Weighted homogeneous singularities and rational homology disk smoothings

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Abstract We classify the resolution graphs of weighted homogeneous surface singularities which admit rational homology disk smoothings. The nonexistence of rational homology disk smoothings is shown by symplectic geometric methods, while the existence is verified via smoothings of negative weight.

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1 Introduction

Suppose that S is the germ of an isolated normal complex surface singularity. For hypersurface and complete intersection singularities, there are natural smoothings (i.e., deformations with smooth generic fibre) given by the defining functions, and their properties have been known for a long time: such a smoothing is topologically a bouquet of 2–spheres. But in general it is not clear whether smoothings of S exist, or, if they do, what their basic topological properties are. It would be natural to try to understand those singularities which possess a smoothing with the 'simplest' possible topology. We say that a smoothing is a *rational homology disk* ($\mathbb{Q}$HD for short) if the underlying smooth 4-manifold has rational homology groups isomorphic to $H_*(D^4;\mathbb{Q})$, where D^4 denotes the 4-dimensional disk. Strong constraints are imposed for a singularity to admit a $\mathbb{Q}$HD smoothing — it is necessarily a *rational* surface singularity, implying among other things that the resolution graph of S must be a (negative definite) *tree*, and the link of S a rational homology sphere. Examples of singularities with $\mathbb{Q}$HD smoothings already appeared in [17]. The $\frac{p^2}{pq-1}$ cyclic quotient singularities ($0<q<p, (p,q)=1$) provide a complete list of cyclic quotients with this property, and [17] also contained some further examples (with resolution graphs given by Figure 1(a)). In fact, throughout the years, a list of such exam-

ples was compiled by J. Wahl, which was known to the experts (cf. the remark in [5, bottom of page 505]) but did not appear in print.

The smooth 4-manifold-theoretic application of certain singularities with $\mathbb{Q}$HD smoothings, through the rational blow-down procedure (introduced by Fintushel and Stern [3] and extended by Park [13]), have put the study of singularities with $\mathbb{Q}$HD smoothings at the forefront of 4-dimensional topology. In [16] a systematic investigation of the resolution graphs of such singularities was initiated, and (relying on Donaldson's famous Theorem A, and some further observations) strong combinatorial constraints have been found for a (negative definite) plumbing tree to be the resolution graph of a singularity admitting a $\mathbb{Q}$HD smoothing. Although [16] did not aim to provide a complete classification of singularities with $\mathbb{Q}$HD smoothings, the examples given there (in hindsight) provided a nearly complete list of weighted homogeneous singularities with $\mathbb{Q}$HD smoothings (the only missing examples from [16] are the ones corresponding to the graphs of Figures 1(h) and (i), which were also known to the authors of [16] to admit $\mathbb{Q}$HD smoothings).

In the present work — resting on results of [16] and on some fundamental theorems in symplectic geometry — we give a complete classification of the resolution graphs of weighted homogeneous singularities admitting $\mathbb{Q}$HD smoothings. Surprisingly enough, the complete list of resolution graphs of weighted homogeneous singularities with $\mathbb{Q}$HD smoothings essentially coincides with the list of examples of Wahl mentioned above. In order to state our results precisely, we need a few preliminary notions and definitions.

The link Y_Γ of a singularity S_Γ with resolution graph Γ is determined by the graph Γ, and according to [2] the 3-manifold Y_Γ admits a (up to contactomorphism) unique contact structure, its *Milnor fillable contact structure* ξ_Γ, given by the 2-plane field of complex tangencies on Y_Γ as a link of S_Γ. Any smoothing of the singularity S_Γ naturally provides a Stein filling of the Milnor fillable contact 3-manifold (Y_Γ, ξ_Γ). (For the definition of various notions of fillings of contact 3-manifolds, see [12, Section 12.1].)

Definition 1.1 We call a normal complex surface singularity S_Γ *spherical Seifert* if the link of the singularity is a Seifert fibred 3-manifold over the sphere S^2. The spherical Seifert singularity S_Γ is *small Seifert* if the link is a small Seifert fibred 3-manifold, i.e., it admits a Seifert fibration over S^2 with exactly three singular fibres.

A normal surface singularity is therefore spherical Seifert if and only if it admits a resolution graph which is a star-shaped tree and the vertices correspond to rational curves; in addition, S_Γ is small Seifert if the central vertex (the unique vertex of valency > 2) in a minimal good resolution is of valency 3. By [11, Theorem 2.6.1], weighted homogeneous singularities with rational homology sphere links are all spherical Seifert singularities (but the converse does not hold). For a definition of weighted homogeneous singularities (also called quasi–homogeneous, or singularities with a good $\mathbb{C}^*$–action) see, for example, [11, p. 206].

Definition 1.2 Define $\mathcal{QHD}^3$ as the set of graphs given by Figure 1.

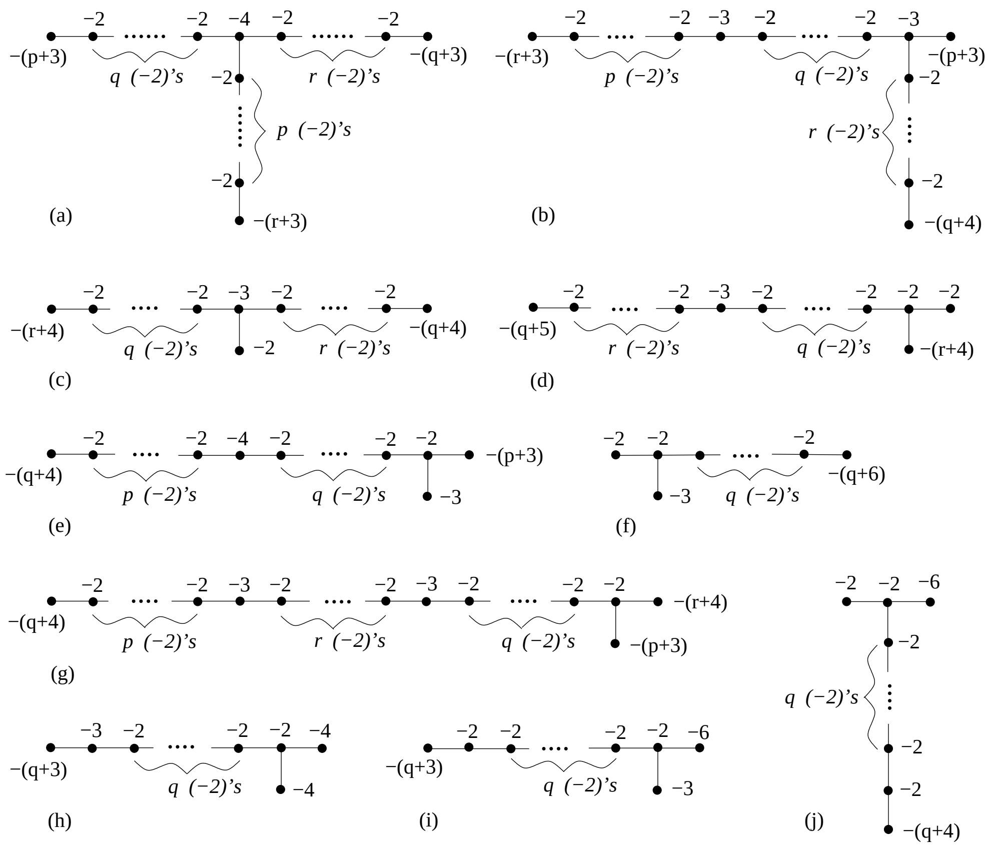

Figure 1: The graphs defining the class $\mathcal{QHD}^3$ of plumbing graphs. We assume that $p, q, r \geq 0$.

Remark 1.3 The graphs given in Figure 1(a) form the set $\mathcal{W}$ of [16], those in Figures 1(b) and (c) form $\mathcal{N}$, while the collection of (d), (e), (f) and (g) were called $\mathcal{M}$ in [16]. The graphs of Figure 1(h) provide the 1-parameter family $\mathcal{B}^3_2$ of certain star-shaped graphs with three legs in the class $\mathcal{B}$ of [16], and the ones of the form (i) and (j) are two 1-parameter families $\mathcal{C}^3_2$ and $\mathcal{C}^3_3$ in $\mathcal{C}$. (For the definition of the classes $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ of graphs see Subsection 2.2. The superscript in the notation is intended to indicate the number of legs; the subscripts in the cases of $\mathcal{B}$ and $\mathcal{C}$ will be explained in Subsections 3.1 and 3.3. With the same line of logic, families $\mathcal{A}^3, \mathcal{B}^3_4$ and $\mathcal{C}^3_6$ could also be defined, but these graphs already appear as (e) (with $p = 0$), (d) (with $r = 0$) and (f) of Figure 1.)

According to [6], normal complex surface singularities corresponding to the resolution trees in $\mathcal{QHD}^3$ are all *taut*, that is, the resolution graph uniquely determines the analytic structure of the corresponding singularity. Since for any star-shaped negative definite plumbing tree of spheres there is a weighted homogeneous singularity with that resolution graph [14, Theorem 2.1], the unique singularity above is necessarily weighted homogeneous. The first main result of the paper is

Theorem 1.4 *Suppose that S_Γ is a small Seifert singularity with link Y_Γ. Assume that Γ is a minimal good resolution graph of S_Γ, and therefore a negative definite star-shaped tree with three branches. Then the following three statements are equivalent:*

(1) *The singularity S_Γ admits a $\mathbb{Q}HD$ smoothing.*

(2) *The Milnor fillable contact structure on Y_Γ admits a weak symplectic $\mathbb{Q}HD$ filling.*

(3) *The graph Γ is in $\mathcal{QHD}^3$.*

For star-shaped diagrams with more than three branches the analytic type of the singularity is not determined by the graph itself, hence the formulation of our result needs a little more care.

Definition 1.5 Define $\mathcal{QHD}^4$ as the union of all graphs given by Figures 2(a), (b) and (c) for $n \geq 2$ in each case.

According to [6], the analytic type of a normal surface singularity with resolution graph in $\mathcal{QHD}^4$ is determined by the analytic type of the four intersection points of the central curve C with the branches, or equivalently, by the cross ratio of these four points in C. In particular, all normal surface singularities with these resolution graphs are weighted homogeneous. With these remarks in place, we are ready to state the second main result of the paper.

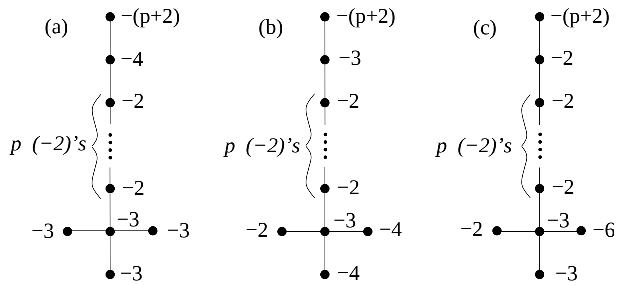

Figure 2: The graphs of (a) define the class $\mathcal{A}^4$, graphs of (b) give the class $\mathcal{B}^4$, while the graphs of (c) give $\mathcal{C}^4$ (in all these cases we assume $p \geq 0$). The union of the above specified classes is, by definition, $\mathcal{QHD}^4$. Once again, the superscript in the notation records the number of legs of these star-shaped graphs.

Theorem 1.6 *Suppose that Γ is a minimal, star-shaped plumbing tree with at least four branches, and the framing (i.e. weight) of the central vertex is less than -2. Then the following statements are equivalent.*

(1) *There is a Seifert singularity S_Γ with resolution graph Γ which admits a $\mathbb{Q}$HD smoothing.*

(2) *The Milnor fillable contact structure on Y_Γ admits a weak symplectic $\mathbb{Q}$HD filling.*

(3) *The graph Γ is in $\mathcal{QHD}^4$.*

Remarks 1.7 **(a)** The assumption on the framing of the central vertex in Theorem 1.6 is needed for our methods to work. In particular, a (-2)–framed central vertex with four legs provides a (-2)–curve in the dual configuration, hence the blow-down operation indicated by the dashed circles of Figures 11, 12 and 13 cannot be started. By accident, this assumption on the framing of the central vertex implies no constraint on the holomorphic result, since a normal surface singularity with $\mathbb{Q}$HD smoothing is necessarily rational, hence the resolution graph does not admit a vertex for which the absolute value of the framing is strictly less than the valency of the vertex minus 1. The question of whether the Milnor fillable contact structure on the link of a normal surface singularity with star-shaped resolution tree, at least four branches and central framing -2 admits a weak symplectic $\mathbb{Q}$HD filling is still open.

(b) The above theorems concern exclusively the cases when the resolution graph is star-shaped. No example of a normal surface singularity with non-star-shaped minimal good resolution graph which admits a $\mathbb{Q}$HD smoothing is known. Partial results regarding the nonexistence of $\mathbb{Q}$HD smoothings follow from [4, 16, 18], but the lack of a convenient and general compactifying divisor prevents us from treating the general case with methods similar to the ones applied in the present paper.

The idea of the proof of the main results can be summarized as follows. First of all, the implication (1) $\Rightarrow$ (2) in both theorems follows from the general principle that any smoothing of a singularity is a weak symplectic filling of the Milnor fillable contact structure on the link of the singularity. The implication (3) $\Rightarrow$ (1) (which was mostly already verified in [16]) in both statements requires the construction of $\mathbb{Q}$HD fillings; in the cases not covered by [16] we will apply the method of smoothings of negative weight. In order to prove (2) $\Rightarrow$ (3) we need to show that for any star-shaped resolution graph outside $\mathcal{QHD}^3$ and $\mathcal{QHD}^4$ the Milnor fillable contact structure admits no symplectic $\mathbb{Q}$HD filling. These nonexistence results rely on deep symplectic geometric theorems (most importantly on McDuff's result regarding symplectic manifolds containing symplectic spheres of self-intersection number 1) and tedious combinatorial arguments. In principle these arguments could be extended to classify other types of symplectic fillings, but the combinatorics (which is already quite delicate for the case of $\mathbb{Q}$HD fillings) can become extremely complex to handle.

Finally a few words about the use of symplectic geometry. In order to show that certain singularities *do not* admit $\mathbb{Q}$HD smoothings, we will apply the following strategy: first we will construct a fixed symplectic manifold for the singularity at hand (which we will call the *compactifying divisor*) and glue the hypothesized $\mathbb{Q}$HD weak symplectic filling to it in a symplectic manner. In the resulting *closed* symplectic manifold we then locate a curve configuration, which will lead to some geometric contradiction unless the singularity had resolution tree from $\mathcal{QHD}^3$ or $\mathcal{QHD}^4$. Although both the compactifying divisor and the hypothesized smoothing are holomorphic objects, we do not know any holomorphic way to glue them together to obtain a globally holomorphic closed manifold, on which then algebro-geometric methods would be applicable. An alternative, algebraic geometric compactification of the smoothings can be achieved by applying the method of deformations of 'weight less than or equal to zero'. As we were informed by J. Wahl [19], the necessary results can be proved using delicate methods of complex algebraic geometry and singularity theory. From that point on, the adaptation of our combinatorial arguments follow in a fairly straightforward manner. We decided to use the symplectic geometric methods, since in this way the resulting theorem becomes stronger in the aspect of getting obstructions even for the existence of $\mathbb{Q}$HD weak fillings. Also, by completing the arguments in the symplectic setting, our result shows yet another instance where objects behave in a parallel manner in the complex analytic and in the symplectic category.

The paper is organized as follows. In Section 2 the symplectic geometric preliminaries used in the proofs of the main results are listed, together with a quick outline of the ideas employed in the later arguments. Section 3 deals with small Seifert singularities, i.e., with those singularities which have star-shaped minimal good resolution graphs with three branches. Finally, in Section 4, we address the general case of spherical Seifert singularities.

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2 Preliminaries

2.1 Symplectic geometric preliminaries

Our results rely on the following fundamental theorem due to McDuff.

Theorem 2.1 *(McDuff, [7, Theorem 1.4]) Let (M, ω) be a closed symplectic 4-manifold. If M contains a symplectically embedded 2-sphere L of self-intersection number 1, then M is a rational symplectic 4-manifold. In particular, M becomes a the complex projective plane after blowing down a finite collection of symplectic (-1)-curves away from L.* □

The following two lemmas are based on the above theorem of McDuff and the details of the proofs can be found in [1]:

Lemma 2.2 *(Cf. [1, Lemma 2.13]) Let (M, ω) be a closed symplectic 4-manifold containing a symplectically embedded 2-sphere L of self-intersection number 1 and a collection of symplectically immersed 2-spheres $C_1, \ldots, C_k$. Suppose that J is a tame almost complex structure for which L, $C_1, \ldots, C_k$ are pseudoholomorphic. Then there exists at least one J-holomorphic (-1)-curve in $M \setminus L$ unless $L \cdot C_i > 0$ and $C_i \cdot C_i = (L \cdot C_i)^2$ for all i.* □

Lemma 2.3 *([1, Lemma 2.5]) Let M be a closed symplectic 4-manifold containing a symplectically embedded 2-sphere L of self-intersection number 1. If C is an irreducible singular or higher genus pseudoholomorphic curve in M, then $C \cdot L \geq 3$. In particular there are no irreducible singular or higher genus pseudoholomorphic curves in $M \setminus L$.* □

This lemma has the following simple corollary.

Corollary 2.4 *Let M be a closed symplectic 4-manifold containing a symplectically embedded 2-sphere L of self-intersection number 1. Then there is no cycle of pseudoholomorphic spheres in the complement L.*

Proof If such a cycle existed, by gluing adjacent components around the nodes we would be able to construct an embedded pseudoholomorphic curve of genus 1 which would contradict Lemma 2.3. □

Another fact which we will frequently use is that for any almost complex structure J on a 4-manifold X any intersection point of two J-holomorphic curves C_1 and C_2 contributes positively to the algebraic intersection number $C_1 \cdot C_2$.

The next lemma easily follows from McDuff's Theorem 2.1.

Lemma 2.5 *Let M be a closed symplectic 4-manifold containing a symplectically embedded 2-sphere L of self-intersection number 1. Then there is no symplectically embedded sphere of nonnegative self intersection number in the complement of L.*

Proof Since M is rational, it follows that $b_2^+(M) = 1$, immediately implying the lemma. (Notice that a symplectic sphere of any self-intersection — including 0 — is homologically essential.) □

Lemma 2.6 *Suppose that $C \subset \mathbb{CP}^2$ is a J-holomorphic curve for some tame almost complex structure J, in the homology class $[C] = d[\mathbb{CP}^1]$, and C has at least two singular points. Then $d \geq 4$.*

Proof The J-holomorphic line passing through two singular points intersects C with multiplicity at least 4, providing the result. □

We record here the following fact which we will apply repeatedly in the sequel: By the adjunction formula, a pseudoholomorphic rational curve representing the class $3[\mathbb{CP}^1]$ in $\mathbb{CP}^2$ must be either immersed with exactly one node (that is a point where two branches of the curve intersect transversely) or it must have exactly one nonimmersed point which is necessarily a $(2,3)$-cusp singularity. (Here a pseudoholomorphic curve in a 4-manifold is said to have a $(2,3)$-cusp singularity if there is a parametrization around the singular point in which the curve has the form $(z^2, z^3) + O(4)$, see [8].) In conclusion, the link of such a curve around its singular point is either connected (and is the trefoil knot) or has two components (and is the Hopf link).

2.2 The families $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$

The three inductively defined families $\mathcal{A}, \mathcal{B}, \mathcal{C}$ of graphs found in [16] will play a central role in our subsequent arguments. For the sake of completeness, we shortly recall the definition of these families below.

Let us define $\mathcal{A}$ as the family of graphs we get in the following way: start with the graph of Figure 3(a), blow up its (-1)-vertex or any edge emanating from the (-1)-vertex and repeat this procedure of blowing up (either the new (-1)-vertex or an edge emanating from it) finitely many times, and finally modify the single (-1)-decoration to (-4). Depending on the number and configuration of the chosen blow-ups, this procedure defines an infinite family of graphs. Define $\mathcal{B}$ similarly, this time starting with Figure 3(b) and substituting (-1) in the last step with (-3), and finally define $\mathcal{C}$ in the same vein by starting with Figure 3(c) and putting (-2) in the place of (-1) in the final step.

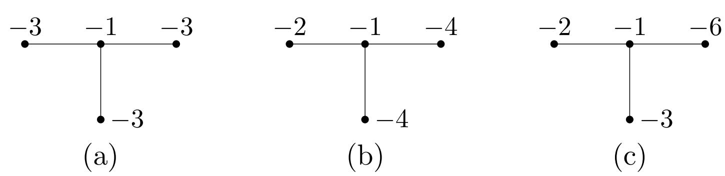

Figure 3: Nonminimal plumbing trees giving rise to the families $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$.

The starting point of the proofs of Theorems 1.4 and 1.6 rests on the main result of [16] which can be summarized as follows. Recall the definitions of $\mathcal{W}, \mathcal{M}, \mathcal{N}$ from Remark 1.3 and let $\mathcal{G}$ denote the set of plumbing chains with framings determined by the negatives of the continued fraction coefficients of the rational numbers of the form $\frac{p^2}{pq-1}$ for all $0 < q < p$ and $(p,q) = 1$.

Theorem 2.7 *([16]) Suppose that Γ is a minimal, negative definite plumbing* *tree. If it gives rise to a surface singularity S_Γ admitting a $\mathbb{Q}$HD smoothing, or if the Milnor fillable contact structure on the corresponding plumbing 3-manifold Y_Γ admits a $\mathbb{Q}$HD filling then Γ is in $\mathcal{G}\cup\mathcal{W}\cup\mathcal{N}\cup\mathcal{M}\cup\mathcal{A}\cup\mathcal{B}\cup\mathcal{C}$.* $\square$

2.3 Outline of the proof of $(2) \Rightarrow (3)$ in the main theorems

Suppose that Γ is a graph of the type considered in Theorems 1.4 or 1.6. Let Y_Γ denote the associated plumbed 3-manifold and ξ_Γ the unique Milnor fillable contact structure on Y_Γ. According to Theorem 2.7, if (Y_Γ, ξ_Γ) admits a symplectic $\mathbb{Q}$HD filling then Γ must be in $\mathcal{W}\cup\mathcal{N}\cup\mathcal{M}\cup\mathcal{A}\cup\mathcal{B}\cup\mathcal{C}$. Since by [16, Section 8] the singularities corresponding to graphs in $\mathcal{W}\cup\mathcal{N}\cup\mathcal{M}$ admit $\mathbb{Q}$HD smoothings, the corresponding links admit symplectic $\mathbb{Q}$HD fillings. Therefore we only need to consider star-shaped graphs in $\mathcal{A}\cup\mathcal{B}\cup\mathcal{C}$; let Γ be such a graph with s legs $\ell_1, \ldots, \ell_s$ and with central framing $-b$. Suppose that the framing coefficients along the leg ℓ_i are given by the negatives of the continued fraction coefficients of $\frac{n_i}{m_i} > 1$. Consider then the "dual" graph Γ' which is star-shaped with s legs $\ell'_1, \ldots, \ell'_s$, central framing $b-s$, and framings along leg ℓ'_i given by the negatives of the continued fraction coefficients of $\frac{n_i}{n_i-m_i}$. Let W_Γ and $W_{\Gamma'}$ denote the corresponding plumbing 4-manifolds.

Lemma 2.8 (Cf., for example, [16]) *Suppose that Γ, Γ' are star-shaped plumbing trees as above. The boundary of W_Γ is orientation preserving diffeomorphic to the link Y_Γ, while $\partial W_{\Gamma'} = -Y_\Gamma$. In addition, $W_\Gamma \cup W_{\Gamma'}$ is a 4-manifold diffeomorphic to $\mathbb{CP}^2 \# m\overline{\mathbb{CP}^2}$ for some positive integer m.*

Proof (sketch) Consider the Hirzebruch surface $\mathbb{F}_b$ with zero-section of self-intersection $-b$ (and hence with infinity-section of self-intersection b). Fix s distinct fibres of the $\mathbb{CP}^1$-fibration and blow up the intersection points of these fibres with the infinity-section. After the appropriate sequence of blow-ups we can identify in the resulting rational surface a configuration of curves intersecting each other according to Γ, and it is easy to see that the complementary curves will intersect each other according to Γ'. Since the curves intersecting according to the graph Γ admit an ω-convex neighbourhood (with the symplectic form ω being the Kähler form on the Hirzebruch surface), with the Milnor fillable contact structure as induced structure on the boundary, the complement (diffeomorphic to $W_{\Gamma'}$) provides a strong concave filling of (Y_Γ, ξ_Γ). Since the complement is also a regular neighbourhood of a configuration K of curves (intersecting each other according to Γ'), we will refer to K (and sometimes, with a slight abuse of notation, to the regular neighbourhood $W_{\Gamma'}$) as the *compactifying divisor.* $\square$

Suppose now that X is a weak symplectic $\mathbb{Q}$HD filling of (Y_Γ, ξ_Γ). Since Y_Γ is a rational homology 3-sphere, we can perturb the symplectic structure on X in a neighbourhood of the boundary so that it becomes a strong symplectic filling of (Y_Γ, ξ_Γ). Glue X and $W_{\Gamma'}$ along Y_Γ to obtain a closed symplectic 4-manifold Z. Let k denote the number of irreducible components of the compactifying divisor K. Then since $W_{\Gamma'}$ is a regular neighbourhood of K, we have that $b_2(W_{\Gamma'}) = k$. Since X is a $\mathbb{Q}$HD, it follows that $b_2(Z) = k$.

In all cases that we consider, it turns out that K (after, possibly, some blow-downs) contains a component which is a sphere that is embedded in $W_{\Gamma'} \subset Z$ with self intersection number 1. (This is the step when the assumption $\Gamma \in \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$, and the constraint of Theorem 1.6 on the framing of the central vertex become crucial.) Let L denote one such component. By McDuff's Theorem 2.1, we conclude that Z is a rational symplectic 4-manifold and hence diffeomorphic to $\mathbb{CP}^2 \# (k-1)\overline{\mathbb{CP}^2}$. In fact, McDuff's Theorem implies that for a generic tame almost complex structure J, in the complement of L we can find $k-1$ disjoint embedded symplectic 2-spheres with self-intersection number -1 (we will refer to these as *symplectic* (-1)*-curves*), and after blowing these down we obtain $\mathbb{CP}^2$. However, we would like to understand how the other components of K descend under the blowing down map. We thus proceed as follows.

We choose a tame almost complex structure J on Z with respect to which all the curves in K are pseudoholomorphic. We assume that J is generic among those almost complex structures for which K is J-holomorphic. Appealing to Lemma 2.2 we can find a pseudoholomorphic (-1)-curve E in Z disjoint from L. By perturbing the almost complex structure J if necessary, we can assume that E intersects each component of K transversely and does not pass through any point where two or more components of K intersect. We choose a maximal family $\{E_j\}$ of such pseudoholomorphic (-1)-curves which are disjoint from L and blow them down. Let Z' denote the resulting symplectic 4-manifold.

By [10, Lemma 4.1], we can find a tame almost complex structure J' on Z' with respect to which the images of all the components of K are pseudoholomorphic. We will again be in the situation where we have a closed symplectic 4-manifold containing a symplectically embedded 2-sphere of self-intersection number 1 and a collection of symplectically immersed 2-spheres (the images of the components of $K - L$). Let K' denote the image of K under the blowing down map. If K' contains a curve disjoint from L (as will always be the case in the situations we consider), then we can again appeal to Lemma 2.2 and find a pseudoholomorphic (-1)-curve E' in $Z' \setminus L$.

Note that E' must be a component of K'. Indeed, assume to the contrary that E' is not a component of K'. Perturbing the almost complex structure slightly, we may assume that E' does not pass through the images of the blown-down (-1)-curves E_j. Hence we may assume that E' is actually a pseudoholomorphic (-1)-curve already in $Z \setminus L$, which contradicts the maximality of $\{E_j\}$.

By suitably perturbing the almost complex structure, we can arrange that E' intersects each component of $K' - E'$ transversely and it does not pass through any point where two or more components of $K' - E'$ meet. We then blow down E'. Let Z'' denote the resulting ambient symplectic 4-manifold and K'' denote the image of K'.

As before, we can again check that there are no pseudoholomorphic (-1)-curves in Z'' except possibly for some components of K''. Perturbing the almost complex structure as before, blowing down these pseudoholomorphic (-1)-curves and proceeding in this way, we must eventually obtain $\mathbb{CP}^2$ together with a symplectically embedded 2-sphere of self-intersection number 1 and a collection of symplectically immersed 2-spheres. Since we are assuming that X is a $\mathbb{Q}$HD, it follows that we must obtain $\mathbb{CP}^2$ after $k-1$ blow downs and the configuration K must descend to a valid configuration in $\mathbb{CP}^2$. This places strong restrictions on the combinatorial structure of K: all components of K which are disjoint from L must be blown down at some point of this procedure (so in particular they must become (-1)-curves at some earlier point), while a component K_0 of K intersecting L must become a J-holomorphic submanifold of $\mathbb{CP}^2$ of degree $K_0 \cdot L$. This condition, for example, determines the homological square of the image of K_0 in $\mathbb{CP}^2$, and for low degrees it also determines the topology of the result. For most graphs Γ we will reach a homological contradiction at some point of this procedure, showing the nonexistence of the hypothesized $\mathbb{Q}$HD filling X.

3 Small Seifert singularities

By Theorem 2.7 and by the fact that all graphs in $\mathcal{W} \cup \mathcal{N} \cup \mathcal{M}$ are known to admit $\mathbb{Q}$HD smoothings [16], we only need to examine the three-legged graphs in $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$. The discussion will be given for each of these classes separately; for technical reasons we start with the case of graphs in $\mathcal{C}$.

3.1 Graphs in $\mathcal{C}$

Recall that graphs in $\mathcal{C}$ are defined by repeatedly blowing up the basic configuration shown by Figure 3(c) and then replacing the (-1)-framing with (-2). To get three-legged graphs, we only blow up *edges* emanating from the (-1)-vertex. There are three cases we distinguish depending on which edge we blow up in the first step in the basic example. The index of the subfamily records the (negative of the) framing of the leaf to which the first blown up edge points. Notice that the families $\mathcal{C}_2^3$ and $\mathcal{C}_3^3$ defined by the graphs of (i) and (j) of Figure 1 are subfamilies of $\mathcal{C}_2$ and $\mathcal{C}_3$, respectively.

The family $\mathcal{C}_6$: Consider the generic member of the family $\mathcal{C}_6$ depicted in Figure 4(a). The dual graph (after possibly repeatedly blowing up the edge

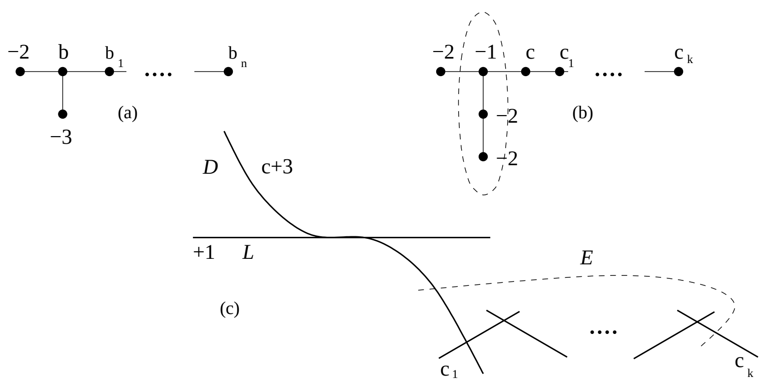

Figure 4: The generic graph, its dual, and the configuration of curves after 3 blow-downs in the family $\mathcal{C}_6$.

emanating from the central vertex towards the long leg until the central framing becomes -1) has the shape given by Figure 4(b). Blowing down the central vertex together with the two (-2)'s (encircled by the dashed circle in Figure 4(b)), we arrive at the diagram of Figure 4(c); here the curves are symbolized by arcs, and the intersection of two arcs means that the two corresponding curves intersect each other. (The dashed arc of Figure 4(c) will be relevant only at some later point of the argument.) The resulting $(+1)$-curve will be denoted by L, while the curves of the long leg (with framings $c, c_1, \ldots, c_k$) will become $D, C_1, \ldots, C_k$, respectively. The tangency between D and L is a triple tangency. (We use a straight line to indicate L and a cubic curve to picture D, which eventually will become a singular cubic in $\mathbb{CP}^2$.) Since $b_n \leq -6$, it is easy to see that $k \geq 3$. Notice also that $c_i \leq -2$ once $i \geq 1$ and c is negative.

By gluing this compactifying divisor to a potentially existing $\mathbb{Q}$HD filling X we get a closed symplectic manifold Z with $b_2(Z) = k + 2$. The symplectic 4-manifold Z obviously contains a symplectic $(+1)$-sphere (namely, the curve L), hence it follows by McDuff's Theorem 2.1 that Z is a rational symplectic 4-manifold, that is, a symplectic blow-up of $\mathbb{CP}^2$ at a finite number of points, hence Z is diffeomorphic to $\mathbb{CP}^2\#(k+1)\overline{\mathbb{CP}^2}$. By repeated applications of Lemma 2.2, we can blow down the pair (Z, L) to obtain $(\mathbb{CP}^2, \text{line})$, while preserving the pseudoholomorphicity of the images of $D, C_1, \ldots, C_k$. Since the curves $C_1, \ldots, C_k$ in the chain are disjoint from the $(+1)$-curve L and are homologically essential, we must blow them down, while the curve D will descend to a cubic curve in $\mathbb{CP}^2$. Since the resulting cubic curve will be the image of a rational curve, it necessarily must contain a singular point. The above observations imply, therefore, that there is a unique additional (-1)-curve E in Z for the chosen almost complex structure, which we have to locate in the diagram. Since J-holomorphic curves intersect positively, the geometric intersections in these cases can be computed via homological arguments.

Proposition 3.1 *Under the above circumstances the exceptional divisor E must intersect the curve D and the curve C_k in the chain in one point each. Consequently, the framings should satisfy $c_i = -2$ for $i = 1, \ldots, k$ and $c = -k+2$. In particular, the resolution graph of the singularity (given by Figure 4(a)) must be of the form given in Figure 1(f).*

Proof Let $\mathcal{J}_K$ denote the nonempty set of tame almost complex structures on Z with respect to which all the curves of $K = L \cup D \cup C_1 \cup \ldots \cup C_k$ are pseudoholomorphic. Choose an almost complex structure J which is generic in $\mathcal{J}_K$. If we blow down all J-holomorphic (-1)-curves away from L, we can show that the chain $C_1, \ldots, C_k$ is transformed into a configuration of curves which can be sequentially blown down. There must be precisely one (-1)-curve E in the complement of L which is not contained in the chain $C_1, \ldots, C_k$; this (-1)-curve E must intersect the chain to start its sequential blow-down. E also must intersect the curve D at least once, since (as D has intersection number 3 with the $(+1)$-curve L) D will become a singular cubic curve in $\mathbb{CP}^2$. By Corollary 2.4 the curve E cannot intersect the long chain twice. With a similar argument we can see that it can intersect the chain only in its endpoints: if it intersects the chain in a curve C_i which is not at one of its ends, then blowing down E we get a curve C_i' which now intersects D and two further curves in the chain. When we blow down C_i', the two neighbours will pass through the same point of D. If, now, the image of C_{i-1} is the next curve of the chain to get blown down, then the images of all curves in the portion $C_1, \ldots, C_{i-1}$ of the chain must get blown down before the image of the curve C_{i+1} is blown down. Otherwise, we will get a singular point on the image of D and at least one further curve of the chain passing through through that singular point. After a slight perturbation of the almost complex structure, when (the image) of one of these curves is eventually blown down we will get a further singular point on the image of D, which (with the aid of Lemma 2.6) provides a contradiction. However, after the images of $C_1, \dots, C_{k-1}$ are blown down, the image of D will become singular, and the same argument again provides a contradiction. If the image of C_{i+1} is the next curve of the chain to get blown down after C'_i, then, as before, we can argue that the images of all curves in the portion $C_{i+1}, \dots, C_k$ of the chain must get blown down before the image of the curve C_{i-1} is blown down. If $i > 3$, then, when the image of C_{i-1} is blown down, we will get a contradiction as before. If $i = 3$, then, when image of C_{i-1} is blown down, we will obtain a singular point on the image of D which has multiplicity greater than 2 and hence its link will not be the trefoil knot or the Hopf link, a contradiction.

If E intersects the chain on its end near D, then after the second blow-down D develops a transverse double point singularity, and the further blow-downs then create more singular points (in the spirit of the argument above), leading to a curve which cannot represent three times the generator in the complex projective plane. Hence the only possibility for the (-1)-curve E is to intersect the chain at its farther end, and intersect D once (as shown by the dashed curve E of Figure 4(c)). In order to blow down all the curves in the chain we must have $c_i = -2$ for $i = 1, \dots, k$, and since the self-intersection of D will become 9 after all the blow-downs, we derive $c = -k + 2$. With this last observation, and a simple computation of the dual graph, the proof is complete. □

The family $\mathcal{C}_3$: The generic member of this family is given by Figure 5(a), together with the dual graph and the result of the triple blow-down. (Once again, we disregard the dashed arcs of Figure 5(c) momentarily.) By gluing the compactifying divisor given by Figure 5(c) to a potentially existing $\mathbb{Q}$HD filling X we get a closed symplectic manifold Z, and a simple count shows that $b_2(Z) = k + 5$. The symplectic 4-manifold Z obviously contains a symplectic $(+1)$-sphere (namely, the curve L), hence, by McDuff's Theorem 2.1, Z is diffeomorphic to $\mathbb{CP}^2 \# (k+4)\overline{\mathbb{CP}^2}$. By repeated applications of Lemma 2.2, we can blow down the pair (Z, L) to obtain $(\mathbb{CP}^2, \text{line})$, while preserving the pseudoholomorphicity of the images of $D, C_1, \dots, C_k, B_1, B_2$. Since the curves $C_1, \dots, C_k$ and B_1, B_2 are disjoint from the $(+1)$-curve L and are homologically essential, we must blow them down. This means that there are two further

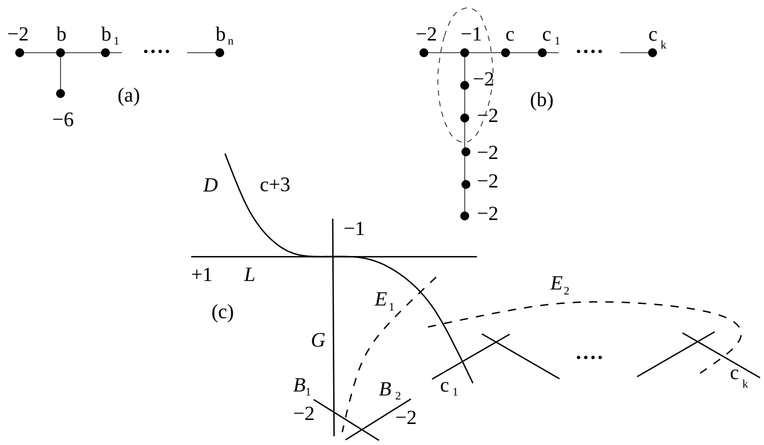

Figure 5: The generic graph, its dual, and the configuration of curves after 3 blow-downs in the family $\mathcal{C}_3$. The curves E_1, E_2 are only shown for the first possibility given by Proposition 3.2.

(-1)-curves E_1 and E_2 which we have to locate in the diagram. For a generic almost complex structure these curves will be (-1)-curves disjoint from each other. Since both B_1 and B_2 have to be blown down (being disjoint from the $(+1)$-curve L), one of them must intersect one of the (-1)-curves, say E_1. Since the complement of the $(+1)$-curve does not contain homologically essential spheres with nonnegative square, E_2 then cannot intersect any of the B_i.

Proposition 3.2 *Under the above circumstances, the existence of a $\mathbb{Q}HD$ smoothing X implies that E_2 intersects D and C_k, and E_1 either intersects B_1 and D or B_2 and C_1. The self-intersections in these two cases are $c = -k-1$ and $c_1 = \ldots = c_k = -2$ or $c = -k+2$, $c_1 = -5$ and $c_2 = \ldots = c_k = -2$. In particular, the resolution graph in the first case is given by Figure 1(j), while in the second case by Figure 1(d) (with $q = k-4$ and $r = 2$).*

Proof Case I: Suppose that $E_1 \cdot B_1 > 0$. After three blow-downs the curve G becomes a $(+1)$-curve, so it cannot be blown down any further: in $\mathbb{CP}^2$ it will be a curve intersecting the $(+1)$-curve once, hence it will be a line with self-intersection number 1. Therefore, to prevent further blow-downs along the points of the vertical curve, $E_2 \cdot G = 0$ and E_1 must be disjoint from the long chain. So E_2 must intersect the long chain, and since the whole chain must be blown down, a simple adaptation of the proof of Proposition 3.1 gives that the only possibility for E_2 is the one described in the statement. Notice that the images of G and D must intersect each other three times after all curves have been blown down, which can be achieved only if E_1 intersects D exactly once. (Recall that E_2 must stay disjoint from G.) This argument shows that the only possibility for E_1 and E_2 (under the assumption $E_1 \cdot B_1 > 0$) is given by the dashed lines of Figure 5(c), providing the first set of values of c and c_i.

Case II: Suppose now that $E_1 \cdot B_2 > 0$. Then after three blow-downs the vertical curve G becomes a 0-curve, so either (a) E_2 intersects G or (b) E_1 intersects a further (-1)-curve in the chain (after it has been partially blown down). If E_1 intersects B_2 and E_2 intersects G then none of the E_i intersect the chain, and since the chain is nonempty, this provides a contradiction.

Therefore E_1 should intersect the long chain, and it should intersect it in the last curve to be blown down from there. Suppose that $E_1 \cdot C_i = 1$. Then E_1 cannot intersect D, since otherwise after blowing down E_1, then sequentially blowing down the images of B_2 and B_1, C_i' (the image of C_i) will intersect the image of D at least three times (counting with multiplicity). When (the image of) C_i' is eventually blown down, the image of D will gain a singularity which is not permitted for a cubic in $\mathbb{CP}^2$. This shows that E_2 has to intersect the chain (and start the sequence of blow-downs) and it also has to intersect D to get a singularity on it. Furthermore, we also know that E_2 must be disjoint from G. The argument of Proposition 3.1 shows that E_2 must intersect the long chain at its farther end and also D. As usual, the framings are dictated by the fact that all curves in the complement of the $(+1)$-curve must be blown down, leading to the second set of values of c and c_i. By determining the dual graphs, the proof is complete. □

The family $\mathcal{C}_2$: The generic case in this family is shown by Figure 6(a). The usual simple calculation shows that by assuming the existence of a $\mathbb{Q}$HD filling for (Y_Γ, ξ_Γ) we have to locate two (-1)-curves in the diagram, which we will denote by E_1 and E_2. Since the curves A_2, A_3 and A_4 must be blown down at some point in the blow-down procedure, one of the (-1)-curves (say E_1) should intersect $A_2 \cup A_3 \cup A_4$.

Proposition 3.3 *In the situation under examination, the existence of a $\mathbb{Q}$HD filling implies that E_2 intersects D and C_k, while E_1 either intersects A_2 and D or A_4 and C_1 or A_4 and C_2. The framings in the three cases are given by $c = -k-2$ and $c_1 = \dots c_k = -2$, or $c = -k+3$, $c_1 = -5$, $c_3 = -3$ and $c_2 = c_4 = \dots = c_k = -2$, or $c = -k+2$, $c_2 = -6$ and $c_1 = c_3 = \dots c_k = -2$. In particular, the resolution graph is as one of the graphs given by Figure 1(i)*

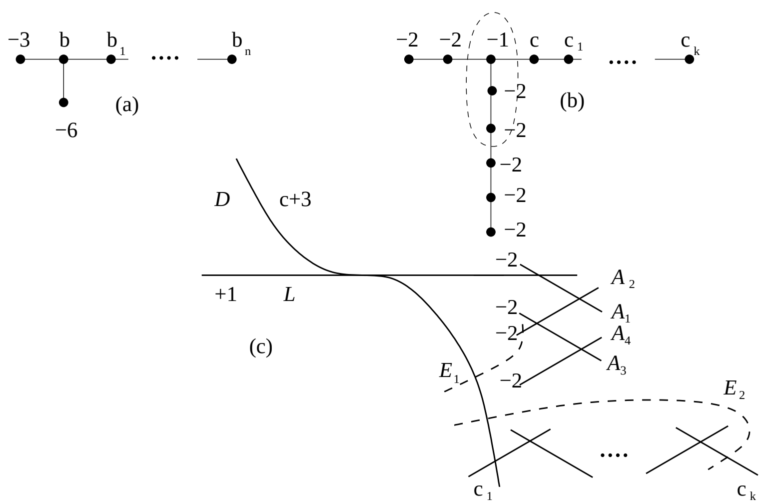

Figure 6: The generic graph, its dual, and the configuration of curves after 3 blow-downs in the family $\mathcal{C}_2$. The curves E_1, E_2 are shown only for the first possibility given by Proposition 3.3.

in the first case, by Figure 1(g) (with $p = 0, r = 2, q = k - 4$) in the second, and by Figure 1(e) ($p = 3, q = k - 4$) in the third.

Proof Notice first that E_1 cannot intersect A_3 (otherwise we will have a self-intersection 0 curve in the complement of L, contradicting Lemma 2.5); hence we have two cases to examine.

Case I: Suppose that $E_1 \cdot A_2 > 0$. In this case, after four blow-downs, the self-intersection of A_1 becomes 1, which cannot go any higher, since in $\mathbb{CP}^2$ the curve A_1 will become a line. Therefore E_1 must be disjoint from the chain and E_2 must be disjoint from all the A_i's. In order for the image of A_1 to intersect D three times, E_1 must intersect D. Since E_2 is disjoint from all the A_i's, and it starts the blow-down of the chain, and is responsible for the singularity on D, the usual argument presented in the proof of Proposition 3.1 locates it. In conclusion, the only possibility for the framings is the one given by the statement.

Case II: Suppose now that E_1 intersects A_4. After blowing down E_1, and then sequentially blowing down the images of A_4, A_3 and A_2, the self-intersection of A_1 will increase to -1. In order to increase it to 1 we have a number of possibilities.

(i) $E_1 \cdot C_i = 0$ for all i, i.e., E_1 is disjoint from the chain. In this case E_2 must intersect A_1 and also the last curve we blow down in the chain. Since then there is no further curve starting the blow-down of the chain, this can happen only if the chain has a single element. If E_2 is disjoint from D, then after all blow-downs have been carried out D remains smooth, which is a contradiction. Therefore E_2 must intersect D. Blowing down E_2 and then the elements in the chain we get that the image of A_1 passes through D three times. Therefore E_1 must be disjoint from D. Computing the self-intersections, however, we see that the curve with framing c (giving rise to D, which will become of self-intersection 9) must have self-intersection $c = 1$ in the dual graph, which is a contradiction.

(ii) Assume now that E_1 intersects the chain in the curve we will blow down last. This implies that E_2 should intersect A_1, but since the blow-down of E_1 (together with the last curve in the chain) increases the self-intersection of A_1 by two, E_2 must be disjoint from the chain. Therefore, once again, the chain must be of length one. Performing the blow-downs, we conclude that D remains smooth and the images of D and A_1 will intersect each other only twice, hence this case does not occur.

(iii) Finally, it can happen that E_1 intersects the chain in the penultimate curve to get blown down. Then E_2 should be disjoint from the A_i's, and since the singularity on D cannot be caused by blowing down E_1, we need that E_2 intersects D. The usual argument given in the proof of Proposition 3.1 shows the position of E_2, leading to two configurations, depending on whether the last curve to be blown down is next to D or is one off. The resulting framings in these two cases are then the ones given by the proposition. □

3.2 Graphs in $\mathcal{A}$

For three-legged graphs in $\mathcal{A}$ there is no need for further subdivisions since the legs in this case are symmetric. As usual, the generic member of the family is shown by Figure 7(a). The usual simple count shows that if we assume the existence of a $\mathbb{Q}$HD filling, then we have to find two (-1)-curves E_1, E_2 in Figure 7(a). The curve A is of self-intersection (-2), and will become a line in $\mathbb{CP}^2$, hence must be hit by one of the (-1)-curves, say by E_1.

Proposition 3.4 *In this case, the curve E_2 intersects D and C_k, while E_1 intersects either A and C_1 or A and C_2. The corresponding framings in both cases are $c = -k+2$, $c_2 = -3$ and $c_1 = c_3 = \dots c_k = -2$. In particular, the resolution graph is of the form of Figure 1(e) with $p = 0$.*

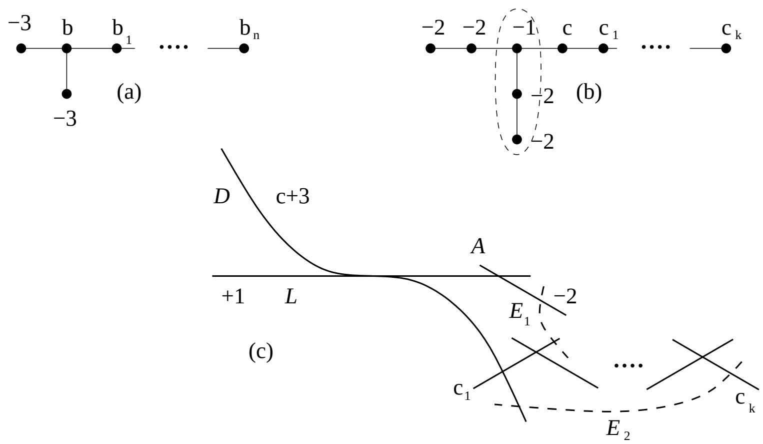

Figure 7: The generic graph, its dual, and the configuration of curves after 3 blow-downs in the family $\mathcal{A}$. Another possibility for E_1 in (c) allowed by Proposition 3.4 is the curve which intersects A and C_2 (instead of C_1). As usual, we do not depict this second possibility.

Proof We are assuming that E_1 intersects A. If E_2 also intersects A, then only one of them (say E_2) can intersect the long chain, and only in the last curve to be blown down, so we cannot start the blow-down process on the chain unless it is of length one. We show that this case never occurs. In fact, to create the singularity on D, the (-1)-curve E_2 must intersect it, and so by blowing down E_2 and the unique element in the chain, we get that the resulting A and D will intersect each other three times, hence E_1 must be disjoint from D. The self-intersection of the resulting singular cubic (which must be equal to 9) is $c+8$, implying that $c=1$, which contradicts the fact that it should be negative. Therefore E_2 cannot intersect A, and so it must intersect the long chain, and to create the singular point on D it must also intersect that curve. The usual argument already discussed in Proposition 3.1 shows that E_2 can intersect the chain only in C_k. In order to raise the self-intersection of A from -2 to 1 we need that E_1 intersect the chain in the penultimate curve to be blown down. Since after the blow-downs the image of A will pass through the singular point of D, E_1 must be disjoint from D. The two very similar possibilities for the (-1)-curves (differing only in the position of the E_1-curve) result the same set of framings, hence the same set of resolution graphs. $\square$

3.3 Graphs in $\mathcal{B}$

Similarly to the case of $\mathcal{C}$, the study of three–legged graphs in the family $\mathcal{B}$ falls into two subcases, of $\mathcal{B}_4$ and $\mathcal{B}_2$, depending on the choice of the first blow-up. The family $\mathcal{B}_2^3$ defined by (h) of Figure 1, for example, is a subfamily of $\mathcal{B}_2$.

The family $\mathcal{B}_4$: The generic member of this family (together with the dual graph and the configuration of curves after three blow-downs) is shown in Figure 8. The usual count of curves shows that we need to locate two (-1)-curves,

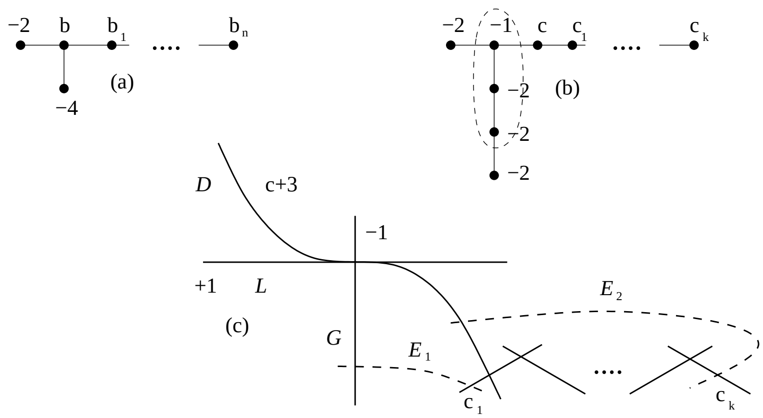

Figure 8: The generic graph, its dual, and the configuration of curves after 3 blow-downs in the family $\mathcal{B}_4$.

denoted by E_1 and E_2. It is clear that one of them, say E_1, must intersect G in order to increase its self-intersection to 1.

Proposition 3.5 *Under the above hypotheses, the existence of a $\mathbb{Q}$HD filling implies that E_2 intersects D and C_k, while E_1 intersects G and C_1. The corresponding framings are $c = -k+2$, $c_1 = -3$ and $c_2 = \ldots c_k = -2$. In particular, the resolution graph is of the form given by Figure 1(d) with $r = 0$.*

Proof If E_2 also intersects G then both E_1 and E_2 must be disjoint from the chain, hence it cannot be blown down. Therefore we can assume that E_2 is disjoint from G, and therefore E_1 must intersect the chain in the last curve to be blown down. The curve E_1 must be disjoint from D, since if E_1 intersects D then after two blow-downs the curves resulting from G and D will intersect at least four times, giving a contradiction. Therefore E_1 must be disjoint from D, hence E_2 intersects the configuration of curves as is found in the proof of

Proposition 3.1. The only possibility for the framings is the one given by the proposition. □

The family $\mathcal{B}_2$: The graphs (with their duals, and the curve configuration we get by the three blow-downs) are shown in Figure 9. The usual curve count shows that for identifying a $\mathbb{Q}$HD filling we must find three (-1)-curves E_1, E_2, E_3 in the diagram. Suppose that E_1 intersects G.

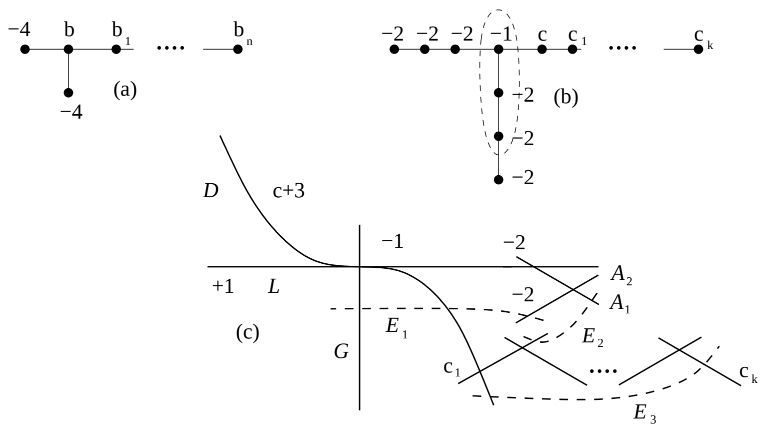

Figure 9: The generic graph, its dual, and the configuration of curves after 3 blow-downs in the family $\mathcal{B}_2$. The curves E_1, E_2 of (c) correspond to the first possibility listed by Proposition 3.6.

Proposition 3.6 *Under the circumstance described above, from the existence of a $\mathbb{Q}$HD filling it follows that either*

- *the curve E_3 intersects D and C_k, E_2 intersects C_1 and A_1 and E_1 intersects G, D and A_2 and therefore the framings satisfy $c = -k$, $c_1 = -3$ and $c_2 = \ldots = c_k = -2$, or*
- *E_3 intersects D and C_k, E_2 intersects A_2 and C_2, and E_1 intersects G and C_1 and therefore the framings are given as $c = -k+2$, $c_1 = -3$, $c_2 = -4$, $c_3 = \ldots = c_k = -2$, or*
- *E_3 intersects D and C_k, E_2 intersects A_2 and C_1, and E_1 intersects G and C_2 and the framings are $c = -k+2$, $c_1 = -3$, $c_2 = -4$, $c_3 = \ldots = c_k = -2$.*

In particular, the resolution graph is of the form of Figure 1(h) in the first case and of Figure 1(g) (with $p = 1, r = 0, q = k-4$) in the second and third cases.

Proof Since G has self-intersection -1 and it intersects the curve L once, its self-intersection must increase to 1, hence either another (-1)-curve, say E_2, intersects G or E_1 intersects either A_2 or the chain. Note that E_1 cannot intersect both A_2 and the chain, since if $E_1 \cdot C_i = 1$ and $E_2 \cdot A = 1$, then after E_1 and the image of A_2 are blown down the image of C_i will become tangent to the image of G. When the image of C_i is eventually blown down, the image of G will gain a singularity, which is impossible for a line in $\mathbb{CP}^2$.

Case I: $E_2 \cdot G = 1$. In this case both E_1 and E_2 must be disjoint from A_2 and the chain, hence E_3 intersects both A_2 and the chain. Also, since G and A_1 will intersect after the blowing down process has been carried out, E_1 or E_2 (say E_1) must intersect A_1. After blowing down the E_i's and the image of A_2, the self-intersection of A_1 will already be zero, hence E_3 can only intersect the chain in the last curve to get blown down, which is possible only if the chain is of length one. If E_3 is disjoint from D then (in order for A_1 to intersect D three times) E_1 must intersect D twice, and hence (in order to avoid $G \cdot D > 3$) the curve E_2 must be disjoint from D. Now we can easily see that the self-intersection of D increases to $c + 8$ after all the blow-downs have been performed, and since it should be equal to 9, we deduce that $c = 1$, contradicting the fact that c is negative. If E_3 intersects D then after blowing down E_3 and then sequentially blowing down the images of A_2 and the unique element in the chain we get a singularity on D of multiplicity 3, a contradiction. This shows that Case I, in fact, cannot occur.

Case II: $E_1 \cdot A_2 = 1$. Then both E_2 and E_3 must be disjoint from G, and one of them (say E_2) intersects A_1. To increase the self-intersection of A_1, the curve E_2 should intersect the chain in the last curve to be blown down. Since the image of G will intersect D, we see that $E_1 \cdot D = 1$. This implies that after blowing down E_1 and A_2, the curve A_1 will intersect D once, therefore E_2 cannot intersect D (since it would add three to $A_1 \cdot D$). Now the usual argument from the proof of Proposition 3.1 shows that E_3 starts the blow-down of the chain, and it also intersects D in one point, leading to the first case of the proposition.

Case III: $E_1 \cdot C_i = 1$. Recall that by the previous argument we can assume that $E_2 \cdot G = E_3 \cdot G = 0$. If E_2 and E_3 are both disjoint from the chain, then the chain must have length one. But then, if $E_1 \cdot D = 0$, then, after completing the blowing down process, the intersection number of the images of G and D will be less than 3 and if $E_1 \cdot D = 1$, then, after completing the blowing down process, the intersection number of the images of G and D will be greater than 3, both contradicting the fact that the intersection number of a line and a cubic in $\mathbb{CP}^2$ is equal to three. So we may assume that E_3 intersects the chain, say $E_3 \cdot C_l = 1$, and, by the preceding argument, that $E_1 \cdot D = 0$. If $E_3 \cdot D = 0$, again we find that, after the blowing down process has been carried out, the intersection number of the images of G and D will be 2, a contradiction. So we must have $E_3 \cdot D = 1$. Now observe that we must have $E_3 \cdot A_2 = 0$. Indeed, if $E_3 \cdot C_j = 1$ and $E_3 \cdot A_2 = 1$, then after E_3 and the image of A_2 are blown down, the image C_j' of C_j will be tangent to the image D. It is now easy to see that after the blowing down process is complete the image of D will have more than one singular point or a singularity of multiplicity greater than 2, both of which are impossible for a cubic in $\mathbb{CP}^2$. Since A_2 must be hit by a (-1)-curve, we deduce that $E_2 \cdot A_2 = 1$. We now check that E_1 and E_3 are disjoint from A_1. If $E_1 \cdot A_1 = 1$, then after blowing down E_1 the images of G and A_1 will intersect in a point and the image of C_i will pass through that point. When the image of C_i is eventually blown down, the intersection number of the images of G and A_2 will be 2, which is impossible for a pair of lines in $\mathbb{CP}^2$. If $E_3 \cdot A_1 = 1$, then the chain must have length one (to prevent the intersection number of the images of A_1 and D going above 3). Usual simple calculation shows that c must be 1 contradicting $c < 0$. We have thus checked that E_1 and E_3 are disjoint from A_1. It follows that, in order for the self-intersection number of the image of A_1 to increase to 1, we must have that E_2 intersects the string in the penultimate curve of the chain to get blown down. Suppose that $E_2 \cdot C_j = 1$. Now if $l < k$, then it is easy to see that we must have $k = 2$, $l = 1$ and $j = 2$. But then, after completing the blowing down process, the intersection number of the images of A_1 and D will be 2, a contradiction. Thus we must have $l = k$. It follows that we must have $j = 1$ or 2. If $j = 1$, then we must have $i = 2$, and if $j = 2$, then we must have $i = 1$. The blowing down process now fixes $c, c_1, \ldots, c_k$, which depends only on k and is independent of j, giving $c = -k + 2, c_1 = -3, c_2 = -4$ and $c_3 = \ldots = c_k = -2$. The two possible configurations of the curves E_1, E_2, E_3 (providing the same dual graphs) are the ones given by the proposition. $\square$

Proof of Theorem 1.4 Consider a small Seifert singularity S_Γ. Since a smoothing of S_Γ provides a weak symplectic filling of the Milnor fillable contact structure (Y_Γ, ξ_Γ) of the link, the implication (1) $\Rightarrow$ (2) follows. The implication (2) $\Rightarrow$ (3) is a direct consequence of the combination of Propositions 3.1, 3.2, 3.3, 3.4, 3.5 and 3.6, together with Theorem 2.7.

In order to verify the implication (3) $\Rightarrow$ (1), we need to produce $\mathbb{Q}$HD smoothings for singularities with resolution graphs in $\mathcal{QHD}^3$. This result follows from [16, Example 8.4] for the graphs of Figure 1(a), (b) and (c), and from [16, Example 8.3] for (d), (e), (f) and (g). For singularities with resolution graphs given by Figure 1(h), (i) and (j) we give an argument resting on the theorem of Pinkham [15] as formulated in [16, Theorem 8.1]. In order to apply this result for a singularity S_Γ, we need to find an embedding of rational curves in a rational surface R intersecting each other according to the dual graph Γ' with the property that $\operatorname{rk} H_2(R;\mathbb{Z}) = |\Gamma'|$.

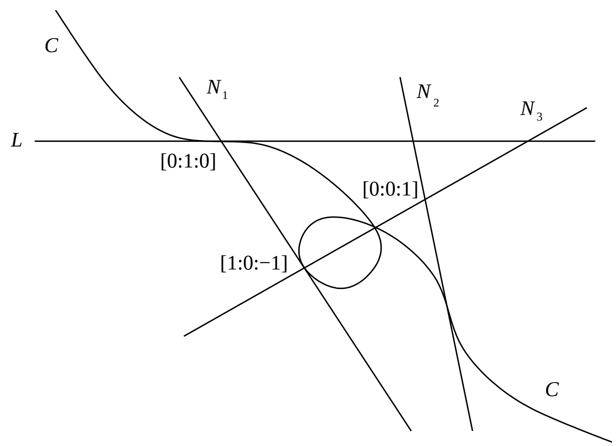

Figure 10: The curves used in the constructions of the embeddings.

To this end, let us consider the singular cubic C given by equation $f(x,y,z) = y^2z - x^3 - x^2z$ in $\mathbb{CP}^2$ and the lines L, N_1, N_2 and N_3 given by the equations $\{z=0\}$, $\{x+z=0\}$, $\{y-(x+\frac{8}{9}z)\sqrt{3}i = 0\}$, and $\{y=0\}$, respectively, cf. Figure 10. (The line L is tangent to C at one of its inflection point $[0:1:0]$; N_1 is tangent to C at $[-1:0:1]$ and further intersects it in $[0:1:0]$; N_2 is tangent to C in another inflection point $[-\frac{4}{3} : -i\frac{4}{3\sqrt{3}} : 1]$.)

By sequentially blowing down the (-1)–curves of Figure 4(c) (starting with the dashed one), we are led to a configuration of curves involving a singular cubic and a tangent at one of its inflection points. Since C and L provide such a configuration, the reverse of the above blow-down procedure gives an embedding of the configuration of Figure 4(c), and therefore of (b) into some blow-up of $\mathbb{CP}^2$. A simple count of the applied blow-ups shows that this embedding is exactly of the type needed to apply Pinkham's result, hence this argument shows that graphs of Figure 1(f) correspond to singularities with $\mathbb{Q}$HD smoothings. The same line of argument, with various starting configurations, then shows that all the remaining graphs of Figure 1 correspond to singularities with $\mathbb{Q}$HD smoothings: a suitable starting configuration for the graphs of Figure 1(h) is the configuration given by the curves L, C, N_1 and N_3 of Figure 10, for (i) L, C, N_2 and for (j) L, C, N_1 will be a convenient choice. With this last step, the proof of Theorem 1.4 is now complete. □

4 Spherical Seifert singularities

Next we turn to the examination of generic spherical Seifert singularities. Since a star-shaped graph in $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$ can have at most four legs, it follows from Theorem 2.7 that if a spherical Seifert singularity admits a $\mathbb{Q}$HD smoothing (or the Milnor fillable contact structure on its link admits a $\mathbb{Q}$HD filling) then the valency of the central vertex is at most four. The three-legged graphs were analyzed in the previous section, so now we will focus on the case of four-legged graphs. Once again, it follows from Theorem 2.7 that we only need to consider graphs in $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$.

4.1 The family $\mathcal{C}$

We start by considering the four-legged graphs in the family $\mathcal{C}$. The generic four-legged member Γ of $\mathcal{C}$ is given in Figure 11(a), with the dual graph given by Figure 11(b). After three blow-downs we obtain the configuration K depicted in Figure 11(c). As before, the horizontal $(+1)$-curve will be denoted L and the two curves which are triply tangent to L will be denoted F and D, with F being the curve with square $+1$. The chain of (-2)-curves connected to the curve F will be denoted $B_1, \ldots, B_4$, with B_1 intersecting F and the chain of curves intersecting D will be denoted $C_1, \ldots, C_k$, with C_1 intersecting D. By symplectically gluing K to a $\mathbb{Q}$HD filling X we get a closed symplectic 4-manifold Z, and the usual elementary homological computation shows that (since X is a $\mathbb{Q}$HD) there must be precisely two (-1)-curves, say E_1 and E_2, in the complement of L that are not contained in the strings $B_1, \ldots, B_4$ and $C_1, \ldots, C_k$. Since the string $B_1, \ldots, B_4$ must be transformed into a configuration which can be sequentially blown down after blowing down E_1 and E_2, it follows that at least one of these (-1)-curves must intersect $B_1 \cup \cdots \cup B_4$. Assume, without loss of generality, that E_1 intersects $B_1 \cup \cdots \cup B_4$.

Proposition 4.1 *By assuming the existence of the $\mathbb{Q}$HD filling X we get that E_1 intersects D, F and B_4, while E_2 intersects D and C_k. The framings then*

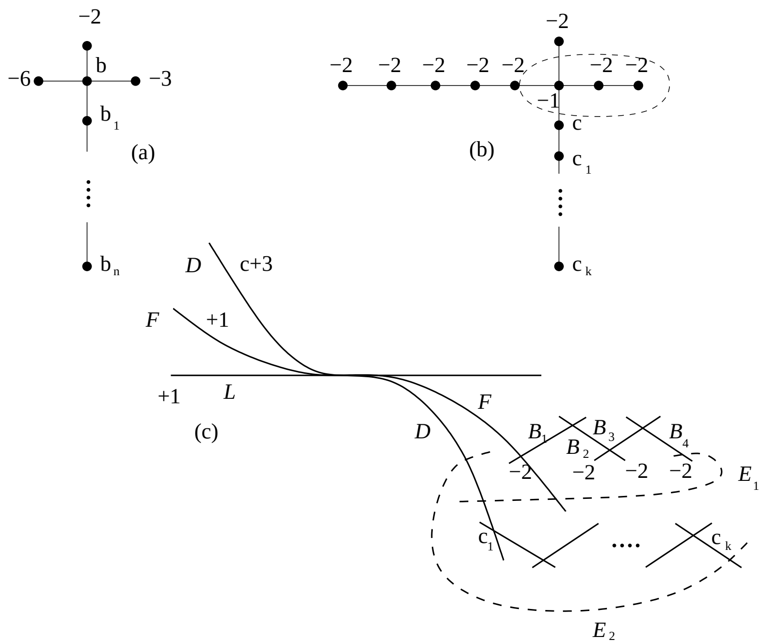

Figure 11: The four-legged graphs in $\mathcal{C}$.

are given by $c = -k - 3$ *and* $c_1 = \ldots = c_k = -2$ *(with* $k \geq 0$*). In particular, the graph of Figure 11(a) should be of the form Figure 2(c).*

Proof If $E_1 \cdot B_2 = 1$ or $E_1 \cdot B_3 = 1$, then blowing down E_1 and then sequentially blowing down the images of B_2 and B_3 leads to a $(+1)$-curve (the image of B_1 or B_4) in the complement of L, contradicting Lemma 2.5. Hence we can assume that either $E_1 \cdot B_1 = 1$ or $E_1 \cdot B_4 = 1$. First we argue that $k > 0$ can be assumed in Figure 11(b). Indeed, $k = 0$ implies that in Figure 11(a) we have $b = -3$, $b_1 = \ldots = b_n = -2$. Among these possibilities only the one with $n = 2$ is in $\mathcal{C}$, and that particular graph appears among the ones of Figure 2(c). For this reason, from now on we will assume that $k > 0$.

Case I: Suppose that $E_1 \cdot B_1 = 1$. Note first that $E_1 \cdot F = 0$. Indeed, suppose that $E_1 \cdot F \geq 1$. If $E_1 \cdot F > 1$, then blowing down E_1 would lead to a point on the image F' of F under the blowing down map through which at least two branches of F' pass. Also the intersection number of the image B_1' of B_1 and F' will be at least three. By perturbing the almost complex structure slightly, we can assume that B_1' and F' intersect transversely. Then blowing down B_1' we see that the image F'' of F' will have two singularities, which by Lemma 2.6

contradicts the fact that F'' will eventually blow down to a cubic in $\mathbb{CP}^2$. A similar contradiction arises if $E_1 \cdot F = 1$, after blowing down both E_1 and B_1'. There are now two possibilities: $E_1 \cdot (C_1 \cup \cdots \cup C_k) = 1$ or $E_1 \cdot (C_1 \cup \cdots \cup C_k) = 0$. Note that $E_1 \cdot (C_1 \cup \cdots \cup C_k) > 1$ is impossible by Corollary 2.4.

IA. $E_1 \cdot (C_1 \cup \cdots \cup C_k) = 1$. Suppose that $E_1 \cdot C_i = 1$. After blowing down E_1 and then sequentially blowing down the images of $B_1, \ldots, B_4$, observe that the image C_i' of C_i will be 4-fold tangent to the image F' of F. Perturbing the almost complex structure, we may assume that C_i' intersects F' transversely. Eventually C_i' will get blown down and this will create a singularity on the image of F that is not allowed for a cubic in $\mathbb{CP}^2$, since the link of its singularity has four components, providing the desired contradiction.

IB. $E_1 \cdot (C_1 \cup \cdots \cup C_k) = 0$. We have $E_1 \cdot D = 0$ or $E_1 \cdot D = 1$. ($E_1 \cdot D > 1$ is not allowed as blowing down E_1, then perturbing the almost complex structure so that B_1', the image of B_1, and D', the image of D, intersect transversely and then blowing down B_1' would create two nodes on the image of D', contradicting Lemma 2.6.) After blowing down E_1 and then sequentially blowing down the images of $B_1, \ldots, B_4$, the intersection number of the images F' and D' of F and D, respectively, will be either 3 or 7. Now, by arguing as in the proof of Proposition 3.1, we can show that E_2 must intersect the last curve C_k in the string $C_1, \ldots, C_k$ and the curve D'. E_2 must also intersect F', otherwise, after the blowing down process has been carried out, the image of F' would be nonsingular and rational, which is impossible for a cubic in $\mathbb{CP}^2$. In fact, it is necessary that $E_2 \cdot F' = 2$, otherwise the image of F' will either be smooth or have the wrong type of singularity. Also it is necessary that the string $C_1, \ldots, C_k$ be empty, otherwise, after blowing down E_2, when the image of C_k is collapsed a further singularity will be introduced in the image of F'. Now the condition that D' gets blown down to a rational cubic in $\mathbb{CP}^2$ forces us to have $E_2 \cdot D' = 2$. Blowing down E_2, we see now that the intersection number of the images of D' and F' will be either 7 or 11 (depending on $E_1 \cdot D = 0$ or 1), which is impossible for a pair of irreducible cubic curves in $\mathbb{CP}^2$. In conclusion, we found that $E_1 \cdot B_1 = 1$ leads to contradiction, hence we can consider

Case II: $E_1 \cdot B_4 = 1$. As before, we distinguish two cases according to the intersection of E_1 with the chain $C_1 \cup \ldots \cup C_k$.

IIA. $E_1 \cdot (C_1 \cup \cdots \cup C_k) = 1$. Suppose that $E_1 \cdot C_i = 1$. Note that $E_1 \cdot F = 0$, otherwise the image of F after completing the blowing down process would have more than one singular points. For a similar reason, $E_1 \cdot D$ must also be 0. We now divide $E_1 \cdot C_i = 1$ into three cases.

(i) Suppose that $i = 1$, i.e., E_1 intersects the chain in the curve intersecting D. Blow down E_1, then sequentially blow down the images of $B_4, \ldots, B_1$ and then the images of $C_1, \ldots, C_l$ until the resulting string $C'_{l+1}, \ldots, C'_k$ attached to D', the image of D, is minimal, that is, contains no (-1)-curves. Let F' denote the image of F. Then $F' \cdot D' = l + 2$, where $0 \leq l \leq k$. First suppose that $l < k$. Then, by arguing as in the proof of Proposition 3.1, one can show that E_2 must intersect the last curve C'_k of the string $C'_{l+1}, \ldots, C'_k$ and the curves F' and D', each once transversally. Now blow down E_2 and then sequentially blow down the images of $C'_k, \ldots, C'_{l+1}$. Then the images of F' and D' will be nodal curves and for the intersection number of them to be 9 we require that $k = 2$. However, to make the self-intersection number of the image of F' equal 9 we require that $k = 3$. This contradiction show that the case $l < k$ cannot occur. Now suppose that $l = k$. Then to introduce singularities of the right type into the images of the curves F' and D' we require that $E_2 \cdot F' = 2$ and $E_2 \cdot D' = 2$. A simple check now shows that, as before, to make the intersection number of the images of F' and D' 9 we require $k = 2$ and to make the image of D' have self-intersection number 9 we require $k = 3$, again a contradiction.

(ii) Suppose next that $1 < i < k$ $(k \geq 3)$. Blow down E_1, then sequentially blow down the images of $B_4, \ldots, B_1$. Suppose first that the image C'_i of C_i under the blowing down map is not a (-1)-curve. Then, arguing as in the proof of Proposition 3.1, one can show that E_2 must intersect the last curve C_k in the string attached to D and it must necessarily intersect F', the image of F. It follows that $i = 1$, otherwise, after blowing down E_2 and then sequentially blowing down the images of $C_k, \ldots, C_1$, the image of F' would have more than one singularity, contradicting Lemma 2.6. Since $i > 1$ is assumed, we reached a contradiction. Thus C'_i must be a (-1)-curve. Now blow down C'_i. Note that the images of the curves C_{i-1} and C_{i+1} must be the last two curves (in some blowing down process) of the string attached to D to get blown down, otherwise the image of F' after completing the blowing down process will have more than one singular point, a contradiction. Now there are two cases to consider: $E_2 \cdot F' = 0$ or $E_2 \cdot F' = 1$.

Suppose that $E_2 \cdot F' = 0$. Then it is easy to see that after the blowing down process has been carried out, the image of F' will have self-intersection number 8, which contradicts the fact that F should blow down to a cubic in $\mathbb{CP}^2$.

Suppose that $E_2 \cdot F' = 1$. Then E_2 must be disjoint from the string attached to D. In order to make D singular, $E_2 \cdot D$ must necessarily be 2. It is now easy to check that, after carrying out the blowing down process, the intersection number of the images of the curves F and D will be less than 9, which contradicts the fact that they should blow down to a pair of cubics in $\mathbb{CP}^2$.

(iii) Finally assume that $i = k$ $(k \geq 2)$. Blow down E_1, then sequentially blow down the images of $B_4, \ldots, B_1$ and then the images of $C_k, \ldots, C_{l+1}$ until the resulting string $C'_1, \ldots, C'_l$ attached to D' (the image of D) is minimal. If a nonempty string remains, then, as before, E_2 must intersect the last curve C'_l in the string and the curves F', the image of F, and D', each once transversally. Then blowing down E_2 and then the image of C'_l, we find that l must be 1, otherwise the image of F', after completing the blowing down process, would have more than one singular point, contradicting Lemma 2.6. It follows that the intersection number of the images of F' and D', after completing the blowing down process, will be 8, contradicting the fact that they should blow down to a pair of cubics in $\mathbb{CP}^2$.

If $l = 1$, that is the whole string attached to D gets sequentially blown down after blowing down E_1, then one can check that the intersection numbers of E_2 and the images of F' and D' must both be 2. Again it follows that, after completing the blowing down process, the intersection numbers of the images of F' and D' will be 8, a contradiction. This completes **IIA** and hence we conclude that

IIB. $E_1 \cdot (C_1 \cup \cdots \cup C_k) = 0$. We claim that $E_1 \cdot F = 1$. To see this, suppose, for a contradiction, that $E_1 \cdot F = 0$. Then we have $E_1 \cdot D = 0$ or 1. Blow down E_1 and then sequentially blow down the images of the curves $B_4, \ldots, B_1$. Then the image F' of F will still be smooth. It is thus necessary to have $E_2 \cdot F' = 2$, otherwise the image of F will be smooth or have the wrong type of singularity. But then the string $C_1, \cdots, C_k$ must be empty, otherwise E_2 would have to intersect it and thus blowing down would create additional singular points on the image of F, a contradiction. It follows that, after completing the blowing down process, the intersection number of the images of F and D will be less than 9, a contradiction. This verifies $E_1 \cdot F = 1$.

Now blowing down E_1 and then sequentially blowing down the B_i, we find that the image of F becomes a rational curve with a single nodal point and having self-intersection number 9. It follows that E_2 cannot intersect F and that E_1 must intersect D once transversally. Let F', D' denote the images of F and D, respectively, after blowing down E_1 and the B_i. It is then easy to check that $F' \cdot D' = 9$. Now the only possibility for E_2, by the argument in the proof of Proposition 3.1, is that $E_2 \cdot C_k = 1$ and $E_2 \cdot D = 1$. For each value of k, the blowing down process now fixes c and $c_1, \ldots, c_k$, providing the result. $\square$

4.2 The family $\mathcal{B}$

We next consider four-legged graphs in the family $\mathcal{B}$: the generic four-legged

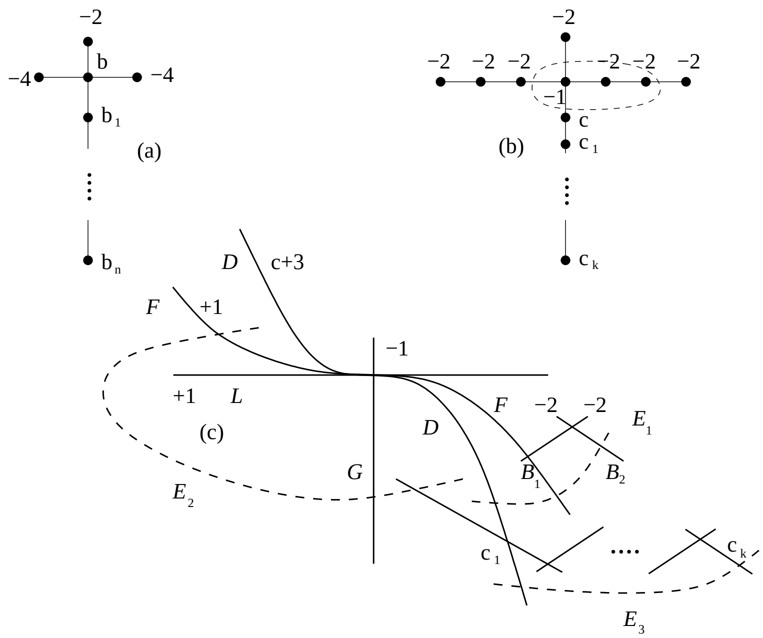

Figure 12: The four-legged graphs in $\mathcal{B}$.

member of this family is given by Figure 12(a) with the dual graph given by Figure 12(b). After three blow-downs we obtain the configuration K depicted in Figure 12(c). As before, Z is the closed symplectic 4-manifold we get by gluing the compactifying divisor $W_{\Gamma'}$ (containing K) to a weak symplectic $\mathbb{Q}$HD filling of (Y_Γ, ξ_Γ). It is easy to check that there must be three (-1)-curves, say E_1, E_2, E_3, not contained in the strings B_1, B_2 and $C_1, \ldots, C_k$, such that, after blowing down these three (-1)-curves, the images of the curves in the strings B_1, B_2 and $C_1, \ldots, C_k$ can be sequentially blown down and in the process F and D will be transformed to a pair of cubics in $\mathbb{CP}^2$ and the images of G and L will be lines. Since in the blowing down process the string B_1, B_2 will eventually transform into a string which can be sequentially blown down, one of the (-1)-curves E_1, E_2, E_3, must intersect $B_1 \cup B_2$; assume that this curve is E_1.

Proposition 4.2 *Under the hypothesis of the existence of a $\mathbb{Q}$HD filling, we get that E_1 intersects D, F and B_2, E_2 intersects F, G and C_1, while E_3*

intersects D and C_k. The corresponding framings are given as $c = -k-2$, $c_1 = -3$ and $c_2 = \ldots = c_k = -2$. In particular, the resolution graph is of the form given by Figure 2(b).

Proof Note that E_1 must be disjoint from G, otherwise blowing down E_1 and then sequentially blowing down the images of B_1 and B_2 the image of G would be either singular or would have self-intersection number 2, which contradicts the fact that G should blow down to a line in $\mathbb{CP}^2$. Since one of the E_i must necessarily intersect G we may assume that $E_2 \cdot G = 1$. We now consider the two possibilities: $E_1 \cdot B_i = 1$ for $i = 1, 2$.

Case I: $E_1 \cdot B_1 = 1$. The curve E_1 must necessarily be disjoint from F, otherwise the image of F after completing the blowing down process would have more than one singular point which is impossible for a cubic in $\mathbb{CP}^2$. We consider the two possibilities: $E_1 \cdot (C_1 \cup \cdots \cup C_k) = 1$ or $E_1 \cdot (C_1 \cup \cdots \cup C_k) = 0$.

IA. $E_1 \cdot (C_1 \cup \cdots \cup C_k) = 1$. Suppose that $E_1 \cdot C_i = 1$. Note that the image of C_i must be the last curve of the string attached to D to get blown down, since blowing down the the image of C_i will make the image of F singular so that if there are any remaining curves in the string then these will create additional singularities on the image of F when they are blown down, a contradiction.

Suppose that $E_2 \cdot (C_1 \cup \cdots \cup C_k) = 0$. Then the condition that G blows down to a $(+1)$-curve in $\mathbb{CP}^2$, forces us to have $E_3 \cdot G = 1$. But then necessarily $E_3 \cdot (C_1 \cup \cdots \cup C_k) = 0$. Thus the string $C_1, \ldots, C_k$ must have length 1. Now E_2 and E_3 must necessarily intersect F, each once transversally, otherwise the intersection number of the images of F and G will not be 3. It is also necessary that the intersection number of one of E_2 or E_3 and D be 2 and the other be 0 to meet the requirements that the image of D be singular and that the images of D and G have intersection number 3. But then after completing the blowing down process we will find that the images of D and F have intersection number 7, a contradiction.

Suppose that $E_2 \cdot (C_1 \cup \cdots \cup C_k) = 1$. Note that E_2 must necessarily intersect C_i, the last curve in the string to get blown down, otherwise the image of G after repeatedly blowing down will have self-intersection number greater than 1, a contradiction. Note also that E_2 must be disjoint from F, otherwise blowing down the image of C_i will lead to a triple point on the image of F, a contradiction. Now consider the (-1)-curve E_3. If E_3 intersects $C_1 \cup \cdots \cup C_k$, then E_3 will be disjoint from F. In such a case, after completing the blowing down process, the image of F will be a 7-curve, a contradiction. If E_3 is disjoint from $C_1 \cup \cdots \cup C_k$, then $E_3 \cdot F$ can be 0 or 1. In either case, after completing the blowing down process, the image of F will have self-intersection number at most 8, again a contradiction. This argument concludes the analysis of the case $E_1 \cdot (C_1 \cup \cdots \cup C_k) = 1$.

IB. $E_1 \cdot (C_1 \cup \cdots \cup C_k) = 0$. Suppose that $E_2 \cdot (C_1 \cup \cdots \cup C_k) = 0$ as well. As before, it implies that $E_3 \cdot G = 1$. It follows that E_1, E_2, E_3 will be disjoint from $C_1 \cup \cdots \cup C_k$. But this means that the string must be empty, which is never the case.

Suppose now that $E_2 \cdot (C_1 \cup \cdots \cup C_k) = 1$. Then E_2 must intersect the last curve of the string to get blown down. Also we must necessarily have $E_3 \cdot G = 0$. If E_3 is disjoint from $C_1 \cup \cdots \cup C_k$, then the string must have length 1. It follows that, after completing the blowing down process, the intersection number of the images of D and G will be either 2 or 4, depending on whether $E_2 \cdot D = 0$ or 1, a contradiction in both cases. So we may assume that $E_3 \cdot (C_1 \cup \cdots \cup C_k) = 1$. Note that the only way an appropriate singularity on the image of D can arise is if $E_3 \cdot D = 1$. It follows that we must have $E_3 \cdot C_k = 1$ and $E_2 \cdot C_1 = 1$. Note also that we necessarily have $E_2 \cdot F = 1$, otherwise the intersection number of the images of F and G will not be 3. If $E_3 \cdot F = 0$, then, after completing the blowing down process, the intersection number of the images of F and D will be at most 8, a contradiction. If $E_3 \cdot F = 1$, then after completing the blowing down process, the intersection number of the images of F and G will be 4, again a contradiction. This last observation concludes the discussion of Case I and shows that $E_1 \cdot B_1 = 1$ is not possible.

Case II: $E_1 \cdot B_2 = 1$. Again we consider the two possibilities: $E_1 \cdot (C_1 \cup \cdots \cup C_k) = 1$ or $E_1 \cdot (C_1 \cup \cdots \cup C_k) = 0$.

IIA. $E_1 \cdot (C_1 \cup \cdots \cup C_k) = 1$. Note that $E_1 \cdot F = 0$, otherwise when the image of C_i is eventually blown down the image of F will develop more than one singularity, a contradiction. For a similar reason we also have $E_1 \cdot D = 0$. Suppose that $E_1 \cdot C_i = 1$. We consider the possibilities for i.

(i) $i = 1$. Suppose that $E_2 \cdot (C_1 \cup \cdots \cup C_k) = 0$. Then the condition that the image of G, after completing the blowing down process, be a $(+1)$-curve forces us to have $E_3 \cdot G = 1$ and $E_3 \cdot (C_1 \cup \cdots \cup C_k) = 0$. Also, the condition that the images of F and D have nodes and that the intersection numbers of the images of F and G, and D and G be 3 forces us to have $E_2 \cdot F = 2$, $E_2 \cdot D = 0$ and $E_3 \cdot F = 0$, $E_3 \cdot D = 2$, or vice-versa. Finally, the condition that F will have self-intersection number 9 forces us to have $k = 3$. But then it follows that the intersection number of the images of F and D, after completing the blowing down process, will be 6, a contradiction.

Suppose that $E_2 \cdot (C_1 \cup \cdots \cup C_k) = 1$. Then E_2 will intersect the last curve of the string to get blown down. Note that $E_2 \cdot D = 0$, otherwise, after completing the blowing down process, the intersection number of the images of D and G will be greater than 3, a contradiction. Similarly $E_2 \cdot F = 0$. Note also that E_3 is necessarily disjoint from G. Thus if E_3 is also disjoint from the string or from D, it follows that the intersection number of D and G after completing the blowing down process will be 2, a contradiction. Thus E_3 necessarily intersects the string and D. In fact, we require that $E_3 \cdot C_k = 1$. Now the condition that the image of F have a singularity forces us to have $E_3 \cdot F = 1$. Also, the condition that the image of F have self-intersection number 9 forces us to have $k = 3$. However, if $k = 3$, then the intersection number of the images of F and D, after completing the blowing down process, will be 10, a contradiction.

(ii) $1 < i < k$ $(k \geq 3)$. If $E_2 \cdot (C_1 \cup \cdots \cup C_k) = 0$, then, as before, we require that $E_3 \cdot G = 1$, $E_3 \cdot (C_1 \cup \cdots \cup C_k) = 0$. It follows that we must have $k = 3$, otherwise, after completing the blowing down process, the image of F will either have a singularity of multiplicity greater than two or will have more than one singular point, neither of which is permitted for a cubic in $\mathbb{CP}^2$. Now the condition that the images of F and G have intersection number 9 forces us to have $E_2 \cdot F = E_3 \cdot F = 1$. But then the image of F, after completing the blowing down process, will have self-intersection number 10, a contradiction. Thus $E_2 \cdot (C_1 \cup \cdots \cup C_k) = 1$ and E_2 intersects the last curve of the string that gets blown down. Note that, as in the previous case, $E_2 \cdot F = 0$, $E_2 \cdot D = 0$.

Suppose that $C_i \cdot C_i = -4$. Then the image of C_i will be a (-1)-curve, after blowing down E_1 and then sequentially blowing down the images of B_2, B_1. It follows that the images of C_{i-1}, C_{i+1} must be the last two curves of the string attached to D to get blown down. Since $E_2 \cdot (C_1 \cup \cdots \cup C_k) = 1$, note that, as before, we require $E_3 \cdot (C_1 \cup \cdots \cup C_k) = 1$, $E_3 \cdot D = 1$. It follows that we must have $E_3 \cdot C_k = 1$. Note that $E_3 \cdot F = 0$, otherwise the image of F after completing the blowing down process would have more than one singular points, a contradiction. Now, after completing the blowing down process, we find that the intersection number of the images of D and F will be 8, a contradiction.

Suppose that $C_i \cdot C_i < -4$. Then after blowing down E_1 and then sequentially blowing down B_2, B_1, the image of C_i will not be a (-1)-curve. As before, we can show that $E_3 \cdot C_k = 1$, $E_3 \cdot D = 1$. The condition that F become singular forces us to have $E_3 \cdot F = 1$. Now after completing the blowing down process we see that the F will have more than one singularity, since $i > 1$, a contradiction.

(iii) $i = k$ $(k \geq 2)$. If $E_2 \cdot (C_1 \cup \cdots \cup C_k) = 0$, then, as before, we require that $E_3 \cdot G = 1$, $E_2 \cdot (C_1 \cup \cdots \cup C_k) = 0$. To obtain the correct types of singularities on the images of F and D and to meet the requirement that the intersection numbers of the images of F and G, and D and G, after completing the blowing down process, be 3, we require that $E_2 \cdot F = 2$, $E_3 \cdot F = 0$ or $E_2 \cdot F = 0$, $E_3 \cdot F = 2$ and likewise for D. It follows that after completing the blowing down process the intersection number of the images of F and D will be 8, a contradiction. So $E_2 \cdot (C_1 \cup \cdots \cup C_k) = 1$ and E_2 intersects the last curve of the string that gets blown down.

Suppose that $E_3 \cdot (C_1 \cup \cdots \cup C_k) = 0$ or $E_3 \cdot D = 0$. Then since $E_3 \cdot G = 0$, after completing the blowing down process the intersection number of the images of D and G will be 2, a contradiction. So $E_3 \cdot (C_1 \cup \cdots \cup C_k) = 0$ and $E_3 \cdot D = 1$. Similarly we can check that $E_3 \cdot F = 1$.

Suppose that $E_3 \cdot C_j = 1$ for $j < k$. Blow down E_1, E_2, E_3 and then sequentially blown down the images of B_2, B_1. Note then that, after the images of $C_k, C_{k-1}, \ldots, C_j$ have been sequentially blown down, the image of F will become singular. Also after the images of $C_j, C_{j-1}, \ldots, C_2$ have been sequentially blown down the image of D will become singular. Since the images of F and D should have exactly one singularity, the image of C_j must necessarily be the last curve of the string to get blown down. It follows that j must be 2. It is now easy to check that, after the blowing down process has been completed, the intersection number of the images of F and D will be 8, a contradiction.

Suppose that $E_3 \cdot C_k = 1$. Then once the image of C_k is blown down the image of F will become singular. It follows that k must be 1, contrary to assumption.

IIB. $E_1 \cdot (C_1 \cup \cdots \cup C_k) = 0$. If $E_2 \cdot (C_1 \cup \cdots \cup C_k) = 0$, then we must have $E_3 \cdot G = 1$ and $E_3 \cdot (C_1 \cup \cdots \cup C_k) = 0$. It follows that the string $C_1, \ldots, C_k$ must be empty, which is never the case. So $E_2 \cdot (C_2 \cup \cdots \cup C_k) = 1$ and E_2 intersects the last curve that gets blown down. We thus necessarily have $E_3 \cdot G = 0$.

Suppose that $E_1 \cdot F = 0$. If $E_2 \cdot F = 0$ also, then the only way that the image of F can have the correct type of singularity is if $E_3 \cdot F = 2$ and $E_3 \cdot (C_1 \cup \cdots \cup C_k) = 0$. But then, after completing the blowing down process, we find that the intersection number of the images of F and G will be 2, a contradiction. So $E_2 \cdot F = 1$. There are now two ways that the image of F can have the correct type of singularity: if $E_3 \cdot F = 1$ and $E_3 \cdot (C_1 \cup \cdots \cup C_k) = 1$ or if $E_3 \cdot F = 2$ and $E_3 \cdot (C_1 \cup \cdots \cup C_k) = 0$. In the former case, after completing the blowing down process, the intersection number of the images of F and G

will be 4, a contradiction. In the latter case, after completing the blowing down process, the intersection number of the images of D and G will be either 2 or 4 depending on whether $E_2 \cdot D = 0$ or 2, a contradiction in either case.

Suppose that $E_1 \cdot F = 1$. If $E_2 \cdot F = 0$, then, after completing the blowing down process, the intersection number of the images of F and G will be either 2 (if $E_3 \cdot F = 1$ and $E_3 \cdot (C_1 \cup \cdots \cup C_k) = 1$) or 1 (if $E_3 \cdot F = 0$ or $E_3 \cdot (C_1 \cup \cdots \cup C_k) = 0$), a contradiction in either case. So $E_2 \cdot F = 1$. Note now that if $E_3 \cdot F = 1$, then the self-intersection number of the image F, after completing the blowing down process, will be greater than 9, which is not possible for a cubic in $\mathbb{CP}^2$, implying that $E_3 \cdot F = 0$. Also if $E_2 \cdot D = 1$, then, after completing the blowing down process, the intersection number of the images of D and G will be greater than 3, a contradiction, hence we conclude that $E_2 \cdot D = 0$. Next note that if $E_3 \cdot (C_1 \cup \cdots \cup C_k) = 0$ or $E_3 \cdot D = 0$, then since $E_3 \cdot G = 0$, after completing the blowing down process, the intersection number of the images of D and G will be 2, a contradiction. So $E_3 \cdot (C_1 \cup \cdots \cup C_k) = 1$ and $E_3 \cdot D = 1$. It follows that we must have $E_3 \cdot C_k = 1$ and $E_2 \cdot C_1 = 1$. Also if $E_1 \cdot D = 0$, then, after completing the blowing down process, the intersection number of the images of D and F will be 5, a contradiction. So we must have $E_1 \cdot D = 1$. Thus the three (-1)-curves E_1, E_2, E_3 must be as given by the Proposition. $\square$

4.3 The family $\mathcal{A}$

Finally we consider four-legged graphs in the family $\mathcal{A}$. The generic four-legged member Γ of $\mathcal{A}$ is given in Figure 13(a) with the dual graph in (b). After three blow-downs we obtain the configuration K indicated in Figure 13(c). Suppose that Z is the closed symplectic 4-manifold we get by symplectically gluing the compactifying divisor $W_{\Gamma'}$ (containing K) to a weak symplectic $\mathbb{Q}$HD filling of Y_Γ. Then it is easy to check that there must be three (-1)-curves, say E_1, E_2, E_3, not contained in the string $C_1, \ldots, C_k$, such that, after blowing down these three (-1)-curves, the image of B can be blown down and the images of the curves in the string $C_1, \ldots, C_k$ can be sequentially blown down so that in the process F and D are transformed to a pair of cubics in $\mathbb{CP}^2$ and the images of L and A are lines. Since in the blowing down process B will be eventually transformed into a curve which can be blown down, one of the three (-1)-curves, call it E_1, must intersect B.

Proposition 4.3 *If a $\mathbb{Q}$HD filling exists in the situation described above, then either Γ' blows down to a 3-legged graph (and was treated earlier), or E_1 intersects D, F and B, E_2 intersects A, F and either C_1 or C_2 and*

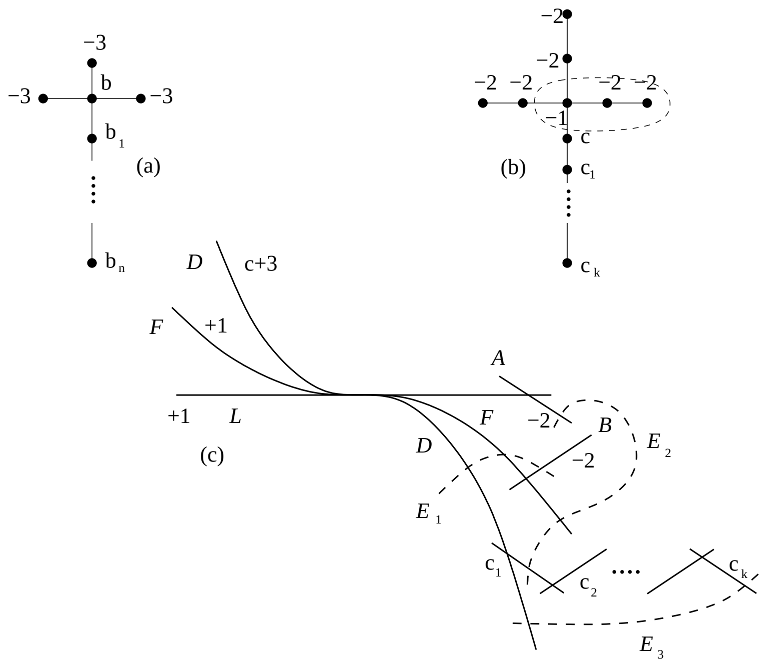

Figure 13: The four-legged graphs in $\mathcal{A}$. Proposition 4.3 allows another configuration for E_2 in (c), where it intersects C_2 instead of C_1.

E_3 intersects D and C_k. The corresponding framings in the latter case are given as $c = -k$, $c_2 = -3$ and $c_1 = c_3 = \ldots = c_k = -2$. In particular, the corresponding resolution graph is of the form given by Figure 2(a).

Proof Note that if $E_1 \cdot A = 1$, then $E_1 \cdot (C_1 \cup \cdots \cup C_k) = 0$, otherwise after blowing down E_1 and then the image of B, the image of A will become singular when the image of C_i is eventually blown down, where $E_1 \cdot C_i = 1$, which contradicts the fact that the image of A in $\mathbb{CP}^2$ will be a line. Thus at least one (-1)-curve different from E_1 should intersect A. Let us call this (-1)-curve E_2. We now begin the case-by-case analysis.

Case I: $E_1 \cdot (C_1 \cup \cdots \cup C_k) = 1$. Suppose that $E_1 \cdot C_i = 1$. In this case, by the argument above, we will necessarily have $E_1 \cdot A = 0$. Note that if $E_1 \cdot F = 1$, then after E_1 and the image of B are blown down, the image F' of F will be singular. However, the intersection number of the image C_i' of C_i and F' will be 2. Thus when the image of C_i' is eventually blown down the image of F' have a second singularity, which contradicts the fact that it must eventually blow down to a cubic in $\mathbb{CP}^2$. Thus $E_1 \cdot F = 0$. Also, we must have $E_1 \cdot D = 0$, otherwise, after repeatedly blowing down, the image of D will eventually have a triple point, which contradicts the fact that the image of D in $\mathbb{CP}^2$ should also be a cubic.

Note that if $E_2 \cdot (C_1 \cup \cdots \cup C_k) = 0$, then we must have have $E_3 \cdot A = 1$ and $E_3 \cdot (C_1 \cup \cdots \cup C_k) = 1$, since, after completing the blowing down process, the image of A should be a smooth curve of self-intersection number 1. Renumbering E_2 and E_3, if necessary, we may assume that $E_2 \cdot (C_1 \cup \cdots \cup C_k) = 1$. Suppose that $E_2 \cdot C_j = 1$. Notice that, in the blowing down process, the image of C_j must either be the last curve of the string attached to D to get blown down or it must be the penultimate curve to get blown down, since otherwise, after the blowing process is complete, the self-intersection number of the image of A will be greater than 1, a contradiction.

(i) $i = 1$.

(ia) Suppose first that the image C_j is the last curve of the string to get blown down. Then we must have $E_3 \cdot A = 1$, and hence $E_3 \cdot (C_1 \cup \cdots \cup C_k) = 0$. Now, since $E_1 \cdot D = 0$, there are two ways that an appropriate singularity can appear on image of D: either $E_2 \cdot D = 1$ or $E_3 \cdot D = 2$. Suppose that $E_2 \cdot D = 1$. Then $E_3 \cdot D = 0$, otherwise, after completing the blowing down process, the intersection number of the images of D and A would be greater than 3, a contradiction. We now have $E_2 \cdot F = 0$ or 1. If $E_2 \cdot F = 0$, then we must have $E_3 \cdot F = 2$, otherwise the image of F, after completing the blowing down process, would be smooth and rational, which is a contradiction. Now the condition that the self-intersection number of the image of F, after completing the blowing down process, will be 9, forces us to have $k = 2$. But then, after completing the blowing down process, the intersection number of the images of F and D will be 7, a contradiction. If $E_2 \cdot F = 1$, then $E_3 \cdot F = 0$, otherwise the intersection number of the images of F and A, after completing the blowing down process, would be greater than 3, a contradiction. Now, again, the condition that the self-intersection number of the image of F, after completing the blowing down process, will be 9, forces us to have $k = 3$. But then, after completing the blowing down process, the intersection number of the images of F and D will be 10, again a contradiction.

Suppose that $E_3 \cdot D = 2$. Then $E_2 \cdot D = 0$. We now have $E_2 \cdot F = 0$ or 1. If $E_2 \cdot F = 0$, then we must have $E_3 \cdot F = 2$. Now, as before, the condition that the self-intersection number of the image of F, after completing the blowing down process, will be 9, forces $k = 3$. But then, after completing the blowing down process, the intersection number of the images of F and D will be 10, a contradiction. If $E_2 \cdot F = 1$, then we must have $E_3 \cdot F = 0$. Thus, again, the condition that the self-intersection number of the image of F, after completing the blowing down process, will be 9, forces $k = 3$. And, this time, after completing the blowing down process, the intersection number of the images of F and D will be 7, again a contradiction.

(ib) The image of C_j is then the penultimate curve of the string to get blown down. Then we must have $E_3 \cdot A = 0$. Also, we must have $E_2 \cdot D = 0$, otherwise, after completing the blowing down process, the intersection number of the images of D and A would be greater than 3, a contradiction. Similarly, we must have $E_2 \cdot F = 0$.

Suppose that $E_3 \cdot (C_1 \cup \cdots \cup C_k) = 0$ or $E_3 \cdot D = 0$. Then, after completing the blowing down process, the intersection number of the images of D and A will be at most 2, a contradiction. So $E_3 \cdot (C_1 \cup \cdots \cup C_k) = 1$ and $E_3 \cdot D = 1$. If $E_3 \cdot C_l = 1$ for $l < k$, then we must have $l = k - 1$ and $j = k$, otherwise, after completing the blowing down process, the image of D will have more than one singular point, a contradiction. However, if $l = k - 1$ and $j = k$, then, after completing the blowing down process, the intersection number of the images of D and A will be 2, a contradiction. So we must have $E_3 \cdot C_k = 1$. Also we must have $E_3 \cdot F = 1$, otherwise the image of F, after completing the blowing down process will be smooth, a contradiction. Now, the condition that the self-intersection number of the image of F, after completing the blowing down process, will be 9, forces us to have $k = 3$. But then, after completing the blowing down process, the intersection number of the images of F and D will be 10, a contradiction.

(ii) $1 < i < k$ $(k \geq 3)$.

(iia) The image of C_j is last curve of the string to get blown down. Then we must have $E_3 \cdot A = 1$, and hence $E_3 \cdot (C_1 \cup \cdots \cup C_k) = 0$. If k is greater than 2, then, after completing the blowing down process, the image of F will either have a point of multiplicity greater than 2 or have more than one singular point, neither of which is possible for a cubic in $\mathbb{CP}^2$. Thus we must have $k = 3$ and thus $j = 1$ or 3. Also, we must have $E_2 \cdot F = 0$, otherwise, after completing the blowing down process, the image of F will have a triple point, a contradiction. Furthermore, we must have $E_3 \cdot F = 1$, otherwise, after completing the blowing down process, the intersection number of the images of F and A will be less that 3, a contradiction. Now the only way a singularity of the appropriate type can appear on the image of D is if $E_2 \cdot D = 1$ or $E_3 \cdot D = 2$.

Suppose first that $E_2 \cdot D = 1$. Then we must have $E_3 \cdot D = 0$, otherwise, after completing the blowing down process, the intersection number of the images of A and D will be greater than 3, a contradiction. It follows that, after completing the blowing down process, the intersection number of the images of F and D will be at most 8, which contradicts the fact that images of F and D in $\mathbb{CP}^2$ are a pair of cubics.

Suppose now that $E_3 \cdot D = 2$. Then we must have $E_2 \cdot D = 0$, otherwise, after completing the blowing down process, the intersection number of the images of D and A will be greater that 3, a contradiction. It follows that, after completing the blowing down process, the intersection number of the images of F and D will be at most 8, a contradiction.

(iib) The image of C_j is the penultimate curve of the string to get blown down. Then we must have $E_3 \cdot A = 0$ and $E_2 \cdot D = E_2 \cdot F = 0$. Also, we must have $E_3 \cdot (C_1 \cup \cdots \cup C_k) = 1$ and $E_3 \cdot D = 1$.

Suppose that $E_3 \cdot C_l = 1$ for $l < k$. Then the image of C_l must be the last curve of the string attached to D to get blown down. Indeed, it is easy to see that after the image of C_l is blown down, the image of the portion $C_{l+1}, \ldots, C_k$ of the the string must be a point, otherwise, after completing the blowing down process, the image of D will have more than one singular point. Thus we must have $i > l$ or $j > l$. In the former case, after the portion $C_l, \ldots, C_k$ of the the string has been collapsed to a point, the image of F will be singular and thus the image of C_l must be the last curve of the string to get blown down. In the latter case, since the image of C_j is the penultimate curve of the string to get blown down, C_l must be the last curve of the string to get blown down. Now again using the assumption that the image of C_j is the penultimate curve of the string to get blown down, we must have either $j < l$ or $j > l$. Suppose that $j < l$. Then we must have $i > l$. Also, we must have $E_3 \cdot F = 0$, otherwise, after completing the blowing down process, the image of F will have a singularity of multiplicity greater than 2, a contradiction. Now, after completing the blowing down process, the intersection number of the images of A and F will be 2, a contradiction. Suppose that $j > l$. Then, after completing the blowing down process, the intersection number of the images of A and D will be 2, again a contradiction.

Suppose that $E_3 \cdot C_k = 1$. Then we must have $E_3 \cdot F = 0$ or 1. Suppose that $E_3 \cdot F = 0$. Then, in the blowing down process, the images of the curves C_{i-1} and C_{i+1} must be the last two curves of the string attached to D to get blown down. It follows that we must have $i = 2$. It is now easy to check that, after completing the blowing down process, the image of F will have self-intersection number 8, a contradiction. Suppose that $E_3 \cdot F = 1$. Then the image of C_i must be the last curve of the string to get blown down. It follows that we must have $i = 2$ and $j = 1$. We now find that, after completing the blowing down process, the intersection number of the images of F and A will be 2, a contradiction.

(iii) $i = k$ $(k \geq 2)$.

(iiia) The image of C_j is last curve of the string to get blown down. Then we must have $E_3 \cdot A = 1$, and hence $E_3 \cdot (C_1 \cup \cdots \cup C_k) = 0$. Also we must have $j = 1$. Now the only way a singularity of the appropriate type can appear on the image of D is if $E_2 \cdot D = 1$ or $E_3 \cdot D = 2$.

Suppose that $E_2 \cdot D = 1$. Then we must have $E_3 \cdot D = 0$. Now we have $E_2 \cdot F = 0$ or 1. If $E_2 \cdot F = 0$, then it is easy to see that, after completing the blowing down process, the intersection number of the images of F and D will be 5, a contradiction. If $E_2 \cdot F = 1$, then one can check that, after completing the blowing down process, the intersection number of the images of F and D will be 8, again a contradiction.

Suppose that $E_3 \cdot D = 2$. Then we must have $E_2 \cdot D = 0$. Again we have $E_2 \cdot F = 0$ or 1. If $E_2 \cdot F = 0$, then we must have $E_3 \cdot F = 2$. It follows that, after completing the blowing down process, the intersection number of the images of F and D will be 8, a contradiction. If $E_2 \cdot F = 1$, then we must have $E_3 \cdot F = 0$. In this case, after completing the blowing down process, the intersection number of the images of F and D will be 5, again a contradiction.

(iiib) The image of C_j is the penultimate curve of the string to get blown down. Then we must have $E_3 \cdot A = 0$ and $E_2 \cdot D = E_2 \cdot F = 0$. Also, we must have $E_3 \cdot (C_1 \cup \cdots \cup C_k) = 1$ and $E_3 \cdot D = 1$. Furthermore, we must have $E_3 \cdot F = 1$, otherwise, after completing the blowing down process, the image of F would be smooth, a contradiction. Now note that if $l \neq 1$, then we must have $l = 2$ and $j = 1$, otherwise, after completing the blowing down process, the image of F will have more than one singular point, a contradiction. If $l = 1$, then C_1 must be the last curve to get blown down, otherwise, after completing the blowing down process, the image of D will have more than one singular point, a contradiction. Thus we must have $j = 2$. It now follows that, after completing the blowing down process, the intersection number of the images of D and A will be 2, a contradiction. If $l = 2$ and $j = 1$, then C_2 must be the last curve to get blown down and in this case it follows that, after completing the blowing down process, the intersection number of the images of F and A will be 2, again a contradiction.

Case II: $E_1 \cdot (C_1 \cup \cdots \cup C_k) = 0$.

IIA. $E_1 \cdot A = 1$. Since we are assuming that $E_2 \cdot A = 1$ also, we will necessarily have $E_2 \cdot (C_1 \cup \cdots \cup C_k) = 0$ and $E_3 \cdot A = 0$. Also, since the string $C_1 \ldots, C_k$ is nonempty for every 4-legged graph Γ in $\mathcal{A}$, we must have $E_3 \cdot (C_1 \cup \cdots \cup C_k) = 1$. Now if $E_1 \cdot D = 0$, then, after completing the blowing down process, the intersection number of the images of D and A will be at most 2, a contradiction. It follows that we must have $E_1 \cdot D = 1$ and thus we must also have $E_2 \cdot D = 1$.

Suppose that $E_1 \cdot F = 1$. Then we must have $E_2 \cdot F = 0$. If $E_3 \cdot F = 0$ also holds, then, after completing the blowing down process, the self-intersection number of the image of F will be 6, a contradiction. So we must have $E_3 \cdot F = 1$ and k must be 2. But then, after completing the blowing down process, the intersection number of the images of F and D will be 10, a contradiction.

Suppose that $E_1 \cdot F = 0$. Then we must have $E_2 \cdot F = 2$. Again we require $E_3 \cdot F = 1$ and $k = 2$. It thus follows again that, after completing the blowing down process, the intersection number of the images of F and D will be 10, a contradiction as before.

IIB. $E_1 \cdot A = 0$. We may now assume $E_2 \cdot (C_1 \cup \cdots \cup C_k) = 1$, since if $E_2 \cdot (C_1 \cup \cdots \cup C_k) = 0$, then we would necessarily have $E_3 \cdot A = 1$ and $E_3 \cdot (C_1 \cup \cdots \cup C_k) = 1$, and we would just renumber the (-1)-curves. Suppose that $E_2 \cdot C_j = 1$. It follows that, in the blowing down process, the image of C_j is either the last curve of the string to get blown down or the penultimate curve to get blown down.

(i) Suppose first that the image of C_j is last curve of the string to get blown down. Then we must have $E_3 \cdot A = 1$ and $E_3 \cdot (C_1 \cup \cdots \cup C_k) = 0$. Since we are assuming that $E_1 \cdot A = 0$, we must have that $k = 1$. Now if $E_2 \cdot F = 0$, then, after completing the blowing down process, the intersection number of the images of A and F will be at most 2, a contradiction. So $E_2 \cdot F = 1$ and thus $E_3 \cdot F = 1$ also. It follows that we must have $E_1 \cdot F = 1$, otherwise, after completing the blowing down process, the image of F would be smooth or have more than one singularity, a contradiction in both cases.

Suppose that $E_2 \cdot D = 1$. Then we must have $E_3 \cdot D = 0$. Note also that we must have $E_1 \cdot D = 1$, otherwise, after completing the blowing down process, the intersection number of the images of F and D will be different from 9, a contradiction. It follows that D must have self-intersection number 2 and C_1 must have self-intersection number -2. It is easy to see that in this case Γ is just the unique three-legged graph in the family $\mathcal{A}$ with four vertices and we already know that in this case the corresponding contact 3-manifold (Y_Γ, ξ_Γ) admits a $\mathbb{Q}$HD filling.

Suppose that $E_2 \cdot D = 0$. Then we must have $E_3 \cdot D = 2$. Again we can check that we must have $E_1 \cdot D = 1$. As in the previous case, it follows that D must have self-intersection number 2 and C_1 must have self-intersection number -2, and this case has already been considered.

(ii) The image of C_j is the penultimate curve of the string to get blown down. Then we must have $E_3 \cdot A = 0$. Note that if $E_2 \cdot D = 1$, then, after completing the blowing down process, the intersection number of the images of A and D will be 4, a contradiction. Thus $E_2 \cdot D = 0$. Also we must have $E_3 \cdot (C_1 \cup \cdots \cup C_k) = 1$ and $E_3 \cdot D = 1$, otherwise, after completing the blowing down process, the intersection number of the images of A and D will be at most 2, a contradiction. Suppose that $E_3 \cdot C_l = 1$. Now if $l < k$, then we must have $k = 2$, $l = 1$ and $j = 2$. But then, after completing the blowing down process, the intersection number of the images of A and D will be 2, a contradiction. So $l = k$. It follows that we must have $j = 1$ or 2. Now note that if $E_2 \cdot F = 0$, then, after completing the blowing down process, the intersection number of the images of A and F will be at most 2, a contradiction. So we must have $E_2 \cdot F = 1$. It also follows that we must have $E_3 \cdot F = 0$, otherwise, after completing the blowing down process, the intersection number of the images of A and F will be greater than 3, a contradiction. We now must have $E_1 \cdot F = 1$, otherwise, after completing the blowing down process, the image of F will be smooth, a contradiction. For each value of k and for $j = 1, 2$, the blowing down process now fixes $c, c_1, \ldots, c_k$, as stated in the Proposition. □

Now we are ready to give the proof of the second main result of the paper.

Proof of Theorem 1.6 Consider a spherical Seifert singularity S_Γ with minimal good resolution graph having at least four legs (and central framing < -2). Once again, the existence of a $\mathbb{Q}$HD smoothing implies the existence of a $\mathbb{Q}$HD filling of the Milnor fillable contact structure ξ_Γ on the link Y_Γ showing the implication (1) $\Rightarrow$ (2). Suppose now that (Y_Γ, ξ_Γ) admits a $\mathbb{Q}$HD filling. By Theorem 2.7, we get that Γ is a 4-legged graph in $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$. Therefore the combination of Propositions 4.1, 4.2 and 4.3 implies (2) $\Rightarrow$ (3). Finally (3) $\Rightarrow$ (1) follows from the results of [16, Examples 8.7, 8.12 and 8.14], cf. also [18]. These existence results then conclude the proof of the theorem. □

For the sake of completeness we provide curve configurations in $\mathbb{CP}^2$ with the property that repeated blow-ups provide the configurations of Figures 11(c), 12(c) and 13(c), and hence, by Pinkham's result [15] (as formulated in [16, Theorem 8.1]), we get an alternative proof of the existence of $\mathbb{Q}$HD smoothings. Below we will restrict ourselves to the description of the curves and their intersection patterns, and leave it to the reader to check that an appropriate blow-up sequence restores the diagrams listed above.

Let D be the cubic curve defined by the equation

$$f(x,y,z)=y^2z-x^3-x^2z$$

and L the line $\{z=0\}$.

In order to find a configuration which can be blown up to Figure 11(c), let us add the cubic D_1 given by the equation

$$f_1(x,y,z)=y^2z+\frac{1}{2}xyz+yz^2-\frac{9}{8}x^3-2x^2z-xz^2$$

to L and D. The curves D and D_1 are rational nodal cubics with nodes at $[0:0:1]$ and $[-\frac{2}{3}:-\frac{1}{3}:1]$, respectively. It is easy to check that both D and D_1 are triply tangent to L at the point $[0:1:0]$ and are also triply tangent to each other at $[0:1:0]$ and have intersection multiplicity 6 at the point $[0:0:1]$.

For finding the configuration providing the base for Figure 12(c), we consider L and D as before, together with D_2 given by the equation

$$f_2(x,y,z)=y^2z+2xyz+2yz^2-2x^3-4x^2z-2xz^2.$$

The curve D_2 is a rational nodal cubic with a node at $[-1:0:1]$, and L, D and D_2 are pairwise triply tangent at $[0:1:0]$. Also D and D_2 intersect at $[0:0:1]$ with intersection multiplicity 4 and at $[-1:0:1]$ with intersection multiplicity 2. Consider, furthermore, L_1 given by the equation $\{x+z=0\}$. It passes through the point $[0:1:0]$ and is tangent to D at $[-1:0:1]$.

Finally we describe a configuration from which repeated blow-ups result in the configuration of Figure 13(c). Once again, consider L and D as before, together with the cubic D_3 given by the equation $f_3(x,y,z)=$

$$y^2z+(1-i\sqrt{3})xyz+\frac{4}{9}(3-i\sqrt{3})yz^2+\frac{1}{2}(-1+i\sqrt{3})x^3+(-2+i\sqrt{3})x^2z-\frac{4}{9}(-3+i\sqrt{3})xz^2.$$

This curve is a rational nodal cubic with a node at $[-\frac{4}{3}:-\frac{4}{9}i\sqrt{3}:1]$. The line L and the curves D, D_3 are pairwise triply tangent at $[0:1:0]$. Also the curves D and D_3 intersect at each of the points $[0:0:1]$ and $[-\frac{4}{3}:-\frac{4}{9}i\sqrt{3}:1]$ with intersection multiplicity 3. Let N be the line $\{y-i\sqrt{3}(x+\frac{8}{9}z)=0\}$; it is triply tangent to D at $[-\frac{4}{3}:-\frac{4}{9}i\sqrt{3}:1]$ and intersects D_3 at the same point with intersection multiplicity 3.

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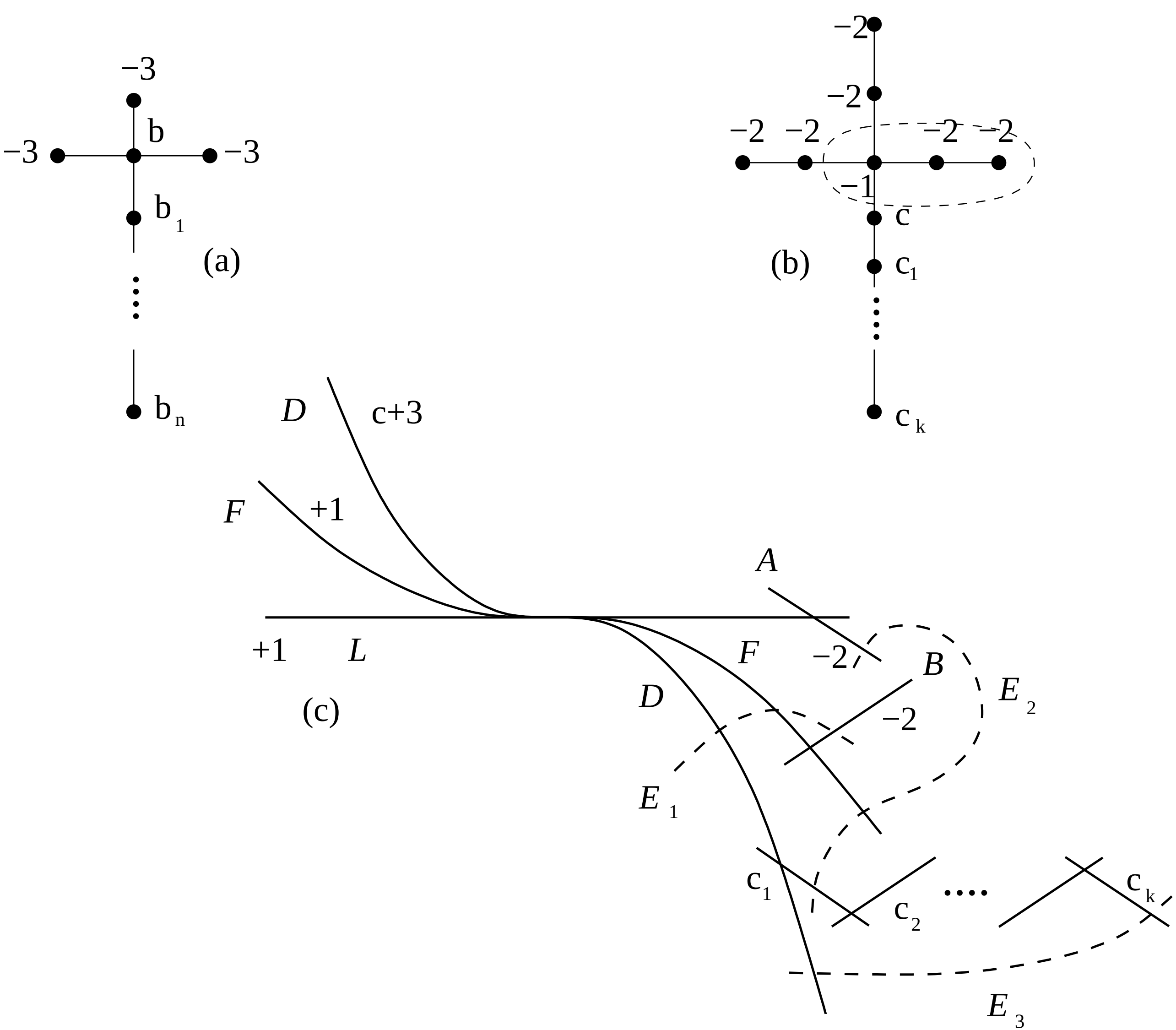

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