

CLUSTER ALGEBRAS AND SYMMETRIZABLE MATRICES

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ABSTRACT. In the structure theory of cluster algebras, principle coefficients are parametrized by a family of integer vectors, called \mathbf{c} -vectors. Each \mathbf{c} -vector with respect to an acyclic initial seed is a real root of the corresponding root system and the \mathbf{c} -vectors associated with any seed defines a symmetrizable quasi-cartan companion for the corresponding exchange matrix. We establish basic combinatorial properties of these companions. In particular, we show that \mathbf{c} -vectors define an admissible cut of edges in the associated diagrams.

1. INTRODUCTION

In the structure theory of cluster algebras, principle coefficients are parametrized by a family of integer vectors, called \mathbf{c} -vectors. Each \mathbf{c} -vector with respect to an acyclic initial seed is a real root of the corresponding root system; furthermore, the \mathbf{c} -vectors associated with any seed defines a symmetrizable quasi-cartan companion for the corresponding exchange matrix [8, Corollary 3.29]. In this paper, we study basic combinatorial properties of these companions. In particular, we show that \mathbf{c} -vectors define an admissible cut of edges in the associated diagrams.

To state our results, we need some terminology. Let us recall that an $n \times n$ integer matrix B is skew-symmetrizable if there is a diagonal matrix D with positive diagonal entries such that DB is skew-symmetric. We denote by \mathbb{T}_n an n -regular tree whose edges are labeled by the numbers $1, \dots, n$ such that the n edges incident to each vertex have different labels. The notation $t \xrightarrow{k} t'$ indicates that vertices $t, t' \in \mathbb{T}_n$ are connected by an edge labeled by k . We fix a vertex t_0 in \mathbb{T}_n and assign the pair (\mathbf{c}_0, B_0) , where \mathbf{c}_0 is the tuple of standard basis and B_0 is a skew-symmetrizable matrix. Then, to every vertex $t \in \mathbb{T}_n$ we assign a pair, called a Y -seed, (\mathbf{c}_t, B_t) , where $\mathbf{c}_t = (\mathbf{c}_1, \dots, \mathbf{c}_n)$ with each $\mathbf{c}_i = \mathbf{c}_{i;t} = (c_1, \dots, c_n) \in \mathbb{Z}^n$ being non-zero and having either all entries nonnegative or all entries nonpositive; we write $\text{sgn}(\mathbf{c}_i) = +1$ or $\text{sgn}(\mathbf{c}_i) = -1$ respectively and call it a \mathbf{c} -vector. Furthermore, for any edge $t \xrightarrow{k} t'$, the Y -seed $(\mathbf{c}', B') = (\mathbf{c}_{t'}, B_{t'})$ is obtained from $(\mathbf{c}, B) = (\mathbf{c}_t, B_t)$ by the Y -seed mutation μ_k defined as follows, where we denote $[b]_+ = \max(b, 0)$:

- The entries of the matrix $B' = (B'_{ij})$ are given by

$$(1.1) \quad B'_{ij} = \begin{cases} -B_{ij} & \text{if } i = k \text{ or } j = k; \\ B_{ij} + [B_{ik}]_+ [B_{kj}]_+ - [-B_{ik}]_+ [-B_{kj}]_+ & \text{otherwise.} \end{cases}$$

Date: February 14, 2018.

2010 Mathematics Subject Classification. Primary: 05E15; Secondary: 13F60.

The author's research was supported in part by the Turkish Research Council (TUBITAK) .

- The tuple $\mathbf{c}' = (\mathbf{c}'_1, \dots, \mathbf{c}'_n)$ is given by

$$(1.2) \quad \mathbf{c}'_i = \begin{cases} -\mathbf{c}_i & \text{if } i = k; \\ \mathbf{c}_i + [\text{sgn}(\mathbf{c}_k)B_{k,i}]_+ \mathbf{c}_k & \text{if } i \neq k. \end{cases}$$

By [4, Corollary 5.5], each $\mathbf{c}'_i = (\mathbf{c}'_1, \dots, \mathbf{c}'_n)$ also has either all entries nonnegative or all entries nonpositive. The matrix B' is skew-symmetrizable with the same choice of D ; we write $B' = \mu_k(B)$ and call the transformation $B \mapsto B'$ the *matrix mutation*. For the Y -seeds, we denote $\mu_k(\mathbf{c}, B) = (\mathbf{c}', B')$; we call (\mathbf{c}_0, B_0) the initial Y -seed. It is well known that mutation is an involutive operation.

Let us also recall that the *diagram* of a skew-symmetrizable $n \times n$ matrix B is the directed graph $\Gamma(B)$ whose vertices are the indices $1, 2, \dots, n$ such that there is a directed edge from i to j if and only if $B_{j,i} > 0$, and this edge is assigned the weight $|B_{ij}B_{ji}|$. The diagram $\Gamma(B)$ is called acyclic if it has no oriented cycles. Then there is a corresponding generalized Cartan matrix A such that $A_{i,i} = 2$ and $A_{i,j} = -|B_{i,j}|$ for $i \neq j$. There is also the associated root system in the root lattice spanned by the simple roots α_i [6]. For each simple root α_i , the corresponding reflection $s_{\alpha_i} = s_i$ is the linear isomorphism defined on the basis of simple roots as $s_i(\alpha_j) = \alpha_j - A_{i,j}\alpha_i$. Then the real roots are defined as the vectors obtained from the simple roots by a sequence of reflections. It is well known that the coordinates of a real root with respect to the basis of simple roots are either all nonnegative or all nonpositive, see [6] for details.

On the other hand, an $n \times n$ matrix A is called symmetrizable if there exists a symmetrizing diagonal matrix D with positive diagonal entries such that DA is symmetric. A *quasi-Cartan companion* (or "companion" for short) of a skew-symmetrizable matrix B is a symmetrizable matrix A such that $A_{i,i} = 2$, $|A_{i,j}| = |B_{i,j}|$ for all $i \neq j$.

A fundamental relation between Y -seeds and symmetrizable matrices has been given in [8, Corollary 3.29] as follows:

Theorem 1.1. [8, Corollary 3.29] *Suppose that the initial seed (\mathbf{c}_0, B_0) is acyclic. Then, for any Y -seed (\mathbf{c}_t, B_t) , $t \in \mathbb{T}_n$, each \mathbf{c} -vector $\mathbf{c}_i = \mathbf{c}_{i;t}$ is the coordinate vector of a real root with respect to the basis of simple roots in the corresponding root system. Furthermore, $A = A_t = (\langle \mathbf{c}_j, \mathbf{c}_i^\vee \rangle)$, the matrix of the pairings between the roots and the coroots, is a quasi-Cartan companion of the skew-symmetrizable matrix $B = B_t$.*

(The matrices A_t are symmetrizable with the same choice of a symmetrizing matrix D , which is also skew-symmetrizing for all B_t .)

An important combinatorial property related to quasi-Cartan companions is admissibility [9, 10], which is a generalization of the notion of a generalized Cartan matrix. More precisely, a quasi-Cartan companion A of a skew-symmetrizable matrix B is *admissible* if, for any oriented (resp. non-oriented) cycle Z in $\Gamma(B)$, there is exactly an odd (resp. even) number of edges $\{i, j\}$ such that $A_{i,j} > 0$. If $\Gamma(B)$ is acyclic, then the associated generalized Cartan matrix is admissible. Our first result generalizes this property by showing that the quasi-Cartan companions defined by \mathbf{c} -vectors are also admissible:

Theorem 1.2. *In the set-up of Theorem 1.1, the quasi-Cartan companion A has the following properties:*

- Every directed path of the diagram $\Gamma(B)$ has at most one edge $\{i, j\}$ such that $A_{i,j} > 0$.
- Every oriented cycle of the diagram $\Gamma(B)$ has exactly one edge $\{i, j\}$ such that $A_{i,j} > 0$.
- Every non-oriented cycle of the diagram $\Gamma(B)$ has an even number of edges $\{i, j\}$ such that $A_{i,j} > 0$.

In particular, the quasi-Cartan companion A is admissible. Furthermore, any admissible quasi-Cartan companion of B can be obtained from A by a sequence of simultaneous sign changes in rows and columns.

The special case of this theorem when B is skew-symmetric was obtained in [10, Theorem 1.4] by the author. Let us also recall from [10] that a set C of edges in $\Gamma(B)$ is called an "admissible cut" if every oriented cycle contains exactly one edge that belongs to C and every non-oriented cycle contains exactly an even number of edges in C . Thus, in the setup of the theorem, the \mathbf{c} -vectors define an admissible cut of edges: the set of edges $\{i, j\}$ in $\Gamma(B)$ such that $A_{i,j} > 0$ is an admissible cut. For skew-symmetric matrices, this notion has been applied to the representation theory of algebras in [5, ?].

Our next result is the following explicit description of the quasi-Cartan companions defined by the \mathbf{c} -vectors:

Theorem 1.3. *In the set-up of Theorem 1.1, the quasi-Cartan companion A has the following properties:*

- If $\text{sgn}(B_{j,i}) = \text{sgn}(\mathbf{c}_j)$, then $A_{j,i} = -\text{sgn}(\mathbf{c}_j)B_{j,i} = -|B_{j,i}|$.
- If $\text{sgn}(B_{j,i}) = -\text{sgn}(\mathbf{c}_j)$, then $A_{j,i} = \text{sgn}(\mathbf{c}_i)B_{j,i} = -\text{sgn}(\mathbf{c}_i)\text{sgn}(\mathbf{c}_j)|B_{j,i}|$.

In particular; if $\text{sgn}(\mathbf{c}_j) = -\text{sgn}(\mathbf{c}_i)$, then $B_{j,i} = \text{sgn}(\mathbf{c}_i)A_{j,i}$.

Let us note that the special case of this theorem when B is skew-symmetric was obtained in [10, Theorem 1.3] by the author. We will prove this more general theorem using [8, Corollary 3.29], which has been given above as Theorem 1.1. (Note that the statement [8, Corollary 3.29] was not present in the earlier versions of [8]).

Corollary 1.4. *In the setup of Theorem 1.3, suppose that $t \xrightarrow{k} t'$ in \mathbb{T}_n . Then, for $\mu_k(\mathbf{c}, B) = (\mathbf{c}', B')$, we have the following: if $\mathbf{c}'_i \neq \mathbf{c}_i$, then $\mathbf{c}'_i = s_{\mathbf{c}_k}(\mathbf{c}_i)$, where $s_{\mathbf{c}_k}$ is the reflection with respect to the real root \mathbf{c}_k and \mathbb{Z}^n is identified with the root lattice.*

Let us also note that Theorem 1.3 could be useful for recognizing mutation classes of acyclic diagrams: a diagram that does not have an admissible quasi-Cartan companion can not be obtained from any acyclic diagram by a sequence of mutations. An example of such a diagram is given in Figure 1. (We refer to [9, Section 2] for properties of diagrams of skew-symmetrizable matrices). Another application of the admissibility property to the corresponding Weyl groups can be found in [11], where a fundamental class of relations have been shown to be satisfied by the reflections of the \mathbf{c} -vectors.

2. PROOFS OF MAIN RESULTS

Let us first recall the following well-known property of root systems: For a generalized Cartan matrix A with symmetrizing matrix $D = \text{diag}(d_1, \dots, d_n)$, there

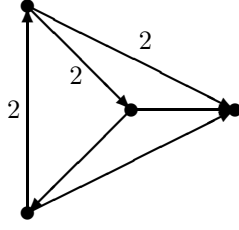


FIGURE 1. a diagram which does not have an admissible quasi-Cartan companion

is an invariant symmetric bilinear form $(,)$ defined on the simple roots as $(\alpha_i, \alpha_j) = d_i A_{i,j} = d_j A_{j,i} = (\alpha_j, \alpha_i)$. Let us note that, for any real root α , the corresponding reflection s_α is defined on the real roots as $s_\alpha(\beta) = \beta - \langle \beta, \alpha^\vee \rangle \alpha$, with $\langle \beta, \alpha^\vee \rangle = \frac{2(\alpha, \beta)}{(\alpha, \alpha)}$. In particular, $s_{\alpha_i}(\alpha_j) = \alpha_j - \langle \alpha_j, \alpha_i^\vee \rangle \alpha_i = \alpha_j - A_{i,j} \alpha_i$.

Let us also recall the mutation of quasi Cartan companions [10, Definition 1.6]. Suppose that B is a skew-symmetrizable matrix and let A be a quasi-Cartan companion of B . Let k be an index. For each sign $\epsilon = \pm 1$, "the ϵ -mutation of A at k " is the quasi-Cartan matrix $\mu_k^\epsilon(A) = A'$ such that for any $i, j \neq k$: $A'_{i,k} = \epsilon \operatorname{sgn}(B_{k,i}) A_{i,k}$, $A'_{k,j} = \epsilon \operatorname{sgn}(B_{k,j}) A_{k,j}$, $A'_{i,j} = A_{i,j} - \operatorname{sgn}(A_{i,k} A_{k,j}) [B_{i,k} B_{k,j}]_+$. In the setup of Theorem 1.1, suppose that $t \xrightarrow{-k} t'$ in \mathbb{T}_n and let A and A' be the associated quasi-Cartan companions. Then $A' = \mu_k^\epsilon(A)$ for $\epsilon = \operatorname{sgn}(\mathbf{c}_k)$.

We first prove Theorem 1.3 for convenience:

Proof of Theorem 1.3. To prove the first part, let us suppose that $\operatorname{sgn}(B_{j,i}) = \operatorname{sgn}(\mathbf{c}_j)$. Let $\mu_j(\mathbf{c}, B) = (\mathbf{c}', B')$ with $B' = \mu_j(B)$. Then $\mathbf{c}'_i = \mathbf{c}_i + [\operatorname{sgn}(\mathbf{c}_j) B_{j,i}]_+ \mathbf{c}_j = \mathbf{c}_i + \operatorname{sgn}(B_{j,i}) B_{j,i} \mathbf{c}_j = \mathbf{c}_i + |B_{j,i}| \mathbf{c}_j$. We denote by $(,)$ the invariant symmetric bilinear form defined by A_0 on the root lattice and let $D = \operatorname{diag}(d_1, \dots, d_n)$ be the symmetrizing matrix for A_0 . Note that, by Theorem 1.1, we have the following: $2d_i = (\mathbf{c}'_i, \mathbf{c}'_i) = (\mathbf{c}_i, \mathbf{c}_i)$, $2d_j = (\mathbf{c}_j, \mathbf{c}_j)$, $(\mathbf{c}_j, \mathbf{c}_i) = (\mathbf{c}_i, \mathbf{c}_j) = d_i A_{i,j} = d_j A_{j,i}$. Then $2d_i = (\mathbf{c}'_i, \mathbf{c}'_i) = (\mathbf{c}_i + |B_{j,i}| \mathbf{c}_j, \mathbf{c}_i + |B_{j,i}| \mathbf{c}_j) = (\mathbf{c}_i, \mathbf{c}_i) + (\mathbf{c}_i, |B_{j,i}| \mathbf{c}_j) + (|B_{j,i}| \mathbf{c}_j, \mathbf{c}_i) + |B_{j,i}|^2 (\mathbf{c}_j, \mathbf{c}_j) = (\mathbf{c}_i, \mathbf{c}_i) + 2|B_{j,i}| (\mathbf{c}_j, \mathbf{c}_i) + |B_{j,i}|^2 (\mathbf{c}_j, \mathbf{c}_j) = 2d_i + 2|B_{j,i}| (\mathbf{c}_j, \mathbf{c}_i) + |B_{j,i}|^2 2d_j = 2d_i + 2|B_{j,i}| d_j A_{j,i} + |B_{j,i}|^2 2d_j = 2d_i + 2|B_{j,i}| d_j (A_{j,i} + |B_{j,i}|)$, implying that $A_{j,i} + |B_{j,i}| = 0$, thus $A_{j,i} = -|B_{j,i}| = -\operatorname{sgn}(B_{j,i}) B_{j,i} = -\operatorname{sgn}(\mathbf{c}_j) B_{j,i}$.

To prove the second part of the theorem, let us suppose that $\operatorname{sgn}(B_{j,i}) = -\operatorname{sgn}(\mathbf{c}_j)$. Let $\mu_i(\mathbf{c}, B) = (\mathbf{c}', B')$ with $B' = \mu_i(B)$. Note that $\operatorname{sgn}(B'_{j,i}) = -\operatorname{sgn}(B_{j,i})$ and $|B'_{j,i}| = |B_{j,i}|$ (by the definition of mutation). Let A' be the quasi-Cartan companion associated to the Y -seed (\mathbf{c}', B') (Theorem 1.1), (Note then that $A' = \mu_i^\epsilon(A)$ where $\epsilon = \operatorname{sgn}(\mathbf{c}_i)$).

For the proof, we first assume that $\operatorname{sgn}(\mathbf{c}_j) = -\operatorname{sgn}(\mathbf{c}_i)$. Then we have $\operatorname{sgn}(\mathbf{c}_i) = \operatorname{sgn}(B_{j,i})$, so $\mathbf{c}'_j = \mathbf{c}_j$ and $\mathbf{c}'_i = -\mathbf{c}_i$, implying $\operatorname{sgn}(\mathbf{c}'_j) = \operatorname{sgn}(\mathbf{c}_j) = -\operatorname{sgn}(B_{j,i}) = \operatorname{sgn}(B'_{j,i})$, i.e. for the Y -seed (\mathbf{c}', B') , we have $\operatorname{sgn}(B'_{j,i}) = \operatorname{sgn}(\mathbf{c}'_j)$. Thus, by the first part of the theorem, we have $-|B'_{j,i}| = A'_{j,i} = -A_{j,i}$. Thus $A_{j,i} = |B'_{j,i}| = |B_{j,i}| = -\operatorname{sgn}(\mathbf{c}_i) \operatorname{sgn}(\mathbf{c}_j) |B_{j,i}|$.

Let us now assume that $\operatorname{sgn}(\mathbf{c}_j) = \operatorname{sgn}(\mathbf{c}_i)$. Then, since we have assumed $\operatorname{sgn}(B_{j,i}) = -\operatorname{sgn}(\mathbf{c}_j)$, we have $\operatorname{sgn}(\mathbf{c}_i) = -\operatorname{sgn}(B_{j,i}) = \operatorname{sgn}(B_{i,j})$. Then, by

the first part of the theorem, we have $A_{i,j} = -|B_{i,j}|$. Thus, since A is symmetrizable and a quasi-Cartan companion, we also have $A_{j,i} = -|B_{j,i}|$, which is equal to $-sgn(\mathbf{c}_i)sgn(\mathbf{c}_j)|B_{j,i}|$.

On the other hand, our assumption $sgn(B_{j,i}) = -sgn(\mathbf{c}_j)$ implies the following: $-sgn(\mathbf{c}_i)sgn(\mathbf{c}_j)|B_{j,i}| = -sgn(\mathbf{c}_i)sgn(\mathbf{c}_j)sgn(B_{j,i})B_{j,i} = -sgn(\mathbf{c}_i)sgn(\mathbf{c}_j)(-sgn(\mathbf{c}_j))B_{j,i} = sgn(\mathbf{c}_i)B_{j,i}$. This completes the proof.

Proof of Corollary 1.4. Let us note that for $\mu_k(\mathbf{c}, B) = (\mathbf{c}', B')$ we have the following: $\mathbf{c}'_k = -\mathbf{c}_k$; $\mathbf{c}'_i = \mathbf{c}_i + [sgn(\mathbf{c}_k)B_{k,i}]_+ \mathbf{c}_k$ if $i \neq k$ by (1.2). On the other hand, $[sgn(\mathbf{c}_k)B_{k,i}]_+ \neq 0$ if and only if $sgn(\mathbf{c}_k)B_{k,i} > 0$ if and only if $sgn(\mathbf{c}_k) = sgn(B_{k,i})$. Then, by Theorem 1.3, we have $[sgn(\mathbf{c}_k)B_{k,i}]_+ = -A_{k,i}$. Thus $\mathbf{c}'_i = \mathbf{c}_i - A_{k,i}\mathbf{c}_k = s_{\mathbf{c}_k}(\mathbf{c}_i)$ by the definition of a reflection. Also $\mathbf{c}'_k = -\mathbf{c}_k = s_{\mathbf{c}_k}(\mathbf{c}_k)$. This completes the proof of the statement.

Proof of Theorem 1.2. As we discussed in Section 1, the special case of this theorem when B is skew-symmetric was obtained in [10, Theorem 1.4] by the author. The proof in [10] uses only the general properties of the mutations of skew-symmetrizable matrices with quasi-Cartan companions and the properties given in Theorem 1.3 (which was obtained for skew-symmetric matrices in [10, Theorem 1.3]; note that in this case the companion A is symmetric and $A_{i,j} = \mathbf{c}_i^T A_0 \mathbf{c}_j$). Since we have proved Theorem 1.3 above for skew-symmetrizable matrices, the proof of [10, Theorem 1.4] also holds for the skew-symmetrizable matrices. Thus, for the proof of Theorem 1.2, we refer the reader to the proof of [10, Theorem 1.4].

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