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A Numerical Study of a Modular Sparse Grad-Div Stabilization Method for Boussinesq Equations

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Abstract. This study presents a modular sparse grad-div stabilization method for solving the Boussinesq equations. Unlike the usual grad-div stabilization which produces fully coupled block matrices, the proposed stabilization method produces block upper triangular matrices. Thus, the proposed method is more attractive in terms of both its computational cost and solution accuracy. We provide unconditional stability results for velocity and temperature. Two numerical experiments are performed to demonstrate the efficiency and accuracy of the method.

1. Introduction

The natural convection flow occurs due to the buoyancy force arising from the temperature. It has a wide range of variety both in nature and engineering applications such as oceanography, geology, free air cooling fluid flows around heat-dissipations fins, etc. [1, 2, 3]. There are numerous studies approximating solutions to Boussinesq equations [4, 5]. Among them, the most popular stabilization method is the grad-div stabilization since they can easily be included into any existing conforming finite element code [8]. Although the grad-div stabilization significantly effects problem accuracy [6, 7], it is inefficient in terms of computational effort. While commonly used discretization methods creates block diagonal matrices, the grad-div method creates fully coupled matrices for velocity which makes the resulting lineer system difficult to solve. To handle this problem, we propose analyze and test a modular sparse grad-div stabilization method for Boussinesq equations by introducing a new stabilization operator [10] based on the ideas of [9]. We aimed to obtain high solution quality with less computational effort.

2. Scheme

Let $X = (H_0^1(\Omega))^d$, $Q = L_0^2(\Omega)$ be the velocity and pressure spaces and $W = H_0^1(\Omega)$ be the temperature space. Let $X^h \subset X, W^h \subset W, Q^h \subset Q$ be finite element spaces where the velocity and pressure spaces fulfil the inf-sup condition. We use the usual $L^2(\Omega)$ norm and the inner product denoted by $\|\cdot\|$ and $(\cdot,\cdot)$, respectively. The skew symmetric trilinear forms are defined by

$$b^*(\mathbf{u},\mathbf{v},\mathbf{w}) = \frac{1}{2}(\mathbf{u}\cdot\nabla\mathbf{v},\mathbf{w}) - \frac{1}{2}(\mathbf{u}\cdot\nabla\mathbf{w},\mathbf{v}) \quad (1)$$

$$c^*(\mathbf{u},T,S) = \frac{1}{2}(\mathbf{u}\cdot\nabla T,S) - \frac{1}{2}(\mathbf{u}\cdot\nabla S,T) \quad (2)$$

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Algorithm: Divide $[0, T]$ time interval into N equal subintervals and set time step size $\Delta t = T/N$. Let the initial conditions $\mathbf{u}^0$ and T^0, the forcing function $\mathbf{f}$ and the heat source γ be given. Define $\mathbf{u}_h^0$ and T_h^0 be the nodal interpolants of $\mathbf{u}^0(\mathbf{x})$ and T^0, respectively.
Step 1 : Given $(\mathbf{u}_h^n, T_h^n) \in (\mathbf{X}^h, \mathbf{W}^h)$, find $(\hat{\mathbf{u}}_h^{n+1}, T_h^{n+1}, p_h^{n+1}) \in (\mathbf{X}^h, \mathbf{W}^h, Q^h)$

$$\left(\frac{\hat{\mathbf{u}}_h^{n+1} - \mathbf{u}_h^n}{\Delta t}, \mathbf{v}_h\right) + b^*(\mathbf{u}_h^n, \hat{\mathbf{u}}_h^{n+1}, \mathbf{v}_h) + Pr(\nabla \hat{\mathbf{u}}_h^{n+1}, \nabla \mathbf{v}_h) - (p_h^{n+1}, \nabla \cdot \mathbf{v}_h) = PrRa(\langle 0, T_n^h \rangle, \mathbf{v}^h) + (\mathbf{f}^{n+1}, \mathbf{v}_h) \quad (3)$$

$$(\nabla \cdot \hat{\mathbf{u}}_h^{n+1}, q_h) = 0 \quad (4)$$

$$\left(\frac{T_h^{n+1} - T_h^n}{\Delta t}, \boldsymbol{\phi}_h\right) + c^*(\hat{\mathbf{u}}_h^{n+1}, T_h^{n+1}, \boldsymbol{\phi}_h) + \kappa(\nabla T_h^{n+1}, \nabla \boldsymbol{\phi}_h) = (\gamma^{n+1}, \boldsymbol{\phi}_h) \quad (5)$$

Step 2 : Given $\hat{\mathbf{u}}_h^{n+1} n \in X^h$, find $\mathbf{u}_h^{n+1} \in X^h$

$$(\mathbf{u}_h^{n+1}, \boldsymbol{\theta}_h) + \gamma \Delta t g(\mathbf{u}_h^{n+1}, \boldsymbol{\theta}_h) = (\hat{\mathbf{u}}_h^{n+1}, \boldsymbol{\theta}_h) \quad (6)$$

for all $(\mathbf{v}_h, \boldsymbol{\phi}_h, \boldsymbol{\theta}_h, q_h) \in (\mathbf{X}^h, \mathbf{W}^h, \mathbf{X}^h, Q^h)$.

3. Numerical Experiments

In this section, we perform two numerical tests to show tje performance of the method (3)-(6). In the first one, we show an optimal convergence rates with a known solution. In the second one, we present the well-known buoyancy driven cavity example. All computations are carried out by using the software FreeFem++ [11]. In both examples, we use conforming Taylor-Hood pairs on uniform triangular grids.

3.1. Chorin's Problem

In this subsection, we test the optimal convergence rates of the scheme (3)-(6) for a known solution:

$$\mathbf{u} = \quad (cos(\pi(y-t)), sin(\pi(x+t)))^T(\exp(t)), \quad (7)$$

$$p = \quad sin(\pi(x+y))(1+t^2), \quad (8)$$

$$T = \quad sin(\pi x) + y\exp(t). \quad (9)$$

on the unit square domain $\Omega = [0, 1]^2$. We choose the parameters $Pr = Ra = \kappa = 1$ and $\gamma = 0.001$. To test the spatial convergence, we fixed the time step $\Delta t = \frac{T}{8}$ with end time $T = 10^{-4}$ and calculate the errors for varying h. Results are shown in Table 1. We observe the optimal rate of convergence for velocity and temperature for the Taylor-Hood finite element space. To test the temporal errors, we fix mesh size $h = \frac{1}{64}$ with final time $T = 0.1$ and calculate errors for varying Δt. Errors and rates are presented in Table 2. As expected, we observe first order convergence in time.

Table 1. Spatial errors and rates of convergence.

h	$\|u-u_h\|_{2,1}$	Rate	$\|T-T_h\|_{2,1}$	Rate
1/4	0.0007140	-	0.0005045	-
1/8	0.0001780	2.004	0.0001256	2.005
1/16	4.356e-5	2.032	3.069e-5	2.037
1/32	1.063e-5	2.034	7.458e-6	2.042
1/64	2.751e-6	1.950	1.844e-6	2.015

Table 2. Temporal errors and rates of convergence.

Δt	$\|u-u_h\|_{2,1}$	Rate	$\|T-T_h\|_{2,1}$	Rate
Δt	0.0246	-	0.0012	-
$\Delta t/2$	0.0124	0.987	0.0005	1.263
$\Delta t/4$	0.0063	0.976	0.0002	1.321
$\Delta t/8$	0.0032	0.976	0.0001	1.000
$\Delta t/16$	0.0017	0.912	7.427e-5	0.919
$\Delta t/32$	0.0008	1.087	3.922e-5	0.920
$\Delta t/64$	0.0004	1.000	2.331e-56	0.750

3.2. Thermal Distribution in Buoyancy Driven Cavity

Calculation of the physical parameter called the Nusselt number (Nu) at a buoyancy driven cavity test example has been widely used in order to verify and validate proposed numerical schemes. Local and avarege Nusselt numbers are given with the formulas

$$Nu_{loc} = \pm \left\{ \frac{\partial T}{\partial x} \right\}_{wall} \tag{10}$$

$$Nu_{av} = \int_{\Omega} Nu_{loc} dy. \tag{11}$$

The domain is a unit square cavity different temperatures at vertical walls, one of them is taken as cold $T_C = 0$ at $x = 1$ and the other one is hot $T_H = 1$ at $x = 0$. The other walls are insulated and assumed allow no heat transfer through. The boundary conditions are no-slip for the velocity at all the boundary and Dirichlet for the temperature at vertical walls as well. The horizontal walls accept the boundary conditions $\frac{\partial T}{\partial n} = 0$. We take flow parameters as $Pr = 1, \gamma = 1$ in this test. In Figure 1, we see temperature contors and velocity streamlines for $Ra = 10^3, 10^4, 10^5$ for the chosen time step $\Delta t = 0.1, 0.01, 0.01$, respectively. Also, we calculate the average Nusselt numbers of the proposed algorithm for $Ra = 10^3, 10^4, 10^5$. As seen in Table 3, acceptable results are obtained for Nu when compared with the literature on much coarser mesh.

Table 3. Comparison of average Nusselt number on hot wall for varying Rayleigh Numbers.

Ra	Proposed Method	Ref [12]	Ref [13]	Ref [14]
10^3	1.118 (11×11)	1.117	1.117	1.12
10^4	2.257 (32×32)	2.254	2.243	2.243
10^5	4.602 (64×64)	4.598	4.521	4.52

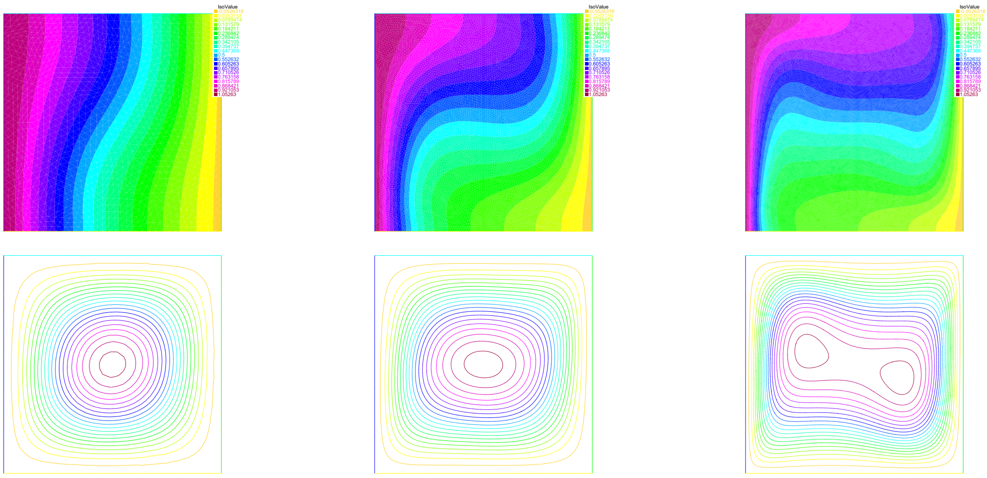

Figure 1. Temperature contours (up) and streamlines (down) for $Ra = 10^3, 10^4, 10^5$ (left to right).

4. Conclusion

In this paper, we proposed and analyzed a modular sparse grad div stabilization method for the solution of Boussinesq equations. Optimal convergece rates are obtained for Chorin's problem. In addition, we demonstrate the accuracy and efficiency of the method on a well known numerical test called buoyancy driven cavity which revealed that the proposed method gives similar accuracy and error advanteges along with a less computational effort when compared with the literature.

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