

FROBENIUS GROUPS OF AUTOMORPHISMS WITH ALMOST FIXED POINT FREE KERNEL

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ABSTRACT. Let FH be a Frobenius group with kernel F and complement H , acting coprimely on the finite solvable group G by automorphisms. We prove that if $C_G(H)$ is of Fitting length n then the index of the n -th Fitting subgroup $F_n(G)$ in G is bounded in terms of $|C_G(F)|$ and $|F|$. This generalizes a result of Khukhro and Makarenko [6] which handles the case $n = 1$.

1. INTRODUCTION

All groups throughout this paper are finite, notation and terminology are standard except as indicated. Let a fixed group A act on the group G by automorphisms. It is well known that the structure of A and the way it acts on the group G has great influence on the structure of G . For example, one of the famous theorems due to Thompson says that G must be nilpotent if A is of prime order and acts fixed point freely on G . This result has played a motivating and stimulating role in studying the structure of a group admitting a group of automorphisms with a prescribed action. There have been a lot of research in this direction some part of which culminated in the work of Turull. We want to state below some of his results ([10], [11]) in order to indicate the great development in this direction after Thompson:

Let G be a solvable group and $f(G)$ denote the Fitting length of G .

(i) If a solvable group A acts coprimely on G , then $f(G) \leq f(C_G(A)) + 2\ell(A)$;

(ii) If A acts coprimely with regular orbits on G , then $f(G) \leq \ell(A) + \ell(C_G(A))$, and the index $|G : F_{\ell(A)}(G)|$ is bounded in terms of $|C_G(A)|$ and $|A|$, where $\ell(A)$ is the length of the longest chain of subgroups of A .

In [4] and [5] Khukhro studied the case where $A = FH$ is a Frobenius group with kernel F and complement H , and obtained very precise results when $C_G(F) = 1$. Namely he proved that $F_k(G) \cap C_G(H) = F_k(C_G(H))$ for any k , in particular $f(G) = f(C_G(H))$. It is worth mentioning here the results of [7], [8], [9] related to the action of a Frobenius group of automorphisms with fixed point free kernel. Although it seems to be essential to assume that $C_G(F) = 1$ in all these results, Khukhro and Makarenko have also considered the action of a Frobenius group $A = FH$ with almost fixed point free kernel when $C_G(H)$ is nilpotent (see [6], Theorem 2.1). They proved namely:

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If a Frobenius group FH with kernel F and complement H acts coprimely on the finite solvable group G in such a way that $C_G(H)$ is nilpotent then the index of the Fitting subgroup is bounded in terms of $|C_G(F)|$ and $|F|$.

The main result of the present paper gives a new proof of the above result and extends it to the case where $C_G(H)$ is of arbitrary Fitting length, that is, the group G has Fitting length at most $f(C_G(H))$ except for some quotient group of G bounded in terms of $|C_G(F)|$ and $|F|$, and hereby answers a question posed by Khukhro. Namely we obtain

Theorem *Let $A = FH$ be a Frobenius group with kernel F and complement H and let $m \in \mathbb{N}$. Then there exists a function $g : \mathbb{N} \rightarrow \mathbb{N}$ which may depend on the parameters $|F|$ and m , but is independent of H , such that for any finite solvable group G on which A acts coprimely by automorphisms with $|C_G(F)| \leq m$ we have $|G : F_n(G)| \leq g(n)$ where $n = f(C_G(H))$.*

2. PRELIMINARIES

Lemma 2.1 ([11], Lemma 2.4). *Let G be a solvable group and suppose that $N_i, i = 1, \dots, r$ be normal subgroups of G such that $\bigcap_{i=1}^r N_i = 1$. Let $n \in \mathbb{N}$. Then*

$$|G : F_n(G)| \leq \prod_{i=1}^r |G/N_i/F_n(G/N_i)|.$$

Lemma 2.2 ([4], Lemma 1.3). *Let FH be a Frobenius group with kernel F and complement H . Suppose that V is a vector space over an arbitrary field on which FH acts by linear transformations. If $[V, F] \neq 0$ then $C_V(H) \neq 0$.*

Lemma 2.3 ([2], Proposition 4.1). *Let FH be a Frobenius group with kernel F and complement H acting on a q -group Q for some prime q coprime to the order of FH . Let V be a $kQFH$ -module where k is a field of characteristic not dividing $|QFH|$. If $C_V(F) = 1$ then*

$$\text{Ker}(C_Q(H) \text{ on } C_V(H)) = \text{Ker}(C_Q(H) \text{ on } V).$$

Definition 2.4. Let G be a group. We call a sequence $P_i = C_i/D_i, i = 1, \dots, \ell$ of sections of G an \mathcal{F} -chain in G if the following are satisfied:

- (a) P_i is a p_i -group for a single prime p_i for $i = 1, \dots, \ell$;
- (b) $p_i \neq p_{i+1}$ for $i = 1, \dots, \ell - 1$;
- (c) $D_\ell = 1$ and $D_i \leq C_{C_i}(P_{i+1})$ for $i = 1, \dots, \ell - 1$;
- (d) $[P_i, P_{i+1}] = P_{i+1}$ for $i = 1, \dots, \ell - 1$.

Lemma 2.5. *In a solvable group G , the Fitting length $f(G)$ of G is equal to the maximum of the set $\{\ell : \ell \text{ is the length of an } \mathcal{F}\text{-chain in } G\}$.*

Proof. Let $m = m(G) = \max\{\ell : \ell \text{ is the length of an } \mathcal{F}\text{-chain in } G\}$ and $f = f(G)$.

It is well known that there is an irreducible tower in G , say $\hat{P}_i, i = 1, \dots, f$, in the sense of [10]. We observe now that the corresponding sequence $P_i, i = 1, \dots, f$, where $P_f = \hat{P}_f$ and $P_i = \hat{P}_i/C_{\hat{P}_i}(P_{i+1})$ for $i = 1, \dots, f - 1$, forms an \mathcal{F} -chain of length f in G in the sense of Definition 2.4. Thus we have $f \leq m$.

We shall prove the reversed inequality by induction on $|G|$. Let $P_i = C_i/D_i, i = 1, \dots, m$, be an \mathcal{F} -chain in G of length m . Now P_m is a p_m -group. Suppose first that there exists a prime p different from p_m such that $O_p(G) \neq 1$. Set $\bar{G} = G/O_p(G)$. Since $\bar{C}_m \neq \bar{D}_m$, the chain $P_i = C_i/D_i, i = 1, \dots, m$, is mapped to an \mathcal{F} -chain in \bar{G} of length m and so $m(\bar{G}) \geq m$. Thus we have

$$m \leq m(\bar{G}) \leq f(\bar{G}) \leq f(G)$$

by induction. Then we may assume that $F(G) = O_{p_m}(G)$. Set now $\tilde{G} = G/F(G)$. Notice that $\tilde{C}_{m-1} \neq \tilde{D}_{m-1}$ and hence the chain $P_i = C_i/D_i, i = 1, \dots, m - 1$, is mapped to an \mathcal{F} -chain in \tilde{G} of length $m - 1$. It follows then by induction that

$$m - 1 \leq m(\tilde{G}) \leq f(\tilde{G}) = f(G) - 1$$

which completes the proof. \square

Finally we want to state a special case of Hartley-Isaacs theorem [[3], Theorem B].

Lemma 2.6. *For any arbitrary group F there is a number $\delta(F)$ depending only on F with the following property: Let F act coprimely on the solvable group G , and let k be any field of characteristic not dividing $|F|$. Then for any completely reducible kGF -module V , we have $\dim_k V \leq \delta(F)\dim_k C_V(F)$.*

3. PROOF OF THEOREM

Let $A = FH$ be a fixed Frobenius group with kernel F and complement H and m be a fixed positive integer and $\mathcal{G} = \mathcal{G}_{A,m}$ be the set of all finite solvable groups G on which A acts coprimely by automorphisms with $|C_G(F)| \leq m$. Clearly, we have $\mathcal{G} = \bigcup_{n \in \mathbb{N}} \mathcal{G}_n$ where

$$\mathcal{G}_n = \{G \in \mathcal{G} : f(C_G(H)) = n\}.$$

Our theorem needs to prove that the subset $\{|G : F_n(G)| : G \in \mathcal{G}_n\}$ of \mathbb{N} has a maximum for any $n \in \mathbb{N}$. If this is known one can define a function $h = h_{A,m} : \mathbb{N} \rightarrow \mathbb{N}$ such that

$$h(n) = \max\{|G : F_n(G)| : G \in \mathcal{G}_n\}$$

for any n . Of course this function, if it exists, may depend on the parameters m, F and H . Our proof will be given by a recursive construction of the function h and will show that h may depend on F , but is actually independent of H . We next define the function $g : \mathbb{N} \rightarrow \mathbb{N}$ by

$$g(n) = \max\{h_{A,m}(n) : A \text{ is a Frobenius group with kernel of order } |F|\}.$$

Notice that this maximum exists as the set of Frobenius groups with kernel of a given order is finite. It is also clear that $g(n)$ depends only on $|F|$ and m , and satisfies $|G : F_n(G)| \leq g(n)$ for any group G on which A acts coprimely by automorphisms with $|C_G(F)| \leq m$ and $f(C_G(H)) = n$.

We proceed now to prove by induction on n that the set $\{|G : F_n(G)| : G \in \mathcal{G}_n\}$ has a maximum $h(n)$. Suppose first that $n = 0$ and let $G \in \mathcal{G}_0$. Now $f(C_G(H)) = 0$, that is $C_G(H) = 1$. This can happen only when $[G, F] = 1$ by Lemma 2.2. Therefore $|G| \leq m$ and so $h(0)$ exists. We assume that for any fixed $n \geq 1$ and any $k < n$ we have $\{|G : F_k(G)| : G \in \mathcal{G}_k\}$ has a maximum $h(k)$. Set $d_n = \max\{g_k : k = 0, \dots, n-1\}$.

Let now G be a fixed, but arbitrary element of \mathcal{G}_n . Notice that if $|F_{n+1}(G) : F_n(G)|$ is bounded by a number which depends only on m then so is $|G : F_{n+1}(G)|$. Therefore there is no loss in assuming that $G = F_{n+1}(G)$. As $|G/[G, F]| \leq m$, without loss of generality we may assume that $[G, F] = G$. Since $F(G)/\Phi(G) = F(G/\Phi(G))$, we may also assume that $\Phi(G) = \Phi(F(G)) = 1$. It is well known that $F(GA)/\Phi(GA)$ is a direct sum of irreducible GA -submodules over possibly different prime fields. Since $F(G)\Phi(GA)/\Phi(GA)$ is a GA -submodule of $F(GA)/\Phi(GA)$ and is isomorphic to $F(G)/\Phi(G)$ as a GA -group we see that $F(G)$ is a direct sum of irreducible GA -modules M_1, \dots, M_k . On the other hand, by Clifford's theorem each M_i is a direct sum of irreducible G -modules $M_{ij}, j = 1, \dots, s_i$. By A.13.8(b) in [1], we see that $F(G) = \bigcap_{i=1}^k \bigcap_{j=1}^{s_i} C_G(M_{ij}) = \bigcap_{i=1}^k C_G(M_i)$.

Without loss of generality we may suppose that $C_{M_i}(F) \neq 0$ for $i = 1, \dots, s$ and $C_{M_i}(F) = 0$ for $i = s+1, \dots, k$. Set $\bar{G} = G/F(G)$ and $\bar{X} = \bigcap_{i=1}^s C_{\bar{G}}(M_i)$ and $\bar{Y} = \bigcap_{i=s+1}^k C_{\bar{G}}(M_i)$. Then \bar{X} and \bar{Y} are subgroups of \bar{G} with $\bar{X} \cap \bar{Y} = 1$. Clearly, by Lemma 2.1

$$|G : F_n(G)| = |\bar{G}/F_{n-1}(\bar{G})| \leq |\bar{G}/\bar{X}/F_{n-1}(\bar{G}/\bar{X})| |\bar{G}/\bar{Y}/F_{n-1}(\bar{G}/\bar{Y})|.$$

On the other hand

$$\prod_{i=1}^s |C_{M_i}(F)| \leq |C_G(F)| = m$$

and so

$$2^s \leq \prod_{i=1}^s |C_{M_i}(F)| \leq m.$$

This gives $s \leq \log_2 m$ as $|C_{M_i}(F)| \geq 2$ for any M_i where $i = 1, \dots, s$. Therefore we have

$$|\bar{G}/\bar{X}| \leq \prod_{i=1}^s |G/C_G(M_i)| \leq \prod_{i=1}^s |Aut M_i| \leq \prod_{i=1}^s |M_i|!.$$

Pick M_i for $i \in \{1, \dots, s\}$ and consider its decomposition into GF -homogeneous components. Note that H acts transitively on the set of GF -homogeneous components, and hence F fixes a point in each component. Then by Lemma 2.6, there is a number $\delta(F)$ depending only on F such that

$$|M_i| \leq |C_{M_i}(F)|^{\delta(F)} \leq m^{\delta(F)}.$$

Thus we have

$$|\bar{G}/\bar{X}| \leq (m^{\delta(F)}!)^{\log_2 m}.$$

Note that this bound depends on F and m , but is completely independent of H . Clearly if $s = k$ then $\bar{Y} = \bar{G}$ and the result follows from the above equality.

Therefore we may assume that $s \neq k$. Set $G_1 = G / \bigcap_{i=s+1}^k C_G(M_i)$. As $C_{\bar{G}}(M_i) = C_G(M_i)/F(G)$ we have $G_1 \cong \bar{G}/\bar{Y}$ and so it suffices to bound $|G_1 : F_{n-1}(G_1)|$ suitably. If $f(C_{G_1}(H)) \leq n - 1$ then it would follow by induction assumption that $|G_1 : F_{n-1}(G_1)| \leq d_n$. Thus we have $f(C_{G_1}(H)) = n = f(G_1)$ as we already have $f(G_1) \leq n$.

Now by Lemma 2.5, there exists a chain of sections $P_i = C_i/D_i, i = 1, \dots, n$, of $C_{G_1}(H)$ satisfying the conditions (a)–(d) of Definition 2.4. Clearly, $P_i = C_i/D_i, i = 1, \dots, n$, is also a chain of sections of G_1 . Furthermore, we have $P_n = C_n \leq F(G_1)$ because otherwise the chain $P_i, i = 1, \dots, n$, is mapped to a chain of the same length in $G_1/F(G_1)$, which is impossible. As a p_n -subgroup, C_n is contained in the Sylow p_n -subgroup, say Q , of $F(G_1)$. Clearly Q is A -invariant. Notice that there exists $M = M_i$ for some $i \in \{s+1, \dots, k\}$ on which C_n acts nontrivially because otherwise C_n is the trivial subgroup of G_1 . We consider now the action of A on the group MQ and apply Lemma 2.3. It follows that

$$\text{Ker}(C_Q(H) \text{ on } C_M(H)) = \text{Ker}(C_Q(H) \text{ on } M).$$

Due to coprimeness, $[C_M(H), P_n, P_n] = [C_M(H), P_n]$. We set now $P_{n+1} = [C_M(H), P_n]$. Clearly the sequence P_1, \dots, P_n, P_{n+1} forms a chain of sections of $C_G(H)$ satisfying the conditions (a)–(d) of Definition 2.4 as $C_{G_1}(H)$ is the image of $C_G(H)$ in G_1 by the coprimeness condition $(|G|, |H|) = 1$. This forces by Lemma 2.5 that $f(C_G(H)) = n + 1$, which is a contradiction completing the proof. \square

REFERENCES

- [1] K. Doerk, T. Hawkes, Finite Soluble Groups, de Gruyter Expositions in Mathematics, Vol.4. (1992) Walter de Gruyter, Berlin and New York.
- [2] G. Ercan, İ. Ş. Güloğlu, Groups of automorphisms with TNI-centralizers, J.Algebra 498 (2018) 38–46.
- [3] B. Hartley, M.I. Isaacs, On characters and fixed points of coprime operator groups, J. Algebra 131 (1990) 342–358.
- [4] E.I. Khukhro, Nilpotent length of a finite group admitting a Frobenius group of automorphisms with fixed-point-free kernel, Algebra Logika 49, (2010) 819–833 (2010); English transl., Algebra Logic 49, (2010) 551–560.
- [5] E.I. Khukhro, Fitting height of a finite group with a Frobenius group of automorphisms, J. Algebra 366 (2012) 1–11.
- [6] E.I. Khukhro, N. Y. Makarenko, Finite groups and Lie rings with a metacyclic Frobenius group of automorphisms, Journal of Algebra 386 (2013) 77–104.
- [7] E.I. Khukhro, N.Y. Makarenko, P. Shumyatsky, Frobenius groups of automorphisms and their fixed points, Forum Math. 26 (1) (2014) 73–112.
- [8] N.Y. Makarenko, P. Shumyatsky, Frobenius groups as groups of automorphisms, Proc. Amer. Math. Soc. 138 (2010) 3425–3436.
- [9] N.Y. Makarenko, E.I. Khukhro, P. Shumyatsky, Fixed points of Frobenius groups of automorphisms, Dokl.Akad.Nauk 437, no.1 (2011) 20–23 ; English transl., Dokl.Math. 83, no.2 (2011).
- [10] A. Turull, Fitting height of groups and of fixed points, J. Algebra 86 (1984) 555–556.
- [11] A. Turull, Groups of automorphisms and centralizers, Math. Proc. Camb. Phil. Soc. 107 (1990) 227–238.

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