Finite mass gravitating Yang monopoles

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(Dated: January 10, 2019)

We show that gravity cures the infra-red divergence of the Yang monopole when a proper definition of conserved quantities in curved backgrounds is used, i.e. the gravitating Yang monopole in cosmological Einstein theory has a finite mass in generic even dimensions (including time). In addition, we find exact Yang-monopole type solutions in the cosmological Einstein-Gauss-Bonnet-Yang-Mills theory and briefly discuss their properties.

PACS numbers: 11.15.-q, 04.50.Gh, 14.80.Hv, 04.50.Kd

I. INTRODUCTION

About 30 years ago, Yang [1], generalizing the Dirac monopole, found a (singular) spherically symmetric solution of the five-dimensional Euclidean Yang-Mills (YM) theory with the $SU(2)$ gauge group. In the same paper, he also showed that his $SU(2)$ monopole does not exist in more than five dimensions. Yang's monopole (on which we shall dwell a bit more in a moment) went pretty much unnoticed up until it emerged in a rather unlikely place, in the study of the four-dimensional analog of the quantum Hall effect [2]. [We have nothing more to say about the Yang monopole in its relevance to the quantum Hall effect, except to remark that no solution of the YM theory seems to be wasted!]

The present work was inspired by and follows closely the recent article by Gibbons and Townsend [3], which does a couple of things at once. It introduces gravity into the picture to get gravitating Yang monopoles, and gives a reconstruction (and reinterpretation) of the higher dimensional versions (with gauge groups other than $SU(2)$) of both the curved and the flat space Yang monopoles. [See [4] for an earlier discussion of the higher dimensional Yang monopoles.] Before we explain how we "improve" on the work of Gibbons-Townsend, let us recapitulate some properties of the Yang monopole.

The way Yang constructed his solution is quite interesting: He considered self-dual, spherically symmetric single instanton (and anti-instanton) solutions on S^4 and showed that they solve the full YM equations in five Euclidean dimensions. As five-dimensional solutions, these instantons have a singularity at the origin just like their three-dimensional cousin, "the Dirac monopole". The action of the single self-dual instanton, $\int \mathcal{F} \wedge \mathcal{F}$ integrated over S^4, becomes a conserved monopole charge of the five-dimensional Yang monopole. [Note that even though there are instantons whose charge can take an arbitrary integral value in four dimensions, none save the ± 1 charge solves the five-dimensional YM equations. Put in another way, there are no Yang multi-monopoles! This is a curious result, but can be shown to be valid by topological arguments [4], as we will also argue.] As summarized in [3], the rare appearance of the Yang monopole in high energy physics literature might

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be due to the fact that unlike the ultra-violet (UV) divergence [$\int d^3xB^2 \to g^2 \int_0^\infty dr/r^2 \to \infty$] of the Dirac monopole, the Yang monopole has an infra-red (IR) divergence, i.e. its mass is IR divergent. We know that when compact Maxwell theory with Dirac monopoles is considered as a low energy limit of, say, a broken $SO(3)$ Georgi-Glashow type theory, finite mass 't Hooft-Polyakov monopoles emerge, which look exactly like Dirac monopoles from a distance. Therefore, UV divergence of the Dirac monopole is not a great concern if some unified theory picture is adopted. In the case of the Yang monopole, one needs to construct a microscopic theory which takes over in the IR limit, which, of course, is quite a difficult task. [See [5, 6] where some higher derivative YM actions with Higgs fields are used to construct regular monopole solutions in higher dimensions.]

Note that all of the discussion about the mass-divergence of the Yang monopole above is in flat space. If we turn on gravity, as we shall do in this paper, the picture changes drastically. Gravity could be blamed for introducing UV divergences, curvature singularities and black holes, but since it clumps matter and fields, it should be a good cure for IR divergences. Gibbons-Townsend [3] introduced the self-gravitating Yang monopole and argued that, in contrast to this expectation, the mass is still IR divergent (beyond four dimensions in their classification). Here, we show that once the proper mass-energy definitions in asymptotically flat and AdS spaces are employed, the Yang monopole does indeed have a finite mass in all dimensions. The main issue here has to do with the choice of a proper background to work out the relevant mass-energy formula.

Our second aim in this paper is to find Yang-monopole type solutions in more generic gravity theories coupled with YM systems. To this end, we consider the cosmological Einstein-Gauss-Bonnet (GB) theory, which appears as a low energy limit of some string theories, and construct new solutions. Compared to General Relativity, GB theory behaves better in the UV region, which is not our main concern here, but exact solutions in this rather complicated theory are always good to have, since there are very few known anyway.

The organization of this paper is as follows. In the next section, we briefly review the Dirac and Yang monopoles in flat space. In section III, we show how the IR divergence mentioned above is overcome. We describe the cosmological Einstein-GB-YM theory in section IV, and present our ansatz for the Yang monopole, our assumptions and the field equations we obtained from these in section V. Section VI is devoted to the solutions found and their properties. Finally we conclude with section VII.

II. DIRAC AND YANG MONOPOLES IN FLAT SPACE

As there can be occasional confusions with regard to gauge symmetries, spacetime symmetries and the charge definitions of higher dimensional singular monopoles, we start by giving a brief recollection of these concepts in flat space, with the help of [7] and [8].

Let us start with Yang's generalization of the three-dimensional Dirac monopole. The latter lives on $\mathbb{R}^3$ with the origin removed. The Maxwell field strength $\mathcal{F}$ is a 2-form whose flux $\int_{S^2} \mathcal{F}$ gives the magnetic charge which can take *any* integral value up to a normalization. Even though the vector potential $\mathcal{A}$ does not reflect it, the physical field $\mathcal{F}$ is spherically symmetric, i.e. it is invariant under the action of $SO(3)$. [This in fact means that spatial rotations can be undone with gauge transformations.] As is well known, the singular Dirac monopole can be described by pure geometry: $\mathbb{R}^3 - \{0\}$ is homotopically equivalent to S^2, therefore one may study the corresponding principal bundle $P(S^2, U(1))$. [For the charge-1 monopole, this is the Hopf fibration of S^3.] Then the transition functions defined on the equator S^1 of S^2 classify the magnetic charge; namely, they map $S^1 \to U(1)$, having $\pi_1(U(1)) = \mathbb{Z}$. A complementary picture is provided by the first Chern character of this "monopole" bundle, i.e. the magnetic flux equals $\int_{S^2} ch_1(\mathcal{F})$.

Let us now look at the "original" Yang monopole [1] in $\mathbb{R}^5 - \{0\}$, which is homotopically equivalent to S^4. Yang considered the field strength $\mathcal{F}$ to be an $\mathfrak{su}(2)$-valued 2-form that "generalizes" the Dirac monopole in the sense that the physically measurable quantities are $SO(5)$ invariant. Now the relevant geometrical object is the principal bundle $P(S^4, SU(2))$. Even though the corresponding homotopy group $\pi_3(SU(2))$ equals $\mathbb{Z}$ which arises from the maps $S^3 \to SU(2)$, the Euclidean YM equation in five dimensions admits only *two* of these solutions. These are just the four-dimensional self-dual and anti self-dual solutions (BPST instanton having the charge ± 1). The charge is now given (up to a normalization) by the integral of the second Chern character

$$\int_{S^4} ch_2(\mathcal{F}) = \int_{S^4} \mathrm{Tr}\,(\mathcal{F} \wedge \mathcal{F}) = \pm 1\,.$$

III. THE EFFECT OF GRAVITY ON MONOPOLES

Here we will show how gravity cures the IR divergence of the mass-energy of the Yang monopole in any even dimensions (time included), just as it cures the UV divergence of the 3+1-dimensional Dirac monopole. [In this context, the latter is nothing but the celebrated Reissner-Nordstrom black hole.] Note that our result about the mass of the Yang monopole is not in agreement with [3], who incorrectly claimed that the divergence persisted in the presence of gravity except for four dimensions. The gist of the problem lies in the correct definition of gravitational mass-energy. For this purpose, we resort to the procedure developed in [9, 10, 11]. Stated briefly, the idea is to define gauge invariant conserved charges in a diffeomorphism invariant theory by employing the generalized "Gauss law" provided there exist asymptotic Killing symmetries of the relevant spacetimes. Put in another way, one chooses a vacuum that satisfies the field equations as the background with respect to which background gauge invariant quantities (such as energy) is calculated. These charges are expressible as surface integrals and, by construction, their value for the background itself is always zero. The latter is quite important.

Given a background Killing vector $\bar{\xi}^\mu$, the corresponding conserved charges can be written as [21] [10, 11]

$$Q^\mu(\bar{\xi}) = \frac{1}{4\,\Omega_{n-2}} \int d^{n-2}x\, \bar{\xi}_\nu\, \mathcal{G}_L^{\mu\nu}\,, \tag{1}$$

where $\mathcal{G}_L^{\mu\nu}$ denotes the linearized Einstein tensor about the background and Ω_{n-2} is the solid angle on the unit $(n-2)$-sphere. As it would be too much of a digression to redrive this formula and its form as a surface integral, we refer the reader to [10, 11] for the details and simply employ it here.

For the gravitating Yang-monopole type solutions found in [3], the spacetime metric in Schwarzschild-like coordinates is simply given by

$$ds^2 = -f^2(r)\,dt^2 + \frac{dr^2}{f^2(r)} + r^2\,d\Omega_{n-2}^2\,, \tag{2}$$

where $d\Omega_{n-2}^2$ is the metric on the $(n-2)$-sphere and the function $f(r)$ reads (in the form presented by [3] but adapted to our conventions for $n \geq 4$)

$$f^2(r) = 1 - \frac{2\,m}{r^{n-3}} - \frac{\mu^2}{r^2} - \frac{2\,\Lambda\,r^2}{(n-2)(n-1)}\,. \tag{3}$$

Here the constant μ is given by

$$\mu^2 = \frac{8\pi(n-3)}{(n-5)\sigma^2}\,,$$

which follows from the normalization choice for the generators Σ_{ij} of the gauge group $SO(n-2)$ (see [3] for details), and cannot be chosen as zero. This is a rather important point. Together with the cosmological term, the μ^2 piece in (3) constitute the background with respect to which any spacetime with nonvanishing m can have a finite and meaningful mass. Otherwise, apart from the special $n=4$ case, one always finds a divergent mass for (2). Thus taking the background to be the spacetime (2) with $m=0$ in (3), which has the timelike Killing vector $\bar{\xi}^\mu = (-1, 0, \dots, 0)$ in the notation and conventions of [10, 11], one finds the total energy of these solutions as

$$E = \frac{1}{4\Omega_{n-2}}\,\Omega_{n-2}\,(2(n-2)\,m) = \frac{m(n-2)}{2}\,.$$

This is the result of the surface integration at $r \to \infty$ in the notation of [10, 11]. To see how gravity modifies the IR divergence of the Yang monopole, let us also compute the (gauge non-invariant) energy contained in a ball of radius R about the origin of spacetime. One then finds

$$E(R) = \frac{(n-2)\,m\,R^{n-5}\,\left[2\,\Lambda\,R^4 + (n-1)(n-2)(\mu^2 - R^2)\right]}{2\left(\left[2\,\Lambda\,R^4 + (n-1)(n-2)(\mu^2-R^2)\right]R^{n-5} + 2(n-1)(n-2)\,m\right)}\,, \tag{4}$$

which is finite in contrast to the flat space result, that goes like R^{n-5} and diverges as $R \to \infty$ for $n \geq 6$ [3].

IV. THE COSMOLOGICAL EINSTEIN-GB-YM THEORY

Let us now describe the cosmological Einstein-GB-YM theory, the assumptions we make and the solutions they lead to in various dimensions. We start with the action

$$I[e, \mathcal{A}] = \int \mathcal{L}\,, \tag{5}$$

where the Lagrangian density n-form

$$\mathcal{L} \;=\; \frac{1}{2}\,R^{ab} \wedge *(e_a \wedge e_b) - \frac{1}{2\sigma^2}\,\mathrm{Tr}(\mathcal{F} \wedge *\mathcal{F}) + \Lambda * 1 + \frac{\gamma}{4}\,R^{ab} \wedge R^{cd} \wedge *(e_a \wedge e_b \wedge e_c \wedge e_d) \tag{6}$$

contains the Einstein-Hilbert term, the YM Lagrangian for the 2-form field $\mathcal{F}$ with coupling constant σ, a cosmological constant Λ and a second order Euler-Poincaré term (the so called GB term in this case) with coupling constant γ.

The basic gravitational field variables are the coframe 1-forms e^a in terms of which the spacetime metric is decomposed as $\mathbf{g} = \eta_{ab}\, e^a \otimes e^b$, where $\eta_{ab} = \mathrm{diag}\,(-,+,+,\dots)$ is the Minkowski metric. The Hodge duality map is specified by the oriented volume element $*1 = e^0 \wedge e^1 \wedge \cdots \wedge e^{n-2} \wedge e^n$. The torsion-free, Levi-Civita connection 1-forms $\omega^a{}_b$ satisfy the first Cartan structure equations

$$de^a + \omega^a{}_b \wedge e^b = 0,$$

where metric compatibility implies $\omega_{ab} = -\omega_{ba}$. The corresponding curvature 2-forms follow from the second Cartan structure equations

$$R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b\,.$$

The GB term in the Lagrangian density (6) can also be written in the alternative form

$$R^{ab} \wedge R^{cd} \wedge *(e_a \wedge e_b \wedge e_c \wedge e_d) = 2\,R_{ab} \wedge *R^{ab} - 4\,P_a \wedge *P^a + \mathcal{R}^2_{(n)} * 1\,,$$

where the Ricci 1-form $P^a = \iota_b R^{ba}$ and the curvature scalar $\mathcal{R}_{(n)} = \iota_a\, \iota_b\, R^{ba}$ have been utilized via the interior product operator $\iota_a = \iota_{X^a}$ for which $\iota_{X^b}(e^a) = \delta_b{}^a$.

Before moving on to the field equations, let us present our setting on the YM sector as well. We take the YM potential $\mathcal{A}$ to be a Lie algebra $\mathfrak{g}$-valued 1-form. The YM 2-form field follows from

$$\mathcal{F} = d\mathcal{A} + \frac{1}{2}\,[\mathcal{A},\mathcal{A}] \tag{7}$$

in the usual way and satisfies the Bianchi identity

$$D\mathcal{F} = d\mathcal{F} + [\mathcal{A},\mathcal{F}] = 0\,. \tag{8}$$

The field equations read

$$\begin{aligned} \frac{1}{2}\, R^{ab} \wedge *(e_a \wedge e_b \wedge e_c) &= -\frac{1}{4\sigma^2}\tau_c[\mathcal{F}] - \Lambda * e_c - \frac{\gamma}{4}\, R^{ab} \wedge R^{dg} \wedge *(e_a \wedge e_b \wedge e_d \wedge e_g \wedge e_c)\,, & (9)\\ D * \mathcal{F} &= d * \mathcal{F} + [\mathcal{A}, *\mathcal{F}] = 0\,. & (10) \end{aligned}$$

Here

$$\tau_c[\mathcal{F}] = 2\,\mathrm{Tr}\,\left(\iota_c\mathcal{F} \wedge *\mathcal{F} - \mathcal{F} \wedge \iota_c * \mathcal{F}\right) \tag{11}$$

is the corresponding stress-energy $(n-1)$-form for the gauge field $\mathcal{F}$.

V. THE ANSÄTZE AND EQUATIONS FOR THE FIELDS

Following [1] and [3], we will consider solutions that have field strengths only on an $(n-2)$-sphere. [Namely, there will be no radial components. In fact, as explained in [3], when radial components are introduced, one usually gets a different class of (numerical) solutions such as the ones obtained by Bartnik-McKinnon in four dimensions [12].] This naturally leads to the choice of the gauge group G to be $SO(n-2)$ (for $n \geq 4$) and the Ansätze for the metric and gauge potential follow accordingly. Let us decompose the local coordinates for the spacetime as

$$x^M = \left\{x^0 \equiv t,\ x^n \equiv r,\ x^i \ \text{ where }\ i = 1, 2, \ldots, (n-2)\right\}\,.$$

We think of x^i as a parameterization of the local coordinates on an $(n-2)$-sphere whose radius equals ρ, i.e. we take $\rho^2 = x_i x^i$, and consider the spacetime metric to be in the form [22]

$$ds^2 = -f^2(r)\,dt^2 + u^2(r)\,dr^2 + g^2(r)\sum_{i=1}^{n-2}\frac{dx_i\,dx^i}{(1+\rho^2/4)^2}\,. \tag{12}$$

We choose the coframe 1-forms for the metric (12) as

$$e^0 = f(r)\,dt, \quad e^n = u(r)\,dr, \quad e^i = g(r)\,\frac{dx^i}{(1+\rho^2/4)}\,,\ \ i = 1, 2, \ldots, (n-2)\,. \tag{13}$$

Levi-Civita connection 1-forms follow easily from the first Cartan structure equations as

$$\omega^0{}_i = 0, \quad \omega^i{}_j = \frac{1}{2g}(x^i e^j - x^j e^i), \quad \omega^0{}_n = \frac{f'}{fu}\,e^0, \quad \omega^i{}_n = \frac{g'}{ug}\,e^i, \tag{14}$$

where prime denotes derivative with respect to r. The curvature 2-forms that follow from these read

$$R^{0n} = B\,e^0 \wedge e^n, \quad R^{ij} = A\,e^i \wedge e^j, \quad R^{0i} = C\,e^0 \wedge e^i, \quad R^{in} = G\,e^n \wedge e^i, \tag{15}$$

where we have used

$$A=\frac{1}{g^2}\Big(1-(\frac{g'}{u})^2\Big),\quad B=-\frac{1}{fu}\Big(\frac{f'}{u}\Big)',\quad C=-\frac{f'g'}{u^2fg},\quad G=\frac{1}{ug}\Big(\frac{g'}{u}\Big)'. \tag{16}$$

As for the YM potential 1-form, we employ the ansatz

$$\mathcal{A}=\frac{1}{2}\,\Sigma_{ij}\,\frac{x^i dx^j-x^j dx^i}{(1+\rho^2/4)}\,, \tag{17}$$

where the matrices Σ_{ij} denote the generators of the gauge group $SO(n-2)$ in the fundamental representation. Specifically, we choose them as

$$\Sigma_{ij}^{\alpha\beta}=\delta_i^\alpha\,\delta_j^\beta-\delta_j^\alpha\,\delta_i^\beta\,, \tag{18}$$

with $1\leq\alpha<\beta\leq n-2$. This choice leads to the $\mathfrak{so}(n-2)$ commutation relations

$$[\Sigma_{ij},\Sigma_{k\ell}]=2\,(\delta_{\ell[i}\,\Sigma_{j]k}-\delta_{k[i}\,\Sigma_{j]\ell})\,, \tag{19}$$

so that one obtains via (7) the YM 2-form field strength to be

$$\mathcal{F}=\frac{1}{2}\,\Sigma_{ij}\,\frac{dx^i\wedge dx^j}{(1+\rho^2/4)^2}=\frac{1}{2g^2}\,\Sigma_{ij}\,e^i\wedge e^j\,. \tag{20}$$

It is not hard to show that $\mathcal{F}$ satisfies (8) and (10) thanks to (19). Our choice (18) also leads to

$$\mathrm{Tr}\,(\Sigma_{ik}\,\Sigma_{kj})=2(n-3)\,\delta_{ij}\quad\text{and}\quad\sum_{i<j}\mathrm{Tr}\,(\Sigma_{ij}\,\Sigma_{ij})=-(n-2)(n-3)\,. \tag{21}$$

Note that (17), and thus (20), satisfy the flat space YM equations as well. Therefore, before moving onto the gravitational field equations, we want to make a digression and consider Yang's problem reviewed in section II for $n=6$ with gravitation still turned off. This time we want to replace the $SU(2)$ gauge group by $SO(4)\simeq(SU(2)\times SU(2))/\mathbb{Z}_2$. Following the discussion above, the corresponding bundle is $P(S^4,SO(4))$ and the relevant homotopy group $\pi_3(SO(4))$ equals $\mathbb{Z}\oplus\mathbb{Z}$, therefore one may be inclined to think that *if* there are solutions, their charges should be labelled by two independent integers. However, this is not the whole story since the gauge fields do have to satisfy the Euclidean YM equations as well. Using the 2-form field strength (20) (with $g=1$) for $n=6$, if one naively calculates the charge as before using the analogous expression, one immediately finds

$$\int_{S^4}\mathrm{Tr}\,(\mathcal{F}\wedge\mathcal{F})=\int_{S^4}ch_2(\mathcal{F})=0\,.$$

Nevertheless, one is saved by the topological quantity that takes the place of the charge which turns out to be the Euler characteristic given by [4, 8]

$$\chi(S^4)=\frac{1}{32\pi^2}\int_{S^4}\epsilon_{\alpha\beta\gamma\delta}\,\mathcal{F}^{\alpha\beta}\wedge\mathcal{F}^{\gamma\delta}=\frac{1}{128\pi^2}\int_{S^4}\epsilon_{\alpha\beta\gamma\delta}\,\Sigma_{ij}^{\alpha\beta}\,\Sigma_{kl}^{\gamma\delta}\,\epsilon^{ijkl}\,\hat{*}1_{(4)}=2\,,$$

where $\hat{*}1_{(4)}$ denotes the volume element of the 4-sphere. We remark that for $n\geq 6$ and n even with the gauge group $SO(n-2)$, a similar argument goes through analogously. Namely

$$\int_{S^{n-2}}\mathrm{Tr}\,\mathcal{F}^{(n-2)/2}=\int_{S^{n-2}}ch_{(n-2)/2}(\mathcal{F})=0\,,$$

and the Euler characteristic reads $\chi(S^{n-2}) = 2$. In fact, for generic n

$$\chi(S^{n-2}) = \begin{cases} 0\,, & n \text{ is odd} \\ 2\,, & n \text{ is even} \end{cases} ,$$

and since the Euler characteristic vanishes for any odd-dimensional manifold, one is urged to set n even. Thus from now on, we always take $n \geq 6$ and even. The solutions thus obtained are what we mean by flat-space Yang monopoles in higher (even) dimensions.

Finally, turning on gravity, the use of (21) in (9) lead to the following system of coupled ordinary differential equations:

$$\begin{aligned} B + (n-3)\Big(\frac{n-4}{2}A + C - G\Big) &= \frac{(n-2)(n-3)}{4\sigma^2 g^4} - \Lambda \\ &\quad -\tilde{\gamma}\Big(AB - 2CG + A(n-5)\big(C - G + \frac{n-6}{4}A\big)\Big)\,, \qquad (22) \\ (n-2)\Big(\frac{n-3}{2}A + C\Big) &= -\frac{(n-2)(n-3)}{4\sigma^2 g^4} - \Lambda - (n-2)\tilde{\gamma}A\Big(\frac{n-5}{4}A + C\Big)\,, \qquad (23) \\ (n-2)\Big(\frac{n-3}{2}A - G\Big) &= -\frac{(n-2)(n-3)}{4\sigma^2 g^4} - \Lambda - (n-2)\tilde{\gamma}A\Big(\frac{n-5}{4}A - G\Big)\,, \qquad (24) \end{aligned}$$

where we have defined and used $\tilde{\gamma} = (n-3)(n-4)\gamma$.

VI. THE SOLUTIONS AND THEIR PROPERTIES

Setting $u(r) = 1/f(r)$ in (16), one finds that (23) and (24) yield $g'' = 0$, and this leads to two independent cases: Either **i)** $g(r) = g_0 =$ constant or **ii)** $g(r) = r$. It follows that (22), (23) and (24) admit two classes of solutions corresponding to each case:

i) The first case leads to a cylindrical metric

$$ds^2 = -f^2(r)\,dt^2 + \frac{dr^2}{f^2(r)} + g_0^2 \sum_{i=1}^{n-2} \frac{dx_i\,dx^i}{(1+\rho^2/4)^2}\,, \qquad (25)$$

where $f^2(r) = C_0\,r^2 + C_1\,r + C_2$. Here C_1 and C_2 are integration constants, and C_0 is given by

$$C_0 = -\frac{1}{g_0^2 + \tilde{\gamma}}\left(\frac{(n-2)(n-3)}{2\sigma^2 g_0^2} + \frac{n(n-3)}{4} + \frac{(n-5)\tilde{\gamma}}{g_0^2}\right).$$

Note that the metric (25) is conformally flat when $C_0 g_0^2 = 1$, which was also observed in [13]. We will not be interested in this solution.

ii) The second case is definitely more interesting and leads to the cosmological Einstein-GB Yang-monopole type solutions

$$ds^2 = -f^2(r)\,dt^2 + \frac{dr^2}{f^2(r)} + r^2 \sum_{i=1}^{n-2} \frac{dx_i\,dx^i}{(1+\rho^2/4)^2}\,, \qquad (26)$$

where

$$f^2(r) = 1 + \frac{r^2}{\tilde{\gamma}}\left(1 \pm \sqrt{1 - 4\tilde{\gamma}\left(\frac{\Lambda}{(n-2)(n-1)} - Mr^{1-n} + \frac{(n-3)}{4\sigma^2(n-5)}r^{-4}\right)}\right) \qquad (27)$$

now. Here, as we will see, the constant M is related to the gravitational mass of the solution.

Before we move on to studying the physical properties of this solution, we should note that there is yet another, perhaps simpler, way of obtaining the solutions (25) and (26). It is based on inserting in the action (5) (and (6)) the gauge fixed, static, spherically symmetric metric (12) with the corresponding YM field content calculated using (20) and (21). This method was originally introduced by Weyl [14] for obtaining the exterior Schwarzschild solution of General Relativity, but was put on solid ground much later in [15]. [See also [16] and [17] for some applications of this technique to various theories of gravitation.] The method considerably simplifies the labor involved in obtaining the relevant field equations. Moreover, one can also use it to show that the Birkhoff's theorem holds for the solution (25): If the functions f and u in the metric (12) are also allowed to depend on the time coordinate t, the Lagrangian density (6) turns out to be t-independent [16, 18]. Thus all spherically symmetric solutions are static in this model.

Let us look at various limits of this solution. For $\gamma \to 0$, we recover the solutions presented in [3] by choosing the $-$ branch. When one takes $\Lambda = 0$ and $\sigma \to \infty$ in (27), one recovers the external solutions of the Einstein-GB theory given in [19]. The branching of the solutions with either a Schwarzschild $f^2(r) = 1 - 2Mr^{3-n}$ or a Schwarzschild-AdS $f^2(r) = 1 + 2Mr^{3-n} + 2r^2/\tilde{\gamma}$ type of asymptotics is recovered. For both sign choices in $f^2(r)$, the gravitational energy is found to be (up to some normalizations) proportional to M by employing the energy definition of [10, 11].

Now we consider the singularity structure of our solution. It is clear that there is a curvature singularity at $r = 0$, which follows from $R_{ab} \wedge *R^{ab} = \mathcal{O}(r^{1-n}) * 1$. There is an event horizon at $r_H > 0$ ($f^2(r_H) = 0$), depending on the choice of parameters. In the most general case, this is a complicated analysis, but can be carried out along the lines of [20]. For simplicity, we concentrate on $n = 6$ (the case of the Yang monopole) with $\Lambda = 0$ and $\gamma \neq 0$ [23]. For this choice the location of the event horizon is given by the roots of the equation

$$r^3 + 3\Big(\frac{1}{2\sigma^2} + \gamma\Big) - 2M = 0\,,$$

which always has a real root r_H if

$$\Big(\frac{1}{2\sigma^2} + \gamma\Big)^3 + M^2 \geq 0\,,$$

and moreover, that root is positive if $M > 0$ and $\gamma > 0$.

Let us now compute the mass of this solution. Given a background Killing vector $\bar{\xi}^\mu$, the corresponding conserved charges of the model (6) can be written as [10, 11]

$$Q^\mu(\bar{\xi}) = \frac{1}{4\,\Omega_{n-2}}\sqrt{1 - \frac{4\,\Lambda\,\tilde{\gamma}}{(n-1)(n-2)}}\int d^{n-2}x\,\bar{\xi}_\nu\,\mathcal{G}_L^{\mu\nu}\,. \tag{28}$$

Note that all the information coming from the GB part is encoded in the coefficient. The correct background to work with is the spacetime (26) with $M = 0$ in (27), and of course, with the timelike Killing vector $\bar{\xi}^\mu = (-1, 0, \ldots, 0)$ again. For convenience we also choose the $-$ branch [24], which is asymptotically flat. One then finds the total energy according to (28) as

$$E = \frac{1}{4\Omega_{n-2}}\,\frac{2(n-2)M}{\sqrt{1 - \frac{4\Lambda\tilde{\gamma}}{(n-1)(n-2)}}}\sqrt{1 - \frac{4\Lambda\tilde{\gamma}}{(n-1)(n-2)}}\,\Omega_{n-2} = \frac{(n-2)M}{2}\,,$$

which is finite.

VII. CONCLUSIONS

We have shown that, contrary to the claim in [3], the Yang monopole defined in even dimensions has a finite mass once gravity is introduced. This has been achieved by employing the method developed in [9, 10, 11] for which a proper choice of background is essential. Specifically, we have shown that out of the three generic parameters m, μ and Λ of the gravitating Yang monopole, the first one can be interpreted as a *mass* once the remaining two are allowed to constitute the background.

We have also extended the family of Yang-monopole type solutions by studying the cosmological Einstein-GB-YM theory in higher even dimensions. We have also shown that these solutions have black hole singularities and event horizons for a proper choice of parameters.

Throughout this work, our discussion has been relying on $SO(n-2)$ gauge theory and on static spherically symmetric n-dimensional metrics. If one abandons spherical symmetry, one ends up with quite a nontrivial task of solving highly complicated differential equations. For example there is no solution describing a *rotating* Yang monopole. As for the case of the (cosmological) Einstein-GB theory, the problem is even harder: Let alone a rotating Yang monopole, there are no known exact rotating black hole solutions.

Acknowledgments

We would like to thank Yıldıray Ozan and Turgut Önder for useful discussions. This work is partially supported by the Scientific and Technological Research Council of Turkey (TÜBİTAK). B.T. is also partially supported by the Turkish Academy of Sciences (TÜBA) and by the TÜBİTAK Kariyer Grant 104T177.

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[21] Throughout, we set the Newton's constant $G_n = 1$.

[22] Note that the change of variable $\chi = \rho/(1+\rho^2/4)$ transforms the metric (12) to the following equivalent form:

$$ds^2 = -f^2(r)\,dt^2 + u^2(r)\,dr^2 + g^2(r)\left(\frac{d\chi^2}{1-\chi^2} + \chi^2\,d\Omega_{n-3}^2\right),$$

where $d\Omega_{n-3}^2$ denotes the metric on the unit $(n-3)$-sphere.
[23] $\gamma = 0$ case was considered in [3].
[24] One can also proceed with the + branch, in which case the spacetime is asymptotically AdS.