Gravitational charges of *transverse* asymptotically AdS spacetimes

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Using Killing-Yano symmetries, we construct conserved charges of spacetimes that asymptotically approach to the flat or Anti-de Sitter spaces *only in certain* directions. In D dimensions, this allows one to define gravitational charges (such as mass and angular momenta densities) of p-dimensional branes/solitons or any other extended objects that curve the transverse space into an asymptotically flat or AdS one. Our construction answers the question of what kind of charges the antisymmetric Killing-Yano tensors lead to.

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I. INTRODUCTION

The use of background symmetries (such as "Killing vector symmetries") for defining conserved charges in gravitational theories - those arising from the extremization of an action or those endowed with "Bianchi identities" - has borne much fruit. Two immediate consequences of such definitions are: the background gauge invariance of the charges and the vanishing of the background charge. One then computes the charges of those spacetimes that asymptotically approach the background. It is important to note that these charges are expressible as surface integrals at the spatial boundary of spacetime. What is implicit in this construction is that the asymptotic structure of the spacetime is either flat or AdS, be it locally or not, and the integrations are carried out on $(D-2)$-dimensional "spheres" or tori (or the products of these) (see e.g. [1–3]).

These definitions allow for the calculation of the charges of spacetimes (such as black holes, localized solitons and so on) that approach to flat or AdS geometries in "all dimensions". However, a naive application of these charge definitions to spacetimes that approach to flat or AdS spaces only in certain *transverse* directions immediately leads to trivial divergences. For example, the "mass" of an extended solution, such as a p-brane, which is infinite in specific directions and curves the *transverse* space into flat or AdS geometry, would be divergent. Thus, the question whether one can construct a reasonable charge definition which only considers the transverse part of such spacetimes is of relevance.

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An important step in this direction, for transverse asymptotically flat case, was taken by Kastor and Traschen [4] who used background antisymmetric Killing-Yano tensors, instead of background Killing vectors. Quite interestingly, unlike the usual Killing charges (such as energy and angular momentum) which are built out of the linearization of the Einstein's equations, Killing-Yano charges are constructed from a new antisymmetric current which makes no reference to Einstein's equations [4]. Kastor and Traschen termed these "new" conserved quantities as Y-ADM charges. For rank n Killing-Yano tensors, they worked out the case when the transverse part of the spacetime is asymptotically flat. In a follow-up paper [5], they discussed the conditions necessary for the positivity of Y-ADM "mass". One very important result of their work is that even though they have constructed these Y-ADM charges from new antisymmetric currents and Killing-Yano tensors, making no reference to Einstein's equations, they obtained the usual ADM expression, except that the integrations are now to be carried out only in transverse directions instead of the usual $(D-2)$-dimensional spheres or tori. Thus, for example, as far as the "mass-like" Y-ADM charge is concerned, what they ended up with is just the mass density of, say, a p-brane that either extends to infinity in p spatial directions or is periodic in certain directions.

We note that Killing-Yano tensors, unlike the Killing vectors, are not related to the isometries of the metric; thus they do not lead to the already known symmetries and conserved quantities. However, in different contexts they have been shown to yield various nontrivial constants of motion which are related to the separability of the Hamilton-Jacobi and wave equations. The most important application of this is the Kerr-Newman metric in four dimensions [6]. [See Gibbons et al. [7] for more details and the connection between Killing-Yano tensors and a nontrivial supersymmetry which is related to the square root of a constant of motion (not the usual Hamiltonian) derived from a Killing-Yano tensor.] Here we shall give another use of these tensors. The main point is that, if a spacetime admits such objects, one can construct new conserved currents [4] whose physical meaning we try to interpret here.

One important question which was mentioned by Kastor and Traschen is finding the surface integral expressions of the Killing-Yano charges for transverse asymptotic AdS spacetimes. Our main task in this paper will be to carry out in full detail the construction of these conserved quantities for such spacetimes admitting rank n Killing-Yano tensors. We shall call these Y-AD (Yano-Abbott-Deser) charge densities. In the next section, which is the bulk of our paper, we give the construction of these and present an example of the long axisymmetric Weyl rod.

II. KILLING-YANO CHARGES OF TRANSVERSE ASYMPTOTICALLY AdS SPACETIMES

Similar to the construction of the background Killing charges for asymptotically AdS spacetimes [2, 3], here we will be interested in transverse asymptotically AdS spacetimes that asymptotically admit completely antisymmetric tensor fields which play an analogous role to the usual background Killing symmetries. For clarity of the discussion, we will first concentrate on the rank 2 case in what follows. Later, we will generalize these results to the case of arbitrary rank n Killing-Yano tensors.

Let us consider a D-dimensional spacetime $\bar{g}_{ab}$, which we call 'the background spacetime'. A completely antisymmetric rank n tensor $\bar{f}_{a_1 a_2 \ldots a_n}$ is a Killing-Yano tensor of this spacetime if it satisfies

$$\bar{\nabla}_a \, \bar{f}_{ba_2 \ldots a_n} + \bar{\nabla}_b \, \bar{f}_{aa_2 \ldots a_n} = 0 \, . \tag{1}$$

Note that when rank equals 1, this reduces to the usual Killing vector definition. The spacetimes g_{ab} whose Killing-Yano charges we will compute do not necessarily admit exact Killing-Yano tensors. However, we demand that the metric g_{ab} can be asymptotically split into a background plus a perturbation:

$$g_{ab} \equiv \bar{g}_{ab} + h_{ab} \, . \tag{2}$$

We require that for certain transverse directions (whose number is less than or equal to $(D-2)$), g_{ab} asymptotically admits Killing-Yano tensors due to this splitting and the assumption that h_{ab} vanishes sufficiently fast at the spatial infinity. Let us take the background spacetime $\bar{g}_{ab}$ to be AdS, which obeys

$$\bar{R}_{abcd} = \frac{2\Lambda}{(D-1)(D-2)} \left(\bar{g}_{ac} \, \bar{g}_{bd} - \bar{g}_{ad} \, \bar{g}_{bc} \right) , \quad \bar{R}_{ab} = \frac{2\Lambda}{(D-2)} \, \bar{g}_{ab} \, , \quad \bar{R} = \frac{2\Lambda D}{(D-2)} \, .$$

In $D = 4$, the AdS spacetime admits 10 Killing-Yano tensors of rank 2 [8, 9]. For the number of Killing-Yano tensors of rank n in D dimensions, see (2.4) of [4].

For the rank 2 case, one can easily prove the following identities, which we freely make use of in the calculation of (10) below:

$$\bar{\nabla}_a \, \bar{f}^{ab} \; = \; 0 \, , \qquad \bar{\nabla}_a \, \bar{f}_{bc} = \bar{\nabla}_b \, \bar{f}_{ca} = \bar{\nabla}_c \, \bar{f}_{ab} \, , \tag{3}$$

$$\bar{\nabla}_d \, \bar{\nabla}_a \, \bar{f}_{bc} \; = \; \frac{2\Lambda}{(D-1)(D-2)} (\bar{g}_{da} \, \bar{f}_{cb} + \bar{g}_{db} \, \bar{f}_{ac} + \bar{g}_{dc} \, \bar{f}_{ba}) \, , \tag{4}$$

$$\bar{\Box} \bar{f}_{ab} \; = \; \frac{2\Lambda}{(D-1)} \, \bar{f}_{ba} \, , \qquad \bar{\nabla}^a \, \bar{\nabla}_b \, \bar{f}_{ac} = \frac{2\Lambda}{(D-1)} \, \bar{f}_{bc} \, . \tag{5}$$

Using the Riemann, the Ricci tensors, the Ricci scalar and the Killing-Yano tensor of rank 2, a covariantly conserved antisymmetric current was introduced in [4]. The 'linearized' version of this antisymmetric current [14]

$$(j^{ab})_L = \bar{f}^{cd} \, (R_{cd}{}^{ab})_L - 2\bar{f}^{ac} \, (R_c{}^b)_L + 2\bar{f}^{bc} \, (R_c{}^a)_L + \bar{f}^{ab} \, R_L \, . \tag{6}$$

given by [4], is background covariantly conserved. [We emphasize that, if the spacetime admits a full Killing-Yano tensor itself, then the full j^{ab} is covariantly conserved. However, for the discussion that follows here, we assume that only the background spacetime has asymptotic Killing-Yano tensors.] Then the 'linearization' process can be carried out in the usual way, by keeping terms linear in the perturbation metric h_{ab}. Since this current is antisymmetric, this covariant conservation leads to an ordinary conservation law via

$$\bar{\nabla}_c (\sqrt{|\bar{g}|} \, (j^{bc})_L) = \partial_c (\sqrt{|\bar{g}|} \, (j^{bc})_L) \, . \tag{7}$$

Out of this linearized current, a conserved charge was constructed in [4] for asymptotically transverse *flat* backgrounds, i.e. they took $\bar{g}_{ab} = \eta_{ab}$. Here we will carry out the analogous calculation for asymptotically transverse AdS backgrounds.

The crucial step in this computation is to express $(j^{ab})_L$ as the divergence of a completely antisymmetric rank 3 tensor, i.e. $(j^{ab})_L = \bar{\nabla}_d \, \bar{\ell}^{abd}$. Then, up to a trivial normalization, the conserved 'charge' can be obtained as

$$Q^{ab} \sim \int_\Sigma dS_i \, \sqrt{|\bar{g}|} \, \bar{\ell}^{abi} \, , \tag{8}$$

where i ranges over the $(D-3)$-dimensional spacelike hypersurface Σ at spatial infinity. [One could as well perform similar steps for asymptotically transverse de Sitter backgrounds, except that one now has to stay inside the cosmological horizon and assume that the p-brane/soliton, if it at all exists for such backgrounds, does not change the location of the horizon.]

In order to find the "potential" $\bar{\ell}^{abd}$ for the current, we need the following expressions for the linearized Riemann, Ricci tensors and the Ricci scalar:

$$\begin{aligned}(R_{cd}{}^{ab})_L &= \frac{1}{2}(\bar{\nabla}^a\,\bar{\nabla}_d\,h^b{}_c - \bar{\nabla}^b\,\bar{\nabla}_d\,h^a{}_c - \bar{\nabla}^a\,\bar{\nabla}_c\,h^b{}_d + \bar{\nabla}^b\,\bar{\nabla}_c\,h^a{}_d) \\ &\quad + \frac{\Lambda}{(D-1)(D-2)}(\delta^a{}_d\,h^b{}_c - \delta^b{}_d\,h^a{}_c - \delta^a{}_c\,h^b{}_d - \delta^b{}_c\,h^a{}_d)\,, \\ (R_c{}^b)_L &= -\frac{2\Lambda}{(D-2)}\,h^b{}_c + \frac{1}{2}(-\bar{\Box}h^b{}_c - \bar{\nabla}_c\,\bar{\nabla}^b\,h + \bar{\nabla}^d\,\bar{\nabla}_c\,h^b{}_d + \bar{\nabla}^d\,\bar{\nabla}^b\,h_{cd})\,, \\ R_L &= -\bar{\Box}h + \bar{\nabla}^c\,\bar{\nabla}^d\,h_{cd} - \frac{2\Lambda}{(D-2)}h\,.\end{aligned}$$

The substitution of these into (6) yield

$$\begin{aligned}(j^{ab})_L &= \bar{f}^{cd}\,\bar{\nabla}^a\,\bar{\nabla}_d\,h^b{}_c - \bar{f}^{cd}\,\bar{\nabla}^b\,\bar{\nabla}_d\,h^a{}_c + \frac{2\Lambda}{(D-1)(D-2)}(\bar{f}^{ca}\,h^b{}_c - \bar{f}^{cb}\,h^a{}_c) \\ &\quad + \frac{4\Lambda}{(D-2)}\bar{f}^{ac}\,h^b{}_c + \bar{f}^{ac}\bar{\Box}h^b{}_c + \bar{f}^{ac}\,\bar{\nabla}_c\,\bar{\nabla}^b h - \bar{f}^{ac}\,\bar{\nabla}^d\,\bar{\nabla}_c\,h^b{}_d - \bar{f}^{ac}\,\bar{\nabla}^d\,\bar{\nabla}^b\,h_{cd} \\ &\quad - \frac{4\Lambda}{(D-2)}\bar{f}^{bc}\,h^a{}_c - \bar{f}^{bc}\bar{\Box}h^a{}_c - \bar{f}^{bc}\,\bar{\nabla}_c\,\bar{\nabla}^a h + \bar{f}^{bc}\,\bar{\nabla}^d\,\bar{\nabla}_c\,h^a{}_d + \bar{f}^{bc}\,\bar{\nabla}^d\,\bar{\nabla}^a\,h_{cd} \\ &\quad - \bar{f}^{ab}\,\bar{\Box}h + \bar{f}^{ab}\,\bar{\nabla}_c\,\bar{\nabla}_d\,h^{cd} - \frac{2\Lambda}{(D-2)}\,h\,\bar{f}^{ab}\,. \end{aligned} \tag{9}$$

After somewhat lengthy but straightforward calculations, we finally obtain

$$(j^{ab})_L = 3!\,\bar{\nabla}_d\left(\bar{f}^{c[a}\,\bar{\nabla}^b\,h^{d]}{}_c + \frac{1}{2}\,\bar{f}^{[ba}\,\bar{\nabla}^{d]}\,h + \frac{1}{2}\,h_c{}^{[b}\,\bar{\nabla}^d\,\bar{f}^{|c|a]} - \frac{1}{2}\,\bar{f}^{[ba}\,\bar{\nabla}_{|c|}\,h^{d]c} + \frac{1}{6}\,h\,\bar{\nabla}^{[b}\,\bar{f}^{da]}\right), \tag{10}$$

where we have used the standard bracket notation to indicate the totally antisymmetric parts of tensors. We give a rederivation of this result in Appendix A using the spin-connection and the vielbein formulation of gravity.

This current leads to the following conserved 'charges'

$$\begin{aligned}Q^{ab} &= \frac{3!}{4\,\Omega_{D-3}\,G_D}\int_\Sigma dS_i\,\sqrt{|\bar{g}|}\,\Big(\bar{f}^{c[a}\,\bar{\nabla}^b\,h^{i]}{}_c + \frac{1}{2}\,\bar{f}^{[ba}\,\bar{\nabla}^{i]}\,h + \frac{1}{2}\,h_c{}^{[b}\,\bar{\nabla}^i\,\bar{f}^{|c|a]} \\ &\qquad - \frac{1}{2}\,\bar{f}^{[ba}\,\bar{\nabla}_{|c|}\,h^{i]c} + \frac{1}{6}\,h\,\bar{\nabla}^{[b}\,\bar{f}^{ia]}\Big)\,,\end{aligned} \tag{11}$$

as an integral over the $(D-3)$-dimensional spatial hypersurface. Our choice of normalization will be clear from the discussion below.

This is our main result for conserved rank 2 'Killing-Yano charges' for asymptotically transverse AdS spacetimes. Of course, the formula (11) covers the flat space ($\Lambda = 0$) limit given by [4], except that our notation is different. Up to now, we have not made a specific choice of coordinates, apart from the assumption that the perturbation part of the metric behaves in such a way that the 'charge' integration does not diverge. Thus, as can also be checked explicitly, (11) is invariant under background diffeomorphisms $\delta_\zeta h_{ab} = \bar{\nabla}_a \zeta_b + \bar{\nabla}_b \zeta_a$.

We shall generalize the formula (11) to the case of rank n Killing-Yano tensors in what follows. However, before we move on to that, let us step back and think about the meaning of the current (6) and the 'charge' (11) it led to. If anything, it is hard to grasp the geometric or the physical meaning of this current. Nevertheless, it is certain that being a conserved current, it leads to a conserved 'charge', whose physical meaning is, unfortunately, again unclear. It was shown in [4, 5] that for asymptotically transverse flat spacetimes, explicit computation using "translational Killing-Yano tensors" indicates that one gets the ADM mass density. However, this is quite puzzling since we know that one gets the ADM mass from Einstein's equation using a timelike Killing vector and the symmetric energy momentum tensor T_{ab}, which is covariantly conserved. Moreover, it is more important to note that the meaning of T_{ab} and its relation to geometry is quite clear. Keeping this discussion in mind and taking the risk of being pedantic, we then ask the following question: Why does one get mass (and angular momenta) from the 'good old' *symmetric* T_{ab} but, mass (and angular momenta) 'densities' from completely new *antisymmetric* currents like j_{ab}? This question, which was not addressed in [4, 5], clearly deserves some attention, whose answer we don't know.

Let us first consider the rather "unique" rank 1 case and go back to the Killing vectors; then the construction above should give the flat space ADM [1] or curved space AD [2, 3] charges. Now, instead of $\bar{f}^{ba}$, suppose that we have a Killing vector $\bar{\xi}^a$. In this case, (6) (or more explicitly (19) below) very naively gives a background conserved current

$$(\bar{j}^a)_L = -2\,\bar{\xi}^b\,(G^a{}_b)_L\,, \tag{12}$$

which is nothing but (up to a constant and the choice of the sign which is a matter of conventions) $\bar{\xi}^b\,(T^a{}_b)_L$. The resulting charge, whose details were given in [3] (similar to the form of (11)), reads

$$Q^a(\bar{\xi}) = \frac{1}{4\,\Omega_{D-2}\,G_D}\int_{\partial\Sigma} dS_i\,\sqrt{|\bar{g}|}\,q^{ai}\,, \tag{13}$$

with

$$\begin{aligned} q^{ai} &= \bar{\xi}_b\bar{\nabla}^a h^{ib} - \bar{\xi}_b\bar{\nabla}^i h^{ab} + \bar{\xi}^a\bar{\nabla}^i h - \bar{\xi}^i\bar{\nabla}^a h + h^{ab}\bar{\nabla}^i\bar{\xi}_b \\ &\quad - h^{ib}\bar{\nabla}^a\bar{\xi}_b + \bar{\xi}^i\bar{\nabla}_b h^{ab} - \bar{\xi}^a\bar{\nabla}_b h^{ib} + h\bar{\nabla}^a\bar{\xi}^i\,, \\ &= 2!\,(\bar{\xi}_b\,\bar{\nabla}^{[a}\,h^{i]b} + \bar{\xi}^{[a}\,\bar{\nabla}^{i]}\,h + h^{b[a}\,\bar{\nabla}^{i]}\,\bar{\xi}_b - \bar{\xi}^{[a}\,\bar{\nabla}_{|b|}\,h^{i]b} + \frac{1}{2}h\,\bar{\nabla}^{[a}\,\bar{\xi}^{i]})\,, \end{aligned} \tag{14}$$

where the integration is over a solid angle of a $(D-2)$-dimensional sphere S^{D-2}. Comparing (14) with (11), we see how the generic rank n case will read; the underlying procedure is nothing but a proper antisymmetrization of (14).

The main question now is whether we could have gotten (11) without referring to the antisymmetric current (6) whose connection to the physical mass and angular momenta is unclear. Of course, the more relevant conserved current to use is the energy momentum tensor. Let us show now that one cannot actually do that. In fact, consider a *generic* gravity theory coupled to a covariantly conserved matter source τ_{ab},

$$\Phi_{ab}(g, R, \nabla R, R^2, \dots) = \tau_{ab}\,, \tag{15}$$

where Φ_{ab} is the "Einstein tensor" of a local, invariant generic gravity action, with $\Phi_{ab}(\bar{g}, \bar{R}, \bar{\nabla}\bar{R}, \bar{R}^2, \dots) = 0$ by assumption. The Bianchi identities of the full theory (just as the background gauge invariance) are now carried on to the background such that $\bar{\nabla}_a \Phi^{ab} = 0$.

Consider now the following:

$$\bar{\nabla}_c(\bar{f}^{ab}\,\Phi_b{}^c) = \bar{\nabla}_c(\bar{f}^{ab}\,\tau_b{}^c) = 0\,,$$

which follows from the background Bianchi identity, (3) and the antisymmetry of the Killing-Yano tensor. Note, however, that

$$\bar{\nabla}_c(\bar{f}^{ab}\,\Phi_b{}^c) = \partial_c(\bar{f}^{ab}\,\Phi_b{}^c) + \bar{\Gamma}^a{}_{cd}\,\bar{f}^{db}\,\Phi_b{}^c + \bar{\Gamma}^c{}_{cd}\,\bar{f}^{ab}\,\Phi_b{}^d\,.$$

Using $\bar{\Gamma}^c{}_{cd} = \partial_d\,(\ln\sqrt{|\bar{g}|})$, this can be written as

$$\bar{\nabla}_c(\sqrt{|\bar{g}|}\,\bar{f}^{ab}\,\Phi_b{}^c) = \partial_c(\sqrt{|\bar{g}|}\,\bar{f}^{ab}\,\Phi_b{}^c) + \sqrt{|\bar{g}|}\,\bar{\Gamma}^a{}_{cd}\,\bar{f}^{db}\,\Phi_b{}^c\,. \tag{16}$$

The last term on the right hand side of (16) does not necessarily vanish and this is the reason why an analogous construction as the one given in [2, 3] with Killing vectors cannot automatically be carried on to the case of Killing-Yano tensors, since the tensor density current $\sqrt{|\bar{g}|}\,\bar{f}^{ab}\,\Phi_b{}^c$ is not *ordinarily* conserved in general. However, if we consider the antisymmetric combination $\bar{f}^{ab}\,\Phi_b{}^c - \bar{f}^{cb}\,\Phi_b{}^a$ [15], then we have

$$\bar{\nabla}_c\Big(\sqrt{|\bar{g}|}(\bar{f}^{ab}\,\Phi_b{}^c - \bar{f}^{cb}\,\Phi_b{}^a)\Big) = \partial_c\Big(\sqrt{|\bar{g}|}(\bar{f}^{ab}\,\Phi_b{}^c - \bar{f}^{cb}\,\Phi_b{}^a)\Big) \neq 0\,,$$

due to the fact that $\bar{\nabla}_a(\bar{f}^{ab}\,\Phi_b{}^c) \neq 0$. Thus, one is forced to use the new antisymmetric current (6).

For the rank n case, these charges are computed as follows: one starts with (14) and first replaces the background Killing vector $\bar{\xi}^a$ with the background Killing-Yano tensor $\bar{f}^{a_1\dots a_n}$. Next thing to do is to fully antisymmetrize the expression thus obtained with respect to the uncontracted indices inside the covariant derivative in such a way that one gets a fully antisymmetric potential $\bar{\ell}^{a_1\dots a_{n+1}}$. For the rank 2 case, one can easily get (11) through this method. For the rank n case, we get

$$\begin{aligned}(\frac{n}{2}\,j^{a_1\dots a_n})_L \;=\; &(n+1)!\,\bar{\nabla}_d\Big(\bar{f}^{c[a_1\dots a_{n-1}}\,\bar{\nabla}^b\,h^{d]}{}_c + \frac{1}{n!}\,\bar{f}^{[ba_1\dots a_{n-1}}\,\bar{\nabla}^{d]}\,h + \frac{1}{n!}\,h_c{}^{[b}\,\bar{\nabla}^d\,\bar{f}^{|c|a_1\dots a_{n-1}]}\\ &-\frac{1}{n!}\,\bar{f}^{[ba_1\dots a_{n-1}}\,\bar{\nabla}_{|c|}\,h^{d]c} + \frac{1}{(n+1)!}\,h\,\bar{\nabla}^{[b}\,\bar{f}^{da_1\dots a_{n-1}]}\Big)\,,\end{aligned} \tag{17}$$

which yields the following conserved 'charges' when integrated over a $(D-1-n)$-dimensional hypersurface at spatial infinity [16]

$$\begin{aligned}Q^{a_1\dots a_n}(\bar{f}) \;=\; &\frac{(n+1)!}{2n\,\Omega_{D-1-n}\,G_D}\int_\Sigma dS_i\,\sqrt{|\bar{g}|}\,\Big(\bar{f}^{c[a_1\dots a_{n-1}}\,\bar{\nabla}^b\,h^{d]}{}_c + \frac{1}{n!}\,\bar{f}^{[ba_1\dots a_{n-1}}\,\bar{\nabla}^{d]}\,h\\ &+\frac{1}{n!}\,h_c{}^{[b}\,\bar{\nabla}^d\,\bar{f}^{|c|a_1\dots a_{n-1}]} - \frac{1}{n!}\,\bar{f}^{[ba_1\dots a_{n-1}}\,\bar{\nabla}_{|c|}\,h^{d]c} + \frac{1}{(n+1)!}\,h\,\bar{\nabla}^{[b}\,\bar{f}^{da_1\dots a_{n-1}]}\Big)\,.\end{aligned} \tag{18}$$

Once again this formula works both for asymptotically transverse AdS and asymptotically transverse flat spacetimes in generic coordinates. We could also have gotten this expression using the following conserved current given by [4]

$$j^{a_1\dots a_n} = (n-1)\,R^{[a_1a_2}{}_{bc}\,f^{a_3\dots a_n]bc} + 4(-1)^n\,R_c{}^{[a_1}\,f^{a_2\dots a_n]c} + \frac{2}{n}\,R\,f^{a_1\dots a_n}\,. \tag{19}$$

Obtaining (18) from (19) by following steps similar to the ones that took us from (6) to (10) (and thus to (11)) is again straightforward but considerably longer.

An explicit example: The long Weyl rod

As has already been mentioned before, our construction works for asymptotically transverse flat or AdS spaces. For the latter, we are not aware of any solution with a *uniform* mass density. However, for the flat case, we can consider the infinitely extended Weyl rod [10]. Since this solution is not widely known, let us first write down the metric for a rod centered at the origin with length L and *total* mass m:

$$ds^2 = -e^{2\phi}\, dt^2 + e^{-2\phi}\left[e^{2\nu}\left(dr^2 + dz^2\right) + r^2\, d\varphi^2\right], \tag{20}$$

where

$$e^{2\phi} = \left(\frac{R_1 + R_2 - L}{R_1 + R_2 + L}\right)^{2m/L}, \quad e^{2\nu} = \left(\frac{\left(R_1 + R_2\right)^2 - L^2}{4\, R_1\, R_2}\right)^{4m^2/L^2}, \tag{21}$$

$$R_1 = \sqrt{r^2 + \left(z - \frac{L}{2}\right)^2}, \qquad R_2 = \sqrt{r^2 + \left(z + \frac{L}{2}\right)^2}. \tag{22}$$

The correct background to work with is the flat Minkowski spacetime written in cylindrical coordinates which is simply obtained by setting $m = 0$. A calculation using (13) with the background Killing vector $\bar{\xi}^\mu = -\delta^\mu{}_0$ indeed yields the mass to be m. [Note that when $L \to -L$, $m \to -m$ and the Schwarzschild solution can be obtained by setting $m = L$.] As $L \to \infty$, the total mass obviously diverges. As for the Killing-Yano charges, one starts with a background Killing-Yano tensor whose only nontrivial component reads $\bar{f}^{tz} = -1$, and uses (11). The result is $Q^{tz} = m/L$, which is simply the constant mass density.

III. CONCLUSIONS AND DISCUSSIONS

Using the antisymmetric background Killing-Yano tensors of rank n, we have constructed conserved gravitational charges of D-dimensional spacetimes that asymptotically approach to Anti-de Sitter or flat spaces only in some transverse directions of an extended gravitational solution. This work generalizes that of Kastor-Traschen [4] which dealt only with asymptotically transverse flat spacetimes. Our construction is based on the linearization of a completely antisymmetric current (19) given by [4], whose physical meaning is somewhat unclear. This current makes no reference to a specific gravity model.

The conserved charge densities are background diffeomorphism invariant. It is somewhat surprising that one can get anything meaningful out of Killing-Yano tensors with regard to the conserved quantities of "extended" gravitational solutions, such as p-branes. To begin with, a naive application of the ADM or AD charges gives divergent results for the total mass and the total angular momenta. However, the mass density and the angular momentum density turn out to be finite which can be expected. What is quite interesting is that these "densities" are expressed as integrals over a $(D-p-2)$-dimensional hypersurface at spatial infinity, in close analogy with the ADM or AD case. In fact, the main difference is that, for Killing-Yano charges parallel directions to the "brane" are left alone in the integration process.

As already mentioned, these conserved gravitational charges, constructed out of the Killing-Yano tensors, turn out to be "charge (mass) densities". One might simply ask why one would bother with these conserved quantities at the first place when one can simply go back to the ADM or the AD formulas and define charge densities out of these. This is, of course, a legitimate question which has two possible answers: Firstly, a naive mass density definition obtained from the ADM or the AD expression would not be diffeomorphism invariant, the use of Killing-Yano tensors renders the result coordinate invariant. This is important. Secondly, it is better to read our result from a different perspective. We have established the following: If a spacetime asymptotically admits antisymmetric Killing-Yano tensors, the conserved quantities they generate are the mass and the angular momentum densities.

As for an explicit example, we have given the mass density of the axisymmetric Weyl rod in $D = 3 + 1$ dimensions. It would, of course, be better to give more examples, especially consider those solutions which are transverse asymptotically AdS and those that also involve rotation, but unfortunately we are not aware of such solutions.

Finally, we have especially refrained from a discussion on the positivity of these "mass-densities" (for the asymptotically transverse flat spacetimes case, such a discussion was given in [5]). In asymptotically AdS spacetimes, there are negative mass objects (see e.g. [12] and the references therein). Thus, in general, we do not expect 'a positive mass theorem' to hold in these spacetimes. It is true that, following e.g. Gibbons et al. [11], one can presumably show the positivity of mass density in certain 4-dimensional spacetimes by rewriting the mass density in terms of the square of certain Witten-Nester spinors. However, for many interesting solutions, such as the AdS soliton, one does not have regular spinors [13] satisfying the requirements of positivity.

APPENDIX A: DERIVATION OF THE KILLING-YANO CHARGES IN THE SPIN-CONNECTION FORMALISM

In this appendix, we will redrive (10) using the first order form of gravity. This is not a redundant task at all, since it maybe relevant to various supergravity models.

Analogous to the definition of a Killing-Yano tensor of rank 2, we define a Killing-Yano 2-form f in terms of orthonormal co-frame 1-forms e^a as

$$f = \frac{1}{2}\, f_{ab}\, e^a \wedge e^b \,. \tag{A1}$$

The current (19) for the case $n = 2$ now can be written as

$$j = f_{ab}\, R^{ab} + 2\iota_a f \wedge P^a + f R \,, \tag{A2}$$

where R^{ab} is the curvature 2-form defined in the usual way as

$$R^{ab} = d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb} \tag{A3}$$

in terms of the connection 1-forms $\omega^a{}_b$; P^a stands for the Ricci 1-forms defined through $P^a = \iota_b R^{ba}$ and R is the curvature scalar $R = \iota_a \iota_b R^{ba}$. The current j^{ab} can be obtained in an orthonormal frame as

$$j^{ab} = f_{cd}\, R^{abcd} - 2f^a{}_c\, P^{cb} + 2f^b{}_c\, P^{ca} + f^{ab}\, R \,, \tag{A4}$$

by noting that

$$P^a = P^a{}_b e^b \,, \qquad R^{ab} = \frac{1}{2} R^{ab}{}_{cd}\, e^c \wedge e^d \quad \text{and} \quad j = \frac{1}{2} j_{ab}\, e^a \wedge e^b \,, \tag{A5}$$

where $f_{ba} = \iota_a \iota_b f$.

Suppose also that the metric tensor $g = \eta_{ab}\, e^a \otimes e^b$ is decomposed such that the 'full' orthonormal coframe 1-form e^a can be written as the sum of a 'background' orthonormal coframe 1-form $\bar{e}^a$ plus a 'deviation' piece as

$$e^a \equiv \bar{e}^a + \varphi^a{}_b\, \bar{e}^b\,, \tag{A6}$$

where the 0-forms $\varphi^a{}_b$ are as usual assumed to vanish sufficiently rapidly at the 'spatial infinity'. The background spacetime geometry is assumed to satisfy the cosmological Einstein field equation

$$\bar{R}^{ab} \wedge \bar{*}(\bar{e}_a \wedge \bar{e}_b \wedge \bar{e}_c) = \Lambda\, \bar{*}\, \bar{e}_c\,, \tag{A7}$$

where $\bar{*}$ is the background Hodge operator. (A7) are solved by a space of constant curvature which satisfies

$$\begin{aligned}
\bar{R}_{abcd} &= \frac{2\Lambda}{(D-1)(D-2)}\left(\eta_{ac}\,\eta_{bd} - \eta_{ad}\,\eta_{bc}\right),\\
\bar{R}_{ab} &= \frac{1}{2}\,\bar{R}_{abcd}\,\bar{e}^c \wedge \bar{e}^d = \frac{2\Lambda}{(D-1)(D-2)}\,\bar{e}_a \wedge \bar{e}_b\,,\\
\bar{P}_a &= \frac{2\Lambda}{D-2}\bar{e}_a\,, \quad \bar{P}_{ab} = \frac{2\Lambda}{D-2}\eta_{ba}\,, \quad \bar{R} = \bar{\imath}_a\,\bar{\imath}_b\,\bar{R}^{ba} = \frac{2\Lambda D}{D-2}\,.
\end{aligned} \tag{A8}$$

Moreover, the relation $\iota_b\, e^a = \delta_b{}^a$ implies that $\iota_b = \bar{\imath}_b - \varphi_{bs}\bar{\imath}^s$ in terms of the inner product operator $\bar{\imath}_b$ of the background spacetime that satisfies $\bar{\imath}_b\, \bar{e}^a = \delta_b{}^a$. Now the Cartan structure equations [in the case of vanishing torsion T^a] $de^a + \omega^a{}_b \wedge e^b = 0$ yield

$$\omega^a{}_b = \bar{\omega}^a{}_b + \bar{e}^c\left(\bar{\imath}_b D\varphi^a{}_c - \bar{\imath}^a D\varphi_{cb}\right) \tag{A9}$$

to first order (linearized form) in $\varphi^a{}_b$, where for any p-form $\Omega^a{}_b$

$$D\Omega^a{}_b = d\varphi^a{}_b + \bar{\omega}^a{}_c \wedge \Omega^c{}_b - \bar{\omega}^c{}_b \wedge \Omega^a{}_c \tag{A10}$$

and $\bar{\omega}^a{}_b$ satisfies the background Cartan structure equation $d\bar{e}^a + \bar{\omega}^a{}_b \wedge \bar{e}^b = 0$. Using (A3) and these preliminaries, the linearized curvature 2-form can be calculated as

$$R^{ab} = \bar{R}^{ab} - \bar{e}^c \wedge D(\bar{\imath}^b D\varphi^a{}_c) + \bar{e}^c \wedge D(\bar{\imath}^a D\varphi_c{}^b)\,, \tag{A11}$$

which yields the following formulas for the Ricci 1-form

$$P^a = \bar{P}^a - D(\bar{\imath}^a D\varphi^b{}_b) + D(\bar{\imath}^b D\varphi_b{}^a) + \bar{\imath}_b D(\bar{\imath}^a D\varphi^b{}_c)\bar{e}^c - \bar{\imath}_b D(\bar{\imath}^b D\varphi_c{}^a)\bar{e}^c - \varphi_{bs}\bar{\imath}^s \bar{R}^{ba}\,, \tag{A12}$$

and the curvature scalar

$$R = \bar{R} - 2\bar{D}_b\,\bar{D}^b\varphi_a{}^a + 2\bar{D}_a\,\bar{D}_b\varphi^{ab} - 2\varphi_{ab}\,\bar{P}^{ab}\,, \tag{A13}$$

where $\bar{D}_b\varphi^{ba} = \bar{\imath}_b D\varphi^{ba}$ and $\bar{D}_a\,\bar{D}_b\varphi^{ba} = \bar{\imath}_a D(\bar{\imath}_b D\varphi^{ba})$.

The use of the background Killing-Yano 2-form $f = \frac{1}{2}\bar{f}_{ab}\bar{e}^a \wedge \bar{e}^b$, and the substitution of (A11), (A12) and (A13) into (A2) gives the following expression for the linearized current 2-form j_L:

$$\begin{aligned}
j_L = {} & \bar{f}_{ab}\bar{e}^c \wedge D(\bar{\imath}^a D\varphi_c{}^b) - \bar{f}_{ab}\bar{e}^c \wedge D(\bar{\imath}^b D\varphi^a{}_c) - \varphi_{an}\bar{f}^n{}_b\bar{R}^{ab} - \varphi_{bl}\bar{f}_a{}^l\bar{R}^{ab}\\
& -2\bar{\imath}_a f \wedge D(\bar{\imath}^a D\varphi^b{}_b) + 2\bar{\imath}_a f \wedge D(\bar{\imath}^b D\varphi_b{}^a) + 2\bar{\imath}_a f \wedge \bar{e}^c\bar{\imath}_b D(\bar{\imath}^a D\varphi^b{}_c)\\
& -2\bar{\imath}_a f \wedge \bar{e}^c\bar{\imath}_b D(\bar{\imath}^b D\varphi_c{}^a) - 2\varphi_{bs}\bar{\imath}_a f \wedge \bar{\imath}^s\bar{R}^{ba} - 2\varphi_{as}\bar{\imath}^s f \wedge \bar{P}^a\\
& +2f\bar{\imath}_a D\left(\bar{\imath}_b D\varphi^{ab}\right) - 2f\bar{\imath}^b D\left(\bar{\imath}_b D\varphi_a{}^a\right) - 2f\varphi_{ab}\bar{P}^{ab}\,.
\end{aligned} \tag{A14}$$

This linearized current can be written in terms of the background orthonormal co-frames as $j_L = \frac{1}{2}(j^{cs})_L\, \bar{e}_c \wedge \bar{e}_s$:

$$\begin{aligned}(j^{cs})_L \;=\;& \bar{f}_{ab}\left(\bar{D}^s\bar{D}^a\varphi^{cb} - \bar{D}^c\bar{D}^a\varphi^{sb} + \bar{D}^c\bar{D}^b\varphi^{sa} - \bar{D}^s\bar{D}^b\varphi^{ca}\right)\\ & -\varphi_{al}\bar{f}^l{}_b\bar{R}^{abcs} - \varphi_{bl}\bar{f}_a{}^l\bar{R}^{abcs} + 2\left(\bar{f}_a{}^c\varphi_{bl}\bar{R}^{abls} - \bar{f}_a{}^s\varphi_{bl}\bar{R}^{ablc}\right)\\ & +2\bar{f}_a{}^c\left(\bar{D}^s\bar{D}^b\varphi_b{}^a - \bar{D}^s\bar{D}^a\varphi^b{}_b + \bar{D}_b\bar{D}^a\varphi^{sb} - \bar{D}^b\bar{D}_b\varphi^{sa}\right)\\ & -2\bar{f}_a{}^s\left(\bar{D}^c\bar{D}^b\varphi_b{}^a - \bar{D}^c\bar{D}^a\varphi^b{}_b + \bar{D}_b\bar{D}^a\varphi^{bc} - \bar{D}^b\bar{D}_b\varphi^{ca}\right)\\ & -2\left(\varphi_{al}\bar{f}^{lc}\bar{P}^{as} - \varphi_{al}\bar{f}^{ls}\bar{P}^{ac}\right) + 2\bar{f}^{cs}\left(\bar{D}_a\bar{D}_b\varphi^{ba} - \bar{D}^b\bar{D}_b\varphi_a{}^a - \varphi_{ab}\bar{P}^{ab}\right). \qquad \text{(A15)}\end{aligned}$$

The next thing to do is to express (A15) in the form $(j^{cs})_L = \bar{D}_a(l^{acs})$, where l^{acs} is totally antisymmetric in its c, s and a indices.

Finally, using the following identities

$$\bar{D}_c\bar{D}_d\bar{f}_{ba} \;=\; -\frac{1}{2}\left(\bar{R}^n{}_{cdb}\bar{f}_{an} + \bar{R}^n{}_{cad}\bar{f}_{bn} + \bar{R}^n{}_{cba}\bar{f}_{dn}\right), \qquad \text{(A16)}$$

$$(\bar{D}_c\bar{D}_d - \bar{D}_d\bar{D}_c)\varphi_{ba} \;=\; -\bar{R}^l{}_{bcd}\varphi_{la} - \bar{R}^l{}_{acd}\varphi_{bl}\,, \quad \bar{D}^b\bar{D}_b\bar{f}_{ca} = -\frac{2\Lambda}{D-1}\bar{f}_{ca}\,, \qquad \text{(A17)}$$

(A15) can be written as

$$(j^{cs})_L = 2\,3!\,\bar{D}_a\Big(\bar{f}^{b[a}\bar{D}^c\varphi^{s]}{}_b + \frac{1}{2}\bar{f}^{[ca}\bar{D}^{s]}\varphi^b{}_b + \frac{1}{2}\bar{D}_b\bar{f}^{[ca}\varphi^{s]}{}_b - \frac{1}{2}\bar{f}^{[ca}\bar{D}_b\varphi^{s]b} + \frac{1}{6}\varphi^b{}_b\bar{D}^{[a}\bar{f}^{cs]}\Big), \qquad \text{(A18)}$$

which corresponds to (10).

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[1] R. Arnowitt, S. Deser and C. W. Misner, Phys. Rev. **117**, 1595 (1960); *Gravitation: An Introduction to Current Research*, edited by L. Witten (Wiley, NY, 1962); "The Dynamics Of General Relativity," [arXiv:gr-qc/0405109].

[2] L. F. Abbott and S. Deser, Nucl. Phys. B **195**, 76 (1982).

[3] S. Deser and B. Tekin, Phys. Rev. Lett. **89**, 101101 (2002), [arXiv:hep-th/0205318]; Phys. Rev. D **67**, 084009 (2003), [arXiv:hep-th/0212292].

[4] D. Kastor and J. Traschen, JHEP **0408**, 045 (2004), [arXiv:hep-th/0406052].

[5] D. Kastor, T. Shiromizu, S. Tomizawa and J. Traschen, Phys. Rev. D **71**, 104015 (2005), [arXiv:hep-th/0410289].

[6] B. Carter, Phys. Rev. D **16**, 3395 (1977).

[7] G. W. Gibbons, R. H. Rietdijk and J. W. van Holten, Nucl. Phys. B **404**, 42 (1993) [arXiv:hep-th/9303112].
[8] L. Howarth, C. D. Collinson, Gen. Rel. Grav. **32**, 1845 (2000).
[9] M. Cariglia, Class. Quant. Grav. **21**, 1051 (2004), [arXiv:hep-th/0305153].
[10] H. Weyl, Annalen Phys. **54**, 117 (1917); R. Bach and H. Weyl, Math. Zeits. **13**, 134 (1922).
[11] G. W. Gibbons, C. M. Hull and N. P. Warner, Nucl. Phys. B **218**, 173 (1983).
[12] H. Cebeci, Ö. Sarıoğlu and B. Tekin, Phys. Rev. D **73**, 064020 (2006), [arXiv:hep-th/0602117].
[13] M. C. N. Cheng and K. Skenderis, JHEP **0508**, 107 (2005), [arXiv:hep-th/0506123].
[14] We note in passing that our normalization differs from that of [4].
[15] We thank D. Kastor for bringing this to our attention.
[16] Note that the 'charges' are gauge/diffeomorphism invariant only at spatial infinity.