system (4), following the so-called Weiss-Kruskal algorithm of singularity analysis [4], [5]. The system (4) is a normal system of four partial differential equations of total order ten, and its general solution must contain ten arbitrary functions of one variable. A hypersurface $\phi\,(x,t) = 0$ is non-characteristic for this system if $\phi_x \phi_t \neq 0$, and we set $\phi_x = 1$ without loss of generality. The substitution of the expansions

$$
\begin{aligned}
&u = u_0\,(t)\,\phi^{\alpha} + \cdots + u_n\,(t)\,\phi^{n+\alpha} + \cdots, \\
&v = v_0\,(t)\,\phi^{\beta} + \cdots + v_n\,(t)\,\phi^{n+\beta} + \cdots, \\
&w = w_0\,(t)\,\phi^{\gamma} + \cdots + w_n\,(t)\,\phi^{n+\gamma} + \cdots, \\
&p = p_0\,(t)\,\phi^{\delta} + \cdots + p_n\,(t)\,\phi^{n+\delta} + \cdots
\end{aligned}
\tag{5}
$$

into the system (4) determines the branches, i.e. the admissible dominant behavior of solutions (values of $\alpha, \beta, \gamma, \delta, u_0, v_0, w_0, p_0$) and the corresponding positions n of the resonances (where arbitrary functions can appear in the expansions (5)).

The system (4) admits many branches, but the presence or absence of most of them depends on values of the parameters a, b, c, d, f, g. For this reason, it is useful to start the analysis from the following singular branch:

$$\begin{aligned}
&\alpha = \beta = -2, \quad \gamma = \delta = -1, \\
&u_0 = -2, \quad v_0 = -2\phi_t, \quad \forall\, w_0\,(t)\,, \quad \forall\, p_0\,(t)\,, \\
&n = -1, 0, 0, 1, 1, 2, 4, 5, 5, 6,
\end{aligned} \tag{6}$$

which is admitted by the system (4) at any choice of its parameters. According to the positions of resonances, the branch (6) is generic, i.e. it represents the general solution of the system (4). Now, constructing the recursion relations for the coefficients of the expansions (5), and checking the consistency of those recursion relations at the resonances of the branch (6), we obtain the compatibility condition $(b-c)\,(w_{0,t}p_0 - w_0 p_{0,t}) = 0$ at $n = 5$. If $c \neq b$, some logarithmic terms must be introduced into the expansions (5). Therefore the system (4) can possess the Painlevé property only if $c = b$. Setting $c = b$ hereafter, we find that the recursion relations are consistent at all the resonances of the branch (6).

Before proceeding to other branches, let us notice that, in the case of $c = b$, we can fix all the free parameters of the system (4) by means of the affine transformation

$$\begin{aligned}
&w \to \xi_1 w + \xi_2 p + \xi_3, \\
&p \to \xi_4 w + \xi_5 p + \xi_6
\end{aligned} \tag{7}$$

with appropriately chosen constants $\xi_1, \ldots, \xi_6$, $\xi_1\xi_5 \neq \xi_2\xi_4$. Certainly, this transformation has no effect on the presence or absence of the Painlevé property. If $ad \neq b^2$, then, using the transformation (7), we can make

$$b = -6, \quad a = d = f = g = 0. \tag{8}$$

If $ad = b^2, a \neq 0, ag \neq bf$, or if $a = b = 0, d \neq 0, f \neq 0$, we can make

$$a = -12, \quad g = 6, \quad b = d = f = 0. \tag{9}$$

If $ad = b^2, a \neq 0, ag = bf$, or if $a = b = 0, d \neq 0, f = 0$, we can make

$$a = -12, \quad b = d = f = g = 0. \tag{10}$$

If $a = b = d = 0, f \neq 0$, or if $a = b = d = f = 0, g \neq 0$, we can make

$$a = b = d = f = 0, \quad g = 6. \tag{11}$$

And the case of

$$a = b = d = f = g = 0 \tag{12}$$

needs no transformation. These five cases (8)–(12) of the system (4) are not related to each other by the transformation (7).

Having reduced the system (4) with $c = b$ to the five distinct cases (8)–(12), we find that the following singular non-generic branches must be studied as well:

$$\begin{array}{l} \alpha = \beta = \gamma = \delta = -2, \\ u_0 = -4, \quad v_0 = -4\phi_t, \quad w_0 p_0 = -8\phi_t, \quad \forall\, w_0\,(t) \text{ or } \forall\, p_0\,(t)\,, \\ n = -2, -1, 0, 2, 2, 3, 4, 6, 7, 8 \end{array} \tag{13}$$

and

$$\begin{array}{l} \alpha = \beta = -2, \quad \gamma = 0, \quad \delta = -4, \\ u_0 = -10, \quad v_0 = -10\phi_t, \quad w_0 p_0 = -80\phi_t, \quad \forall\, w_0\,(t) \text{ or } \forall\, p_0\,(t)\,, \\ n = -5, -4, -1, 0, 2, 4, 6, 7, 8, 12 \end{array} \tag{14}$$

in the case (8);

$$\begin{array}{l} \alpha = \beta = -2, \quad \gamma = 0, \quad \delta = -4, \\ u_0 = -10, \quad v_0 = -10\phi_t, \quad p_0 = 80\phi_t, \quad \forall\, w_0\,(t)\,, \\ n = -5, -4, -1, 0, 2, 4, 6, 7, 8, 12 \end{array} \tag{15}$$

in the case (9);

$$\begin{array}{l} \alpha = \beta = \gamma = \delta = -2, \\ u_0 = -4, \quad v_0 = -4\phi_t, \quad w_0^2 = -8\phi_t, \quad \forall\, p_0\,(t)\,, \\ n = -2, -1, 0, 2, 2, 3, 4, 6, 7, 8 \end{array} \tag{16}$$

in the cases (9) and (10); and

$$\begin{array}{l} \alpha = \beta = -2, \quad \gamma = \delta = -4, \\ u_0 = -10, \quad v_0 = -10\phi_t, \quad p_0 = 80\phi_t, \quad \forall\, w_0\,(t)\,, \\ n = -5, -1, 0, 2, 4, 4, 6, 8, 11, 12 \end{array} \tag{17}$$

in the case (11). Then, using the *Mathematica* system [6], we prove that the recursion relations are consistent at the resonances of the branches (13)–(17), and therefore no logarithmic terms should be introduced into the expansions (5).

Now, we can conclude that the generalized Ito system (4) passes the Painlevé test for integrability if, and only if, $c = b$, or, up to the equivalence (7), in the five distinct cases (8)–(12). The cases (8) and (9) correspond to the Tam-Hu-Wang systems (2) and (1), respectively.

The obtained results of the singularity analysis are highly suggestive that the system (4) with $c = b$ must be integrable in the Lax sense. Moreover, we *conjecture* that, for any constant k_{ij} and l_i, $i, j = 1, \ldots, m$, and any integer

m, the following system of $m+2$ coupled nonlinear evolution equations for $u, v, q_1, \ldots, q_m$, a vector generalization of the Ito system (3),

$$
\begin{aligned}
&u_t = v_x, \\
&v_t = -2v_{xxx} - 6\,(uv)_x + \Big(\textstyle\sum_{i,j} k_{ij} q_i q_j + \sum_i l_i q_i\Big)_x\,, \\
&q_{i,t} = q_{i,xxx} + 3uq_{i,x}, \quad i = 1, \ldots, m,
\end{aligned}
\tag{18}
$$

passes the Painlevé test for integrability and possesses a parametric zero-curvature representation.

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