On the existence of κ-existentially closed groups

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Abstract. We prove that a κ-existentially closed group of cardinality λ exists whenever $\kappa \leq \lambda$ are uncountable cardinals with $\lambda^{<\kappa} = \lambda$. In particular, we show that there exists a κ-existentially closed group of cardinality κ for regular κ with $2^{<\kappa} = \kappa$. Moreover, we prove that there exists no κ-existentially closed group of cardinality κ for singular κ. Assuming the Generalized Continuum Hypothesis, we completely determine the cardinals $\kappa \leq \lambda$ for which a κ-existentially closed group of cardinality λ exists.

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Following [Sco51], we define a group G with $|G| \geq \kappa$ to be κ-*existentially closed* if every system of less than κ-many equations and inequations with coefficients in G which has a solution in some supergroup $H \geqslant G$ already has a solution in G. The existence of $\aleph_0$-existentially closed groups is established in [Sco51]. For the rest of this paper, let κ always denote an uncountable cardinal. In [KK17, Proposition 2.8], the authors gave the following characterization of κ-existentially closed groups.

Proposition 1. *Let G be a group and κ be an uncountable cardinal. Then G is κ-existentially closed if and only if*

- *G contains an isomorphic copy of every group of cardinality less than κ, and*
- *every isomorphism between two subgroups of G of cardinality less than κ is induced by an inner automorphism of G.*

Moreover, they proved the uniqueness of κ-existentially closed groups of a given cardinality and the existence of κ-existentially closed groups of cardinality κ for inaccessible cardinals κ. In this paper, we extend the latter

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result and prove the existence and non-existence of κ-existentially closed groups of various cardinalities. We refer the reader to [Jec03] for basic facts regarding cardinal arithmetic.

Before we state our main theorem, we would like to recall two facts. First, every group is contained in some κ-existentially closed group. This fact is stated in [Sco51, Theorem 3] without proof and can be also proven by imitating the construction in [KK17, Section 4] as follows.

Lemma 2. *For any group G and uncountable cardinal κ, there exists a κ-existentially closed group $H \geqslant G$.*

Proof Sketch. Let $G_0 = G$. By transfinite recursion, for each ordinal $\alpha \geq 0$, define $G_{\alpha+1} = Sym(G_\alpha)$, and for limit ordinals γ, define $G_\gamma = \varinjlim G_\alpha$ where the direct limit is taken over the system of groups $\{G_\alpha\}_{\alpha<\gamma}$ each of which embeds into next one via its right regular representations. Then the group $H = G_{2^\kappa}$ satisfies the requirements as in [KK17, Section 4]. □

Second, it follows from [KK17, Lemma 2.5] that if there exists a κ-existentially closed group of cardinality λ, then $2^\theta \leq \lambda$ whenever $\theta < \kappa$. Consequently, we have the following.

Proposition 3. *If there exists a κ-existentially closed group of cardinality λ, then $2^{<\kappa} \leq \lambda$.*

As we shall see later, for regular $\lambda = \kappa$, this is the only restriction for the existence of κ-existentially closed groups of cardinality κ. We are now ready to state our main theorem.

Theorem 4. *Let $\kappa \leq \lambda$ be uncountable cardinals with $\lambda^{<\kappa} = \lambda$. Then there exists a κ-existentially closed group G of cardinality λ.*

Proof. By assumption, the number of isomorphism types of groups of cardinality less than κ is at most $2^{<\kappa} \leq \lambda$. Let $(G_\alpha)_{\alpha<\lambda}$ be a sequence of groups which contains a representative from each such isomorphism type. By transfinite recursion, for each ordinal $\alpha < \lambda$, define the following sequence of groups:

- $H_0 = G_0$,
- For every ordinal $\alpha \geq 0$, define $H_{\alpha+1} = H_\alpha * G_{\alpha+1}$ where $*$ denotes the free product of groups, and
- For every limit ordinal $\gamma > 0$, define $H_\gamma = \bigcup_{\alpha<\gamma} H_\alpha$

By Lemma 2, we can find a κ-existentially closed group K containing $\left(\bigcup_{\alpha<\lambda} H_\alpha\right) * H$ where H is some group of cardinality λ. By transfinite recursion, for each ordinal $\alpha < \lambda$, define the following sequence of subgroups of K:

- $K_0 = \left(\bigcup_{\alpha<\lambda} H_\alpha\right) * H$,
- For every ordinal $\alpha \geq 0$, define

$$K_{\alpha+1} = \langle K_\alpha, g_\varphi \mid \varphi \in \Omega_\alpha \rangle$$

where Ω_α is the set of isomorphisms between subgroups of K_α of cardinality less than κ and g_φ is an element of K such that the isomorphism φ is the restriction of the inner automorphism of K induced by g_φ. Observe that the existence of such g_φ is guaranteed by Proposition 1.

- For every limit ordinal $\gamma > 0$, define $K_\gamma = \bigcup_{\alpha<\gamma} K_\alpha$

For $\alpha = 0$, K_α clearly has cardinality λ. For an ordinal $0 \leq \alpha < \lambda$, if K_α has cardinality λ, then so does $K_{\alpha+1}$. This is because the cardinality of the set Ω_α at stage α is at most

$$\sum_{\theta<\kappa} \lambda^\theta \cdot \lambda^\theta \cdot \theta^\theta = \sum_{\theta<\kappa} \lambda^\theta = \lambda^{<\kappa} = \lambda$$

For a limit ordinal $0 < \gamma < \lambda$, if K_α has cardinality λ for all ordinals $\alpha < \gamma$, then K_γ has cardinality λ since it is a union of less than λ-many sets of size λ. Hence, by transfinite induction, each K_α has cardinality λ for $\alpha < \lambda$ and so the group $G = \bigcup_{\alpha<\lambda} K_\alpha$ has cardinality λ. We claim that G is as desired.

It is easily seen that G, being a supergroup of K_0, contains an isomorphic copy of every group of cardinality less than κ. Thus, it suffices to show that G satisfies the second property in Proposition 1. Let $\varphi : A \to B$ be an isomorphism between two subgroups of G of cardinality $\theta < \kappa$. Since we have $\lambda^{cf(\lambda)} > \lambda$, it follows from $\lambda^{<\kappa} = \lambda$ that $cf(\lambda) \geq \kappa > \theta$ and hence, there exists an ordinal $\alpha < \kappa$ such that $A, B \leqslant K_\alpha$. By construction, $\varphi \in \Omega_\alpha$ and hence it is the restriction of the inner automorphism induced by $g_\varphi \in G$. Therefore, G is κ-existentially closed and of cardinality λ. □

Corollary 5. *Let κ be an uncountable regular cardinal with $2^{<\kappa} = \kappa$. Then there exists a κ-existentially closed group G of cardinality κ.*

Proof. This follows from Theorem 4 by letting $\kappa = \lambda$, in which case all hypotheses are satisfied. □

Our proof uses the group-theoretic characterization of κ-existential closedness in Proposition 1. We would like to remark that, if one prefers to work with the definition of κ-existential closedness, then one may obtain these existence theorems in a more general setting from a model-theoretic argument that mimics the construction of saturated models, as follows. First, start with a model of size λ and transfinitely list all partial quantifier-free types (possibly involving infinitely many variables) over subsets of size less than κ. Next, build a new model as the union of a transfinite chain of models realizing these types one-by-one. Then, repeat this whole procedure along a transfinite chain of length λ of models. The assumption on cardinalities will ensure that the resulting model is of size λ and realizes all quantifier-free types over itself of size less than κ. Such a construction would work for any theory whose class of models is closed under increasing unions, such as the theory of groups.

We next provide a necessary condition of the existence of a κ-existentially closed group of cardinality λ under GCH.

Proposition 6 (GCH). *If there exists a κ-existentially closed group of cardinality $\lambda \geq \kappa$, then $cf(\lambda) \geq \kappa$.*

Proof. Let G be a κ-existentially closed group of cardinality λ. Assume towards a contradiction that $cf(\lambda) = \delta < \kappa$. We first show that there exists a sequence of groups $\{G_\alpha\}_{\alpha<\delta}$ such that $G_0 \cong Sym(\delta)$ and $G = \bigcup_{\alpha<\delta} G_\alpha$ where $G_\alpha \subsetneq G_\beta$ for ordinals $\alpha < \beta$. Since

$$|Sym(\delta)| = 2^\delta \leq 2^{<\kappa} = \kappa$$

by GCH, it follows from [KK17, Lemma 2.6] that $Sym(\delta)$ embeds into G, say, $G_0 \leqslant G$ is an isomorphic copy of $Sym(\delta)$. We claim that there are λ elements in $G - G_0$. If $\kappa < \lambda$, then we clearly have λ elements in $G - G_0$ as $2^\delta \leq \kappa < \lambda$. If $\kappa = \lambda$, then we have $2^\delta < \kappa$, since otherwise we would have $\kappa = 2^\delta = (2^\delta)^\delta = \kappa^{cf(\kappa)} > \kappa$; and hence there are still $\kappa = \lambda$ elements in $G - G_0$. It is now straightforward to construct such a sequence $\{G_\alpha\}_{\alpha<\delta}$ by adding elements of G one by one transfinitely.

Let $H = \langle h_\alpha : \alpha < \delta \rangle$ where h_α is some element in $G_{\alpha+1} - G_\alpha$. Since H is of cardinality at most δ, it embeds into $Sym(\delta)$ and hence is isomorphic to a subgroup of G_0. By κ-existential closedness, there exist $\beta < \delta$ and $h \in G_\beta$ such that $H^h \leqslant G_0$. However, $h_{\beta+1} \notin G_\beta = G_\beta^{h^{-1}} \geqslant G_0^{h^{-1}}$, which leads to a contradiction. □

We would like to remark that if $\kappa = \lambda$, then the inequality where the GCH assumption is used in the proof above is automatically satisfied by Proposition 3 and the rest of the proof goes through without GCH. Therefore, even without GCH, we have that

Corollary 7. *If there exists a κ-existentially closed group of cardinality κ, then κ is regular.*

Therefore, no κ-existentially closed group of cardinality κ exists for singular κ. Using Proposition 6, one can completely characterize the cardinals $\kappa \leq \lambda$ for which there exists a κ-existentially closed group of cardinality λ, under GCH.

Corollary 8 (GCH). *Let $\kappa \leq \lambda$ be uncountable cardinals. Then there exists a κ-existentially closed group of cardinality λ if and only if $cf(\lambda) \geq \kappa$. In particular, if λ is a successor cardinal, then there exists a κ-existentially closed group of cardinality λ.*

Proof. Recall that, under GCH, we have that $\theta < cf(\lambda)$ implies $\lambda^\theta = \lambda$. Thus, the right-to-left direction follows from Theorem 4 and the left-to-right direction is Proposition 6. For the latter proposition, recall that successor cardinals are regular. □

Finally, we would like to say some words regarding uniqueness. Neumann proved that there are uncountably many non-isomorphic countable $\aleph_0$-existentially closed groups [Neu73]. On the contrary, the back-and-forth argument provided in [KK17, Theorem 2.7] shows that any two κ-existentially closed groups of the same cardinality are isomorphic. Even though this argument does not work for the case $\kappa = \aleph_0$, it can be made to work under assumptions regarding the existence of certain local systems. Recall that a collection Σ of subgroups of a group G is called a *local system* for G if we have $G = \bigcup_{H\in\Sigma} H$ and for every $H, K \in \Sigma$ there exists $L \in \Sigma$ such that $\langle H, K\rangle \leqslant L$. The following can easily be proven imitating the argument in [KK17, Theorem 2.7].

Theorem 9. *Any two countable $\aleph_0$-existentially closed groups that have local systems consisting of finitely presented subgroups are isomorphic.*

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