LEFSCHETZ FIBRATIONS AND AN INVARIANT OF FINITELY PRESENTED GROUPS

MUSTAFA KORKMAZ

ABSTRACT. Every finitely presented group is the fundamental group of the total space of a Lefschetz fibration. This follows from results of Gompf and Donaldson, and was also proved by Amoros-Bogomolov-Katzarkov-Pantev. We give another proof by providing the monodromy explicitly. We then define the genus of a finitely presented group Γ to be the minimal genus of a Lefschetz fibration with fundamental group Γ. We also give some estimates of the genus of certain groups.

1. INTRODUCTION

Every finitely presented group is the fundamental group of some closed symplectic 4-manifold [3]. By the work of Donaldson [2], every closed oriented symplectic 4-manifold admits a Lefschetz pencil. Thus such a manifold admits a Lefschetz fibration over $\mathbb{S}^2$, perhaps after blowing up many times. It follows that any finitely presented group is realized as the fundamental group of the total space of a Lefschetz fibration over $\mathbb{S}^2$. Conversely, if $g \geq 2$, then the total space of a genus g Lefschetz fibration is symplectic (c.f. [4]).

The paper [1] gives another construction of a symplectic 4-manifold as the total space of a Lefschetz fibration with the given fundamental group. In this construction, the genus of the Lefschetz fibration depends on the number of intersection points of some curves on some surface representing the relators of the group. Thus it is quadratic in the lengths of the relators.

The purpose of this paper is to give yet another construction of a Lefschetz fibration with the prescribed finitely presented group. In our construction, we give the monodromy explicitly. The genus of the Lefschetz fibration depends linearly on the number of generators and on the syllable lengths of the relators of the presentation of the given finitely presented group. More precisely, given a finitely presented group Γ with n generators and k relators, if ℓ is the sum of the syllable lengths of the relators, then for every $g \geq 2(n+\ell-k)$ we construct a genus-g Lefschetz fibration over $\mathbb{S}^2$ whose fundamental group is isomorphic to Γ.

In [1], the authors first represent the relators of the finitely presented group by loops on a certain surface Σ and then for each intersection point of

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these loops they increase the genus of Σ by one as illustrated in Figure 1 (a). This makes the genus of the fiber of the Lefschetz fibration quadratic in the lengths of the relators. However, in our proof, we increase the genus of Σ by one for each set of consecutive self intersection points of a loop as illustrated in Figure 1 (b).

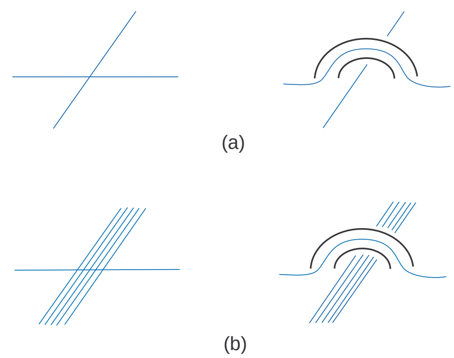

FIGURE 1. Two ways of resolving the self intersection points of a loop.

Here is an outline of the paper. In Section 2, we fix the notations, state the main theorem, Theorem 2.1, and give the necessary tool from the theory of mapping class groups to prove it. Section 3 reviews the theory of Lefschetz fibrations and proves Theorem 3.2, which is used throughout the paper. Section 4 is devoted to the proof of Theorem 2.1. In Section 5, we define an invariant of a finitely presented group Γ, which we call the genus of Γ. It turns out that the genus of the fundamental group of a closed orientable surface of genus g is g. We compute the genera of the groups $\mathbb{Z}, \mathbb{Z}\oplus\mathbb{Z}_n, \mathbb{Z}_m\oplus\mathbb{Z}_n, \mathbb{Z}_n$, which are all equal to 2. We also give upper and lower bounds for the genus for any finitely presented group and we obtain some results of the genera of some groups related to topology; free groups, finitely generated abelian groups, fundamental groups of closed nonorienrable surfcaes, braid groups and the group $SL(z, Z)$. In the final section, Section 6, we compare our invariant with Kotschick's invariants p and q, and pose some problems motivated by the content of the paper.

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2. PRELIMINARIES AND THE STATEMENT OF MAIN RESULT

Let Γ be a finitely generated group generated by a set $X = \{x_1, x_2, \ldots, x_n\}$. For an element $w \in \Gamma$, let the *syllable length* $\ell(w)$ of w be defined as follows:

$$\ell(w) = \min\{\, s \mid w = x_{i_1}^{m_1} x_{i_2}^{m_2} \cdots x_{i_s}^{m_s},\ 1 \le i_j \le n,\ m_j \in \mathbb{Z}\}.$$

Suppose now that Γ is a finitely presented group with a presentation

$$\Gamma = \langle x_1, x_2, \ldots, x_n | \ r_1, r_2, \ldots, r_k\rangle, \tag{1}$$

so that Γ is the quotient F/N, where F is the free group (nonabelian for $n \geq 2$) freely generated by $\{x_1, \ldots, x_n\}$ and N is the normal subgroup of F generated normally by the elements $r_1, r_2, \ldots, r_k$. That is, N is the subgroup of F generated by all conjugates of $r_1, r_2, \ldots, r_k$.

Define $\ell = \ell(r_1) + \ell(r_2) + \cdots + \ell(r_k)$, which depends on the presentation. We always assume that the relators r_i are cyclically reduced.

The main result of this paper is the following theorem.

Theorem 2.1. *Let Γ be a finitely presented group with a presentation (1). Then for every $g \geq 2(n + \ell - k)$ there exists a genus-g Lefschetz fibration $f : X \to \mathbb{S}^2$ such that $\pi_1(X)$ is isomorphic to Γ.*

For a closed oriented surface Σ_g of genus g, the mapping class group Mod_g of Σ_g is defined to be the group of isotopy classes of orientation preserving diffeomorphism $\Sigma_g \to \Sigma_g$. If a is a simple closed curve on Σ_g, then cutting Σ_g along a and gluing the two boundary components back after twisting one of the sides to the right by 2π give a diffeomorphism $\Sigma_g \to \Sigma_g$. The isotopy class of this diffeomorphism, denoted by t_a, is called the right Dehn twist about a. The mapping class t_a depends only on the isotopy class of a. Dehn twists are the simplest diffeomorphisms of Σ_g. They are the main building blocks in Mod_g; they generate the mapping class group and each representation of the identity element 1 in the group Mod_g as a product of right Dehn twists gives a Lefschetz fibration, whose total space is a symplectic 4-manifold.

We now describe such a relation in Mod_g which is the main ingredient in the proof of our result. This relation was obtained in [6] as an extension of Matsumoto's relation in [8] and was used to construct noncomplex smooth symplectic 4-manifolds admitting genus-g Lefschetz fibrations. It was also used by Ozbagci and Stipsicz in [9] in their construction of infinitely many Stein fillings of a certain contact 3-manifold.

Let Σ_g denote the closed oriented surface of genus g standardly embedded in the 3-space as in Figures 3, so that it is the boundary a 3-dimensional handlebody. Let $\Sigma_{g,2}$ denote the surface of genus g with two boundary components obtained from Σ_g by deleting the interior of two disjoint discs (c.f. Figure 2). Let us define a word

$$W = \begin{cases} (t_c^2 t_{B_g} t_{B_{g-1}} \cdots t_{B_2} t_{B_1} t_{B_0})^2 & \text{if } g \text{ is even,} \\ (t_a^2 t_b^2 t_{B_g} t_{B_{g-1}} \cdots t_{B_2} t_{B_1} t_{B_0})^2 & \text{if } g \text{ is odd,} \end{cases} \tag{2}$$

in the mapping class group of $\Sigma_{g,2}$, where the simple closed curves B_j and a, b, c are given in Figure 2. It was shown in [6] that the word W represents the identity element in the mapping class group of Σ_g. It can be shown easily that the word W is equal to the product $t_{\delta_1} t_{\delta_2}$ in the mapping class group of $\Sigma_{g,2}$. Here, δ_1 and δ_2 are the boundary components of $\Sigma_{g,2}$. When there is one boundary component δ, it was shown in [10] that the word W is equal to the Dehn twist t_δ, and this will be sufficient for us.

We note that Dehn twists in the relation W above is given in the reversed order in [6]. There the functional notation was used for the composition of functions. However, in this paper, the composition fg of two diffeomorphism f and g means that we first apply f and then g.

Consider the standard set of generators $a_1, b_1, a_2, b_2, \ldots, a_g, b_g$ of $\pi_1(\Sigma_g)$ as shown in Figure 3. The subgroup of $\pi_1(\Sigma_g)$ generated by $a_1, a_2, \ldots, a_g$ is a free group of rank g.

Throughout the paper a loop and its homotopy class will be denoted by the same notation. Similarly, a diffeomorphism and its isotopy class or a simple closed curve and its isotopy class will be denoted by the same symbol. We even denote a simple loop and a simple closed curve by the same symbol, and this will not cause any problem as it will be clear from the context which one we mean.

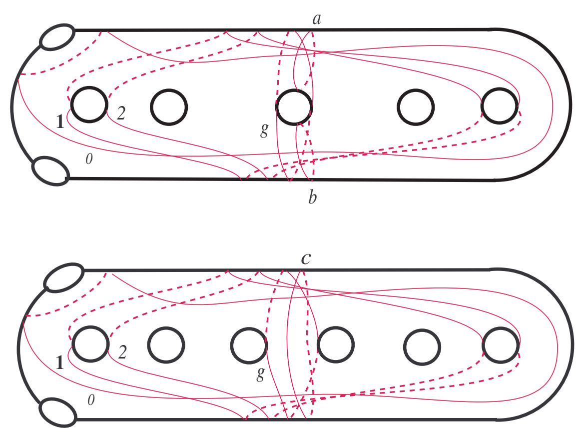

FIGURE 2. The simple closed curve labelled j is B_j.

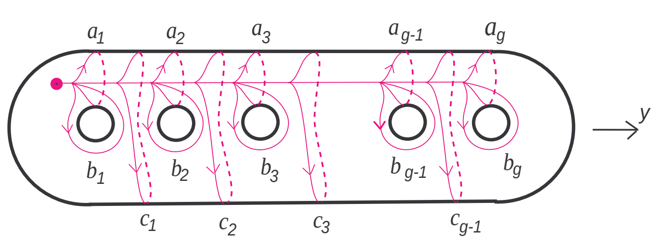

FIGURE 3. Generators of the fundamental group.

If x and y are two elements of a group, we write $[x, y] = x^{-1}y^{-1}xy$ and $x^y = y^{-1}xy$. We recall that if f is in mapping class group of an oriented surface Σ and t_a is a Dehn twist about a simple closed curve a on Σ, then it easily follows from the definition of a Dehn twist that $f^{-1}t_a f = t_{f(a)}$, so that for a mapping class (in particular for a Dehn twist) f, the conjugation $W^f = 1$ is a representation of the identity as a product of right Dehn twists in Mod_g.

3. Lefschetz fibrations

Let us review briefly the theory of Lefschetz fibrations on 4-manifolds. For the details the reader is referred to [4]. Let X be a closed connected oriented smooth 4-manifold. A Lefschetz fibration on X is a smooth map $f : X \to \mathbb{S}^2$ such that if q is a critical point, then q has a neighborhood U with complex coordinates (z_1, z_2) agreeing with the orientation of X and $f(q)$ has a neighborhood V with complex coordinate z agreeing with the orientation of $\mathbb{S}^2$ such that the restriction $f| : U \to V$ of f is of the form $f(z_1, z_2) = z_1^2 + z_2^2$.

It follows that the number of critical points of f is finite. We can assume that each singular fiber contains only one critical point. Suppose that $Q = \{q_1, q_2, \ldots, q_s\}$ are critical points and $f(Q) = \{p_1, p_2, \ldots, p_s\}$ are the critical values so that $f(q_i) = p_i$. Removing the singular fibers from the total space X and the critical values from the base, we get a surface bundle

$$X - f^{-1}(f(Q)) \longrightarrow \mathbb{S}^2 - f(Q).$$

Therefore, all regular fibers of f are diffeomorphic to a closed connected oriented smooth surface of genus g for some g. We will assume that all Lefschetz fibrations are relatively minimal, i.e. no fiber contains an embedded 2-sphere with self-intersection number -1.

Let us fix a base point p_0 on $\mathbb{S}^2$ which is a regular value and let Σ denote the fiber over it; $\Sigma = f^{-1}(p_0)$. If C is a simple loop on $\mathbb{S}^2 - f(Q)$ encircling only one of the critical values, then $f^{-1}(C)$ is surface bundle over a circle with the monodromy a right Dehn twist about a simple closed curve on Σ, which is called a vanishing cycle.

The topology of the total space of a Lefschetz fibration is determined by the monodromy representation $\varphi : \pi_1(\mathbb{S}^2 - f(Q)) \to \mathrm{Mod}_g$. For a simple loop C encircling only one critical value, if c is the corresponding vanishing cycle, then $\varphi(C)$ is the Dehn twist t_c.

Let D be a disc on $\mathbb{S}^2$ containing all critical values in the interior and the base point p_0 on the boundary. For each $i = 1, 2, \ldots, s$ let e_i be a small circle in the interior of D encircling the critical value p_i but no other critical value. Let d_i be a simple arc connecting p_0 to a point e_i. We assume that d_i and d_j intersect only at p_0 for $i \neq j$. We choose d_i in such a way that, on the boundary ∂ of a small regular neighborhood of p_0, the intersection of ∂ and $d_1, d_2, \ldots, d_s$ are read in the counter clockwise order. Now let α_i be the loop obtained first travelling from p_0 to e_i along d_i, then travelling along e_i once counter clockwise and then back to p_0 along d_i. Clearly, the fundamental group $\pi_1(\mathbb{S}^2 - f(Q))$ has a presentation with generators $\alpha_1, \alpha_2, \ldots, \alpha_s$ and with a single defining relation

$$\alpha_1\alpha_2\cdots\alpha_s = 1.$$

Thus if c_i is the vanishing cycle corresponding to α_i, then the Dehn twists t_{c_i} satisfy the relation

$$t_{c_1}t_{c_2}\cdots t_{c_s} = 1 \tag{3}$$

in the mapping class group Mod_g of the surface Σ.

Conversely, given a decomposition of the identity as a product of right Dehn twists of the form (3) in the mapping class group Mod_g, we can construct a genus-g Lefschetz fibration as follows: Consider the product $\Sigma_g \times D$, where Σ_g is a closed connected oriented surface of genus g and D is a 2-disc. For each $i = 1, 2, \ldots, s$, attach a 2-handle to $\Sigma_g \times D$ along the simple closed curve c_i with framing -1 relative to the product framing in $\Sigma_g \times \partial D$. The resulting 4-manifold Y admits a Lefschetz fibration over the 2-disc whose monodromy around the boundary is the product $t_{c_1}t_{c_2}\cdots t_{c_s}$, which is equal to the identity in the mapping class group of Σ_g. Therefore, the boundary of Y is diffeomorphic to the product $\Sigma_g \times S^1$. Now glue $\Sigma_g \times D$ and Y along their boundaries to get the closed connected oriented smooth 4-manifold X admitting a Lefschetz fibration with generic fiber Σ_g.

A section of a Lefschetz fibration $f : X \to \mathbb{S}^2$ is a map $\sigma : \mathbb{S}^2 \to X$ such that $f\sigma$ is the identity map of $\mathbb{S}^2$. Since the word W represents the element $t_{\delta_1}t_{\delta_2}$ in the mapping class group of $\Sigma_{g,2}$, the Lefschetz fibration of genus g with the monodromy $W = 1$ admits two disjoint sections of self intersection -1 (c.f. [11]).

Lemma 3.1. ([4]) *Let $f : X \to \mathbb{S}^2$ be a genus-g Lefschetz fibration with global monodromy given by the relation (3). Suppose that f has a section. Then the fundamental group of X is isomorphic to the fundamental group of Σ_g divided out by the normal closure of the simple closed curves $c_1, c_2, \ldots, c_s$, considered as elements in $\pi_1(\Sigma_g)$. In particular, there is an epimorphism $\pi_1(\Sigma_g) \to \pi_1(X)$*

Suppose that V is a product of right Dehn twists representing the identity in the mapping class group Mod_g of the closed oriented surface Σ_g. If $f_1, f_2, \ldots, f_m$ are arbitrary elements in Mod_g, then the product

$$V^{f_1}V^{f_2}\cdots V^{f_m}$$

is also a product of right Dehn twists representing the identity in Mod_g. Here, V^f denotes the conjugation $f^{-1}Vf$. Let us denote by $X_V^g(f_1, f_2, \ldots, f_m)$ the total space of the Lefschetz fibration over $\mathbb{S}^2$ whose monodromy is $V^{f_1}V^{f_2}\cdots V^{f_m}$. For simplicity, if d_i is a simple closed curve on Σ, we denote the 4-manifold $X_V^g(t_{d_1}, t_{d_2}, \ldots, t_{d_m})$ by $X_V^g(d_1, d_2, \ldots, d_m)$. We note that $X_V^g(d_1, d_2, \ldots, d_m)$ is a fiber sum

$$X_V^g(1)\#_f X_V^g(1)\#_f \cdots \#_f X_V^g(1)$$

of m copies of $X_V^g(1)$. Clearly, if $X_V^g(1)$ has a section, then $X_V^g(d_1, d_2, \ldots, d_m)$ has a section as well.

The main result of this section is the following theorem, which is used throuhout the paper.

Theorem 3.2. *Let $V = t_{c_1} t_{c_2} \cdots t_{c_n}$ be a product of right Dehn twists representing the identity in the mapping class group of the closed oriented surface Σ_g. Suppose that the Lefschetz fibration $X_V^g(1) \to \mathbb{S}^2$ has a section. Let $d_1, d_2, \ldots, d_m$ be simple closed curves on Σ_g with the property that for each $j = 1, 2, \ldots, m$, there exists an i_j such that d_j intersects c_{i_j} transversely at only one point. Then the fundamental group of $X_V^g(1, d_1, d_2, \ldots, d_m)$ is isomorphic to the group $\pi_1(X_V^g(1))$ divided out by the normal closure of $\{d_1, d_2, \cdots, d_m\}$.*

Proof. By Lemma 3.1, the group $\pi_1(X_V^g(1))$ admits a presentation with generators $a_1, b_1, \ldots, a_g, b_g$ and with relations

- $[a_1, b_1]^{y_1} \cdots [a_g, b_g]^{y_g} = 1$ and
- $c_1 = c_2 = \cdots = c_n = 1$,

where y_i are some words in a_j, b_j. Since $V^f = t_{f(c_1)} t_{f(c_2)} \cdots t_{f(c_n)}$, in order to obtain a presentation of the group $\pi_1(X_V^g(1, d_1, d_2, \ldots, d_m))$ one needs to add the relations $t_{d_j}(c_i) = 1$ to this presentation. That is, the fundamental group $\pi_1(X_V^g(1, d_1, d_2, \ldots, d_m))$ admits a presentation with generators $a_1, b_1, \ldots, a_g, b_g$ and with relations

$$\begin{aligned} &[a_1, b_1]^{y_1} \cdots [a_g, b_g]^{y_g} = 1, \\ &c_1 = c_2 = \cdots = c_n = 1, \text{ and} \\ &t_{d_j}(c_i) = 1, \ 1 \le i \le n, \ 1 \le j \le m. \end{aligned} \tag{4}$$

Let us fix some $1 \le j \le m$. By assumption, the simple closed curve d_j intersects c_l transversely at one point for some $1 \le l \le n$. Fix an orientation of d_j, Then the curve $t_{d_j}(c_l)$ is equal (in fact conjugate) to $c_l \, d_j^{\varepsilon}$, where $\varepsilon = \pm 1$. Since $c_l = 1$ in this presentation, we may replace the relator $t_{d_j}(c_l) = 1$ by $d_j = 1$. Suppose that $i \neq l$. If d_j is disjoint from c_i, then $t_{d_j}(c_l) = c_l$. Suppose now that d_j intersects c_i at t points. It can be seen easily that there are elements $x_1, x_2, \ldots, x_{t+1}$ in $\pi_1(\Sigma_g)$ such that $c_i = x_1 x_2 \ldots x_{t+1}$ and that the curve $t_{d_j}(c_l)$ is conjugate to $x_1 d_j^{\varepsilon_1} x_2 d_j^{\varepsilon_2} \ldots x_t d_j^{\varepsilon_t} x_{t+1}$, where each ε_i is equal to 1 or -1. Since $d_j = 1$ and $c_i = 1$, we can delete the relators $t_{d_j}(c_i) = 1$ from the presentation (4).

Repeating this argument for each $j = 1, 2, \ldots, m$ shows that the relators $t_{d_j}(c_i) = 1$, $1 \le i \le n$, $1 \le j \le m$, may be replaced by $d_j = 1$.

This proves the theorem. □

4. The proof of Theorem 2.1

In this section, we prove Theorem 2.1. For the proof, we take appropriate fiber sums of the Lefschetz fibration $X_W^g(1)$ with global monodromy W with itself.

4.1. **The proof of Theorem 2.1 for free groups.** From the presentation point of view, the simplest groups are free groups. So we assume first that Γ is a free group.

Proposition 4.1. *Let Γ be a free group of rank n. Then for every $g \geq 2n$ there is a genus-g Lefschetz fibration $f : X \to \mathbb{S}^2$ with $\pi_1(X) \cong \Gamma$.*

Proof. Assume first that $g = 2r$ is even, so that $r \geq n$. Let $a_1, b_1, \ldots, a_g, b_g$ be the standard generators of the fundamental group $\pi_1(\Sigma_g)$ of the surface Σ_g as shown in Figure 3. It can easily be shown that in $\pi_1(\Sigma_g)$, up to conjugation, the following equalities hold:

- $B_0 = b_1 b_2 \cdots b_g$,
- $B_{2k-1} = a_k b_k b_{k+1} \cdots b_{g+1-k} c_{g+1-k} a_{g+1-k}$, $1 \leq k \leq r$,
- $B_{2k} = a_k b_{k+1} b_{k+2} \cdots b_{g-k} c_{g-k} a_{g+1-k}$, $1 \leq k \leq r$,
- $c = c_r$.

Note that $c_k = [a_1, b_1]^{y_1} \cdots [a_k, b_k]^{y_k}$ for some elements $y_1, \ldots, y_k$ in the group $\pi_1(\Sigma_g)$.

Let us consider the symplectic 4-manifold

$$X_W^g(1, b_1, b_2, \ldots, b_g, a_{n+1}, a_{n+2}, \ldots, a_r)$$

and let us denote it simply by X. Notice that for each $i = 1, 2, \ldots, g$, the simple closed curve b_i intersects two of the curves B_j transversely at only one point (and is disjoint from the others). Similarly, the curve a_i intersects at least one B_j transversely at one point.

By Theorem 3.2, the fundamental group of X admits a presentation with generators $a_1, b_1, \ldots, a_g, b_g$ and with relations

- $[a_1, b_1]^{y_1} [a_2, b_2]^{y_2} \cdots [a_g, b_g]^{y_g} = 1$,
- $B_0 = B_1 = \cdots = B_g = c = 1$,
- $b_1 = b_2 = \cdots = b_g = 1$, and
- $a_{n+1} = a_{n+2} = \cdots = a_r = 1$,.

It is easy to show that this is a presentation of the free group of rank n, with a free basis $a_1, a_2, \ldots, a_n$. Thus the group $\pi_1(X)$ is a free group of rank n.

Suppose now that $g = 2r + 1$ is odd and $g > 2n$. A similar computation shows that the fundamental group of

$$X_W^g(1, b_1, b_2, \ldots, b_g, a_{n+1}, a_{n+2}, \ldots, a_r)$$

is, again, a free group of rank n, finishing the proof of the proposition. □

This proves Theorem 2.1 if Γ is a free group.

Remark 4.2. The number of singular fibers of the Lefschetz fibration in above proposition is

$$(2g + 4)(1 + g + (\frac{g}{2} - n)) = (2g + 4)(1 + 3r - n)$$

if $g = 2r$, and

$$(2g + 10)(1 + g + (\frac{g-1}{2} - n)) = (2g + 10)(2 + 3r - n)$$

if $g = 2r + 1$. It is possible to reduce the number of singular fibers. In fact, one can show that the fundamental group of the 4-manifold

$$X^g_W(1, b_{r+1}, b_{r+2}, \ldots, b_g, a_{n+1}, a_{n+2}, \ldots, a_r)$$

is also isomorphic to F_n.

4.2. **The proof of Theorem 2.1 for arbitrary finitely presented group.** Recall that the elements $a_1, a_2, \ldots, a_n$ of $\pi_1(\Sigma_n)$ shown in Figure 3 generate a free subgroup of rank n. In fact, any subset of the standard generators with cardinality less than $2n$ generate a free subgroup.

Let us begin with the following proposition, which is a part of the proof but it deserves to be stated separately, as it is interesting in itself.

Proposition 4.3. *Let F_n denote the subgroup of $\pi_1(\Sigma_n)$ generated by the generators $a_1, a_2, \ldots, a_n$, so that F_n is a free group of rank n. Let $r_1, r_2, \ldots, r_k$ be arbitrary k elements in F_n represented as words in $a_1, a_2, \ldots, a_n$. Let ℓ denote $\ell(r_1) + \ell(r_2) + \cdots + \ell(r_k)$, the sum of the syllable lengths of r_i, and let $h = n + \ell - k$. Then, on the closed connected orientable surface Σ_h of genus h, there are loops $R_1, R_2, \ldots, R_k$ satisfying the following conditions:*

(a) *Each R_i is a simple loop on Σ_h.*
(b) *Each R_i is freely homotopic to a simple closed curve intersecting a_h transversely at exactly one point.*
(c) *If $\Phi : \pi_1(\Sigma_h) \to \pi_1(\Sigma_n)$ denotes the map defined by $\Phi(a_j) = a_j$ for $1 \leq j \leq n$ and $\Phi(\alpha) = 1$ for $\alpha \in \{a_{n+1}, \ldots, a_h, b_1, \ldots, b_h\}$, then $\Phi([R_i]) = r_i$ for each i, where $[R_i] \in \pi_1(\Sigma_h)$ is the homotopy class of R_i.*

FIGURE 4. Annuli $P(3)$ and $P(-3)$.

Proof. For an integer m, let $P(m)$ denote the annulus $[0, 1] \times S^1$ together with the arc $\gamma_m(t) = (t, e^{-2m\pi i t})$, $0 \leq t \leq 1$, on it.

Let us consider the surface Σ_n embedded in $\mathbb{R}^3$ as in Figure 3. Note that for each $i = 1, 2, \ldots, n$ there is a constant ν_i such that the intersection of Σ_n with the plane $y = \nu_i$ is the disjoint union of two oriented simple closed curves, one of which is freely homotopic to the loop a_i. Let us denote these simple closed curves by α_i and α_i' so that α_i is freely homotopic to a_i and

α_i' is obtained from α_i by rotating the surface Σ_n by π about the y-axis. Note that in general we denote α_i by a_i; we will use distinct notations only in this proof.

Let L be an embedded arc on Σ_n connecting the base point to a point on α_n such that L intersects each α_i at a single point and is disjoint from each α_i'. We can assume that the arc L lies in a plane z=constant.

Now let $r = a_{i_1}^{m_1} a_{i_2}^{m_2} \cdots a_{i_d}^{m_d}$ be an element of the free group F_n, where $d = \ell(r)$ is the syllable length of r. Choose pairwise disjoint simple closed curves $\tilde{\alpha}_{i_1}, \tilde{\alpha}_{i_2}, \ldots, \tilde{\alpha}_{i_d}$ such that $\tilde{\alpha}_{i_j}$ is isotopic to α_{i_j}. We can assume further that $\tilde{\alpha}_{i_j}$ is the intersection of a plane $y = \nu_{i_j}$ with the surface Σ_n, and if $i_j = i_k$ for some $j < k$, then $\nu_{i_j} < \nu_{i_k}$. Moreover, we assume that each $\tilde{\alpha}_{i_j}$ intersects L at exactly one point.

In what follows, we glue various copies of the annulus $P(m)$ to Σ_n minus the interiors of disjoint open discs and annuli in such a way that the resulting surface is orientable, and the obvious orientation of $P(m)$ and that of Σ_n agree.

For each $j = 1, 2, \ldots, d$, let us choose a regular neighborhood N_j of $\tilde{\alpha}_{i_j}$ so that the closures of any two distinct N_j are pairwise disjoint and do not contain the base point. We now identify N_j and the annulus $P(m_j)$ so that $[0,1] \times 1$ is identified with a subarc of L and the y-coordinate of 0×1 is less than that of 1×1. This gives us a bunch of disjoint arcs on Σ_n as shown in Figure 5(a).

On the annulus N_j, let us label the starting point of the curve γ_{m_j} by A_j and the terminal point by B_j. We note that the $y-$coordinate of A_j is less than that of B_j. For each $1 \leq j \leq d-1$, let δ_j denote the subarc of L from the point B_j to the point A_{j+1}. Then

$$\beta = \gamma_{m_1} \star \delta_1 \star \gamma_{m_2} \star \delta_2 \star \cdots \star \delta_{d-1} \star \gamma_{m_d}$$

is an arc on Σ_n connecting A_1 to B_d, and if we delete all δ_j from β, the result is the disjoint union of d simple arcs. Furthermore, if δ_0 is the subarc of L from the base point to A_1 and δ_d is the subarc of L from B_d to the base point, then the loop $\delta_0 \star \beta \star \delta_d$ is a representative of r. By perturbing β slightly, we assume that $\delta_0, \delta_1, \delta_2, \ldots, \delta_d$ are pairwise disjoint.

It is now clear from our construction that the base point can be joined to the point A_1 by an arc δ' and the point B_d can be joined to the base point by an arc δ'' such that interiors of β, δ' and δ'' are pairwise disjoint and that the loop $\delta' \star \beta \star \delta''$ represents the element

$$b_1 b_2 \cdots b_{i_1 - 1}\, r\, b_{i_d}^{-1} \cdots b_2^{-1} b_1^{-1}$$

in $\pi_1(\Sigma_n)$ (c.f. Figure 5(b)). We can assume that the arc δ' intersects $\alpha_1', \ldots, \alpha_{i_1-1}'$ at one point and is disjoint from all other α' and all α. Similarly, the arc δ'' intersects only $\alpha_1', \ldots, \alpha_{i_d}'$ at one point.

Let $D_1, D_2, \ldots, D_{2d-2}$ be pairwise disjoint discs on Σ_n such that the interior $\mathrm{Int}(D_i)$ of each D_i is disjoint from β, δ' and δ'', and for each $j = 1, 2, \ldots, d-1$, $\partial D_{2j-1} \cap \beta = \{B_j\}$ and $\partial D_{2j} \cap \beta = \{A_{j+1}\}$.

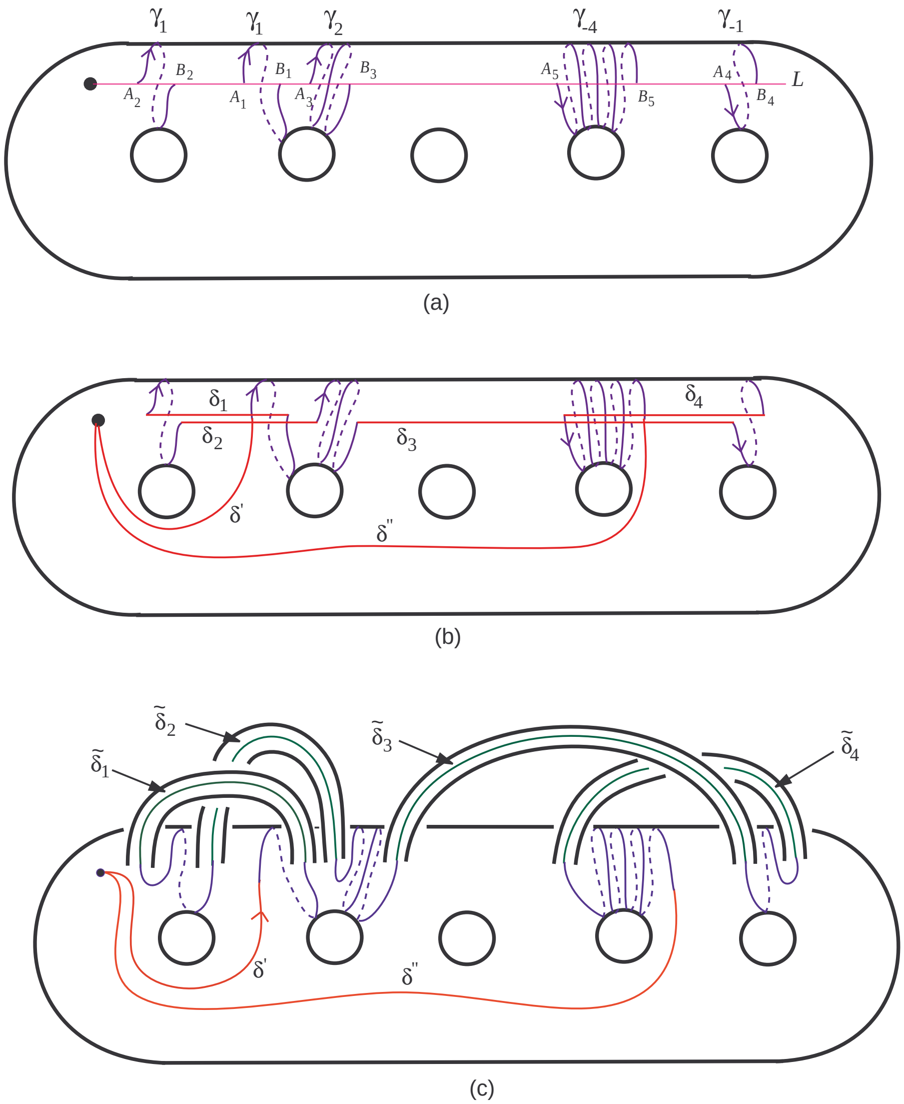

FIGURE 5. Construction of R on Σ_{n+d-1} for $r = x_2x_1x_2^2x_5^{-1}x_4^{-4}$ and for $n = 5$.

We now remove $2d-2$ open discs $\mathrm{Int}(D_i)$ from Σ_n and for each $1 \leq j \leq d-1$ we glue a copy of the annulus $P(0)$ to the surface

$$\Sigma_n \setminus \bigcup_{i=1}^{2d-2} \mathrm{Int}(D_i)$$

in such a way that $0 \times S^1$ (respectively $1 \times S^1$) is identified with ∂D_{2j-1} (respectively ∂D_{2j}) and the point 0×1 (respectively 1×1) is identified with B_j (respectively A_{j+1}). In this way, we obtain the closed orientable surface

$$\left(\Sigma_n \setminus \bigcup_{i=1}^{2d-2} \mathrm{Int}(D_i) \right) \cup \left(\bigcup_{j=1}^{d-1} P(0) \right)$$

of genus $n+d-1$, which is denoted by Σ_{n+d-1}. The orientation on Σ_n gives an orientation on Σ_{n+d-1}.

Let R be the loop on Σ_{n+d-1} obtained as follows. For each $j = 1, 2, \dots, d-1$, let $\tilde{\delta}_j$ denote the arc $\gamma_0 = I \times 1$ on the jth annulus $P(0)$, so that it is a simple arc joining the point B_j to A_{j+1} on Σ_{n+d-1}. Then the loop

$$R = \delta' \star \gamma_{m_1} \star \tilde{\delta}_1 \star \gamma_{m_2} \star \tilde{\delta}_2 \star \cdots \star \tilde{\delta}_{d-1} \star \gamma_{m_d} \star \delta''$$

obtained from $\delta' \star \beta \star \delta''$ by "replacing" δ_j by $\tilde{\delta}_j$ is simple on Σ_{n+d-1} (c.f. Figure 5(c)).

We note that it follows from the construction that $\tilde{\delta}_j \star \delta_j^{-1}$ is a simple closed curve on Σ_{n+d-1}. Collapsing each $P(0)$ onto the arc δ_j gives a map $\Sigma_{n+d-1} \to \Sigma_n$ and the induced map between the fundamental groups takes $[R]$ to $b_1 b_2 \cdots b_{i_1-1}\, r\, b_{i_d}^{-1} \cdots b_2^{-1} b_1^{-1}$, which is mapped to r under Π.

Clearly, the above construction can be done for all elements r_i simultaneously instead of a single element of F_n. For each $i = 1, 2, \dots, k$, we increase the genus of the surface by $\ell(r_i) - 1$. Thus the resulting surface is a closed orientable surface Σ_h of $h = n+\ell-k$. We have k simple loops $R_1, R_2, \dots, R_k$ on Σ_h such that the homomorphism $\Phi : \pi_1(\Sigma_h) \to \pi_1(\Sigma_n)$ maps $[R_i]$ to r_i for all i.

Now slide the $\ell - k$ cylinders that we attached on Σ_n to bring the surface into the standard position as shown in Figure 3 so that for each $j = 1, 2, \dots, \ell - k$ the simple closed curve $\delta_j' {\delta_j}^{-1}$ becomes isotopic to b_{n+j} and the core $\frac{1}{2} \times S^1$ of the jth handle becomes isotopic to a_{n+j}. Note that exactly one of R_i intersects the curve a_h transversely only once. It is clear that those R_i that does not intersect a_h can be modified to intersect a_h at one point at the expense of multiplying $[R_i]$ by some elements b_j for $j > n$, which are mapped to the identity under Φ.

This finishes the proof of the proposition. □

Finishing the proof. We now continue with the proof of Theorem 2.1. Let $g \geq 2h$ be an integer, where $h = n + \ell - k$.

Suppose first that g is even, say $g = 2r$. Suppose that for each $i = 1, 2, \dots, k$, the relator r_i in the presentation (1) is represented by the word $V_i(x_1, x_2, \dots, x_n)$. Let us denote the word $V_i(a_1, a_2, \dots, a_n)$ also by r_i.

Consider the surface Σ_h and the loops R_j constructed in Proposition 4.3 using these r_i. Let us remove the interior of a small disc from Σ_h near the curve a_h and disjoint from all R_j. Denote by $\Sigma_{h,1}$ the resulting surface of genus h with one boundary component. Embed $\Sigma_{h,1}$ into our standard surface Σ_g so that simple loops $a_1, a_2, \ldots, a_h, b_1, b_2, \ldots, b_h$ on the surface $\Sigma_{h,1}$ with one boundary component coincide with the loop on Σ_g labelled by the same letters. Thus, the loops R_j on Σ_g are disjoint from the simple closed curves $a_{h+1}, b_{h+1}, \ldots, a_g, b_g$.

It follows that, on the surface Σ_g, the simple closed curve B_{2h} intersects each R_j transversely once.

We note that the element $[R_i] \in \pi_1(\Sigma_g)$ is contained in the subgroup generated by $a_1, \ldots, a_r$ and $b_1, \ldots, b_r$. Moreover, if we set $a_s = 1$ for $n+1 \leq s \leq r$ and $b_j = 1$ for $1 \leq j \leq r$ in a word representing $[R_i] \in \pi_1(\Sigma_g)$, we get a word representing the element r_i.

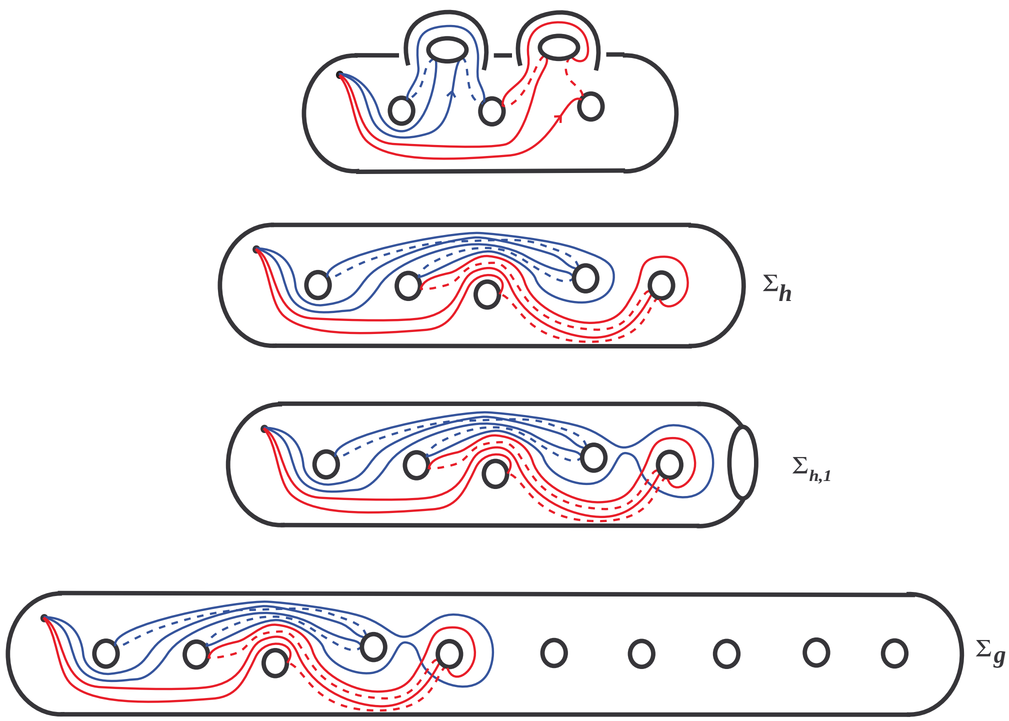

FIGURE 6. Construction of R_1 and R_2 for $r_1 = x_2 x_1^{-1}$ and $r_2 = x_3^{-1} x_2^{-1}$ in the case $n = 3$ and $g = 10$.

Let X denote the 4-manifold

$$X^g_W(1, b_1, \ldots, b_g, a_{n+1}, \ldots, a_r, R_1 \ldots, R_k)$$

which admits the structure of a Lefschetz fibration. Since for each of the simple closed curves $b_1, \ldots, b_g, a_{n+1}, \ldots, a_r, R_1 \ldots, R_k$ there is at least one B_j such that they intersect precisely at one point, by Theorem 3.2 $\pi_1(X)$ admits a presentation with generators $a_1, \ldots, a_g, b_1, \ldots, b_g$ and with the defining relations

- $[a_1, b_1]^{y_1}[a_2, b_2]^{y_2} \cdots [a_g, b_g]^{y_g} = 1$;
- $B_0 = B_1 = \cdots = B_g = c = 1$;
- $b_i = 1$ for $1 \leq i \leq g$;
- $a_i = 1$ for $n+1 \leq i \leq r$;
- $R_i = 1$ for $1 \leq i \leq k$.

This presentation is equivalent to the presentation (1) of Γ.

It can also be shown similarly that the fundamental group of

$$X^g_W(1, b_1, \ldots, b_g, a_{n+1}, \ldots, a_r, R_1 \ldots, R_k)$$

is again isomorphic to Γ if g is odd and greater than h.

This completes the proof of Theorem 2.1.

Remark 4.4. It is clear from the above proof that, our bound $2(n + \ell - k)$ is not optimal; depending on the presentation, the genus of the Lefschetz fibration can be made smaller. We give an example of this sort when we given an upper bound for the genus of finitely generated abelian groups in Theorem 5.7 and of the braid group in the next section.

Remark 4.5. For each positive integer m, let X_m denote the 4-manifold

$$X^g_W(1, 1, \ldots, 1, b_1, \ldots, b_g, a_{n+1}, \ldots, a_r, R_1 \ldots, R_k),$$

where 1 is repeated m times. Then the fundamental group of X_m is isomorphic to Γ for all m. Moreover, if $m \neq l$ then X_m is not diffeomorphic to X_l, because, for example, their Euler characteristics are different.

5. An invariant of finitely presented groups

Using the fact that every finitely presented group is the fundamental group of the total space of a Lefschetz fibration, we define an invariant of finitely presented groups. We then compute the invariant of some groups and give bounds for certain groups; free groups, abelian groups, braid groups and and the group $SL(2\mathbb{Z})$.

5.1. **Definition and the genus of certain groups.**

In this section, we consider only those Lefschetz fibrations which have sections. For a finitely presented group Γ, we define the *genus* $g(\Gamma)$ of Γ to be the minimum g such that there exists a genus-g Lefschetz fibration X over $\mathbb{S}^2$ such that $\pi_1(X)$ is isomorphic to Γ. Since for each finitely presented group Γ there is a Lefschetz fibration $f : X \to \mathbb{S}^2$ with a section such that $\pi_1(X) \cong \Gamma$, the number $g(\Gamma)$ always exists and is a nonnegative integer. We note that in the definition, we allow a Lefschetz fibration not to have any singular fibers. In that case, the total space is just the product of the fiber surface with the base $\mathbb{S}^2$.

Theorem 5.1. *Let π_g denote the fundamental group of a closed orientable surface of genus g. Then*

(a) $g(\pi_g) = g$.
(b) $g(\Gamma) = 0$ *if and only if* Γ *is the trivial group.*

(c) $g(\Gamma) = 1$ *if and only if* Γ *is isomorphic to* $\mathbb{Z} \oplus \mathbb{Z}$.
(d) $g(\Gamma) = 2$ *if* $\Gamma \in \{\mathbb{Z}, \mathbb{Z}\oplus\mathbb{Z}_n, \mathbb{Z}_m\oplus\mathbb{Z}_n, \mathbb{Z}_n\}$, *where* m *and* n *are positive integers and* $\mathbb{Z}_k$ *denotes the group* $\mathbb{Z}/k\mathbb{Z}$.

Proof. If $X \to \mathbb{S}^2$ is a genus-h Lefschetz fibration $X \to \mathbb{S}^2$ with $\pi_1(X) = \pi_g$, then there is an epimorphism $\pi_h \to \pi_g$. Since no map $\pi_h \to \pi_g$ can be surjective unless $h \geq g$, it follows that $g \leq g(\pi_g)$. On the other hand, since the projection $\Sigma_g \times \mathbb{S}^2 \to \mathbb{S}^2$ is a genus-g Lefschetz fibration, we conclude that $g(\pi_g) = g$. This proves (a).

In particular, the genus of the trivial group is zero. Suppose that $g(\Gamma) = 0$. Let $X \to \mathbb{S}^2$ be a genus-0 Lefschetz fibration with $\pi_1(X) = \Gamma$. Then $\pi_1(X)$ is trivial, proving (b).

Let $X \to \mathbb{S}^2$ be a genus-1 Lefschetz fibration. If there is no singular fibers then $X = T^2 \times \mathbb{S}^2$. If there are singular fibers then $X = E(n)$ for some n. Hence X is simply connected. It follows that $g(\Gamma) = 1$ if and only if $\Gamma = \mathbb{Z} \oplus \mathbb{Z}$.

Thus, if Γ is a nontrivial group other than $\mathbb{Z} \oplus \mathbb{Z}$, then $g(\Gamma) \geq 2$. On the other hand, there are genus-2 Lefschetz fibrations with fundamental groups $\mathbb{Z} \oplus \mathbb{Z}_n$, $\mathbb{Z}$, $\mathbb{Z}_m \oplus \mathbb{Z}_n$ and $\mathbb{Z}_n$. More precisely, the Lefschetz fibrations $X^2_W(1, t^n_{b_1})$, $X^2_W(1, t_{b_1})$, $X^2_W(1, t^m_{a_1}, t^n_{b_1})$ and $X^2_W(1, t_{a_1}, t^n_{b_1})$ have fundamental groups isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_n$, $\mathbb{Z}$, $\mathbb{Z}_m \oplus \mathbb{Z}_n$ and $\mathbb{Z}_n$, respectively. The fact that the fundamental group of $X^2_W(1, t^n_{b_1})$ is isomorphic to $\mathbb{Z}\oplus\mathbb{Z}_n$ was shown in [9]. This proves (d). $\square$

5.2. **Upper and lower bounds.** Let Γ be a finitely presented group with a given presentation with n generators and with k relators. Suppose that ℓ is the sum of the syllable lengths of the relators. Set $d(\Gamma) = n + \ell - k$. The integer $d(\Gamma)$ depends on the presentation.

Theorem 5.2. *Let* Γ *be a nontrivial finitely presented group. Let* $m(\Gamma)$ *denote the minimal number of generators for* Γ. *Then*

(a) $g(\Gamma) \leq \inf\{2d(\Gamma)\}$, *where the infimum is taken over all presentations of* Γ.
(b) $\frac{m(\Gamma)}{2} \leq g(\Gamma)$, *with the equality if and only if* Γ *is a surface group.*

Proof. By Theorem 2.1, for a given presentation pf Γ there is a Lefschetz fibration whose fundamental group is isomorphic to Γ and whose fiber genus is $2d(\Gamma)$. Thus, $g(\Gamma) \leq 2d(\Gamma)$ for any finite presentation of Γ, proving (a).

Let $X \to \mathbb{S}^2$ be a Lefschetz fibration of genus $h = g(\Gamma)$ with $\pi_1(X) \cong \Gamma$. Then there is an epimorphism $\pi_1(\Sigma_h) \to \Gamma$. Since $\pi_1(\Sigma_h)$ is generated by $2h$ elements, it follows that minimal number of generator of Γ satisfies $m(\Gamma) \leq 2h$.

If $\Gamma = \pi_1(\Sigma_h)$ is a surface group, then the minimal number of generators of Γ is $m(\Gamma) = 2h = 2g(\Gamma)$. If Γ is not a surface group, then the Lefschetz fibration $X \to \mathbb{S}^2$ of genus $h = g(\Gamma)$ has singular fibers. Since the monodromy group of any Lefschetz fibration cannot be contained in the Torelli group by the work of Smith [11], there must be at least one nonseparating vanishing cycle. It follows that Γ can be generated by $2h-1$ elements, that is, $m(\Gamma) \leq 2g(\Gamma) - 1$. □

Corollary 5.3. *Let Γ be a nontrivial finitely presented group. Let $b_1(\Gamma)$ denote the first Betti number of Γ, the dimension of the first homology of Γ with rational coefficients. Then $\frac{b_1(\Gamma)}{2} \leq g(\Gamma)$.*

5.3. **Free groups.** As usual, let F_n denote the free group of rank n freely generated by $x_1, x_2, \ldots, x_n$.

Next theorem is due to Zieschang [12].

Theorem 5.4. ([12]) *If there is an epimorphism $\varphi : \pi_1(\Sigma_g) \to F_n$, then $n \leq g$.*

Proof. The induced map $\varphi^* : H^1(F_n; \mathbb{Z}) \to H^1(\pi_1(\Sigma_g); \mathbb{Z})$ is injective. Since the cup product is zero in $H^1(F_n; \mathbb{Z})$, its image is Lagrangian in $H^1(\pi_1(\Sigma_g); \mathbb{Z})$. Thus $n \leq g$. □

Theorem 5.5. *Let F_n denote the nonabelian free group of rank n. Then $n \leq g(F_n) \leq 2n$.*

Proof. The group F_n has a presentation with n generators and with no relations. Hence, $d(F_n) = n$ for this presentation. Thus, $g(F_n) \leq 2n$ by Theorem 5.2 (a). On the other hand, if there is an epimorphism from $\pi_1(\Sigma_g)$ onto the free group F_n, then $n \leq g$ by Theorem 5.4. It follows that $n \leq g(F_n)$. □

Corollary 5.6. *Let $m \geq 2$ be an integer and let F_n denote the free group of rank n. Let $F_n * \mathbb{Z}_m$ be the free product of F_n with the cyclic group of order m. Then $n+1 \leq g(F_n * \mathbb{Z}_m) \leq 2n+2$.*

Proof. It can be shown that the fundamental group of

$$X_W^{2n+2}(1, b_1, b_2, \ldots, b_{2n+2}, a_1^m)$$

is isomorphic to $F_n * \mathbb{Z}_m$. Hence, $g(F_n * \mathbb{Z}_m) \leq 2n+2$.

Suppose that $g(F_n * \mathbb{Z}_m) = h$, so that there is an epimorphism $\varphi : \pi_1(\Sigma_h) \to F_n * \mathbb{Z}_m$. The kernel of the epimorphism $\phi : F_n * \mathbb{Z}_m \to \mathbb{Z}_m$ mapping F_n to zero and a generator of $\mathbb{Z}_m$ to a generator of $\mathbb{Z}_m$ is a free group F of rank mn. Since the index of F in $F_n * \mathbb{Z}_m$ is m, so is the index of $\phi^{-1}(F)$ in $\pi_1(\Sigma_h)$. Thus $\phi^{-1}(F)$ is isomorphic to the fundamental group of a closed orientable surface of genus $m(h-1)+1$, an m-sheeted covering of Σ_h. By Theorem 5.4, $m(h-1)+1 \geq mn$. It follows now that $h \geq n+1$. □

5.4. **Abelian groups.** Let $\mathbb{Z}^n$ denote the free abelian group of rank n and let $\mathbb{Z}_m$ denote the cyclic group of order m for $m \geq 2$. By considering the standard presentation

$$\langle x_1, x_2, \dots, x_n \mid x_i x_j x_i^{-1} x_j^{-1}, 1 \leq i < j \leq n \rangle$$

of the free abelian group $\mathbb{Z}^n$, one can conclude from Theorem 5.2 (a) that $g(\mathbb{Z}^n) \leq 3n^2 - n$. By reexamining the proof of Theorem 2.1 for this special case we can get a better estimate. In fact, we will give better bound for any finitely presented abelian group.

Let $m_i \geq 2$. For the abelian group $\mathbb{Z}^n \oplus \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_k}$, consider the presentation

$$\left\langle x_1, x_2, \dots, x_{n+k} \mid x_i x_j x_i^{-1} x_j^{-1}, (x_{n+s})^{m_s} \right\rangle, \tag{5}$$

where $1 \leq i < j \leq n+k$ and $1 \leq s \leq k$.

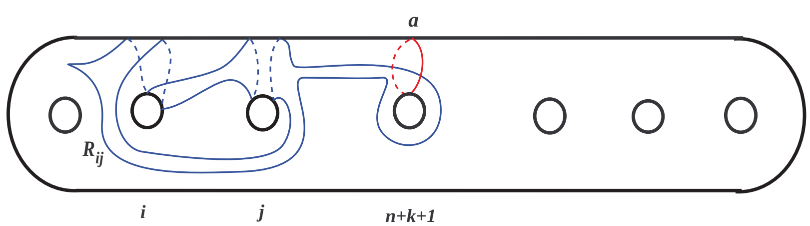

FIGURE 7. The curve R_{ij} on the surface $\Sigma_{(2n+2k+1)}$.

Theorem 5.7. *Let n and k be nonnegative integers with $n+k \geq 3$ and let $m_i \geq 2$. Let $\Gamma = \mathbb{Z}^n \oplus \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_k}$. Then $\frac{n+k+1}{2} \leq g(\Gamma) \leq 2n+2k+1$.*

Proof. The first inequality follows from Theorem 5.2(b) using the fact that the minimal number of generators for Γ is $n+k$ and that Γ is not a surface group.

For the second inequality, we set $g = 2n+2k+1$. For $i < j$, consider the simple loop R_{ij} on the surface Σ_g of genus g as shown in Figure 7. Clearly, the loop R_{ij} represents the element

$$a_i a_j a_i^{-1} (b_i \cdots b_j) a_j^{-1} b_{n+k+1}^{-1} (b_i \cdots b_j)^{-1}$$

in $\pi_1(\Sigma_g)$. Note that each R_{ij} intersects the simple closed curve a transversely only once, where a is the curve appearing in the word W for $g = 2n+2k+1$. After setting all $b_k = 1$, this word reduces to $a_i a_j a_i^{-1} a_j^{-1}$. Also, for each $s = 1, \dots, k$, the element $(a_{n+s})^{m_s} b_{n+s}^{-1}$ is represented by a simple loop T_s which intersects B_{n+s} only once. It follows now from Theorem 3.2 that the fundamental group of

$$X = X_W^g(1, b_1, b_2, \dots, b_g, T_1, \dots, T_k, R_{ij})$$

is isomorphic to Γ. Here, all R_{ij} are included, so that the manifold X is a fiber sum of $(g+1+k+\frac{(n+k)(n+k+1)}{2})$ copies of $X_W^g(1)$.

This shows that $g(\Gamma) \le g$, finishing the proof of Theorem. □

5.5. **The fundamental groups of closed nonorientable surfaces.** Let N_g denote be closed connected nonorientable surface of genus $g \ge 1$, that is the connected sum of g copies of the real projective plane. The fundamental group $\pi_1(N_g)$ of N_g has a presentation

$$\langle\, c_1, c_2, \dots, c_g \mid c_1^2 c_2^2 \cdots c_g^2 \,\rangle.$$

If $g = 1$, then $\pi_1(N_g)$ is isomorphic to $\mathbb{Z}_2$. Hence, $g(\pi_1(N_g)) = 2$.

Suppose that $g \ge 2$. Since $\pi_1(N_g)$ cannot be generated by less than g elements and since it is not isomorphic to the fundamental group of a closed orientable surface, we get from Theorem 5.2 (b) that $g(\pi_1(N_g)) \ge \frac{g+1}{2}$.

On the other hand, the loop $a_1^2 a_2^2 \cdots a_g^2 (b_1 b_2 \cdots b_g)^{-1}$ can be represented by a simple closed curve R, which intersects the simple closed curve B_g only once. It follows that the fundamental group of $X_W^{2g}(1, b_1, \dots, b_g, R)$ is isomorphic to $\pi_1(N_g)$. Therefore, $g(\pi_1(N_g)) \le 2g$.

5.6. **The braid group.** The braid group B_n on n strands has a presentation

$$\left\langle \sigma_1, \sigma_2, \dots, \sigma_{n-1} \mid \sigma_i\sigma_{i+1}\sigma_i\sigma_{i+1}^{-1}\sigma_i^{-1}\sigma_{i+1}^{-1}, \sigma_j\sigma_k\sigma_j^{-1}\sigma_k^{-1} \;\; |j-k| \ge 2 \right\rangle.$$

On the oriented surface Σ_{2n+1}, consider the simple loops as in Figure 7 for the relations of the type $\sigma_j\sigma_k\sigma_j^{-1}\sigma_k^{-1}$ and the simple loops in Figure 8 for the relations of the type $\sigma_i\sigma_{i+1}\sigma_i\sigma_{i+1}^{-1}\sigma_i^{-1}\sigma_{i+1}^{-1}$. It can now easily be shown that there is a Lefschetz fibration of genus $2n+1$ whose fundamental group is isomorphic to B_n. It follows that $2 \le g(B_n) \le 2n+1$.

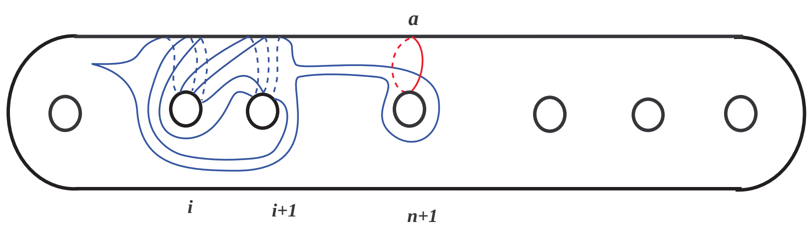

FIGURE 8.

5.7. **The group $SL(2,\mathbb{Z})$.** The group $SL(2,\mathbb{Z})$ has a presentation with two generators x and y and with two relators x^2y^3 and x^4. For this presentation, we have $d(SL(2,\mathbb{Z}))=3$. Since $SL(2,\mathbb{Z})$ is not a surface group, by Theorem 5.2 we get the bounds $2\leq g(SL(2,\mathbb{Z}))\leq 6$. However, by repeating the proof of Theorem 2.1 for $SL(2,\mathbb{Z})$ can see that there is a genus 4 Lefschetz fibration whose total space has fundamental group isomorphic to $SL(2,\mathbb{Z})$. The construction of the loop R corresponding to the relator x^2y^3 can be done without increasing the genus. Just take R to be a simple representative of the loop $a_1^2a_2^3b_2^{-1}b_1^{-1}$.

6. Other invariants and problems

In [7], Kotschick defines two invariants for finitely presented groups. For a finitely presented group Γ, these invariants are defined as

$$q(\Gamma)=\inf\{\chi(X)\}$$

and

$$p(\Gamma)=\inf\{\chi(X)-|\sigma(X)|\},$$

where infimums are taken aver all closed orientable 4-manifolds X with $\pi_1(X)\cong\Gamma$. Here, $\chi(X)$ and $\sigma(X)$ denote the Euler characteristic and the signature of X.

For a finitely presented group Γ, the invariants $q(\Gamma)$ and $p(\Gamma)$ satisfy the following inequalities (cf. [7]):

$$2-2b_1(\Gamma)+b_2(\Gamma)\leq q(\Gamma)\leq 2(1-d(\Gamma)),$$

and

$$2-2b_1(\Gamma)\leq p(\Gamma)\leq q(\Gamma).$$

It follows that

$$2-4g(\Gamma)\leq p(\Gamma)\leq q(\Gamma).$$

Here are some problems motivated by the discussions in this paper and the properties of the invariants p and q proved in [7].

Problem 1. Find the exact values of $g(F_n)$ and $g(\mathbb{Z}^n)$.

Problem 2. Compare $g(\Gamma_1\times\Gamma_2)$ and $g(\Gamma_1*\Gamma_2)$ with $g(\Gamma_1)$ and $g(\Gamma_2)$ for any two finitely presented groups Γ_1 and Γ_2, where $\Gamma_1*\Gamma_2$ is the free product of Γ_1 with Γ_2.

Problem 3. For a finitely presented group Γ and a subgroup Γ' of finite index k in Γ, compare the values $g(\Gamma')$ and $g(\Gamma)$.

Problem 4. In all examples I know, Lefschetz fibrations of genus 2 having singular fibers have abelian fundamental group. Is this true for all genus-2 Lefschetz fibrations?

References

[1] J. Amoros, F. Bogomolov, L. Katzarkov, T. Pantev, *Symplectic Lefschetz fibrations with arbitrary fundamental groups*, J. Differential Geom. **54** (2000), 489–545.

[2] S. K. Donaldson, *Lefschetz fibrations in symplectic geometry* Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998). Doc. Math. Extra Vol. II **1998**, 309–314

[3] R. E. Gompf, *A new construction of symplectic manifolds*, Ann. of Math. (2) 142 (1995), no. 3, 527–595.

[4] R. E. Gompf and A. I. Stipsicz, *4-manifolds and Kirby calculus*, Graduate Studies in Mathematics, vol. **20**, American Math. Society, Providence 1999.

[5] P. de la Harpe, *Topics in geometric group theory* Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 2000.

[6] M. Korkmaz, *Noncomplex smooth 4-manifolds with Lefschetz fibrations*, International Mathematics Research Notices **2001**, 115-128.

[7] D. Kotschick, *Four-manifold invariants of finitely presentable groups*, Topology, geometry and field theory, 89–99, World Sci. Publishing, River Edge, NJ, 1994.

[8] Y. Matsumoto, *Lefschetz fibrations of genus two—a topological approach*, Topology and Teichmller spaces (Katinkulta, 1995), 123–148, World Sci. Publ., River Edge, NJ, 1996.

[9] B. Ozbagci, A. I. Stipsicz, *Noncomplex smooth* 4*-manifolds with genus-*2 *Lefschetz fibrations*, Proc. Amer. Math. Soc. **128** (2000), 3125–3128.

[10] B. Ozbagci, A. I. Stipsicz, *Contact 3-manifolds with infinitely many Stein fillings*, Proc. Amer. Math. Soc. **132** (2004), 1549–1558.

[11] I. Smith, *Lefschetz fibrations and the Hodge bundle*, Geom. Topol. **3** (1999), 211–233.

[12] H. Zieschang *Über einfache Kurven auf Vollbrezeln*, Abh. Math. Sem. Univ. Hamburg **25** 1961/1962 231–250.

Department of Mathematics, Middle East Technical University, 06531 Ankara, Turkey

E-mail address: korkmaz@metu.edu.tr