

# Holographic renormalization group flow, Wilson loops and field-theory $\beta$ -functions

Ian I. Kogan<sup>1</sup>, Martin Schvellinger<sup>2</sup>, Bayram Tekin<sup>3</sup>

*Theoretical Physics, Department of Physics, University of Oxford, 1 Keble Road, Oxford,  
OX1 3NP, UK*

We study the Renormalization Group (RG) flow of critical bosonic background fields in the framework of the RG approach to string theory. In this approach quantum field theory  $\beta$ -functions are the extra inputs in solving the string theory sigma-model equations. We study two different situations, the first one is the Yang-Mills theory where the coupling constant diverges in the infrared limit. The second case corresponds to a type of theories where the  $\beta$ -function has a pole in the infrared limit and it changes sign through the pole (as in  $\mathcal{N} = 1$  super-Yang-Mills theory). For this case in the strong coupling branch, in the infrared, there is an interval of values of the coupling in which the theory only leads to confinement. We have obtained this range. We also mention the theories with conformal-fixed points and their relation to theories with a pole in the  $\beta$ -functions. We calculate the Wilson loops in these theories.

---

<sup>1</sup>kogan@thphys.ox.ac.uk

<sup>2</sup>martin@thphys.ox.ac.uk

<sup>3</sup>tekin@thphys.ox.ac.uk

# 1 Introduction

The large  $N$ -limit of super-conformal gauge field theories has been proven to have dual representations in terms of the weakly coupled super-gravity or string theory via the Maldacena's conjecture [1, 2, 3, 4]. In such a duality the radius of the space  $R_c$  behaves like  $\Delta^{1/4}$ , where  $\Delta = g_{YM}^2 N$  is the 't Hooft parameter which is fixed in the large- $N$  limit [5]. Naturally for large  $\Delta$  this leads to a large radius which allows us to describe strings propagating in the background fields [6]. This is a particular example of an idea suggested by Polyakov [7] about the description of gauge theories in  $D$  dimensions in terms of certain non-critical string theory in  $D + 1$  dimensions, where an extra dimension is due to the Liouville field. In this approach the properties of the geometry give us information about the properties of the gauge theory. On the other hand it should be possible to describe the weakly coupled regime of gauge theories by the strong coupling regime of string theory.

One of the simplest questions that one can ask is about the derivation of the RG flow in gauge theories from the geometry, including the weakly coupled regime. Here we are not going to deal with this in the full 10-dimensional theory. Instead of that we are going to study an alternative approach due to Álvarez and Gómez [8] where the idea is to model the renormalization group equations of gauge theories. Basically their proposal implies to associate to the couplings of the gauge field theories some background fields of a closed string theory. Then one demands that the string  $\beta$ -functions for those backgrounds have to coincide with the RG equations of the gauge theories. In this approach the geometry dictates the properties of the gauge theory. In particular, as it has been shown by Álvarez and Gómez, for one-loop  $\beta$ -function in pure gluodynamics the space-time curvature is a continuously increasing function of the running-energy scale from the IR to the UV limit. The curvature behaves like an inverse power of the coupling. It implies that the theory runs continuously from the strongly coupled regime to the weakly coupled one. In this framework they have calculated the Wilson loops and shown that both confinement and over-confinement occur depending only on the area of the world-sheet of the fundamental string, at every value of the coupling. In the present paper we address the following situation. Consider a theory with a  $\beta$ -function which has a pole at some energy scale  $\Lambda$  in the infrared. The existence of the pole leads to two branches, one corresponds to the super-strongly coupled regime while the other one is the asymptotically-free regime. The point  $\Lambda$  behaves like an infrared attractive point since the RG flow of the theory goes from

the UV limit to the IR one in both branches [9]. Once the Wilson loops are computed for this kind of theories, one obtains an interesting new result: in the super-strong coupling, in the infrared, there is a range of values of the coupling in which the theory only leads to confinement and not over-confinement. We have calculated that range.

Starting from the bosonic string action it is possible to derive the one loop  $\beta$ -functions which yield the equations of motion of the background fields

$$\begin{aligned}\beta^\Phi &= \frac{D-26}{48\pi^2} + \frac{\alpha'}{16\pi^2} \left( 4(\nabla\Phi)^2 - 4\nabla^2\Phi - R + \frac{1}{12}H^2 \right) + \mathcal{O}(\alpha'^2) , \\ \beta_{\mu\nu}^G &= R_{\mu\nu} - \frac{1}{4}H_\mu^{\lambda\sigma}H_{\nu\lambda\sigma} + 2\nabla_\mu\nabla_\nu\Phi + \mathcal{O}(\alpha') , \\ \beta_{\mu\nu}^B &= \nabla_\lambda H_{\mu\nu}^\lambda - 2(\nabla_\lambda\Phi)H_{\mu\nu}^\lambda + \mathcal{O}(\alpha') ,\end{aligned}\tag{1}$$

where  $G_{\mu\nu}$  and  $B_{\mu\nu}$  are the symmetric and antisymmetric fields, respectively.  $R$  is the scalar curvature and  $R_{\mu\nu}$  is the Ricci tensor of the  $D$ -dimensional space-time.  $H_{\mu\nu\lambda}$  is the antisymmetric tensor-field strength derived from  $B_{\nu\lambda}$ . These equations were derived using the world-sheet background-field perturbation theory from the non-linear sigma-model action plus the renormalizable action for the dilaton field

$$S_{dilaton} = \frac{1}{4\pi} \int d\sigma d\tau \sqrt{\gamma} R^{(2)} \Phi(X) .\tag{2}$$

Here  $\gamma$  is the determinant of the 2-dimensional world-sheet metric,  $R^{(2)}$  is the scalar curvature of the world-sheet and  $\Phi(X)$  is the dilaton background field in the  $D$ -dimensional target space-time  $X$ . In the low-energy theory instead of strings one can deal with the background fields.

Following Álvarez and Gómez we will solve the equations of motion which come from the vanishing of the  $\beta$ -functions, Eq.(1). For simplicity we will turn off all the fields but the symmetric tensor and the dilaton. We will work in the critical dimension ( $D = 26$ ) for which the equations reduce to those derived from the action of gravity coupled to the dilaton. Using the Liouville ansatz [8, 10, 11] the equations of motion are trivially satisfied for any dilaton. In the metric we identify 4 of the 26 dimensions as the usual Euclidean (or Minkowski) space-time where the gauge theory is placed. One of the remaining 22-spatial coordinates is identified as the running-energy scale ( $\mu$ ) associated to the gauge field theories. In this prescription the gauge coupling is related to the dilaton by using the association<sup>4</sup>  $g_s = e^\Phi = g_{YM}^2$ . Since we know how the coupling constants run in gauge

---

<sup>4</sup>The second part of this relation is an assumption related to the soft-dilaton theorem [8].

theories through the  $\beta$ -functions, the above correspondence determines the  $\mu$ -dependence of the dilaton. This procedure is known as the Renormalization Group approach to the string theory. See for example [8, 12, 13, 14, 15] and references therein.

We describe some features of the space-time and calculate the Wilson loops. There are two different types of  $\beta$ -functions that we will work on. The first is the one-loop  $\beta$ -function of pure gluodynamics where the coupling constant diverges in the infrared. The second case is a  $\beta$ -function which changes sign through a pole in the infrared. For instance we will consider in particular the Novikov-Shifman-Vainstein-Zakharov (NSVZ) [16]  $\beta$ -function of  $\mathcal{N} = 1$  super-Yang-Mills theory, which has those properties. It was conjectured that because of this pole this theory has two phases [9]. One is the super-strongly coupled phase and the other one is the asymptotically-free phase, both of which flow to the infrared attractive point at some finite scale. Finally in the discussions we mention about the theories with conformal-fixed points at the finite values of the coupling constants.

In section 2 we study the properties of the background fields and the corresponding confining geometry derived from them. In section 3 the Wilson loops are studied by calculating the area of a fundamental string world-sheet in the background fields. In sections 4 and 5 we apply this framework to investigate the Yang-Mills type  $\beta$ -function and the  $\mathcal{N} = 1$  super-Yang-Mills type one, respectively.

## 2 The background fields

Let us consider the RG equations of the bosonic strings [6]. The vacuum configurations of string theory at one loop are determined by the sigma-model RG  $\beta$ -functions which, at leading order in  $\alpha'$ , read as

$$\beta_{\mu\nu}^G = R_{\mu\nu} + 2\nabla_\mu \nabla_\nu \Phi \ , \quad (3)$$

and

$$8\pi^2\beta^\Phi = \frac{D-26}{6} - \alpha' \nabla^2 \Phi + 2\alpha' (\nabla\Phi)^2 . \quad (4)$$

Here  $\nabla_\mu \Phi = \partial_\mu \Phi$  and  $\nabla_\mu \nabla_\nu \Phi = \partial_\mu \partial_\nu \Phi - \Gamma_{\mu\nu}^\alpha \partial_\alpha \Phi$ . Conformal invariance implies that  $\beta_{\mu\nu}^G = \beta^\Phi = 0$ . For all orders in  $\alpha'$ -expansion these  $\beta$ -functions were shown to vanish [17]. For the critical dimension these equations become

$$R_{\mu\nu} + 2\nabla_\mu \nabla_\nu \Phi = 0 \ , \quad (5)$$

and

$$\nabla^2\Phi - 2(\nabla\Phi)^2 = 0 . \quad (6)$$

As we have said before the four-dimensional space-time is embedded into 26 dimensions and one of the remaining 22-spatial coordinates is identified as the running-energy scale  $\mu$ . We will use the following form for the metric

$$ds^2 = a(\mu) (\pm dt^2 + d\vec{x}^2) + b(\mu) d\mu^2 + c(\mu) d\vec{y}^2 , \quad (7)$$

where the  $a(\mu)$  term is the Euclidean (Minkowski) metric in four dimensions,  $\mu$  is the fifth-coordinate, while  $d\vec{y}^2$  corresponds to an 21-dimensional hyper-plane. The Ricci tensor is computed to be

$$R_{ij} = - \left( \frac{a'b'}{4b^2} - \frac{21a'c'}{4bc} - \frac{a'^2}{2ab} - \frac{a''}{2b} \right) \eta_{ij} , \quad (8)$$

for  $i, j = 1, \dots, 4$ , where we choose the mostly plus signature for the Minkowski metric,  $\eta_{ij} = (-, +, +, +)$ ,

$$\begin{aligned} R_{55} &= -\frac{a'^2}{a^2} - \frac{a'b'}{ab} + \frac{2a''}{a} - \frac{21c'^2}{4c^2} - \frac{21b'c'}{4bc} + \frac{21c''}{2c} , \\ R_{\alpha\beta} &= \left( \frac{a'c'}{ab} - \frac{b'c'}{4b^2} + \frac{19c'^2}{4bc} + \frac{c''}{2b} \right) \delta_{\alpha\beta} , \end{aligned} \quad (9)$$

for  $\alpha, \beta = 6, \dots, 26$ , and the scalar curvature reads as

$$R = -\frac{2b'a'}{ab^2} - \frac{21b'c'}{2b^2c} + \frac{189c'^2}{2bc^2} + \frac{a'^2}{a^2b} + \frac{4a''}{ab} + \frac{42a'c'}{abc} + \frac{21c''}{bc} , \quad (10)$$

where prime denotes derivative with respect to  $\mu$ . We will use the following ansatz to solve the equations of motion

$$\begin{aligned} a(\mu) &= e^{2\Phi} , \\ b(\mu) &= 4e^{4\Phi}\Phi'^2 , \end{aligned} \quad (11)$$

and  $c(\mu) = 1$ . In this case the scalar curvature is  $R = -e^{-4\Phi} = -g_{YM}^{-8}$ , where we have used  $e^\Phi = g_{YM}^2$ . There is a naked singularity in the space-time which corresponds to the weakly-coupled regime of the gauge theory. On the other hand the strongly-coupled regime of the gauge theory corresponds to the weakly-curved spaces. This allows us to make a straightforward analysis of the geometry in terms of the quantum field theory  $\beta$ -functions.

It is important to mention that we did not excite the tachyon field here. In the general case the metric will be more complicated. The geometry in this framework is universal in the sense that the metric and the scalar curvature depend only on the coupling constant. Choosing different types of theories, as long as the coupling diverges at certain point and goes to zero at another one, corresponds to essentially choosing different coordinates in this geometry. However once the gauge theory is introduced on a 4-dimensional hyper-plane and the prescription of computing the Wilson loops in terms of the minimal surfaces is given, the flow of the coupling constant becomes important and therefore different theories, in principle, can behave differently. With respect to this, and as we already mentioned before in the introduction, the important fact here is that the geometry can distinguish between one-loop  $\beta$ -function and  $\beta$ -functions with a pole at some finite value of the running-energy scale.

### 3 The Wilson loop

In this section we discuss the calculation of the Wilson loops. First one should notice that the metric which we are dealing with can be trivially rewritten just by using the ansatz given in Eq.(11)

$$ds^2 = e^{2\Phi}(\pm dt^2 + dx_i dx_i) + 4l_c^2 e^{4\Phi} (d\Phi)^2 + d\vec{y}^2 \ , \quad (12)$$

where  $x_i$ ,  $i = 1, 2, 3$ , and  $l_c$  is an arbitrary scale of dimension of length. In this framework  $l_c$  is related to the running-energy scale of the field theory studied<sup>5</sup>.

Since we want to deal with Wilson loops placed in the 4-dimensional Euclidean (Minkowski) space-time, the extra 21 dimensions are irrelevant. Therefore this problem is equivalent to the problem of 5-dimensional gravity coupled to the dilaton.

Let us consider the Nambu-Goto action

$$S_{NG} = \frac{1}{2\pi l_s^2} \int d\sigma d\tau \sqrt{\det G_{MN} \partial_\alpha X^M \partial_\beta X^N} \ , \quad (13)$$

where  $X^M$  is a generic coordinate on the 5-dimensional space-time. In the static configuration, for  $\tau = t$  and  $\sigma = x$  we have

$$ds^2 = \pm e^{2\Phi} d\tau^2 + (e^{2\Phi} + 4l_c^2 e^{4\Phi} \Phi_\sigma^2) d\sigma^2 \ , \quad (14)$$

---

<sup>5</sup> Moreover one can also define  $\rho = e^{2\Phi}$  and rewrite the above metric in the Liouville form  $ds^2 = \rho(\pm dt^2 + dx_i dx_i) + l_c^2 (d\rho)^2 + d\vec{y}^2$ . In this parameterization  $\rho = \infty$  is the horizon and  $\rho=0$  is the singularity. Therefore this space-time allows the description of zigzag invariant Wilson loops [18].

where  $\Phi_\sigma = \frac{\partial\Phi}{\partial\sigma}$ . Therefore the action is

$$S_{NG} = \frac{T}{2\pi l_s^2} \int_0^L d\sigma e^{2\Phi} \sqrt{1 + 4l_s^2 e^{2\Phi} \Phi_\sigma^2} , \quad (15)$$

where  $T$  is the time. Basically we are considering the configurations as it is shown in figure 1.

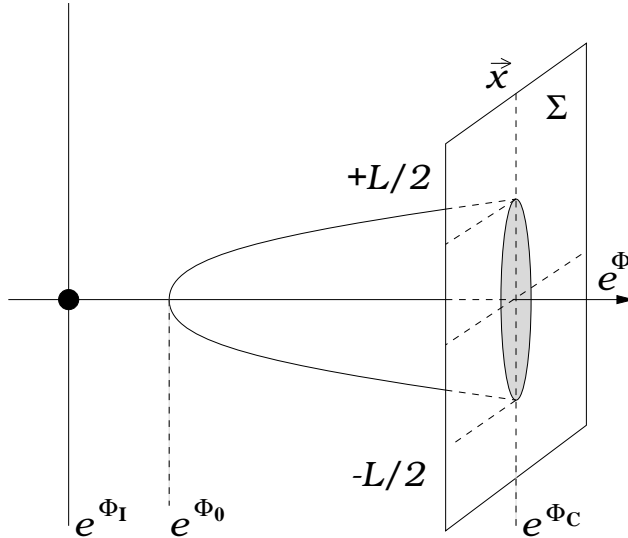


Figure 1: Schematic representation of the prescription of calculating the Wilson loops in terms of Liouville coordinates.

In this figure we show the shape of the string world-sheet in terms of the Liouville field  $e^\Phi$ . In particular  $e^{\Phi_I}$  is the limit when  $\Phi \rightarrow -\infty$ . Since the scalar curvature  $R$  goes like  $e^{-4\Phi_I}$  it implies that at this point the metric has a naked singularity (the scalar curvature blows up). The point at the origin represents the naked singularity. However in terms of the running-energy scale  $\mu$  the location of this singularity depends on the particular field-theory  $\beta$ -function. We can put the 4-dimensional hyper-plane  $\Sigma$  at any point which corresponds to choosing a particular value of the coupling constant. Here we denote a generic choice by  $e^{\Phi_c}$ . Once the hyper-surface is chosen the string can fluctuate in either of the directions orthogonal to  $\Sigma$ . However the minimal surfaces come from the strings that fluctuate in the direction of the naked singularity as it is depicted in the figure. In terms of the gauge theory language this means that semi-classically we allow

the RG flow from strong coupling to the weak coupling regime. On the other hand, if an infrared attractive point occurs, the RG flow will be from the ultraviolet limit to the infrared one <sup>6</sup>. We will see that this is what happens for  $\mathcal{N} = 1$  super-Yang-Mills theory.

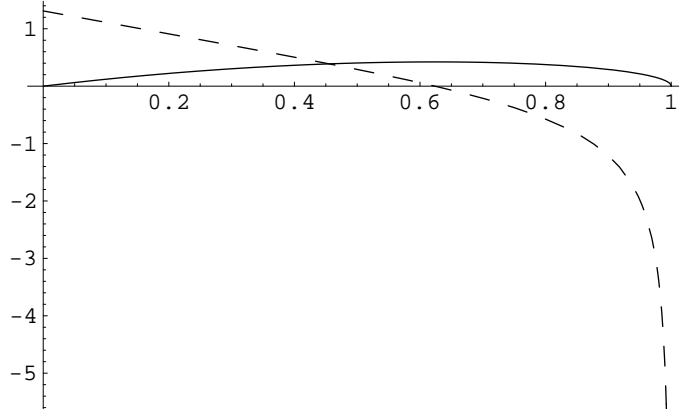


Figure 2: The solid line indicates the scaled Wilson-loop size  $L/e^{\Phi_c}$  (in units of  $l_c$ ) in terms of the dimensionless variable  $e^{\Phi_0}/e^{\Phi_c}$ . The dashed curve shows the derivative of the scaled Wilson-loop size.

Taking the vertical axis as the  $\vec{x}$ -direction we will consider a Wilson-loop of size  $L$ . This means that  $L$  is the static separation of the very heavy ”*quark-anti-quark pair*”. In the symmetric configuration  $\vec{x} = 0$  corresponds to a minimum of  $e^\Phi$ , let us call it  $e^{\Phi_0}$ , and since the dilaton is a function of  $\mu$  we will call  $\Phi_0 = \Phi(\mu_0)$ . For classical solutions one gets

$$\frac{e^{4\Phi}}{1 + 4l_c^2 e^{2\Phi} \Phi_\sigma^2} = e^{4\Phi_0} . \quad (16)$$

By inverting this expression

$$\frac{L}{2} = 2l_c e^{\Phi_0} \int_1^{e^{\Phi_c}/e^{\Phi_0}} \frac{1}{\sqrt{v^4 - 1}} dv , \quad (17)$$

where  $v = \frac{e^\Phi}{e^{\Phi_0}}$ . After integration we obtain

$$L = l_c e^{\Phi_0} \left( \sqrt{\pi} \frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})} - 4 \frac{{}_2F_1(\frac{1}{4}, \frac{1}{2}; \frac{5}{4}; e^{4(\Phi_0 - \Phi_c)})}{e^{\Phi_c - \Phi_0}} \right) , \quad (18)$$

where  ${}_2F_1(\frac{1}{4}, \frac{1}{2}; \frac{5}{4}; e^{4(\Phi_0 - \Phi_c)})$  is a hypergeometric function [19].  $L$  is a function of  $e^{\Phi_c}$  and  $e^{\Phi_0}$ . It is convenient to study the function  $L/e^{\Phi_c}$  in terms of  $e^{\Phi_0}/e^{\Phi_c}$ . It has a

<sup>6</sup>Observe that if one considers the Liouville coordinate to be time-like with a metric of the form  $ds^2 = \rho (dt^2 + dx_i dx_i) - l_c^2 (d\rho)^2$  the direction of RG flow is from weak coupling to strong coupling.



maximum in the interval between 0 and 1 which indicates that there are two regions (I and II) to consider when one calculates the Wilson loops [8]. For the interval below the maximum in the  $e^{\Phi_0}/e^{\Phi_c}$ -axis, region I, one considers large world-sheets. In figure 2 we plot  $L/e^{\Phi_c}$  in terms of  $e^{\Phi_0}/e^{\Phi_c}$ . There is a maximum at  $e^{\Phi_0^M}/e^{\Phi_c} \approx 0.62$  and at this point  $L/e^{\Phi_c}|_M = 0.42 l_c$ .

Let us calculate the energy for the static configuration. The Nambu-Goto action and the corresponding energy are related by

$$E = \frac{S_{NG}}{T} = \int \mathcal{L} d\sigma = \int \mathcal{L} \left( \frac{dv}{d\sigma} \right)^{-1} dv , \quad (19)$$

where the lagrangian is given by

$$\mathcal{L} = \frac{e^{2\Phi}}{2\pi l_s^2} \sqrt{1 + 4l_c^2 e^{2\Phi} \Phi_\sigma^2} = \frac{1}{2\pi l_s^2} e^{2\Phi_0} v^4 . \quad (20)$$

Then the integral becomes

$$E = \frac{2l_c}{\pi l_s^2} e^{3\Phi_0} \int_1^{e^{\Phi_c}/e^{\Phi_0}} \frac{v^4}{\sqrt{v^4 - 1}} dv , \quad (21)$$

which can be integrated as

$$E = \frac{2l_c}{\pi l_s^2} e^{3\Phi_0} \left( \frac{\sqrt{\pi}}{12} \frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})} + \frac{1}{3} e^{3(\Phi_c - \Phi_0)} {}_2F_1 \left( \frac{1}{2}, \frac{-3}{4}; \frac{1}{4}; e^{4(\Phi_0 - \Phi_c)} \right) \right) . \quad (22)$$

Expanding Eqs.(18) and (22) in series of powers of  $e^{\Phi_0}/e^{\Phi_c}$  we obtain

$$\frac{L}{l_c} = \sqrt{\pi} e^{\Phi_0} \frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})} - 4 e^{2\Phi_0 - \Phi_c} + \mathcal{O}(e^{6\Phi_0 - 5\Phi_c}) , \quad (23)$$

and

$$\frac{\pi l_s^2 E}{2l_c} = \frac{e^{3\Phi_c}}{3} + \frac{\sqrt{\pi}}{12} \frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})} e^{3\Phi_0} - \frac{1}{2} e^{4\Phi_0 - \Phi_c} + \mathcal{O}(e^{8\Phi_0 - 5\Phi_c}) . \quad (24)$$

Replacing  $L$  in terms of  $e^{\Phi_0}$  in Eq.(24) we get an expression which shows that there is over-confining.

$$E = \frac{1}{6\pi^2 l_s^2 l_c^2} \left( \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} \right)^2 L^3 + \frac{2}{3\pi} \frac{l_c}{l_s^2} e^{3\Phi_c} . \quad (25)$$

In the limit of extremely strong coupling, which corresponds to  $e^{\Phi_c} \rightarrow \infty$ , the above expression becomes divergent. However the divergent part is independent of the size of the Wilson loop. In the context of AdS/CFT duality this kind of divergence was regularized by a mass renormalization [20] or by considering the Legendre transform of the minimal

area [21]. In our case observing that zero-size Wilson loops diverge we will drop the  $L$ -independent divergent term and measure the energy with respect to the zero-size Wilson loops. In the extremely strong coupling limit  $L$  becomes

$$L_\infty = \sqrt{\pi} l_c e^{\Phi_0} \frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})} , \quad (26)$$

while the energy for the static configuration reads as follows

$$E_\infty = \frac{l_c e^{3\Phi_0}}{6\sqrt{\pi} l_s^2} \frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})} , \quad (27)$$

so that the over-confining is shown by the relation

$$E_\infty = \frac{1}{6\pi^2 l_s^2 l_c^2} \frac{\Gamma(\frac{3}{4})^2}{\Gamma(\frac{1}{4})^2} L_\infty^3 . \quad (28)$$

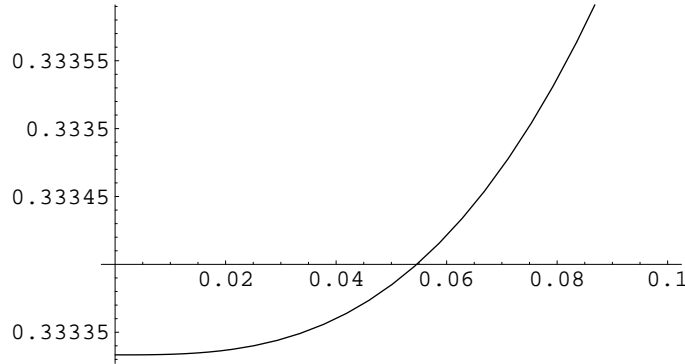


Figure 3: Energy (in units of  $\frac{l_c^2}{l_s^2}$ ) for the static configuration as a function of  $\frac{e^{\Phi_0}}{e^{\Phi_c}}$ , in a small interval around 0.

In figure 3 we plot the energy versus  $e^{\Phi_0}/e^{\Phi_c}$  in a small interval around zero (region I). It shows the cubic behaviour of the potential. A quite different situation arises when the region II is analyzed. In this region  $e^{\Phi_0}$  is close to  $e^{\Phi_c}$ . In such a case we have to do the integration between 1 and  $1 + \epsilon$  so that we have

$$\frac{L_\epsilon}{2} = 2l_c e^{\Phi_0} \left( \sqrt{\pi} \frac{\Gamma(\frac{5}{4})}{\Gamma(\frac{3}{4})} - (1 + \epsilon) {}_2F_1 \left( \frac{1}{4}, \frac{1}{2}; \frac{5}{4}; \frac{1}{(1 + \epsilon)^4} \right) \right) = 2l_c e^{\Phi_0} \left( \sqrt{\epsilon} + \mathcal{O}(\epsilon^{\frac{3}{2}}) \right) , \quad (29)$$

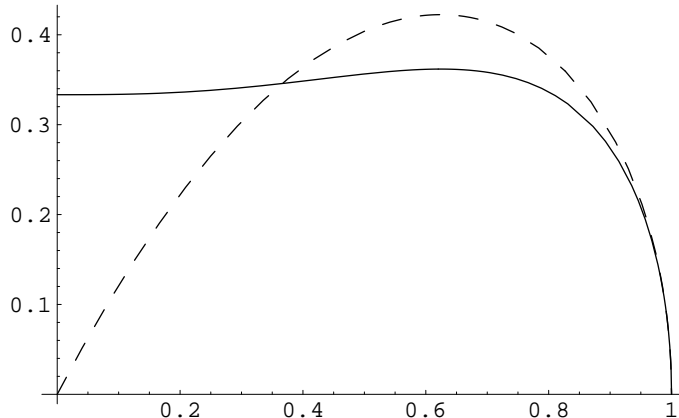
and

$$\begin{aligned} E_\epsilon &= \frac{2l_c}{\pi l_s^2} e^{3\Phi_0} \left( \frac{\sqrt{\pi}}{12} \frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})} + \frac{(1 + \epsilon)^3}{3} {}_2F_1 \left( \frac{1}{2}, \frac{-3}{4}; \frac{1}{4}; \frac{1}{(1 + \epsilon)^4} \right) \right) \\ &= \frac{2l_c}{\pi l_s^2} e^{3\Phi_0} \left( \sqrt{\epsilon} + \mathcal{O}(\epsilon^{\frac{3}{2}}) \right) . \end{aligned} \quad (30)$$

At order  $\sqrt{\epsilon}$  it gives

$$E_\epsilon = \frac{e^{2\Phi_0}}{2\pi l_s^2} L_\epsilon \ , \quad (31)$$

indicating confinement at region II. Figure 4 shows the whole picture of  $\frac{E}{e^{3\Phi_c}}$  (solid line) and  $\frac{L}{e^{\Phi_c}}$  (dashed line), as functions of  $e^{\Phi_0}/e^{\Phi_c}$ . In the region II we observe the area law ( $S \approx LT$ ).



pt

Figure 4:  $\frac{E}{e^{3\Phi_c}}$  (solid line) and  $\frac{L}{e^{\Phi_c}}$  (dashed line), as functions of  $e^{\Phi_0}/e^{\Phi_c}$ .

## 4 One-loop $\beta$ -function

In this section we will study the one-loop  $\beta$ -function of pure gluodynamics and compute the Wilson loops in this theory. This case was studied in [8]. The  $\beta$ -function is

$$\mu \frac{dg}{d\mu} = -\frac{11N}{24\pi^2} g^3 \ . \quad (32)$$

The dilaton field is

$$\Phi(\mu) = -\log \log \left( \frac{\mu}{\Lambda} \right) - \log \left( \frac{11N}{24\pi^2} \right) \ , \quad (33)$$

where  $\Lambda$  is a renormalization scale. The solutions for the metric are

$$a(\mu) = \frac{576\pi^4}{121N^2} \frac{1}{\log^2(\mu/\Lambda)} \ , \quad (34)$$

and

$$b(\mu) = \frac{3^4 2^{14} \pi^8}{(11N)^4} \frac{1}{\mu^2 \log^6(\frac{\mu}{\Lambda})} . \quad (35)$$

The curvature reads as

$$R(\mu) = - \left( \frac{11}{3} \right)^4 \frac{N^4}{2^{12} \pi^8} \log^4(\mu/\Lambda) . \quad (36)$$

Figure 5 shows a picture for the Yang-Mills coupling constant in terms of the running energy scale  $\mu$ .

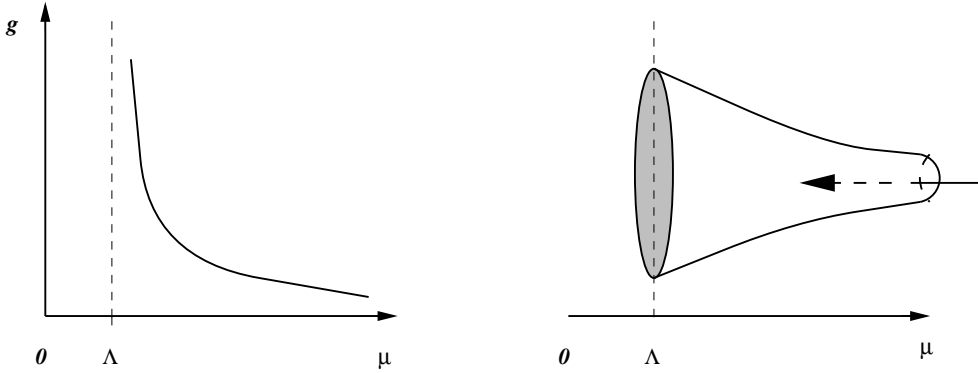


Figure 5: Coupling constant for pure gluodynamics as a function of the running-energy scale  $\mu$ . The picture on the right shows the qualitative behaviour of the four-dimensional space-time.

The picture on the right shows the behavior of the space-time radius (inverse of the curvature) as  $\mu$  approaches to the IR limit. At  $\mu = \Lambda$  the space-time becomes flat. In the weak coupling there is a naked singularity.

We will use the previous analysis for the Wilson loop calculation. In figure 6 we show the shape of the Wilson loop. The 4-dimensional hyper-plane  $\Sigma$  where the Wilson loop is drawn can be placed at any value of  $\mu$ , let us say  $\mu_c$ , while the minimum  $e^{\Phi_0}$  corresponds to  $\mu_0$ . The naked singularity, represented by a dot, is the point at infinity and it is labeled as  $\mu_I$ . The corresponding point to the maximum in figure 4, *i.e.*  $\frac{e^{\Phi_0^M}}{e^{\Phi_c}}$ , is for this case  $\mu_0^M \approx \Lambda \left( \frac{\mu_c}{\Lambda} \right)^{1.61}$ . For larger world-sheets it leads to over-confinement

$$E = \frac{1}{6\pi^2 l_s^2 l_c^2} \left( \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} \right)^2 L^3 . \quad (37)$$

Note that we have dropped the divergent part. While for smaller world-sheets we get

$$E_\epsilon = \frac{288\pi^3}{121N^2 l_s^2 \log^2(\frac{\mu_0}{\Lambda})} L_\epsilon \ , \quad (38)$$

indicating the confinement in region II. The factor in front of  $L_\epsilon$  can be interpreted as an effective tension.

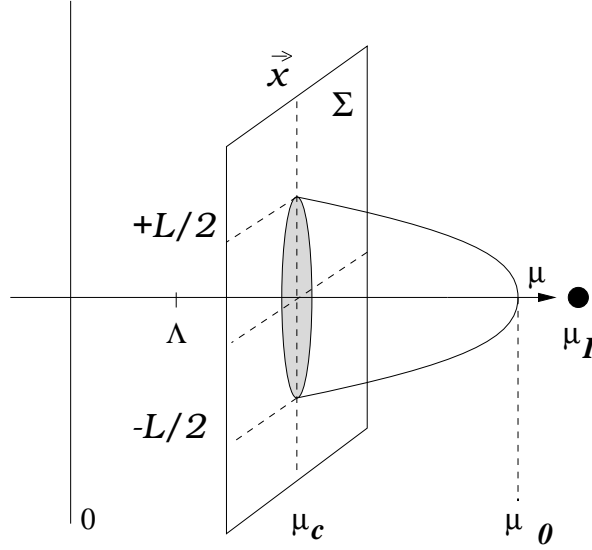


Figure 6: Schematic representation of the Wilson loop calculation for pure gluodynamics.

## 5 $\beta$ -function with a pole

On the other hand, for a  $\beta$ -function with a pole at some finite value of the coupling the situation is quite different. For instance let us consider the  $\mathcal{N} = 1$  super-Yang-Mills theory. In this case we will study NSVZ  $\beta$ -function [16]

$$\beta(g) = \mu \frac{dg}{d\mu} = \frac{g^3}{16\pi^2} \frac{(\mathcal{N} - 4)N}{\left(1 - \frac{(2-\mathcal{N})Ng^2}{8\pi^2}\right)} \ , \quad (39)$$

where  $N$  is the corresponding label to the gauge group  $SU(N)$  while  $\mathcal{N}$  labels the number of super-symmetries. By integrating this equation one gets a transcendental equation

$$\frac{\mu}{\Lambda} = e^{\frac{8\pi^2}{(4-\mathcal{N})Ng^2}} \left(\frac{g^2}{4\pi}\right)^{\frac{2-\mathcal{N}}{4-\mathcal{N}}} \ . \quad (40)$$

For  $\mathcal{N} = 1$  super-Yang-Mills theory the  $\beta$ -function is exact both perturbatively and non-perturbatively and it reads

$$\beta(g) = \mu \frac{dg}{d\mu} = -\frac{3}{16\pi^2} \frac{g^3 N}{\left(1 - \frac{Ng^2}{8\pi^2}\right)} . \quad (41)$$

This  $\beta$ -function has a pole at  $g^2 = 8\pi^2/N$  and it changes sign through the pole. As it is shown in figure 7 the coupling constant is a double-valued function of  $\mu$ . The pole is an infrared attractive point.

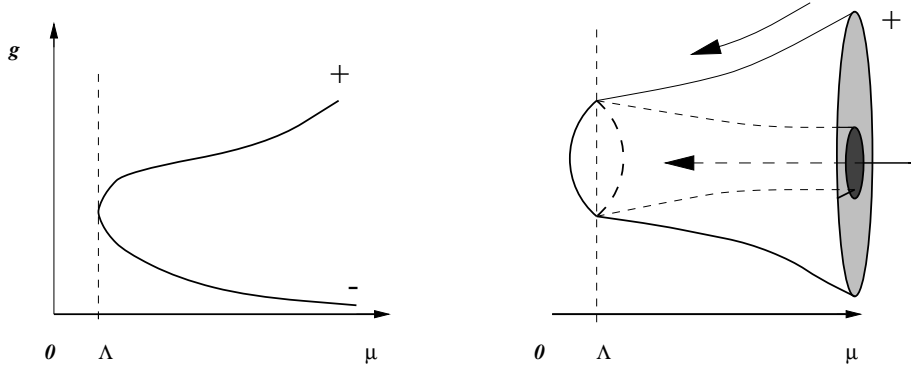


Figure 7: Coupling constant for  $\mathcal{N} = 1$  super-Yang-Mills theory as a function of the running-energy scale  $\mu$ . The picture on the right shows the qualitative behaviour of the four-dimensional space-time.

The theory can flow, both from the asymptotically-free phase where the coupling is small at large  $\mu$  (lower branch (-)), and from the super-strongly coupled phase where the coupling is large (upper branch (+)), to the infrared attractive point. Since we can not invert the transcendental equation (40) we will analyze the behaviour around the infrared point  $\Lambda$ . By expanding the Eq.(40) around  $\Lambda$  one gets

$$g_{\pm}^2 = \frac{8\pi^2}{N} \left( 1 \pm \sqrt{3 \frac{\mu - \Lambda}{\Lambda}} \right) . \quad (42)$$

Since  $\Phi = \log(g^2)$  it follows that

$$\Phi_{\pm} = \log(8\pi^2/N) + \log(1 \pm \zeta) , \quad (43)$$

where  $\zeta = \sqrt{3(\mu - \Lambda)/\Lambda}$ . The solutions are

$$\begin{aligned} a_{\pm}(\mu) &= \frac{64\pi^4}{N^2}(1 \pm \zeta)^2 , \\ b_{\pm}(\mu) &= 9\frac{2^{12}\pi^8}{N^4\Lambda^2}\frac{(1 \pm \zeta)^2}{\zeta^2} . \end{aligned} \quad (44)$$

We set  $c_{\pm}(\mu) = 1$ . The scalar curvature is given by

$$R_{\pm} = -\frac{N^4}{2^{12}\pi^8(1 \pm \zeta)^4} \quad \zeta \ll 1 . \quad (45)$$

In figure 8 we show the picture of the Wilson loop for the present case. Notice that we use the same coordinates as in figure 1. In these coordinates the picture becomes more clear.

The vertical line at the point  $e^{\Phi_{\Lambda}}$  represents the infrared attractive point. Here it acts like an effective horizon. This is because in the super-strongly coupled phase the direction of the RG flow is from strong coupling to the weak coupling, however since we have an infrared attractive point we can not continue the flow to the smaller couplings. On the other hand in the asymptotically-free phase the flow is from weak coupling to the strong coupling. The singularity placed at the origin corresponds to the limit of weak coupling shown in the lower-branch of figure 7. Again confinement and over-confinement have to be understood as properties of the world-sheet size. Close to effective horizon, for both branches we have

$$E_{\epsilon\pm} = \frac{32\pi^3 \left(1 \pm \sqrt{3(\mu_0 - \Lambda)/\Lambda}\right)^2}{N^2 l_s^2} L_{\epsilon} . \quad (46)$$

In figure 8 the left side of the vertical line represents the weakly-coupled phase and the right side represents the super-strongly coupled phase.

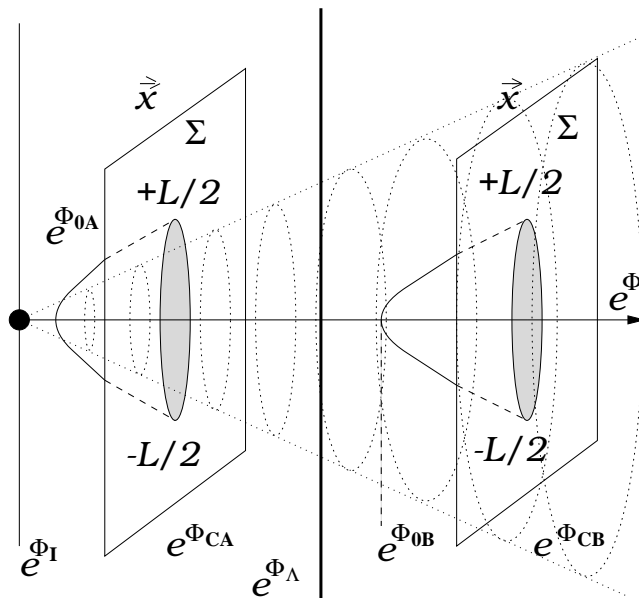


Figure 8: Schematic representation of the Wilson loop for  $\mathcal{N} = 1$  super Yang-Mills theory. The dashed ellipses represent the behaviour of the 4-dimensional space-time as it has seen in the right picture of figure 8.

When one studies the theory in the super-strongly coupled phase and sufficiently close to the attractive point one always gets the area law for the Wilson loops. In fact there is a critical coupling  $g_{critical} = 1.27 g_{\Lambda}$ . For all values of the coupling between  $g_{critical}$  and  $g_{\Lambda}$  we have the area law.

## 6 Discussion and conclusions

In this paper, using the RG approach to string theory, we studied two types of confining theories. As it was shown by Álvarez and Gómez [8] a theory which has a  $\beta$ -function with a zero at UV-limit is confining or over-confining, depending on the minimal surfaces we use. We have considered the theories which have  $\beta$ -functions with a pole at some finite value of the coupling. For example  $\mathcal{N} = 1$  super-Yang-Mills theory is of this form. It has two phases, the super-strongly coupled phase and the asymptotically-free phase which flow to an infrared attractive point. In this theory one has also both confinement and over-confinement. We showed that in the super-strongly coupled regime of the theory there is a critical value below which one has confinement only. In this regime large world-sheets are not allowed and one does not have over-confinement. It is interesting to stress



that this fact is due to the existence of a pole at some finite scale in the  $\beta$ -function. This leads to a new scale  $g_{critical} = 1.27 g_\Lambda$  below which these theories satisfy the area law for their Wilson loops.

Concerning to the flow of the space-time for the one-loop  $\beta$ -function we have seen that it goes from the strong coupling to the weak coupling regime. In the case of  $\beta$ -functions with a pole the situation changes in the sense that now the RG flow goes from the UV limit to the IR limit, in both branches.

The geometry we have considered is universal in the sense that it leads to confinement. So presumably this geometry is not suitable for conformal theories and Abelian theories. For the conformal case we know that the space-time should be anti-de Sitter. However let us assume that, by using the metric in this paper, we can describe theories with conformal-fixed points. The  $\beta$ -function changes sign through zero instead of a pole but the geometry and the analysis of the Wilson loops will be similar to the case of  $\beta$ -functions with a pole which was discussed in section 5. Although this is surprising, an argument given by Damgaard and Haangensen [22] may help one to understand this. They showed a theory with a self-dual point is either conformal at the self-dual point or its  $\beta$ -function has a pole at this point. Here we reproduce their argument. Let us consider a theory with a coupling  $g$ . If this theory is self-dual there is a relation between  $g$  and the coupling of its dual theory  $g^*$

$$g^* = f(g) . \quad (47)$$

The only essential requirement is that the interaction part in the action and its dual have to be of the same form. The map in Eq.(47) is assumed to be *one-to-one* and  $f \cdot f = 1$ , which implies that its derivative  $\frac{\partial f}{\partial g}$  is negative. We also assumed that  $g$  and  $g^*$  are positive-valued functions. At the self-dual point, let us call it  $g_{self}$  we have

$$g_{self} = f(g_{self}) . \quad (48)$$

On the other hand we know that the RG flow is dictated by the  $\beta$ -function

$$\beta(g) = \mu \frac{\partial g}{\partial \mu} , \quad (49)$$

by applying the operator  $\mu \frac{\partial}{\partial \mu}$  to the map of Eq.(47) we obtain a consistency relation

$$\mu \frac{\partial g^*}{\partial \mu} = \mu \frac{\partial g}{\partial \mu} \frac{\partial g^*}{\partial g} = \beta(g) \frac{\partial f}{\partial g} , \quad (50)$$

if one identifies the left hand side of the above equation with the  $\beta$ -function for the dual theory one obtains

$$\beta(g^*) = \beta(g) \frac{\partial f}{\partial g} . \quad (51)$$

From the condition  $f \cdot f = 1$  at the self-dual point results that  $\frac{\partial f}{\partial g} = -1$ , and therefore there are two solutions of Eq.(50), *i.e.* the  $\beta$ -function can be zero at this point or it is discontinuous. The most natural way to realize the discontinuity is through a pole in the  $\beta$ -function. Then one has at the self-dual point  $\beta(g_{self} + \epsilon) = -\beta(g_{self} - \epsilon)$ .

Finally we would like to speculate that when one deforms  $\mathcal{N} = 4$  theory down to  $\mathcal{N} = 1$  to obtain a dual description in terms of string theory or gravity, a prescription which works only in the strong coupling regime of the gauge theory, one might be dealing with the super-strongly coupled regime, the upper branch, of  $\mathcal{N} = 1$  theory.

## Acknowledgments

We benefited from useful discussions with Alex Kovner. M.S. also acknowledges discussions with Gastón Giribet, Esteban Moro and Carlos Núñez, and the organizers of the Spring Workshop on Superstrings and Related Matters (2000) at the Abdus Salam ICTP for kind hospitality where a part of this work was carried out. The work of I.K. and B.T. was supported by PPARC Grant PPA/G/O/1998/00567. The work of M.S. was supported by the CONICET of Argentina, the Fundación Antorchas of Argentina and The British Council.

## References

- [1] J. Maldacena, “*The large  $N$  limit of super-conformal field theories and super-gravity*”, Adv. Theor. Math. Phys. **2** (1998) 231, hep-th/9711200.
- [2] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, “*Gauge theory correlators from non-critical string theory*”, Nucl. Phys. **B 526** (1998) 393, hep-th/9802109.
- [3] E. Witten, “*Anti de Sitter space and holography*”, Adv. Theor. Math. Phys. **2** (1998) 253, hep-th/9802150.
- [4] O. Aharony, S.S. Gubser, J. Maldacena, H. Ooguri, Y. Oz, “*Large  $N$  field theories, string theory and gravity*”, Phys. Rept. **323** (2000) 183, hep-th/9905111.
- [5] G. 't Hooft, “*A planar diagram theory for strong interactions*”, Nucl. Phys. **B 72** (1974) 461.
- [6] C. G. Callan, D. Friedan, E. J. Martinec, M. J. Perry, “*Strings in background fields*”, Nucl. Phys. **B 262** (1985) 593.  
E.S. Fradkin, A.A. Tseytlin, “*Quantum string theory effective action*”, Nucl. Phys. **B 261** (1985) 1.
- [7] A.M. Polyakov, “*Confining strings*”, Nucl. Phys. **B 486** (1997) 23, hep-th/9607049.
- [8] E. Álvarez, C. Gómez, “*On the string description of confinement*”, hep-th/0003205.  
E. Álvarez, C. Gómez, “*The renormalization group approach to confining string*”, hep-th/9911215.  
E. Álvarez, C. Gómez, “*The confining strings from the soft-dilaton theorem*”, Nucl. Phys. **B 566** (2000) 363, hep-th/9907158.  
E. Álvarez, C. Gómez, “*Geometric holography, the renormalization group and the  $c$ -theorem*”, Nucl. Phys. **B 541** (1999) 441, hep-th/9807226.
- [9] I.I. Kogan, M. Shifman, “*Two phases of super-symmetric gluodynamics*”, Phys. Rev. Lett. **75** (1995) 2085.
- [10] A.M. Polyakov, “*String theory and quark confinement*”, Proceedings of ”Strings 97”, hep-th/9711002.

- [11] A.M. Polyakov, “*The wall of the Cave*”, J. Mod. Phys. **A 14** (1999) 645, hep-th/9809057.
- [12] E.T.Akhmedov, “*A remark on the AdS/CFT correspondence and the renormalization group flow*”, Phys. Lett. **B 442** (1998) 152, hep-th/9806217.
- [13] V. Balasubramanian, P. Kraus, “*Space-time and holographic renormalization group*”, Phys. Rev. Lett. **83** (1999) 3605, hep-th/9903190.
- [14] D.Z. Freedman, S.S. Gubser, K. Pilch, N.P. Wagner, “*Renormalization group flows from holography-supersymmetry and a c-theorem*”, hep-th/9904017.
- [15] V. Sahakian, “*Holography, a covariant c-function, and the geometry of the renormalization group*”, hep-th/9910099.
- [16] V.A. Novikov, M. Shifman, A.I. Vainshtein, V.I. Zakharov, “*Exact Gell-Mann-Low function of super-symmetric Yang-Mills theories from instanton calculus*”, Nucl. Phys. **B229** (1983) 381.
- [17] G. Curci and G. Paffuti, “*Consistency between the string background field equation of motion and the vanishing of the conformal anomaly*”, Nucl. Phys. **B 286** (1987) 399.
- [18] A. Polyakov and V. Rychkov, “*Gauge fields-string duality and the loop equation*”, hep-th/0002106.
- [19] M. Abramowitz and I.A. Stegun, “*Handbook of Mathematical Functions*”, Dover Publications, NY, (1970).
- [20] J. Maldacena, “*Wilson loops in large N field theories*”, Phys. Rev. Lett. **80** (1998) 4859, hep-th/9803002.
- [21] N. Drukker, D.J. Gross and H. Ooguri, “*Wilson loops and minimal surfaces*”, Phys. Rev. **D 60** (1999) 125006, hep-th/9904191.
- [22] P.H. Damgaard, P.E. Haagensen, “*Constraints on beta functions from duality*”, J. Phys. **A 30** (1997) 4681, cond-mat/9609242.