QUANTUM VACUUM ENERGY FOR MASSLESS CONFORMAL SCALAR FIELD IN EINSTEIN AND CLOSED FRIEDMANN UNIVERSES

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ABSTRACT

QUANTUM VACUUM ENERGY FOR
MASSLESS CONFORMAL SCALAR FIELD
IN EINSTEIN AND CLOSED FRIEDMANN UNIVERSES

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The quantum vacuum energy of the massless conformal scalar field in Einstein and closed Friedmann universes are discussed. Ford's results for the massless conformal scalar field in an Einstein universe are reproduced by considering mode sums and by renormalizing the divergent vacuum energy by introducing a cutoff function. Adiabatic regularization is applied to the massless conformal scalar field in a closed Friedmann universe. Explicit expressions for the vacuum expectation values of the components of the energy momentum tensor are obtained.

Key Words: Vacuum energy, Regularization, Renormalization, Cutoff, Mode sum, Adiabatic regularization, Energy momentum tensor.

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ÖZET

EINSTEIN VE FRIEDMANN EVRENİNDE KÜTLESİZ KONFORMAL SAYISAL ALANIN TANECİK SIFIR NOKTASI ENERJİSİ

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Bu tezde; Einstein ve Friedmann evreninde kütlesiz, konformal sayısal alanın, tanecik sıfır noktası enerjisi tartışıldı. Mod toplamları göz önüne alınarak ve kesilim fonksiyonuyla yeniden normalize edilmiş ıraksak sıfır noktası enerjisi kullanılarak Ford'un Einstein evreni için daha önce hesapladığı kütlesiz konformal sayısal alan için tanecik sıfır noktası enerjisi yeniden elde edildi. Kapalı, genişleyen Friedmann evreninde kütlesiz konformal sayısal alan için ısısız düzenleyici yaklaşımı uygulandı. Enerji momentum tansör bileşenlerinin sıfır noktasındaki beklenti değerleri için açık ifadeler elde edildi.

Anahtar Kelimeler: Sıfır noktası enerjisi, Düzenleyici, Yeniden normalleştirme, Kesilim , Modları toplamı, Isısız düzenleyici, Energy momentum tansörü.

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Chapter 1

INTRODUCTION

One of the basic questions concerning the relationship of quantum field theory and gravitation is the zero-point or vacuum energy. Does the zero-point energy of a quantized field act as a source of the gravitatonal field? In other areas of physics the zero-point energy represents a real property of the ground state, and the divergent zero-point energy is usually dispensed on the grounds that only differences in energy are observable. Hence, the origin of the energy scale could be made to start wherever we wish. On the other hand, in gravitational physics this luxury does not exist i.e. the actual value of the energy- momentum tensor determines the geometry of spacetime, and the law of conservation of energy is violated unless this energy is included in the source of the gravitational field.

In this thesis we are going to study quantum vacuum energy in curved spacetime and we will concentrate on a useful object; the energy momentum tensor $T^{\mu\nu}$. This quantitity has the advantage of being defined locally and also in field theories on curved background spacetime we will substitute its expectation value as the source term to the right hand side of the Einstein field equation.

First let us briefly review the case of a real scalar field $\Phi(\vec{x}, t)$ in a four dimensional Minkowski spacetime which satisfies the field equation [1]

$$(\Box + m^2)\Phi = 0. \tag{1.1}$$

Where $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$ and $\eta^{\mu\nu}$ is the Minkowski metric tensor (We take the signature as + - - -.). The quantity m is defined as the mass of the field quanta when the theory is quantized. The Lagrangian density leading to the equation (1.1) is;

$$\mathcal{L} = \frac{1}{2}(\eta^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - m^2\Phi^2). \tag{1.2}$$

We construct the action

$$S = \int \mathcal{L} d^4x, \tag{1.3}$$

and we take the variation of equation (1.3) with respect to Φ to get

$$\frac{\delta S}{\delta \Phi} = 0. \tag{1.4}$$

This leads us to the field equation (1.1). We have a set of solutions of the equation (1.1) which could be given as

$$U_{\vec{k}} = A_0 e^{i\vec{k}.\vec{x} - iwt}, \tag{1.5}$$

where A_0 is the a normalization constant , and

$$w = (k^2 + m^2)^{\frac{1}{2}}. \tag{1.6}$$

$U_{\vec{k}}$ are the positive frequency mode functions and they form an orthonormal set with respect to the scalar product which is defined as [1]

$$(\Phi_1, \Phi_2) = -i \int \left[\Phi_1(x)\partial_t\Phi_2^*(x) - [\partial_t\Phi_1(x)]\Phi_2^*(x)\right] d^3x \ , \tag{1.7}$$

hence

$$(U_{\vec{k}'}, U_{\vec{k}}) = \delta(\vec{k} - \vec{k}'). \tag{1.8}$$

In canonical quantization scheme the scalar field Φ behaves like an operator and we impose the following equal-time commutation rules:

$$[\Phi(t,\vec{x}),\Phi(t,\vec{x}')] = 0, \tag{1.9}$$

$$[\Pi(t,\vec{x}),\Pi(t,\vec{x}')] = 0, \tag{1.10}$$

$$[\Phi(t,\vec{x}),\Pi(t,\vec{x}')] = i\delta^3(\vec{x}-\vec{x}'), \tag{1.11}$$

where Π is the canonically conjugate momentum which is defined by

$$\Pi = \frac{\partial \mathcal{L}}{\partial(\partial_t\Phi)} = \partial_t\Phi. \tag{1.12}$$

The field modes given in equation (1.5) and their respective complex conjugate form a complete and orthonormal basis with the scalar product (1.7). Hence, Φ may be expanded as

$$\Phi(t,\vec{x}) = \sum_{\vec{k}}[a_{\vec{k}}U_{\vec{k}} + a_{\vec{k}}^{\dagger}U_{\vec{k}}^{*}]. \tag{1.13}$$

($\dagger$ denotes Hermitian conjugate , * denotes complex conjugate.) The equal time commutation relations for Φ and Π are now equivalent to

$$[a_{\vec{k}}, a_{\vec{k}'}] = 0, \tag{1.14}$$

$$[a_{\vec{k}}^{\dagger}, a_{\vec{k}'}^{\dagger}] = 0, \tag{1.15}$$

$$[a_{\vec{k}}, a_{\vec{k}'}^{\dagger}] = \delta_{\vec{k}\vec{k}'}. \tag{1.16}$$

Where, $a_{\vec{k}}$ is referred to as an annihilation operator and $a_{\vec{k}}^{\dagger}$ as a creation operator for quanta in the mode $\vec{k}$. Now we discuss the Hamiltonian for the scalar field. The Hamiltonian is obtained from the energy momentum tensor $T_{\mu\nu}$. This tensor for a scalar field in curved spacetime could be obtained from variation of the action with respect to the metric as

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}}, \tag{1.17}$$

where g is the determinant of the metric tensor.For our case $g^{\mu\nu} = \eta^{\mu\nu}$ and using equations (1.2) and (1.3) one obtains

$$T_{\mu\nu} = \partial_\mu \Phi \partial_\nu \Phi - \frac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta}\partial_\alpha \Phi \partial_\beta \Phi + \frac{1}{2}m^2\Phi^2\eta_{\mu\nu}. \tag{1.18}$$

The Hamiltonian density becomes

$$T_{tt} = \frac{1}{2}[(\partial_t \Phi)^2 + \sum_{i=1}^{3}(\partial_i \Phi)^2 + m^2\Phi^2], \tag{1.19}$$

where i=1,2,3 in terms of the Minkowski space coordinates. Substituting Φ from (1.13) into (1.19), and integrating over all space, yields

$$H = \int_t T_{tt} d^3x = \frac{1}{2}\sum_{\vec{k}}(a_{\vec{k}}^{\dagger}a_{\vec{k}} + a_{\vec{k}}a_{\vec{k}}^{\dagger})w_{\vec{k}}. \tag{1.20}$$

By using the commutation relations (1.14),(1.15),and (1.16), equation (1.20) becomes

$$H = \sum_{\vec{k}}(a_{\vec{k}}^{\dagger}a_{\vec{k}} + \frac{1}{2})w_{\vec{k}},$$
$$H = \sum_{\vec{k}} w_{\vec{k}}(n_{\vec{k}} + \frac{1}{2}). \tag{1.21}$$

Where $a_{\vec{k}}^{\dagger}a_{\vec{k}} = n_{\vec{k}}$ is called the number operator that has eigenvalues

$$n_{\vec{k}} = 0, 1, 2, 3, ..., \tag{1.22}$$

and $\vec{k}$ is the mode number.

An important point is that even the ground state, which has the lowest energy ($n_{\vec{k}} = 0$) has a non-zero energy. To understand the vacuum energy we can construct the vacuum state from the normalized basis ket vectors denoted by $|\,0>$. Where, $|\,0>$ is called the vacuum or no particle state. The state $|\,0>$ has the property that it is annihilated by all the $a_{\vec{k}}$ operators:

$$a_{\vec{k}}\,|\,0>=0, \text{ for all } \vec{k}$$
$$<0\,|\,a^{\dagger}_{\vec{k}}=0,$$
$$<0\,|\,a_{\vec{k}}a^{\dagger}_{\vec{k}'}\,|\,0>=\delta_{\vec{k}\vec{k}'}. \tag{1.23}$$

By taking the vacuum expectation value of the Hamiltonian we find that

$$\begin{aligned} <0\,|\,H\,|\,0> &= <0\,|\,0>\sum_{\vec{k}}\frac{1}{2}w_{\vec{k}}, \\ E_0 &= \sum_{\vec{k}}\frac{1}{2}w_{\vec{k}}. \end{aligned} \tag{1.24}$$

We have used the normalization condition $<0\,|\,0>=1$.

Since there are infinite number of normal modes of increasingly high frequency E_o is infinite. The basic problem occurs at the vacuum energy. The fact that Eq. (1.24) is divergent apparently indicates that the vacuum contains an infinite density of energy. As $w_{\vec{k}}$ has no upper bound the zero-point energy can be arbitrarily large.On flat spacetime the problem of infinite vacuum energy is usually bypassed by introducing normal ordering. Defining a normal ordering operator ::,which demands that destruction operator $a_{\vec{k}}$ should always be written on the right hand side of $a^{\dagger}_{\vec{k}}$ wherever they appear in pairs.Hence,

$$:a_{\vec{k}}a^{\dagger}_{\vec{k}}:=a^{\dagger}_{\vec{k}}a_{\vec{k}}, \tag{1.25}$$

and

$$:a^{\dagger}_{\vec{k}}a_{\vec{k}}:=a^{\dagger}_{\vec{k}}a_{\vec{k}}. \tag{1.26}$$

Now the normal ordered Hamiltonian becomes

$$:H:=\sum_{\vec{k}}a^{\dagger}_{\vec{k}}a_{\vec{k}}w_{\vec{k}}. \tag{1.27}$$

By using Eq.(1.24) one obtains

$$(H' =) : H := H - < 0 \mid H \mid 0 > . \tag{1.28}$$

One observes that replacing the original Hamiltonian H by the normal-ordered Hamiltonian H' is equivalent to a formal subtraction of the infinite zero-point energy. This has the effect that the energy of the corresponding free vacuum state becomes zero [2]. However, the presence of boundaries for the field induces a change in the energy spectrum and therefore also modifies the zero-point energy. When fields are confined to a finite volume which can change, the zero-point energy cannot be removed by the simple normal-ordering prescription used in conventional field theories.

The first zero-point energy calculation in the presence of boundaries was done by Casimir in (1948) [3]. Casimir evaluated the quantum vacuum energy for the electromagnetic field bounded by two parallel plates. Casimir showed that this energy is finite, independent of any cutoff, and depends only on the distance between the plates. Since the finite quantum vacuum energy he found was negative he concluded that there must exist an atractive force between the plates. Later in 1958 existence of this force was verified by experiments [4,5]. Hence, it became clear that Casimir effect is real and has to be taken seriously. Encouraged by the existence of this atractive force, Casimir conjectured that one may even construct an electron model where the repulsive electric forces will be balanced by the atractive Casimir effect. Later the vacuum energy inside a spherical shell was calculated by T. Boyer [6], and B.Davies [7]. They found it to be positive and depending only on the radius of the shell. This result did not confirm Casimir's suggestion. The fact that the vacuum energy in a spherical shell is positive means that the force on the shell is repulsive. This being quite different from the case of two parellel plates violates the

Casimir's idea, at least in its original form.

We may take the Casimir's beautiful calculation [3] of the quantum vacuum energy as a model for the calculations in the case of a quantized field in a curved background spacetime. Ford [8,9] discussed the vacuum energy of a massless conformal scalar field in an Einstein universe by the mode sum method. He emphasized the analogy between this energy and the quantum vacuum energy of the massless scalar field in the presence of a pair of parallel plates. Ford showed that the renormalized energy of the vacuum state of the quantized scalar field in a one-dimensional box of length L is equal to

$$E = -\frac{\pi}{6L}. \tag{1.29}$$

Renormalization is a method where the infinities are absorbed into the physical constants, such as charge and mass, or are canceled by a suitable counterterm in the Lagrangian. A massless, conformal scalar field in an Einstein universe also possesses a non-zero vacuum energy.Ford obtained the renormalized vacuum energy density in this case as

$$\rho = \frac{1}{480\pi^2 a_0^4}. \tag{1.30}$$

Where, a_0 is the radius of the universe and the pressure is given by $P = \frac{1}{3}\rho$.

Thus the energy momentum tensor is in the same form as that for the classical radiation. In Ford's work on the Casimir effect in an Einstein universe, he showed that the energy is associated with the closed spatial topology. Ford's approach consisted of removing the divergences of the energy-momentum tensor by a suitable cutoff in the mode sum and then isolating and subtracting the cutoff dependent terms.In section ll we will discuss details of Ford's paper [8].

In section lll we will discuss the energy-momentum tensor for the closed Friedmann universe. In theories involving quantized fields the formal

formal expressions for the observables of the theory often possess infinite expectation values. Method for obtaining suitable finite observables from the formally divergent expressions is called regularization[10]. Where, the divergent quantities are replaced by well-defined expressions in a manner consistent with the physical basis of the theory. The expectation values of the energy-momentum tensor of the quantized scalar field are formally divergent. Adiabatic regularization is a method of finding the finite parts of expectation values of the components of the energy momentum tensor for the scalar field in homogeneous cosmological spacetimes [10-14].The essential point of adiabatic regularization is the identification of the infinite cotributions to the vacuum, which are later subtracted to obtain a finite result.

P.R.Anderson and L.Parker [14] showed that among the possible ways to apply adiabatic regularization in a closed Robertson-Walker universe, only one yields the accepted trace anamoly and the vacuum energy for the massless conformal scalar field. Anderson and Parker calculation yielded the terms which are to be subtracted from the divergent mode-sum expressions for the expectation value of the energy-momentum tensor to obtain the finite renormalized energy-momentum tensor.They also mentioned that in calculating the vacuum subtraction, one should use the flat space measure in the mode sum.

Our sign conventions are such that the metric signature is $(+ - - -)$, $R^{\alpha}_{\beta\gamma\delta} = \Gamma^{\alpha}_{\beta\gamma,\delta} - \ldots$, and $R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu}$. The usual summation convetion is in effect over Greek (spacetime) indices. Summations over Latin (three-space) indices are indicated explicitly , but an index may be omitted from the summation sign when there is no chance of confusion. The units are such that $\hbar = c = 1$. We abbreviate the spacetime point $(t, \vec{x}) = (x^0, \vec{x})$ as x.

Chapter 2

RENORMALIZED VACUUM ENERGY OF THE MASSLESS CONFORMAL SCALAR FIELD IN AN EINSTEIN UNIVERSE

We begin with the basic properties of a massless conformal scalar field considered on curved background Einstein geometry, where the metric is given by

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = dt^2 - R_0^2[d\chi^2 + sin^2\chi(d\theta^2 + sin^2\theta d\phi^2)]. \qquad (2.1)$$

Here, χ and θ run from 0 to π, while ϕ runs from 0 to 2π, and R_0 represent the radius of the universe, and it is a constant.

The conformally invariant Klein-Gordon equation for a massless scalar field is [1];

$$\Box\Psi - \frac{1}{6}R\Psi = 0. \qquad (2.2)$$

Here $\Box$ is the D'Alembertian operator, which is given by

$$\Box\Psi = \frac{1}{\sqrt{-g}}(\sqrt{-g}g^{\mu\nu}\Psi_{;\mu})_{;\nu}. \qquad (2.3)$$

Where, g is the determinant of the metric tensor.

In equation(2.2) $R = 6R_0^{-2}$ is the scalar curvature corresponding to the metric

(2.1).

Solution of equation (2.2) could be easily found as [8]

$$\Psi(x) = c_0 X(\chi) P_l^m(cos\theta) e^{im\phi} e^{-iwt}, \tag{2.4}$$

where P_l^m is the associated Legendre function, c_0 is an appropriate normalization constant, and

$$\begin{aligned} l &= 0, 1, 2, ..., and \\ m &= -l, -l+1, ..., 0, ..., (l-1), l. \end{aligned} \tag{2.5}$$

$X(\chi)$ satisfies the following differential equation;

$$\frac{d}{d\chi}(sin^2\chi \frac{dX}{d\chi}) + (N^2 - 1)\sin^2\chi X(\chi) - l(l+1)X(\chi) = 0, \tag{2.6}$$

where $N^2 = R_0^2 w^2$. Making the following substitution:

$$X(\chi) \propto sin^l\chi C(cos\chi), \tag{2.7}$$

the equation to be solved for $C(cos\chi)$ becomes

$$(1 - x^2)\frac{d^2C}{dx^2} - x(2l+3)\frac{dC}{dx} + [(N^2 - 1) - l(l+2)]C = 0, \tag{2.8}$$

where, $x = cos\chi$, and $x \in [-1, 1]$.

This differential equation could be solved by the method of Frobenius [15]. Hence, we express the function $C(x)$ by a power series as

$$C(x) = \sum_{n=0}^{\infty} a_n x^{n+\alpha}. \tag{2.9}$$

The coefficients a_n are chosen so that equation (2.8) is formally satisfied. Substituting equation (2.9) into equation(2.8) one finds that

$$\begin{aligned} \sum_{n=0}^{\infty} \quad & \{(n+\alpha+1)(n+\alpha+2)a_{n+2} - [(n+\alpha)(n+\alpha-1) \\ & +(n+\alpha)(2l+3) - ((N^2-1) - l(l+2))]a_n\}x^{n+\alpha} = 0. \end{aligned} \tag{2.10}$$

For this equation to be satisfied for all x, the coefficient of each power of x must vanish. The lowest power of x is $(\alpha - 2)$ and the corresponding coefficient is $\alpha(\alpha-1)a_0$; hence setting this coefficient to zero gives us the indical equation

$$\alpha(\alpha-1)a_0 = 0 \qquad\qquad (a_0 \neq 0). \tag{2.11}$$

Therefore, the constant α could be 0 or 1. If $\alpha = 0$ is chosen the above Eq.(2.10) becomes [15]

$$\sum_{n=0}^{\infty}\{(n+1)(n+2)a_{n+2} - [(n+l)^2 + 2(n+l) - (N^2-1)]a_n\}x^n = 0. \tag{2.12}$$

This leads us to the following recursion relation:

$$\frac{a_{n+2}}{a_n} = \frac{(n+l)^2 + 2(n+l) - (N^2-1)}{(n+1)(n+2)}, \tag{2.13}$$

which could be simplified as

$$\frac{a_{n+2}}{a_n} = \frac{(n+l+1)^2 - N^2}{(n+1)(n+2)}. \tag{2.14}$$

The above recursion relation gives us the following series solution for $C(x)$:

$$C(x) = \sum_{n=0}^{\infty} a_{2n}x^{2n} + \sum_{n=0}^{\infty} a_{2n+1}x^{2n+1}, \tag{2.15}$$

$$\begin{aligned} C(x) = \quad & a_0\{1 + \frac{(l+1)^2 - N^2}{2!}x^2 + \dots\dots\dots\} \\ & + a_1\{x + \frac{(l+2)^2 - N^2}{3!}x^3 + \dots\dots\dots\dots\}. \end{aligned} \tag{2.16}$$

This is in the form

$$C(x) = a_0 C_1(x) + a_1 C_2(x), \tag{2.17}$$

where $C_1(x)$ and $C_2(x)$ are linearly independent. Convergence of the series at the end points of our interval could be checked easily by using the Raabe test which says that if $\sum_{n=0}^{\infty} u_n$ is a series of positive terms and if the $\lim_{n\to\infty} n(\frac{u_n}{u_{n+1}} - 1) = A$ the series is divergent for $A < 1$, convergent for $A > 1$

and the test fails for A=1 [16]. At the end points we apply this test to the series with even powers i.e. $C_1(x)$ .We write

$$\lim_{n\to\infty} n(\frac{a_{2n}}{a_{2n+2}} - 1) = A, \tag{2.18}$$

and using

$$\frac{a_{2n}}{a_{2n+2}} = \frac{(2n+1)(2n+2)}{(2n+l+1)^2 - N^2}, \tag{2.19}$$

one obtains A as $\frac{1}{2} - l$ $(l \geq 0)$. Hence, the infinite series $C_1(x)$ diverges at the end points. One could similarly check that $C_2(x)$ is also divergent at the end points of our interval. To obtain a regular solution in the entire interval $x \in [-1, 1]$ we terminate the infinite series after a finite number of terms by restricting N to have integer values given as

$$N = (n + l) + 1, \tag{2.20}$$

where $n = 0, 1, 2, 3, ... \; and \; l = 0, 1, 2, 3, ...$. Redefining a new index n' as $(n + l)$ we write

$$N = n' + 1, \tag{2.21}$$

where $n' = 0, 1, 2, 3, ...$, and $l = 0, 1, 2, 3, ...,$ n' . Dropping primes one now obtains the eigenfrequencies as

$$w_n = \frac{(n+1)}{R_0}. \tag{2.22}$$

The polynomial solutions obtained this way are the well known Gegenbauer polynomials, hence we could write $C(x)$ in terms of the Gegenbauer polynomials as [8]

$$C(x) = C_{n-l}^{l+1}(x). \tag{2.23}$$

For fixed n, l takes on the values 0,1,....,n. Thus, the degeneracy of each eigenfrequency is

$$(g_n =) \sum_{l=0}^{n} (2l + 1) = (n + 1)^2. \tag{2.24}$$

Formal quantization may be carried out in the usual way by defining the creation and annihilation operators $a_\lambda^\dagger$ and a_λ as [8]

$$\Psi(x) = \sum_\lambda (a_\lambda F_\lambda + a_\lambda^\dagger F_\lambda^*). \tag{2.25}$$

where λ stands for n, l and m. Here $F_\lambda = f_\lambda e^{-i\omega_n t}$ is a solution of equation (2.2).

$$f_{\lambda(n,l,m)}(\chi, \theta, \phi) = C_0 C_{n-l}^{l+1}(\cos\chi) P_l^m(\cos\theta) e^{im\phi}. \tag{2.26}$$

The operators a_λ and $a_\lambda^\dagger$ satisfy the usual commutation relations and may be used to define a Fock space [1]. The existence of a global timelike killing vector leads us naturally to a unique choice for the vacuum state, which is defined as

$$a_\lambda \mid 0 >= 0, \qquad \text{for all } \lambda \text{ (n,l,m)}. \tag{2.27}$$

The energy momentum tensor is given as [8]

$$\begin{aligned} T_\alpha^\beta &= -\Psi_{,\alpha}\Psi^{,\beta} + \frac{1}{2}\delta_\alpha^\beta g^{\rho\sigma}\Psi_{,\rho}\Psi_{,\sigma} + \frac{1}{6}(\Psi^2)_{;\alpha}^{;\beta} \\ &\quad -\frac{1}{6}\delta_\alpha^\beta(\Psi^2)_{;\rho}^{;\rho} - \frac{1}{6}G_\alpha^\beta\Psi^2, \end{aligned} \tag{2.28}$$

where $G_\alpha^\beta = R_\alpha^\beta - \frac{1}{2}\delta_\alpha^\beta R$ is the Einstein tensor. Now setting $\alpha = 0$ and $\beta = 0$, we obtain T_0^0 as

$$T_0^0 = \frac{1}{2}(\Psi_{,0}^2 - \Psi\Psi_{,00}). \tag{2.29}$$

The Hamiltonian is

$$H = \int T_0^0 \sqrt{h} d^3x. \tag{2.30}$$

The spatial functions f_λ may be normalized so that

$$\int f_{\lambda_1} f_{\lambda_2} \sqrt{h} \; d^3x = \delta_{\lambda_1\lambda_2}. \tag{2.31}$$

h is the determinant of h_{ij}, where the line element (2.1) is written as

The Hamiltonian now takes the familiar form

$$H = \frac{1}{2}\sum_{\lambda} w(a_\lambda a_\lambda^\dagger + a_\lambda^\dagger a_\lambda), \tag{2.33}$$

and vacuum energy is given as

$$E_0 = \sum_n \frac{1}{2} w_n g_n. \tag{2.34}$$

This divergent expression must now be regularized by the insertion of a cutoff function. Even though the result is going to be cutoff independent [8,9,17] a convenient choice for the cutoff function is;

$$f(\alpha, w_n) = e^{-\alpha w_n}. \tag{2.35}$$

Where α is a cutoff parameter and w_n is given in equation (2.22). Hence, the cutoff zero-point energy is defined as

$$\begin{aligned} E_0 &= \sum_{\lambda} \frac{g_n}{2} w_n e^{-\alpha w_n}, \\ &= \frac{1}{2R_0}\sum_{n=0}^{\infty}(n+1)^3 e^{-(n+1)\frac{\alpha}{R_0}}. \end{aligned} \tag{2.36}$$

where α is a cutoff parameter which will be set to zero at the end.

Before we proceed, we recall the Euler-Maclaurin sum formula, which could be given as [9,18]

$$\begin{aligned} \sum_{j=0}^{n} F(j) = & \int_0^n F(x)dx + \frac{1}{2}F(0) + \frac{1}{2}F(n) \\ & + \sum_{s=1}^{m-1} \frac{B_{2s}}{(2s)!}(F^{(2s-1)}(n) - F^{(2s-1)}(0)) \\ & + \int_0^n \frac{B_{2m} - B_{2m}(x-[x])}{(2m)!} F^{(2m)}(x)dx \\ = & \; F(0) + F(1) + ... + F(n). \end{aligned} \tag{2.37}$$

where ,m and n are integers such that n>0 and m>0 and

$$F^{(2m)}(x) = \frac{d^{2m}F(x)}{dx^{2m}} \tag{2.38}$$

is absolutely integrable over the interval (0,n). Also [x] denotes the integer in the interval (x-1,x]; in consequence, as a function of x, $B_{2m}(x-[x])$ is periodic and continuous, with period 1. B_{2m} are the Bernoulli numbers. Significance of the Euler-Maclaurin sum formula is that it could be used to evaluate a given sum interms of the integral of a continuous and differentiable function, plus some correction terms.

Using the Euler-Maclaurin sum formula (2.37) the vacuum energy given in equation (2.36) becomes

$$\begin{aligned} E_0 = & \frac{1}{2R_0}\{\int_0^\infty F(n)dn + \frac{1}{2}F(0) \\ & +\frac{1}{2}F(\infty) + \frac{B_2}{2!}(F^{(1)}(\infty) - F^{(1)}(0)) \\ & +\frac{B_4}{4!}(F^{(3)}(\infty) - F^{(3)}(0)) \\ & +\frac{B_6}{6!}(F^{(5)}(\infty) - F^{(5)}(0)) + ...\}. \end{aligned} \quad (2.39)$$

one could easily check that as m→ ∞ the remainder term → 0. In equation (2.39) F(n) is given as

$$F(n) = (n+1)^3 e^{-(n+1)\frac{\alpha}{R_0}}. \quad (2.40)$$

The continuous and differentiable function F(n) and its derivatives up to the fifth order are given as

$$\begin{aligned} F(0) &= 0 & ;F(\infty) &= 0 \\ F^{(1)}(0) &= 0 & ;F^{(1)}(\infty) &= 0 \\ F^{(2)}(0) &= 0 & ;F^{(2)}(\infty) &= 0 \\ F^{(3)}(0) &= 6 & ;F^{(3)}(\infty) &= 0 \\ F^{(4)}(0) &= -24\frac{\alpha}{R_0} & ;F^{(4)}(\infty) &= 0 \\ F^{(5)}(0) &= 60\frac{\alpha^2}{R_0^2} & ;F^{(5)}(\infty) &= 0, \end{aligned}$$

(2.41)

and

$$\int_0^\infty F(x)dx = \frac{6R_0^4}{\alpha^4}. \tag{2.42}$$

The vacuum energy given in equation (2.36) becomes

$$E_0 = 3\frac{R_0^3}{\alpha^4} + \frac{1}{240R_0} + \frac{17\alpha^2}{360R_0^2} + \{\text{ terms in positive powers of } \alpha\}. \tag{2.43}$$

If one divides E_0 by the spatial volume of the universe $2\pi^2 R_0^3$ one gets the divergent vacuum energy density in an Einstein universe. It is appeerent that the cutoff zero-point enegy density in an equal volume of Minkowski space is just

$$\bar{\rho}_0 = \frac{3}{2\pi^2\alpha^4}. \tag{2.44}$$

This quantity is independent of the physical properties of the system and diverges in the limit as $\alpha \to 0$. Hence, we define the physical vacuum energy density in an Einstein universe as

$$\rho = \lim_{\alpha\to 0}(\rho_0 - \bar{\rho}_0). \tag{2.45}$$

$$\rho = \frac{1}{480\pi^2 R_0^4}. \tag{2.46}$$

The corresponding pressure may be obtained from the requirements that the conformally invariant energy-momentum tensor be traceless. Since our renormalization procedure consists in substracting the Minkowski-space energy-momentum tensor from the Einstein universe energy-momentum tensor, and since both are traceless, the renormalized energy-momentum tensor is also traceless.

Hence the pressure is [8]

$$P = \frac{1}{3}\rho. \tag{2.47}$$

Chapter 3

RENORMALIZED QUANTUM VACUUM ENERGY MOMENTUM TENSOR OF THE MASSLESS CONFORMAL SCALAR FIELD IN A CLOSED FRIEDMANN UNIVERSE

In this section we will write the Robertson-Walker metric as

$$ds^2 = a^2(\eta)[d\eta^2 - h_{ij}dx^i dx^j], (i = 1, 2, 3) \tag{3.1}$$

where

$$h_{ij}dx^i dx^j = (1 - r^2)^{-1}dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2). \tag{3.2}$$

Defining a new variable

$$r = sin\chi. \tag{3.3}$$

Equation (3.2) becomes

$$h_{ij}dx^i dx^j = d\chi^2 + sin^2\chi(d\theta^2 + sin^2\theta d\phi^2). \tag{3.4}$$

Here χ and θ run from 0 to π, while ϕ runs from 0 to 2π , while $a(\eta)$ is the scale factor.

The Lagrangian density is given as [1]

$$\mathcal{L}(\vec{x}) = \frac{1}{2}\sqrt{-g}(g^{\mu\nu}\Phi_{;\mu}(x)\Phi_{;\nu}(x) - m^2\Phi^2(x) - \frac{R}{6}\Phi^2(x)). \tag{3.5}$$

We construct the action

$$S = \int \mathcal{L}(\vec{x})d^4x. \tag{3.6}$$

Varying S with respect to Φ, we obtain the scalar field equation,

$$0 = \frac{\delta S}{\delta \Phi} = \sqrt{-g}(\Phi^{\mu}_{;\mu} + (\frac{1}{6}R + m^2)\Phi). \tag{3.7}$$

where $\frac{\delta}{\delta\Phi}$ indicates functional differentiation, g is the determinant of the background metric $g_{\mu\nu}$. Hence, the equation of motion is given as

$$\Box\Phi + m^2\Phi + \frac{R}{6}\Phi = 0. \tag{3.8}$$

Where Φ is the scalar field, m is the scalar field's mass, and R is the scalar curvature , which is given as

$$R = \frac{6}{a^2}(\frac{a''}{a} + 1). \tag{3.9}$$

We will let $m \to 0$ in equation (3.8) for the massless conformal scalar field. The solution of equation (3.8) could be obtained by separation of variables as

$$U_{\vec{k}}(\eta, \chi, \theta, \phi) = \frac{1}{a(\eta)}\Psi_{\vec{k}}(\eta)\mathcal{Y}_{\vec{k}}(\chi, \theta, \phi). \tag{3.10}$$

The general solution of the field equation could be written as a sum over these modes in the form

$$\Phi(\vec{x}) = \int d\,\tilde{\mu}\,(\vec{k})(a_{\vec{k}}U_{\vec{k}} + a^{\dagger}_{\vec{k}}U^{*}_{\vec{k}}). \tag{3.11}$$

The measure $d\,\tilde{\mu}\,(\vec{k})$ for closed Friedmann models is given as

$$\int d\,\tilde{\mu}\,(\vec{k}) = \sum_{k,l,M}. \tag{3.12}$$

The function$\mathcal{Y}_{\vec{k}}$ is written as

$$\mathcal{Y}_{\vec{k}}(\chi,\theta,\phi) = \Pi^{(+)}_{kl}(\chi)Y_{lM}(\theta,\phi), \tag{3.13}$$

where

$$k = 1,2,3,...$$
$$l = 1,2,3,..,(k-1)$$
$$M = -l,-l+1,.....,0,....,l. \tag{3.14}$$

Here, the $Y_{lM}(\theta,\phi)$'s are the standart spherical harmonics, and $\Pi^{(+)}_{kl}(\chi)$ satisfies the following differential equation:

$$\frac{d}{d\chi}(sin^2\chi\frac{d\Pi^{(+)}_{kl}}{d\chi}) + \{sin^2\chi(k^2-1) - l(l+1)\}\Pi^{(+)}_{kl}(\chi) = 0. \tag{3.15}$$

We have seen that the solution of the above differential equation is given as [15]

$$\Pi^{(+)}_{kl}(\chi) = C_0 sin^l\chi\ C^{l+1}_{k-l-1}(cos\chi), \tag{3.16}$$

where C_0 is a constant and $C^{l+1}_{k-l-1}(cos\chi)$ are the Gegenbauer polynomials:

$$C^{l+1}_{k-l-1}(cos\chi) \propto \frac{d^{l+1}}{d(cos\chi)^{l+1}}cosk\chi. \tag{3.17}$$

The functions $\mathcal{Y}_{\vec{k}}(\chi,\theta,\phi)$ are the eigenfunctions of the three-space Laplacian $\Delta^{(3)}$ such that

$$\Delta^{(3)}\mathcal{Y}_{\vec{k}}(\chi,\theta,\phi) = -(k^2-1)\mathcal{Y}_{\vec{k}}(\chi,\theta,\phi). \tag{3.18}$$

The time-dependent mode functions $\Psi_k(\eta)$ satisfy

$$\frac{d^2\Psi_k(\eta)}{d\eta^2} + (k^2 + m^2a^2)\Psi_k(\eta) = 0. \tag{3.19}$$

The energy-momentum tensor is defined by functional differentiation of the action with respect to the metric tensor $g^{\mu\nu}$:

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}}, \tag{3.20}$$

where

$$\delta S = \int \delta \mathcal{L} d^4x. \tag{3.21}$$

Using the relation [1,19]

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}, \tag{3.22}$$

we obtain $\delta\mathcal{L}$:(with respect to $g^{\mu\nu}$)as

$$\begin{aligned}\delta\mathcal{L} &= \frac{1}{2}(-\frac{1}{2}\sqrt{-g}g_{\mu\nu}g^{\mu\nu}\Phi_{;\mu}\Phi_{;\nu}\delta g^{\mu\nu} + \sqrt{-g}\Phi_{;\mu}\Phi_{;\nu}\delta g^{\mu\nu} \\ &+ \frac{1}{12}\sqrt{-g}g_{\mu\nu}R\Phi^2\delta g^{\mu\nu} - \frac{1}{6}\sqrt{-g}\delta R\Phi^2 + \frac{1}{2}m^2\sqrt{-g}g_{\mu\nu}\Phi^2\delta g^{\mu\nu}).\end{aligned} \tag{3.23}$$

Using the following relations [1,19]

$$\delta R = -R^{\mu\nu}\delta g_{\mu\nu} + g^{\rho\sigma}g^{\mu\nu}(\delta g_{\rho\sigma;\mu\nu} + \delta g_{\rho\mu;\sigma\nu}), \tag{3.24}$$

$$\begin{aligned}\delta g^{\mu\nu} &= -g^{\mu\rho}g^{\nu\sigma}\delta g_{\rho\sigma}, \\ g^{\mu\nu}\delta g_{\mu\nu} &= -g_{\mu\nu}\delta g^{\mu\nu},\end{aligned}$$

and

$$g^{\rho\sigma}\Phi_{;\rho\sigma} = \Box\Phi, \tag{3.25}$$

we obtain

$$\begin{aligned}\delta\mathcal{L} &= \frac{1}{2}\sqrt{-g}\delta g^{\mu\nu}(\frac{2}{3}\Phi_{;\mu}\Phi_{;\nu} - \frac{1}{6}g_{\mu\nu}g^{\rho\sigma}\Phi_{;\rho}\Phi_{;\sigma} \\ &- \frac{1}{3}\Phi_{;\mu\nu}\Phi - \frac{1}{6}R_{\mu\nu}\Phi^2 + \frac{1}{36}g_{\mu\nu}R\Phi^2 + \frac{1}{6}m^2g_{\mu\nu}\Phi^2).\end{aligned} \tag{3.26}$$

Inserting (3.26) into (3.20) and rearranging, we obtain $T_{\mu\nu}$ [11,14] as

$$\begin{aligned}T_{\mu\nu} &= \frac{2}{3}\Phi_{;\mu}\Phi_{;\nu} - \frac{1}{6}g_{\mu\nu}g^{\rho\sigma}\Phi_{;\rho}\Phi_{;\sigma} - \frac{1}{3}\Phi_{;\mu\nu}\Phi \\ &- \frac{1}{6}R_{\mu\nu}\Phi^2 + \frac{1}{36}g_{\mu\nu}R\Phi^2 + \frac{1}{6}m^2g_{\mu\nu}\Phi^2.\end{aligned} \tag{3.27}$$

Using the equation of motion (3.8), one rewrites this as

$$\begin{aligned}T_{\mu\nu} &= \frac{2}{3}\Phi_{;\mu}\Phi_{;\nu} - \frac{1}{6}g_{\mu\nu}g^{\rho\sigma}\Phi_{;\rho}\Phi_{;\sigma} - \frac{1}{3}\Phi_{;\mu\nu}\Phi \\ &+ \frac{1}{12}g_{\mu\nu}\Phi\Box\Phi - \frac{1}{6}R_{\mu\nu}\Phi^2 + \frac{1}{24}Rg_{\mu\nu}\Phi^2 + \frac{1}{4}m^2g_{\mu\nu}\Phi^2.\end{aligned} \tag{3.28}$$

The trace of $T_{\mu\nu}$ is defined as

$$(T =)T^{\mu}_{\mu} = g^{\mu\nu}T_{\mu\nu}. \tag{3.29}$$

Also using

$$R = g^{\mu\nu}R_{\mu\nu}, \tag{3.30}$$

one obtains

$$\begin{aligned} T &= \frac{2}{3}g^{\mu\nu}\Phi_{;\mu}\Phi_{;\nu} - \frac{1}{6}g^{\rho\sigma}\Phi_{;\rho}\Phi_{;\sigma} \\ &\quad -\frac{1}{3}g^{\mu\nu}\Phi_{;\mu\nu}\Phi + \frac{1}{3}\Phi\Box\Phi - \frac{1}{12}R\Phi^2 + \frac{1}{2}m\Phi^2. \end{aligned} \tag{3.31}$$

Using the following relations [1,19];

$$\begin{aligned} -\frac{1}{12}R\Phi^2 &= \frac{1}{2}\Phi\Box\Phi + \frac{1}{2}m^2\Phi^2, \\ g^{\mu\nu}\Phi_{;\mu\nu}\Phi &= \Phi\Box\Phi, \end{aligned}$$

and

$$g^{\mu\nu}\Phi_{;\mu}\Phi_{;\nu} = (g^{\mu\nu}\Phi_{;\mu}\Phi)_{;\nu} - g^{\mu\nu}\Phi_{;\mu\nu}\Phi. \tag{3.32}$$

We find that

$$T = m^2\Phi^2. \tag{3.33}$$

Taking the vacuum expectation value $< T >$ and using the fact that the renormalized energy-momentum tensor should also reflect the symmetries of the underlying space-time one could write $< 0 \mid T \mid 0 >$ [1,12] as

$$< 0 \mid T \mid 0 >= (\int d^3x\sqrt{h})^{-1} < 0 \mid \int d^3x\sqrt{h}T \mid 0 > . \tag{3.34}$$

Volume of the universe is given as

$$\int d^3x\sqrt{h} = 2\pi^2 a^3, \tag{3.35}$$

where h is the determinant of h_{ij}.

We use the following mode decomposition for Φ;

$$\Phi = \sum_{\lambda(k,l,M)} (a_{\vec{\lambda}} U_{\vec{\lambda}} + a^{\dagger}_{\vec{\lambda}} U^{*}_{\vec{\lambda}}). \tag{3.36}$$

where $U_{\vec{k}}$ are given in equation (3.10), and we shall take the functions to be normalized so that

$$\int d\chi d\theta d\phi sin^2\chi sin\theta \mid \mathcal{Y}_{\vec{k}}(\chi,\theta,\phi) \mid^2 = 1. \tag{3.37}$$

Also, we use the canonical commutation relations

$$[a_{\vec{k}}, a_{\vec{k}'}] = 0, \tag{3.38}$$

and

$$[a_{\vec{k}}, a^{\dagger}_{\vec{k}'}] = \delta_{\vec{k}\vec{k}'}. \tag{3.39}$$

We denote by $\mid 0 >$ a normalized vector, which is annihilated by all the a_k. A basis for a Fock space may be built up by operating on $\mid 0 >$ with $a^{\dagger}_k$. The a_k do not depend on time. In general, no choice of the a_k corresponds to the annihilation operators of physical particles, since the particle number is not constant in time-dependent gravitational fields [1]. Taking the vacuum expectation value of the trace of $T_{\mu\nu}$ (3.33) one obtains

$$< 0 \mid T \mid 0 >= \frac{1}{2\pi^2 a^4} \sum_{k,l,M} m^2 a^2 \mid \Psi_k \mid^2, \tag{3.40}$$

where

$$\sum_{k,l,M} = \sum_{k=1}^{\infty} k^2. \tag{3.41}$$

Thus

$$< 0 \mid T \mid 0 >= \frac{1}{2\pi^2 a^2} \sum_{k=1}^{\infty} k^2 m^2 \mid \Psi_k \mid^2 . \tag{3.42}$$

For the massless conformal scalar field $m = 0$, hence the classical expression for $< 0 \mid T \mid 0 >$ becomes exactly zero. However, it is composed of formally divergent expressions i.e. it contains ultraviolet divergences.

In adiabatic regularization one subtracts off these divergences leaving a finite result. In identifying the divergences one considers slowly varying spacetimes, where the particle number is an adiabatic invariant [1,10-14], and remains constant in the limit of infinitely slow change in $a(\eta)$. In this method one evaluates the adiabatic approximations to the mode functions and evaluates the vacuum energy for slowly changing $a(\eta)$. Since the ultraviolet divergences are included in this unobservable vacuum, its substraction from the formally divergent expression (obtained by using exact modes) gives the finite observable result.

Following Parker and Fulling [10] if we introduce a slowness parameter such that each time derivative of $a(\eta)$ brings out one more factor of this parameter, then the particle number will be constant to any finite power of the adiabatic parameter. Hence, the vacuum contribution that needs to be substracted could be obtained to any desired order in this slowness parameter.

Let us now consider the time-dependent mode function $\Psi_k(\eta)$ which satisfies,

$$\frac{d^2\Psi_k}{d\eta^2} + w^2\Psi_k = 0, \tag{3.43}$$

where $w^2 = k^2 + m^2a^2$. $a(\eta)$ varies slowly under this condition, the time-dependent mode functions $\Psi_k(\eta)$ can be approximated by

$$\Psi_k = (2W_k)^{-1/2} e^{-i\int^{\eta} W_k(\eta) d\eta}. \tag{3.44}$$

This is the basis for the WKB method (The adiabatic approximation is a generalized WKB approximation [10,14]) and then expanding to fourth order in the adiabatic parameter. The equation satisfied by W_k is obtained by substituting (3.44) into (3.43) which gives:

$$W_k^2 = w^2 - \frac{1}{2}\left[\frac{W_k''}{W_k} - \frac{3}{2}\frac{W_k'^2}{W_k^2}\right], \tag{3.45}$$

where $w^2 = k^2 + m^2a^2$. Here, primes denote derivatives with respect to η. The WKB solution is obtained by solving (3.45) iteratively, taking the zeroth-order WKB solution to be

$$\begin{aligned} W_k^{(0)} &= w, \\ &= (k^2 + m^2a^2)^{1/2}. \end{aligned} \tag{3.46}$$

The first iterated WKB solution is

$$W_k^{(1)^2} = w^2 - \frac{1}{2}\left[\frac{W_k^{(0)''}}{W_k^{(0)}} - \frac{3}{2}\frac{W_k^{(0)'^2}}{W_k^{(0)^2}}\right]. \tag{3.47}$$

Now we calculate the above first iterated WKB solution:

$$\begin{aligned} W_k^{(0)} &= (k^2 + m^2a^2)^{1/2} = w, \\ W_k^{(0)'} &= \frac{m^2aa'}{w}, \\ W_k^{(0)''} &= \frac{m^2}{w}(a'^2 + aa'') - \frac{m^4a^2a'^2}{w^3}. \end{aligned} \tag{3.48}$$

Hence, the first iterated solution becomes;

$$W_k^{(1)^2} = w^2(1 - \frac{m^2(a'^2 + aa'')}{2w^4} + \frac{5m^4a^2a'^2}{4w^6}). \tag{3.49}$$

This could also be written as

$$W_k^{(1)^2} = w^2(1 + \epsilon), \tag{3.50}$$

where

$$\epsilon = -\frac{m^2(a'^2 + aa'')}{2w^4} + \frac{5m^4a^2a'^2}{4w^6}. \tag{3.51}$$

Using the binomial expansion to get

$$W_k^{(1)} = w(1 + \frac{1}{2}\epsilon - \frac{1}{8}\epsilon^2 + \text{..........}). \tag{3.52}$$

Where, ϵ is small compared with 1. Since all the divergences are contained in terms up to fourth order we do not consider higher order terms.

Thus, Eq. (3.52) becomes

$$\begin{aligned} W_k^{(1)} &= w + \frac{5m^4a^2a'^2}{8w^5} - \frac{m^2}{4w^3}(a'^2 + aa'') \\ &-\frac{25m^8a^4a'^4}{128w^{11}} \\ &-\frac{m^4}{32w^7}(a'^4 + 2aa'^2a'' + a^2a''^2) \\ &+\frac{10m^6}{64w^9}(a^2a'^4 + a^3a'^2a''). \end{aligned} \tag{3.53}$$

The second iterated WKB solution is

$$W_k^{(2)^2} = w^2 - \frac{1}{2}\left[\frac{W_k^{(1)''}}{W_k^{(1)}} - \frac{3}{2}\frac{W_k^{(1)'^2}}{W_k^{(1)^2}}\right]. \tag{3.54}$$

Taking the Eq. (3.53) first and second derivative with respect to η and neglecting terms than fourth order , ones obtain

$$W_k^{(1)'} = \frac{m^2aa'}{w} - \frac{m^2}{4w^3}(3a'a'' + aa''') + \frac{2m^4}{w^5}(aa'^3 + a^2a'a'') - \frac{25m^6a^3a'^3}{8w^7}, \text{and} \tag{3.55}$$

$$\begin{aligned} W_k^{(1)''} &= \frac{m^2}{w}(a'^2 + aa'') - \frac{m^4a^2a'^2}{w^3} \\ &-\frac{m^2}{4w^3}(aa'''' + 4a'a''' + 3a''^2) \\ &+\frac{m^4}{4w^5}(8a'^4 + 49aa'^2a'' + 8a^2a''^2 + 11a^2a'a''') \\ &-\frac{155m^6}{8w^7}(a^2a'^4 + a^3a'^2a'') + \frac{175m^8a^4a'^4}{8w^9}. \end{aligned} \tag{3.56}$$

Let us now consider $W_k^{(1)'^2}$, $W_k^{(1)''}$, $W_k^{(1)}$, and $W_k^{(1)^2}$ for the second iterated WKB solution.

$$W_k^{(1)'} = \frac{m^2aa'}{w}(1+\tau) \tag{3.57}$$

where

$$\tau = -\frac{1}{4w^2aa'}(3a'a'' + aa''') + \frac{2m^2}{w^4aa'}(aa'^3 + a^2a'a'') - \frac{25m^4a^2a'^2}{8w^6}, \tag{3.58}$$

Again using the binomial expansion, one obtains

$$W_k^{(1)'^2} = \frac{m^4a^2a'^2}{w^2}(1 + 2\tau + ...). \tag{3.59}$$

Thus,

$$W_k^{(1)'^2} = \frac{m^4 a^2 a'^2}{w^2} - \frac{m^4}{2w^4}(3aa'^2 a'' + a^2 a' a''') + \frac{4m^6}{w^6}(a^2 a'^4 + a^3 a'^2 a'') - \frac{50 m^8 a^4 a'^4}{8w^8} \tag{3.60}$$

Now, let us turn to Eq.(3.50) for the calculation of $\frac{1}{W_k^{(1)^2}}$ and $\frac{1}{W_k^{(1)}}$, and again use the binomial expansion to obtain

$$\begin{aligned} \frac{1}{W_k^{(1)^2}} &= \frac{1}{w^2}(1 - \epsilon + \ldots\ldots\ldots\ldots), \\ \frac{1}{W_k^{(1)}} &= \frac{1}{w}(1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 + \ldots\ldots\ldots\ldots). \end{aligned} \tag{3.61}$$

Thus,

$$\frac{1}{W_k^{(1)^2}} = \frac{1}{w^2} + \frac{m^2(a'^2 + aa'')}{2w^6} - \frac{5m^4 a^2 a'^2}{4w^8}, \tag{3.62}$$

$$\frac{1}{W_k^{(1)}} = \frac{1}{w} + \frac{m^2(a'^2 + aa'')}{4w^5} - \frac{5m^4 a^2 a'^2}{8w^7}. \tag{3.63}$$

Subsitute Eqs. (3.56), (3.60),(3.62) and (3.63) into Eq.(3.54), obtaining second iterated WKB solution as

$$\begin{aligned} W_k^{(2)^2} = \; & w^2 - \frac{m^2}{2w^2}(a'^2 + aa'') + \frac{5m^4 a^2 a'^2}{4w^4} + \frac{m^2}{8w^4}(aa'''' + 4a'a''' + 3a''^2) \\ & - \frac{m^4}{8w^6}(9a'^4 + 60aa'^2 a'' + 9a^2 a''^2 + 14a^2 a' a''') \\ & + \frac{216m^6}{16w^8}(a^2 a'^4 + a^3 a'^2 a'') - \frac{540 m^8 a^4 a'^4}{32 w^{10}} + [\text{contain higher} \\ & \text{than fourth order terms}]. \end{aligned} \tag{3.64}$$

Having determined $W_k^{(1)}$ to fourth order from (3.53), a second iteration yields $W_k^{(2)}$ to fourth order. Further iterations only yield terms of higher adiabatic order so one obtains the following result (which could also have been obtained by using binomial expansion) [10,14].

$$W_k^{(2)} = w - \frac{m^2}{4w^3}(a'^2 + aa'') + \frac{5m^4a^2a'^2}{8w^5} + \frac{m^2}{16w^5}(aa'''' + 4a'a''' + 3a''^2)$$
$$-\frac{m^4}{32w^7}(19a'^4 + 122aa'^2a'' + 19a^2a''^2 + 28a^2a'a''')$$
$$+\frac{221m^6}{32w^9}(a^2a'^4 + a^3a'^2a'') - \frac{1105m^8a^4a'^4}{128w^{11}}. \qquad (3.65)$$

Now we return to Eq. (3.42), and calculate $|\Psi_k|^2$, which is given as

$$|\Psi_k|^2 = \Psi_k\Psi_k^*. \qquad (3.66)$$

Here, star denotes complex conjugate. Using the Eq. (3.44), one obtains

$$|\Psi_k|^2 = \frac{1}{2W_k^{(2)}}. \qquad (3.67)$$

We rewrite Eq.(3.64) as

$$W_k^{(2)^2} = w^2(1+\mu), \qquad (3.68)$$

where

$$\mu = -\frac{m^2}{2w^4}(a'^2 + aa'') + \frac{5m^4a^2a'^2}{4w^6} + \frac{m^2}{8w^6}(aa'''' + 4a'a''' + 3a''^2)$$
$$-\frac{m^4}{8w^8}(9a'^4 + 60aa'^2a'' + 9a^2a''^2 + 14a^2a'a''')$$
$$+\frac{216m^6}{16w^{10}}(a^2a'^4 + a^3a'^2a'') - \frac{540m^8a^4a'^4}{32w^{12}}. \qquad (3.69)$$

and use the binomial expansion in Eq. (3.68) ,we obtain

$$\frac{1}{2W_k^{(2)}} = \frac{1}{2w}(1 - \frac{1}{2}\mu + \frac{3}{8}\mu^2 + ...). \qquad (3.70)$$

Thus using Eq. (3.69) in Eq. (3.70), we obtain

$$
\begin{aligned}
\mid \Psi_k \mid^2 \; = \; & \frac{1}{2w} + \frac{m^2}{8w^5}(a'^2 + aa'') - \frac{5m^4a^2a'^2}{16w^7} \\
& - \frac{m^2}{32w^7}(aa'''' + 4a'a''' + 3a''^2) \\
& + \frac{m^4}{64w^9}(21a'^4 + 126aa'^2a'' + 21a^2a''^2 + 28a^2a'a''') \\
& - \frac{231m^6}{64w^{11}}(a^2a'^4 + a^3a'^2a'') + \frac{1155m^8a^4a'^4}{256w^{13}}.
\end{aligned} \tag{3.71}
$$

The sum in equation (3.42) is replaced by an integral over k is given as

$$
<0 \mid T \mid 0>_A = \frac{1}{2\pi^2a^2}\int_0^\infty k^2m^2 \mid \Psi_k \mid^2 dk. \tag{3.72}
$$

Substitution of the fourth-order expression Eq. (3.71) into Eq. (3.72) gives the following adiabatic vacuum contributions:

$$
\begin{aligned}
<0 \mid T \mid 0>_A \; = \; & (4\pi^2a^4)^{-1}\int_0^\infty k^2dk[\frac{m^2a^2}{w} + \frac{m^4a^4}{4w^5}(\frac{a''}{a} + \frac{a'^2}{a^2}) \\
& - \frac{5m^6a^6a'^2}{8w^7a^2} - \frac{m^4a^4}{16w^7}(\frac{a''''}{a} + \frac{4a'a'''}{a^2} + \frac{3a''^2}{a^2}) \\
& + \frac{m^6a^6}{32w^9}(\frac{28a'''a'}{a^2} + \frac{126a''a'^2}{a^3} + \frac{21a''^2}{a^2} + \frac{21a'^4}{a^4}) \\
& - \frac{231m^8a^8}{32w^{11}}(\frac{a''a'^2}{a^3} + \frac{a'^4}{a^4}) + \frac{1155m^{10}a^{10}a'^4}{128w^{13}a^4}].
\end{aligned} \tag{3.73}
$$

where $w^2 = k^2 + m^2a^2$, and $\mid 0>_A$ denotes adiabatic vacuum state. In equation (3.73) all of the integrals, except the first one, could be taken by using the formula

$$
\int_0^\infty \frac{k^2dk}{(k^2+m^2a^2)^{p/2}} = \frac{1}{2(ma)^{p-3}}\frac{\Gamma(3/2)\Gamma(\frac{p-3}{2})}{\Gamma(p/2)}. \tag{3.74}
$$

In the limit as $m \to 0$ they all contribute as finite terms. On the other hand, the first integral in Eq. (3.73) diverges as $m \to 0$, which we could get rid of by a cosmological constant renormalization [11] hence will not be discussed further. Hence, Eq. (3.73) becomes

$$
\begin{aligned}
<0\,|\,T\,|\,0>_A &= \frac{6}{2880\pi^2}[-\frac{a''''}{a^5}+\frac{4a'''a'}{a^6}+\frac{3a''^2}{a^6} \\
&- \frac{8a''a'^2}{a^7}+\frac{2a'^4}{a^8}].
\end{aligned} \tag{3.75}
$$

Now the complete renormalized trace of the massless conformal scalar field in an expanding closed Friedmann universe is given as

$$
\begin{aligned}
<0\,|\,T\,|\,0>_{\text{ren}} &= 0-<0\,|\,T\,|\,0>_A \\
&= \frac{6}{2880\pi^2}[\frac{a''''}{a^5}-\frac{4a'''a'}{a^6}-\frac{3a''^2}{a^6} \\
&+ \frac{8a''a'^2}{a^7}-\frac{2a'^4}{a^8}].
\end{aligned} \tag{3.76}
$$

where "ren" denotes renormalized. This is identical to the result obtained by Anderson and Parker and is known as the trace anamoly [14]. As we have mentioned the trace anamoly vanishes for static models. As far as the other components of the renormalized energy-momentum tensor is concerned, in general, the renormalized energy-momentum tensor will have the following components [1]:

$$
\begin{aligned}
&< T_1^1 >_{\text{ren}}=< T_2^2 >_{\text{ren}}=< T_3^3 >_{\text{ren}}, \\
&< T_\nu^\mu >_{\text{ren}}= 0, \qquad \text{where} \quad \mu \neq \nu
\end{aligned} \tag{3.77}
$$

Because of the symmetry of the Robertson-Walker universes, the energy-momentum tensor will have only two independent components, the 00 component, T_{00} and 11 component T_{11}. The nonzero components of the renormalized energy-momentum tensor could be obtained from the trace anomoly by using the fact that the renormalized energy-momentum tensor is conserved [13] i.e.

$$
<0\,|\,T_\mu^\nu\,|\,0>_{\text{ren};\nu}= 0, \tag{3.78}
$$

The conservation condition may be written as

$$
T^0_{0,0}+\frac{a'}{a}T^0_0-3\frac{a'}{a}T^1_1=0. \tag{3.79}
$$

But

$$T^{\alpha}_{\alpha} = T^{0}_{0} + 3T^{1}_{1}. \tag{3.80}$$

Substituting Eq. (3.80) into Eq. (3.79) , one obtains

$$T^{0}_{0,0} + 2\frac{a'}{a}T^{0}_{0} = \frac{a'}{a}T^{\alpha}_{\alpha}. \tag{3.81}$$

Using

$$\begin{aligned} T^{0}_{0} &= g^{00}T_{00}, \\ T^{0}_{0,0} &= g^{00}T_{00,0}, \end{aligned} \tag{3.82}$$

we find

$$T_{00,0} + 2\frac{a'}{a}T_{00} = a'aT^{\alpha}_{\alpha}, \tag{3.83}$$

or

$$\frac{d}{d\eta}(a^{2}T_{00}) = a'a^{3}T^{\alpha}_{\alpha}, \tag{3.84}$$

and so [13]

$$\frac{d}{d\eta}(a^{4} < 0 \mid T^{0}_{0} \mid 0 >_{\rm ren}) = a'a^{3} < 0 \mid T \mid 0 >_{\rm ren} . \tag{3.85}$$

We may bring Eq. (3.76) into the form

$$< 0 \mid T \mid 0 >_{\rm ren} = \frac{6}{2880\pi^{2}a'a^{3}}\frac{d}{d\eta}\left(\frac{a'''a'}{a^{2}} - \frac{a''^{2}}{2a^{2}} - \frac{2a''a'^{2}}{a^{3}} + \frac{a'^{4}}{2a^{4}}\right). \tag{3.86}$$

Inserting (3.86) into (3.85) and rearranging, we obtain

$$\frac{d}{d\eta}(a^{4} < 0 \mid T^{0}_{0} \mid 0 >_{\rm ren}) = \frac{6}{2880\pi^{2}}\frac{d}{d\eta}\left(\frac{a'''a'}{a^{2}} - \frac{a''^{2}}{2a^{2}} - \frac{2a''a'^{2}}{a^{3}} + \frac{a'^{4}}{2a^{4}}\right). \tag{3.87}$$

Now we can determine the complete renormalized energy-momentum tensor for the massless conformal scalar field in an expanding closed Friedmann model (valid up to adiabatic order four). Equation (3.87) could be integrated to yield [13,14,18]

$$< 0 \mid T^0_0 \mid 0 >\text{ren}= \frac{6}{2880\pi^2 a^4}\left(\frac{a'''a'}{a^2} - \frac{a''^2}{2a^2} - \frac{2a''a'^2}{a^3} + \frac{a'^4}{2a^4} + C_0\right). \qquad (3.88)$$

Where, C_0 is an integration constant to be determined from the static case. Using the result of section II (2.46) we see that for closed Friedmann models C_0 is actually 1. These expressions (2.46) and (3.88) are in complete agreement with the results obtained by using dimensional regularization and point splitting [1]. Thus one obtains both the trace anomaly and the Casimir energy. The remaining nonzero component of $T_{\mu\nu}$ is given as

$$< 0 \mid T^1_1 \mid 0 >\text{ren}= \frac{1}{3}(< 0 \mid T \mid 0 >\text{ren} - < 0 \mid T^0_0 \mid 0 >\text{ren}), \qquad (3.89)$$

hence

$$< 0 \mid T^1_1 \mid 0 >\text{ren}= \frac{1}{1440\pi^2 a^4}\left(\frac{a''''}{a} - \frac{5a'a'''}{a^2} - \frac{5a''^2}{2a^2} + \frac{10a''a'^2}{a^3} - \frac{5a'^4}{2a^4} - 1\right). \quad (3.90)$$

Chapter 4

DISCUSSION

In this thesis we considered quantum vacuum energy of the massless conformal scalar field in curved background Einstein and closed Friedmann geometries. Calculations we have presented are based on the approximate methods of quantum field theory on curved background geometry i.e. gravity is still considered as a classical field [1]. However, in the early days of the development of quantum electrodynamics we have experienced that some of the results that emerge from such an approximate theory may still survive and remain to be valid even in the exact theory. We preferred to work with the massless conformal scalar field since it is the simplest yet closest analogy we have to electromagnetic theory [20]. We concentrated on the renormalization of the vacuum expectation value of the energy momentum tensor $< 0 \mid T_{\mu\nu} \mid 0 >$. This expression contains divergences and we discussed how to extract finite meaningfull quantities from these divergent expressions.

In section II we reproduced Ford's result for the massless conformal scalar field in an Einstein universe by considering mode sums and renormalized the divergent vacuum energy by introducing a cutoff function. This result is in agreement with the results obtained by dimensional regularization, zeta function renormalization and covariant point splitting method [1].

In section III following Anderson and Parker [14] we considered slowly expanding Einstein geometries (closed Friedmann models) and by using Adiabatic regularization we reproduced the trace anomaly. By using the renormalized trace and the fact that the renormalized $< 0 \mid T_{\mu\nu} \mid 0 >_{\rm ren}$ be conserved we obtained all the other nonzero components of $< 0 \mid T_{\mu\nu} \mid 0 >_{\rm ren}$. Again, these results agree with those obtained by dimensional regularization and covariant point splitting.

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