A REVIEW OF EADY MODEL

A MASTER'S THESIS

in

Physics

Middle East Technical University

By

EROL, Cemil Berin

September 1991

15467

Aproval of the Graduate School of Natural and Applied Sciences.

Prof.Dr. O. Alpay Ankara
Director

I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science.

Prof.Dr. Şinasi Ellialtıoğlu
Chairman of the Department

We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Assoc.Prof.Dr. Davut Köşker
Supervisor

Examining Committee in Charge:

Prof.Dr. Cevdet Tezcan(Chairman) ....

Assoc.Prof.Dr. Davut Köşker ....

Assist.Prof.Dr. Erdoğan Apaydın ....

ABSTRACT

A REVIEW OF EADY MODEL

EROL, Cemil Berin

M.S. in Physics

Supervisor: Assoc.Prof.Dr. Davut Köşker

September 1991, 34 pages

Within the limitations of Eady model, the hydrodynamical properties of a baroclinic flow is examined. The model equations are linearized about a prespecified basic state onto which a large scale perturbation field of small amplitude is superimposed. The solutions of the resulting differential equation show that such a basic flow is unstable and thus there are amplifying and decaying perturbation modes. The growth rate curves, calculation of the wavelengths of the unstable perturbation waves and their structural analysis demonstrate that the growing waves have physical behaviour very similar to the observed cyclone waves in midlatitudes.

Keywords :Hydrodynamics, Baroclinic Instability, Basic Uniform Flow Field, Amplifying and Decaying Perturbation Waves, Cyclone Waves, Eady Model

Science Code :404.01.01

ÖZET

EADY MODELİ' NİN İNCELENMESİ

EROL, Cemil Berin

Yüksek Lisans Tezi, Fizik Bölümü

Tez Yöneticisi: Assoc.Prof.Dr. Davut Köşker

Eylül 1991, 34 sahife

Baroklinik bir akışın hidrodinamik özellikleri Eady modeli sınırları içinde incelenmiştir. Önceden belirlenen bir temel durum etrafında liniarize edilen model denklemleri seti üzerine genliği küçük olan, büyük ölçekli bir ek-katkı alanı bindirilmiştir. Elde edilen diferansiyel denklemin çözümleri, temel akış alanının kararsızlığını, büyüyen ve bozunan ek-katkı dalgalarının varlığını gösteriyor. Büyüyen dalgaların büyüme oran eğrileri, dalgaboylarının hesaplamaları ve yapısal analizleri, bu dalgaların fiziksel davranışları bakımından orta enlemlerde gözlenen siklon dalgalarına çok benzediklerini ortaya koyuyor.

Anahtar Kelimeler :Hidrodinamik, Baroklinik Kararsızlık, Temel Düzgün Akış Alanı, Büyüyen ve Bozunan Ekkatkı Dalgaları, Siklon Dalgaları, Eady Modeli

Bilim Sayısal Kodu :404.01.01

ACKNOWLEDGEMENTS

I am very grateful to Assoc.Prof.Dr. Davut Köşker for suggesting the subject of this thesis and for his invaluable aid and guidance throughout the work.

I would like to thank Mr. Sinan Kaan Yerli for his help for drawing graphs in Chapter 3.

Finally, I also thank my family for their patience and support throughout this work.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

Chapter 1

INTRODUCTION

1.1 Baroclinic Instability

As is well known, various instability modes are exhibited by the earth's atmosphere. But the most important instability mechanism in the middle latitudes is the baroclinic instability. This kind of atmospheric instability can take place whenever a mean state, possessing vertical shear, is perturbed on a sufficiently large scale. Any disturbance which is superposed on such a basic flow will start amplifying, deriving its energy from the mean flow. The subsequent growth and development of such disturbances continue until they reach their mature stage as fully developed baroclinic waves. These waves are the observed midlatitudinal cyclone waves and are mainly responsible for the poleward transport of sensible heat and momentum. Thus their role in the maintenance of the general circulation of the atmosphere, particularly in the middle and high latitudes against frictional dissipation is a fundamental one.

The criteria for the baroclinic instability and the details of the developing wave structure depend on both internal and external parameters. The internal ones are the vertical shear of the velocity field, its meridional variation, the lateral and vertical variations of the static stability, the variation of the Coriolis term with latitude and the zonal length scales of the waves. The dispersion relations, the stability conditions and the three dimensional structures of baroclinic waves are all sensitive to any changes in these internal parameters. Also involved, on the other hand, are external factors such as frictional dissipation, diabatic heating and cooling and orography of the earth's surface.

1.2 Historical Background

The first important theoretical studies of baroclinic instability of atmospheric flows were those of Charney [1] and Eady [2]. In their classical papers, they investigated the dynamics of amplifying waves on a zonal current. Charney included the beta effect and derived instability criteria and showed that the rate of growth of baroclinic waves depends on the vertical shear of the velocity field, the static stability, the wave-number and the latitude. His results suggested that these unstable waves are similar in many respects to the observed disturbances in the atmosphere. The speed of propagation is generally towards the east and is approximately equal to the mean speed of the zonal current. The waves exhibit a westward tilt with height.

However, Eady model did not include the beta effect and is more elegant, revealing that the f-plane approximation is the only requirement for the baroclinic instability to occur. By solving his quasi-geostrophic model equations, Eady showed that a steady baroclinic atmosphere is hydrodynamically unstable. In all the cases examined, he found that there exists one particular wave-number which grows faster than any other. He suggested that by a process analogous to natural selection, this component would tend to dominate the observed wave spectrum. He identified his wave structures as the ideal forms of the observed cyclone waves of middle and high latitudes.

Kuo [3] also investigated the stability properties of baroclinic disturbances. He found out that when friction is neglected, all waves shorter than a critical wavelength are unstable and there exists a most stable wave. The critical wavelength and the zonal length scale of the wave of maximum instability both depend on the vertical shear of the zonal current and static stability. Moreover, he showed that there are two entirely different types of disturbances in the atmosphere, the very long planetary waves and the very short surface waves which occupy, respectively, the upper and the lower part of the troposphere.

Green [4], on the other hand, studied the classical stability analysis of Charney and Eady and, in particular, examined the effects of the upper boundary conditions on the analytical solutions. He found out that with the removal upper lid in the Eady model and of the beta effect in the Charney model, the problems become identical and all waves are neutral. Thus, he concluded that the motion becomes unstable by the action of these constraints. He also showed that, in the case of Eady model, there are two solutions corresponding to each wavelength. For long waves, this pair of solutions corresponds to amplifying and decaying waves, travelling with the mean speed of the undisturbed flow.

Pedlosky [5] investigated the stability properties of two-layer systems and proved that there exist upper and lower limits to, both the speed of propagation and the growth rate of the unstable waves.

Wiin-Nielsen [6] also studied some of the aspects of the problem of baroclinic instability. He computed the instability criteria for a two-level model of the large-scale motion and compared for the results of a quasi-geostrophic model with constant static stability. He deduced that the effect of the variable static stability is to shift the wavelength of maximum instability toward shorter waves.

Using a quasi-geostrophic model of high spatial resolution, Brown [7] examined certain planetary zonal flows and their associated hydrodynamic instability properties. He determined that some zonal flows which consist of both baroclinic and barotropic components are unstable with respect to large-scale quasi-geostrophic perturbations.

Multi-level numerical atmospheric models have also been constructed for the purpose of studying baroclinic waves. The spectral method has been used to integrate the model equations as well as the finite-difference method. In particular, Simmons and Hoskins [14], studied the growth of a baroclinic perturbation on a mid-latitude jet using such models. Their main concern was to compare the results of the finite-difference formulation and the spectral method. Their conclusions can be briefly summarized as follows: As far as the structural details of baroclinic waves are concerned, the results of the two different types of models are virtually the same. The spectral integrations, however, require less computer time storage and also predict the lower order harmonics; i.e the larger scale features of the numerical solutions with greater accuracy. On the other hand, the finite-difference model is superior in describing the smaller-scale features of the the synoptic scale wave phenomena.

Simmons and Hoskins [8] determined the growth rate, phase speed, structure and transfer of normal modes of the primitive and quasi- geostrophic equations by applying an initial value technique to global non-linear atmospheric models. Many properties of the unstable modes are found to be the same as in simpler models of baroclinic instability. They also stated that spherical geometry has a significant effect on the location of disturbances, particularly those of low zonal wave- number.

Blumen [9] studied the physical process for short-wave baroclinic instability (zonal wave-number > 10) by using a two-layer Eady model. He was able to show that how the short- and long-wave baroclinic instabilities depend on the relative layer depths as well as on the jump in static stability between the two layers.

Blumen [10] examined the nonlinear evolution of unstable two- dimensional Eady waves by means of a two-layer version of the Hoskins and Bretherton model [19]. In this model, the upper layer is characterized by a higher static stability than the lower layer. He realized two types of solutions: the relatively long-wave solution has a vertical structure that extends throughout the depth of the fluid and is the counterpart of the solution for a single layer system, while the shorter wave is essentially confined to the lower fluid layer.

Mechoso [11] investigated that in a two-layer quasi-geostrophic model with boundaries sloping perpendicular to the basic flow, the ratio of the slopes of the bottom and the top of the interface between the fluid layers in the basic state are important parameters in the expression of the growth rate of unstable waves. He also showed that when Eady model is extented to include sloping bottom and top boundaries, the growth rates of unstable waves depend on the ratios of the slopes of the bottom and the top to that of the isentropes of the basic state.

Farrell [12] examined the growth of perturbations in a baroclinic flow as an initial value problem. He assumed that the perturbations have infinite horizontal extent and a fixed wave-number in a short time limit. He also concluded that the vertical structure of perturbations emerges as an important influence on initial growth and showed physically realistic disturbances grow to amplitudes where nonlinear effects are important before obtaining normal mode form.

Branscome [13] studied the classical Charney baroclinic stability problem through perturbation techniques in the short-wave limit, near the first neutral curve separating Charney and Green modes, and near the second neutral curve separating long and short Green modes. He also represented structures of heat potential vorticity fluxes by approximate solutions and examined their dependence on wave-number.

Bannon [15] seeked to study the impact of the geostrophic momentum approximation on the linear Eady baroclinic instability problem through comparison with quasi-geostrophic and nongeostrophic theory. He solved the linear Eady model of baroclinic instability with the geostrophic momentum approximation analytically in physical space and showed that it is identical to linear three-dimensional semigeostrophic theory. He found out that both the growth rates and the wave-number of the short-wave cut-off are larger than those predicted by quasi- geostrophic theory. Morever, he stated that this behaviour arises because the effective static stability is reduced in the geostrophic momentum case. He also analyzed the effect of geostrophy for the meridional and for the zonal components of the perturbation wind field.

Nakamura and Held [16] examined the initial value problem for Eady model using a two-dimensional (x-z) primitive equation model. They said that as a result of enhanced potential vorticity in the frontal region that is mixed into the interior from the boundaries, diffusion prevents the frontal discontinuity from forming and the wave amplitude eventually stops growing and begins to oscillate. They analyzed this equilibration and presented scaling arguments to determine the three- dimensional flows for which this equilibration mechanism should be important.

1.3 Purpose Of The Present Study

In the present work, the dynamic instability of a baroclinic atmosphere on a rotating earth is examined. To avoid complexity, the analysis is made within the framework of the linear classical instability theory which is based on the study of Eady model. The solutions of the governing model equations show that a basic uniform shearing flow is dynamically unstable and there exist amplifying and decaying wave solutions which are superimposed on this basic flow field. The waves which would grow most rapidly would have wavelengths of 4000-5000km whose structure resembles, in basic respects, the structure of observed growing cyclone waves.

In chapter two, dynamical equations together with the approximations and assumptions made about the model atmospheric flow are presented. In particular, geostrophic approximation is discussed in some detail and the corresponding geostrophic equations are obtained. Next, the thermal wind equation is derived and the associated terms are physically interpreted. Finally, a qualitative discussion on the validity of the hydrostatic assumption for the model atmosphere is made.

In the following chapter, linear baroclinic instability theory is examined. First the quasi-geostrophic forms of the dynamical equations are given and then Eady model is described. The properties of developing baroclinic waves are studied and the dependence of the growth rates of these waves on the wave-number is determined. Attention is focused on the wave of maximum instability and structural characteristics of this most unstable wave are investigated.

Finally, in chapter four, a brief discussion of the important results of Eady model is presented. In particular, emphasis is laid on the properties of the most unstable wave.

Chapter 2

DYNAMICAL EQUATIONS

2.1 Dynamical Equations

Here we take as our starting point the adiabatic, inviscid primitive equations of motion written in general vector form with geometric height z as the vertical coordinate. Horizontal momentum equations

$$(\frac{\partial}{\partial t} + \vec{V}_h \cdot \vec{\nabla}_h)\vec{V}_h + w\frac{\partial \vec{V}_h}{\partial z} + f\ \hat{k} \times \vec{V}_h + \frac{1}{\rho}\vec{\nabla}_h p = 0 \tag{2.1}$$

Hydrostatic equation

$$\frac{\partial p}{\partial z} + \rho g = 0 \tag{2.2}$$

Continuity equation

$$(\frac{\partial}{\partial t} + \vec{V}_h \cdot \vec{\nabla}_h)\rho + \rho\vec{\nabla}_h \cdot \vec{V}_h + \frac{\partial}{\partial z}(\rho w) = 0 \tag{2.3}$$

Thermodynamic equation

$$(\frac{\partial}{\partial t} + \vec{V}_h \cdot \vec{\nabla}_h)\ln\theta + w\frac{\partial}{\partial z}(\ln\theta) = 0 \tag{2.4}$$

Equation of state

$$\ln\theta = \frac{1}{\gamma}\ln p - \ln\rho - \ln(Rp_s^{-(1-1/\gamma)}) \tag{2.5}$$

Here $\vec{V}_h$ is the two dimensional horizontal velocity vector, $\vec{V}_h = (u, v, 0)$, w is the vertical velocity, p the pressure, ρ the density and θ the potential temperature. t

is time, z geometric height and $\vec{\nabla}_h$ the two dimensional horizontal gradient operator. f is the Coriolis parameter, twice the vertical component of the earth's angular rotation, $\hat{k}$ is the unit vector along the z-direction, g the magnitude of the acceleration due to gravity, γ the ratio of the specific heat of air at constant pressure to that at constant volume, R the gas constant and p_s the mean reference surface pressure used in the definition of θ:

$$\theta = T(\frac{p}{p_s})^{-(1-1/\gamma)} \tag{2.6}$$

where T is temperature.

2.2 Geostrophic Approximation

For the atmospheric waves in middle latitudes, as a result of scaling analysis, it can be shown that the Coriolis force and the pressure gradient force are in approximate balance with each other and the remaining terms are comparatively small enough to be neglected. Therefore under normal atmospheric conditions, for mid-latitude systems called synoptic scale systems, horizontal momentum equations may be approximated by the following equation

$$f\ \hat{k} \times \vec{V}_h = -\frac{1}{\rho}\vec{\nabla}_h p \tag{2.7}$$

This approximation is referred to as the geostrophic approximation and the horizontal velocity field which satisfies (2.7) is called the geostrophic wind. (2.7) is a diagnostic expression being equivalent to the following two scalar equations

$$f\ v = \frac{1}{\rho}\frac{\partial p}{\partial x} \tag{2.8}$$

$$f\ u = -\frac{1}{\rho}\frac{\partial p}{\partial y} \tag{2.9}$$

which are also known as geostrophic equations. Thus, given the pressure distribution at any time, it is possible to use (2.8) and (2.9) to derive the geostrophic wind. However, the geostrophic equations contain no reference to time and therefore cannot be used to predict the evolution of the velocity field.

In order to obtain approximate prognostic equations which are to be used as prediction equations, it is necessary to retain the acceleration terms in (2.1) and represent the velocity field by its geostrophic value. The resulting approximate horizontal momentum equations are

$$\frac{du}{dt} - f\,v = -\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{2.10}$$

$$\frac{dv}{dt} + f\,u = -\frac{1}{\rho}\frac{\partial p}{\partial y} \tag{2.11}$$

and they form the basis of the quasi-geostrophic theory. Scale analysis of (2.10) and (2.11) for synoptic scale systems show that the acceleration terms are about an order of magnitude smaller than the Coriolis force and the pressure gradient force.

A convenient measure of the magnitude of the acceleration compared to the Coriolis force may be obtained by forming the ratio of the characteristic scales for the acceleration and the Coriolis force terms

$$\frac{U^2/L}{f_0 U} \tag{2.12}$$

This ratio is a nondimensional number called the Rossby number and is designated by

$$R_0 \equiv U/f_0 L \tag{2.13}$$

where f_0 is the value of the Coriolis parameter at the latitude where $\phi = 45^o$. Thus the smallness of the Rossby number is a measure of the validity of the geostrophic approximation.

2.3 The Thermal Wind Equation

Making use of the equation of state in the form $\rho = p/RT$, density ρ can be eliminated in the geostrophic expressions (2.8) and (2.9) and hydrostatic condition (2.2) and the following relations can be obtained

$$\frac{f\,v}{T} = R\frac{\partial \ln p}{\partial x}, \qquad \frac{f\,u}{T} = -R\frac{\partial \ln p}{\partial y}, \qquad \frac{g}{T} = -R\frac{\partial \ln p}{\partial z} \tag{2.14}$$

Now eliminating pressure p among these equations and rearranging the resulting terms one gets

$$\frac{\partial v}{\partial z} = \frac{g}{fT}\frac{\partial T}{\partial x} + \frac{v}{T}\frac{\partial T}{\partial z} \tag{2.15}$$

$$\frac{\partial u}{\partial z} = -\frac{g}{fT}\frac{\partial T}{\partial y} + \frac{u}{T}\frac{\partial T}{\partial z} \tag{2.16}$$

These are the thermal wind equations in differential form. The first terms on the right are the contribution of the horizontal temperature gradient and the second terms on the right are correction terms involving the slopes of the isobaric surface and vertical temperature gradient. The correction terms are indeed relatively small. This can be seen by dividing (2.16) by u so that an expression for the fractional rate of change of u with height is obtained. The correction term is then $(1/T)\partial T/\partial z$, which, even if the lapse rate is as high as the dry adiabatic, is about 4 percent per kilometer. The normal shear of west wind with height in the troposphere in middle latitudes is approximately 25 percent per kilometer. Thus it is apparent that the additional term in question does not usually make a major contribution to vertical wind shear, although there are times and places where it should be taken into account. Therefore, in order to simplify matters, the correction terms are neglected and the thermal wind equations reduce to the following forms

$$\frac{\partial v}{\partial z} = \frac{g}{f\,T}\frac{\partial T}{\partial x} \tag{2.17}$$

$$\frac{\partial u}{\partial z} = -\frac{g}{f\,T}\frac{\partial T}{\partial y} \tag{2.18}$$

These equations require that for v to increase with height temperature must increase to the east, and for u to increase with height temperature must increase to the south, in the Northern Hemisphere. (2.17) and (2.18) can be combined to yield the following vector equation

$$\frac{\partial \vec{V}}{\partial z} = -\frac{g}{fT}\hat{k} \times \vec{\nabla} T \tag{2.19}$$

which is known as the thermal wind equation. In vector form, the physical meaning of the thermal wind equation is much more clearer and may be restated as follows: The vertical shear of the geostrophic wind is a vector which lies parallel to the isotherms on a level surface with low temperatures on the left in the Northern Hemisphere, and low temperatures on the right in the Southern Hemisphere. The fact that actual westerly winds in middle latitudes normally increase in strength going upward through the troposphere is to be explained, therefore, as a result of the normal decrease of the temperature towards the poles in the troposphere.

2.4 The Hydrostatic Approximation

Since for the synoptic-scale systems in mid-latitudes, the vertical scales of atmospheric motions are much smaller than their horizontal scales, then the statement that the gravitational force in approximate balance with the vertical pressure gradient force term is a quite valid approximation. This means that the vertical acceleration which is a measure of the vertical net force experienced by each fluid particle, is about five orders of magnitude smaller than the acceleration due to the gravitational pull of the earth. Thus the vertical pressure gradient force almost exactly balances the gravitational force in virtually all cases. This means one needs to consider only the simple case of equilibrium between these two forces; that is

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = -g \tag{2.20}$$

The minus sign is necessary because $\partial p/\partial z$ is intrinsically negative and g is a positive number. (2.20) is called the hydrostatic equation, which is one of the best approximations in theoretical atmospheric physics. It may be also applied safely to all atmospheric phenomena except those, like the tornado, in which appreciable vertical accelerations may exist.

Chapter 3

EADY MODEL

3.1 The Quasi-Geostrophic Equations

Within the limits of quasi-geostrophic theory, it can be shown that for a quasi-geostrophic, nearly nondivergent, incompressible and dry atmospheric flow, the dynamical equations given in section 2.1 take the following forms :

Vorticity equation[1]

$$\frac{\partial \zeta}{\partial t} + \vec{V}_h \cdot \vec{\nabla}_h \zeta = f_0 \frac{\partial w}{\partial z} \tag{3.1}$$

Thermodynamic equation

$$\frac{\partial \phi}{\partial t} + \vec{V}_h \cdot \vec{\nabla}_h \phi + wB = 0 \tag{3.2}$$

Continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{3.3}$$

Equation of state

$$\ln \theta = \frac{1}{\gamma} \ln p - \ln \rho - \ln(R p_s^{-(1-1/\gamma)}) \tag{3.4}$$

where $\phi = ln\theta$, $\zeta = \hat{k} \cdot (\vec{\nabla}_h \times \vec{V}_h)$ the vertical component of the vorticity vector and $B = \partial\phi/\partial z$ is the static stability taken as constant. In deriving the above set of equations the approximations that have been made are all consistent with the condition

$$R_0 \ll 1 \tag{3.5}$$

[1]As shown in appendix A

and f_0 is taken as a constant.

3.2 Eady's Model Of An Amplifying Wave In A Baroclinic Atmosphere

The starting point of the theory is the specification of an initial basic state for the atmospheric flow, which is, in this case a uniform shearing flow $U_0(z)$ between rigid lids at $z = \pm H/2$ as shown in Fig.3.1. The flow is unbounded in the horizontal.

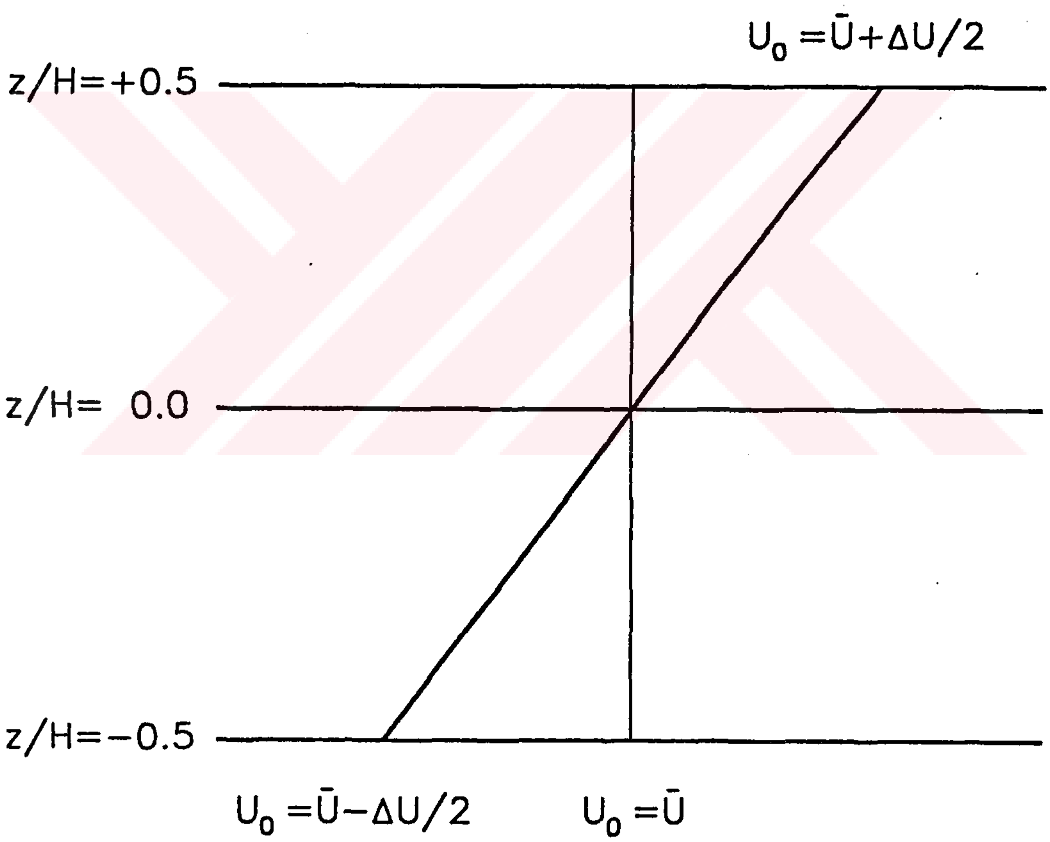

Figure 3.1. The vertical variation of the basic flow field

In order to satisfy the thermal wind equation (2.18) the temperature field must have a uniform gradient in the y-direction. Now writing any field variable, say Q in general, as the sum of its basic part $Q_0 = Q_0(y, z)$ and its perturbation component

$Q' = Q'(x,y,z,t)$; i.e. $Q = Q_0 + Q'$, the initial basic thermal state must have the following form

$$\ln \theta_0 = M + Ny + Jz \tag{3.6}$$

where M, N and J are dimensional constants. Thus the thermal wind equation[2] is

$$\frac{\partial U_0}{\partial z} = -\frac{g}{f_0}\frac{\partial \ln \theta_0}{\partial y} \tag{3.7}$$

Let u' and v' denote perturbations of the basic velocity field. Then (3.3) enables a perturbation stream function ψ' to be defined ; i.e.

$$\begin{aligned} u' &= -\frac{\partial \psi'}{\partial y} \\ v' &= \frac{\partial \psi'}{\partial x} \\ \zeta' &= \nabla^2 \psi' \end{aligned}$$

Also let θ' denote the perturbation in potential temperature θ. Then if the perturbation field is assumed to be approximately geostrophic ; i.e.

$$\frac{\partial u'}{\partial z} = -\frac{g}{f_0}\frac{\partial ln\theta'}{\partial y}$$

and

$$\frac{\partial v'}{\partial z} = \frac{g}{f_0}\frac{\partial ln\theta'}{\partial x}$$

Thus

$$\ln \theta' = \frac{f_0}{g}\frac{\partial \psi'}{\partial z} \tag{3.8}$$

The perturbed form of (3.2)[3] is

$$\frac{\partial \ln \theta}{\partial t} + U_0 \frac{\partial \ln \theta}{\partial x} + v\frac{\partial \ln \theta_0}{\partial y} + wB = 0 \tag{3.9}$$

---

[2] As shown in appendix B

[3] Hereafter the primes will be dropped

where $B \equiv \partial ln\theta_0/\partial z$. From (3.8) and (3.9) one can obtain

$$\frac{\partial w}{\partial z} = -\frac{f_0}{gB}[\frac{\partial}{\partial t} + U_0\frac{\partial}{\partial x}]\frac{\partial^2 \psi}{\partial z^2} \tag{3.10}$$

From (3.10) and (3.1), one can show that the perturbation stream function is seen to satisfy the linear partial differential equation

$$(\frac{\partial}{\partial t} + U_0\frac{\partial}{\partial x})(\nabla^2\psi + \frac{f_0^2}{gB}\frac{\partial^2\psi}{\partial z^2}) = 0 \tag{3.11}$$

The nature of the solutions for ψ are of the form

$$\psi_{k,l} = F(z)\exp[ik(x - ct)]\exp(ily)$$

where $c = c_r + ic_i$, in general, complex and $F(z)$ is, in general, a complex function of z; i.e. representing the perturbation as a wave of wavelength $2\pi/k$ in the x-direction, $2\pi/l$ in the y-direction and moving in the x-direction with phase velocity c_r ; ψ is also, of course, in general complex[4].

(3.11) then gives

$$(-ikc + ikU_0)[-(k^2 + l^2)F + \frac{f_0^2}{gB}F'']\exp[ik(x - ct)]\exp(ily) = 0$$

where prime denotes the total derivative with respect to z.

This can be satisfied only if

$$F'' - \nu^2 F = 0 \tag{3.12}$$

where $\nu^2 = gB(k^2 + l^2)/f_0^2$.

The general solution of (3.12) is

$$F = A\cosh\nu z + B\sinh\nu z \tag{3.13}$$

where A and B are arbitrary (complex) constants.

[4]The suffices k, l are now dropped.

To determine A and B it is necessary to use the boundary conditions in z. Specifying rigid boundaries at $z = \pm H/2$ implies

$$w = 0 \quad at \quad z = \frac{H}{2} \tag{3.14}$$

and

$$w = 0 \quad at \quad z = -\frac{H}{2} \tag{3.15}$$

To apply these conditions to determine A and B it is necessary to express w in terms of ψ. By using Fig.3.1, (3.8) and (3.9) one can easily find

$$w = -\frac{f_0}{gB}\left(\frac{\partial}{\partial t} + U_0\frac{\partial}{\partial x}\right)\frac{\partial \psi}{\partial z} + \frac{f_0}{gB}\frac{\partial \psi}{\partial x}\frac{\partial U_0}{\partial z} \tag{3.16}$$

$$w = -\frac{ikf_0}{gB}[(U_0 - c)F' - \frac{\Delta U}{H}F] \exp[ik(x - ct)] \exp(ily) \tag{3.17}$$

Applying the upper boundary condition one gets

$$A[(\overline{U} - c)\nu s' + \frac{\Delta U}{2}\nu s' - \frac{\Delta U}{H}c'] + B[(\overline{U} - c)\nu c' + \frac{\Delta U}{2}\nu c' - \frac{\Delta U}{H}s'] = 0 \tag{3.18}$$

and applying the lower boundary condition one obtains

$$A[-(\overline{U} - c)\nu s' + \frac{\Delta U}{2}\nu s' - \frac{\Delta U}{H}c'] + B[(\overline{U} - c)\nu c' - \frac{\Delta U}{2}\nu c' + \frac{\Delta U}{H}s'] = 0 \tag{3.19}$$

where $s' = sinh(\nu H/2)$ and $c' = cosh(\nu H/2)$. (3.18) and (3.19) imply

$$-\frac{A}{B} = \frac{G_2}{G_1} = \frac{G_4}{G_3} \tag{3.20}$$

where

$$\begin{aligned}
G_1 &= [(\overline{U} - c)\nu s' + \frac{\Delta U}{2}\nu s' - \frac{\Delta U}{H}c'] \\
G_2 &= [(\overline{U} - c)\nu c' + \frac{\Delta U}{2}\nu c' - \frac{\Delta U}{H}s'] \\
G_3 &= [-(\overline{U} - c)\nu s' + \frac{\Delta U}{2}\nu s' - \frac{\Delta U}{H}c'] \\
G_4 &= [(\overline{U} - c)\nu c' - \frac{\Delta U}{2}\nu c' + \frac{\Delta U}{H}s']
\end{aligned}$$

Hence

$$G_2G_3 = G_1G_4 \tag{3.21}$$

Multiplying out and collecting terms [5] gives the equation in c:

$$(\overline{U} - c)^2 = (\Delta U)^2 f(\chi) \tag{3.22}$$

where $\chi = \nu H$ and

$$f(\chi) = \frac{1}{\chi^2} - \frac{\coth \chi}{\chi} + \frac{1}{4} \tag{3.23}$$

$f(\chi)$ has a single root(for $\chi > 0$) at $\chi = \chi_c \simeq 2.4$. For $\chi \leq \chi_c$, $f(\chi) < 0$ as shown in Fig.3.2 and hence the right hand side of (3.22) is less than zero i.e. for $\chi \leq \chi_c$,

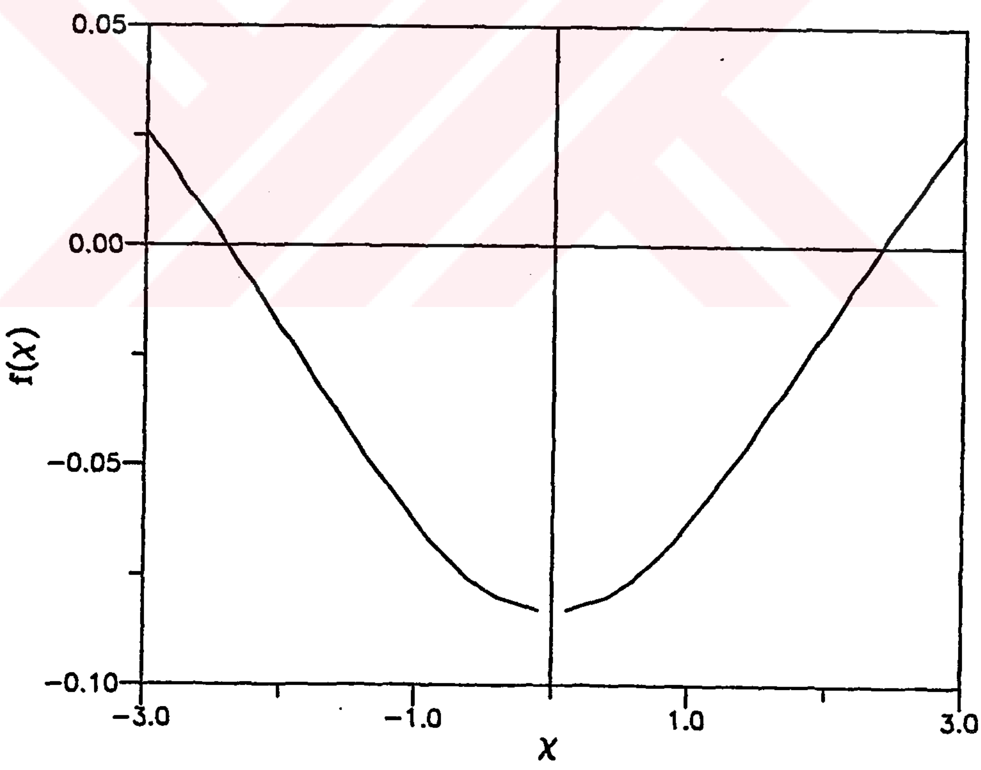

Figure 3.2. f(χ) versus χ

[5]As shown in appendix C

$$c = \overline{U} \pm i\Delta U\sqrt{-f(\chi)} \tag{3.24}$$

i.e. $c_r = \overline{U}$ and $c_i = \Delta U\sqrt{-f(\chi)}$, where $\chi = \nu H = (H/f_0)\sqrt{(k^2+l^2)gB}$. This provides the first basic result of the theory; for values of $\nu < \chi_c/H$, i.e. for sufficiently long wavelengths, wave perturbations will amplify; i.e the basic baroclinic flow field is hydrodynamically unstable.

The growth rate is kc_i i.e. $k\Delta U\sqrt{-f(\chi)}$ for $\chi < \chi_c$, and from (3.24), $k = \sqrt{\frac{f_0^2\chi^2}{gBH^2} - l^2}$. The maximum growth rate will thus occur for waves with $l = 0$ and with a value of k corresponding to the maximum value of $\chi\sqrt{-f(\chi)}$. The curve of $\chi\sqrt{-f(\chi)}$ is shown in Fig.3.3, and indicates a maximum value of 0.309 at $\chi = 1.61$. This

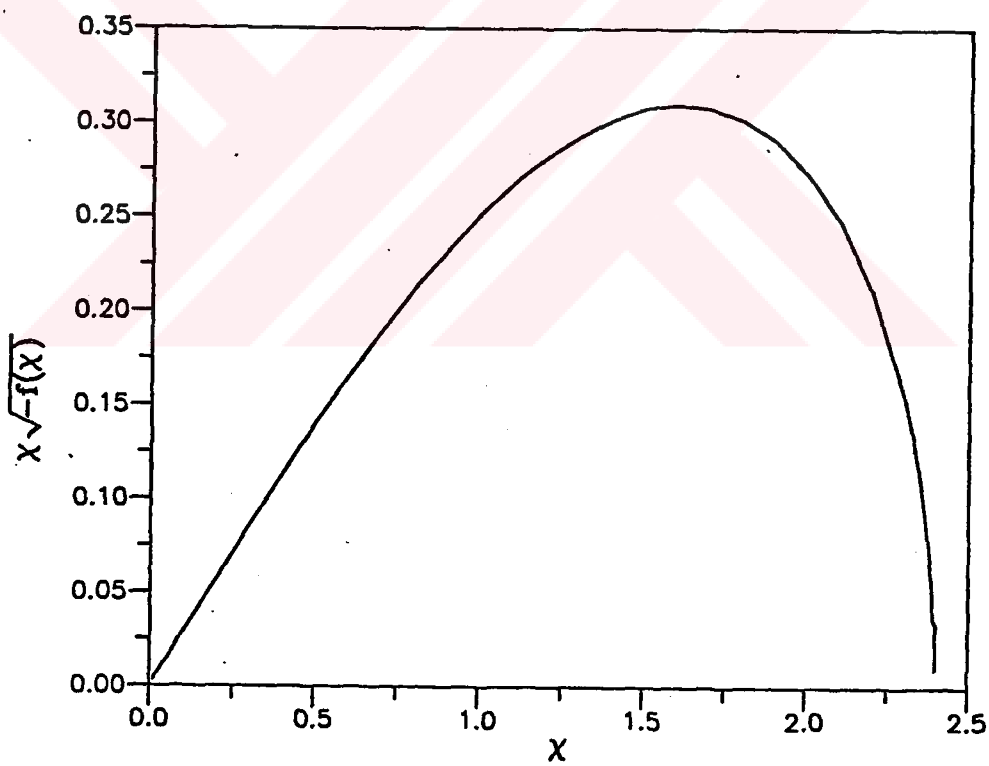

Figure 3.3. $\chi\sqrt{-f(\chi)}$ versus χ

states that in a baroclinic atmosphere, the waves which will be observed to dominate

Table 3.1. Wavelengths of some of the amplifying waves

χ	$k(m^{-1}) \times 10^{-6}$	$\lambda(m) \times 10^{3}$
1.50	1.50	4189
1.54	1.54	4080
1.58	1.58	3977
1.61	1.61	3902
1.64	1.64	3832
1.68	1.68	3740

the spectrum are those of the wavelength corresponding to maximum growth rate, as can be seen from Table 3.1.

Now the structure of the wave of maximum growth rate will be examined. From (3.13) and the assumed form of the stream function ψ, it follows that, for the wave of maximum growth [6]

$$\psi_1 = (A\cosh\nu_1 z + Bsinh\nu_1 z)\exp(k_1 c_{i1} t)\exp[ik_1(x - \overline{U}t)]$$

where A/B is given by (3.20) with $c = c_1$ substituted in G_1, G_2. This expression is, of course, a complex function of z. Carrying through this latter process gives

$$\frac{A}{B} = \frac{(0.309i - 0.805)c' + s'}{-[(0.309i - 0.805)s' + c']} \tag{3.25}$$

Hence as far as its dependence on z is concerned, ψ_1 is proportional to

$$[(0.309i - 0.805)c' + s']\cosh\nu_1 z + [-(0.309i - 0.805)s' + c']\sinh\nu_1 z$$

where $\nu_1 = \chi_1/H = 1.61/H$; i.e. to

$$0.805\cosh[1.61(\frac{z}{H} - 0.5)] + \sinh[1.61(\frac{z}{H} - 0.5)] - i0.309\cosh[1.61(\frac{z}{H} - 0.5)]$$

[6] Denoted by suffix 1

Writing $\psi_1 \propto f_R(z) + if_I(z)$, this expression can be converted to modulus argument (i.e. amplitude-phase) form by writing

$$\psi_1 \propto R(z) \exp i\varepsilon(z)$$

where

$$R(z) = \sqrt{f_R^2 + f_I^2}$$

and

$$\varepsilon(z) = \arctan[\frac{f_I(z)}{f_R(z)}]$$

Hence

$$\psi_1(x,z,t) = \Lambda R(z) \exp(k_1 c_{i1} t) \exp[ik_1(x - \overline{U}t + \varepsilon(z)/k_1)] \tag{3.26}$$

where Λ is a real constant. To interpret the expression (3.26) for ψ_1, there is no loss of generality in considering just the real part (the imaginary part is simply the same expression out of phase by $\pi/2$). Also the dependence on t is easy to interpret a simple translation of distribution of $\psi_1(x,z)$ with velocity $\overline{U}$ in the positive x-direction together with an amplification with a e-folding time $1/k_1 c_{i1}$.

Thus now the pattern at $t = 0$ can be constructed, i.e.

$$\psi_1(x,z,0) \propto R(z) \cos[k_1 x + \varepsilon(z)]$$

Clearly maxima of ψ_1 at each value of z occur where $k_1 x + \varepsilon(z) = 0, \pm 2\pi, \ldots$ and minima at $k_1 x + \varepsilon(z) = \pm\pi, \pm 3\pi, \ldots$. The loci of these maxima and minima for the phase variations of the stream function ψ_1 and potential temperature θ_1 in the vertical, for the fastest growing perturbation wave, are shown Fig.3.4.

Since ψ_1 is proportional to perturbation pressure, and the perturbation is assumed to be geostrophic, the perturbation wind field v may be deduced. Also, from (3.8), the temperature perturbation is proportional to $\partial\psi_1/\partial z$.

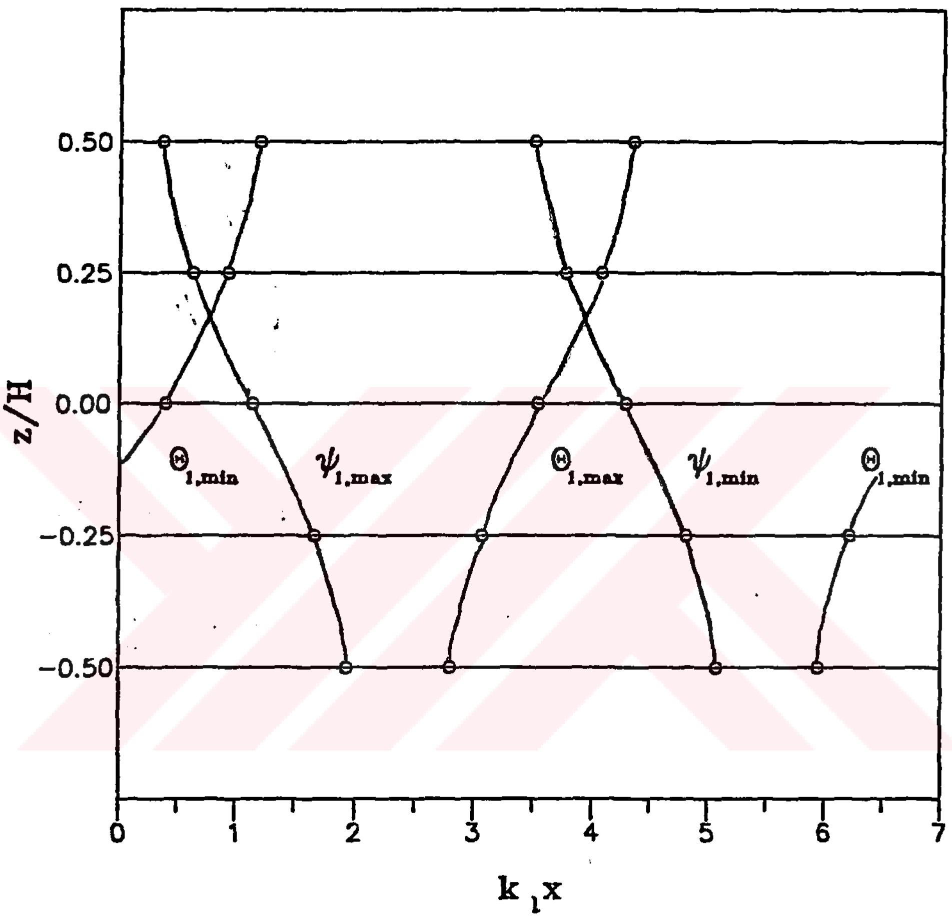

Figure 3.4. Phase variations of θ_1 and ψ_1 in the vertical

Finally, for w, (3.16) yields

$$w = -\frac{f_0}{gB}[(k_1 c_{i1} + ik_1(U_o - \overline{U}))(\frac{\partial \psi_1}{\partial z}) - \frac{\Delta U}{H}\frac{\partial \psi_1}{\partial x}] \qquad (3.27)$$

where

$$\frac{\partial \psi_1}{\partial z} = [\frac{1}{R}\frac{dR}{dz} + i\frac{d\varepsilon}{dz}]\psi_1$$

and

$$\frac{\partial \psi_1}{\partial x} = ik_1\psi_1$$

Considering only x at $z = 0$ where $U_0 = \overline{U}$,

$$w = -\frac{f_0}{gB}[k_1 c_{i1}(\frac{\partial \psi_1}{\partial z}) - \frac{\Delta U}{H}\frac{\partial \psi_1}{\partial x}]$$

$$\begin{aligned} \frac{dR}{dz} &= 0 \\ \frac{d\varepsilon}{dz} &\simeq \frac{\varepsilon(\frac{H}{2}) - \varepsilon(-\frac{H}{2})}{H} \simeq -\frac{\pi}{2H} \end{aligned}$$

Also $k_1 c_{i1} = k_1 \Delta U 0.309/1.61$. So

$$\begin{aligned} w(z=0) &\simeq -\frac{if_0 k_1}{gB}\frac{\Delta U}{H}[-\frac{\pi}{2} \times \frac{0.309}{1.61} - 1]\psi_1 \\ w(z=0) &\simeq 1.301\frac{if_0 k_1}{gB}\frac{\Delta U}{H}\psi_1 \end{aligned}$$

Thus w at $z = 0$ is proportional to ψ_1 in amplitude, but is displaced by $\pi/2$ to the west in phase. The variations of the amplitude of ψ_1 and w_1 in the vertical are also given in Fig.3.5.

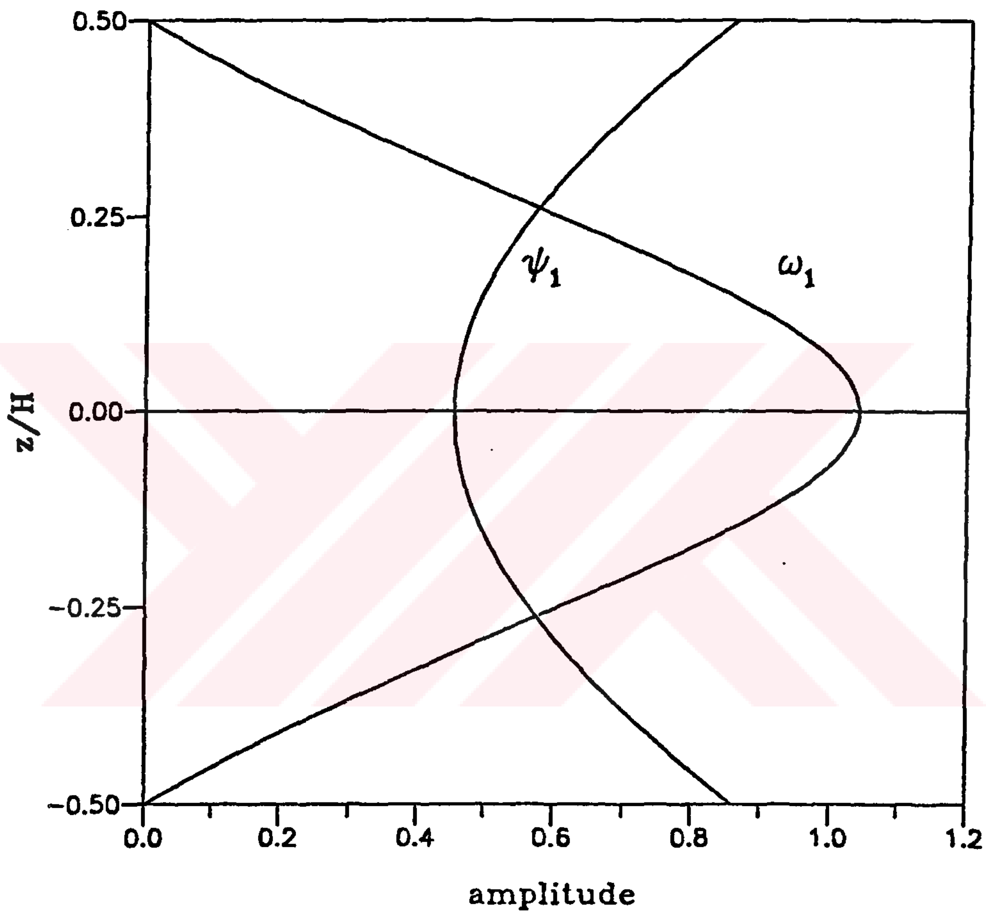

Figure 3.5. Amplitude variations of ψ_1 and w_1 in the vertical

Chapter 4

CONCLUSION

In this study, the hydrodynamical properties of a baroclinically unstable basic flow field is investigated. For that purpose, a linear instability analysis is made which is based on a detailed study of Eady model. In this model, a baroclinic basic flow having a uniform shear in the vertical is chosen. The model atmosphere is confined between two rigid lids at $z = -H/2$ and $z = H/2$ which correspond to the ground and tropopause levels in the earth's atmosphere respectively. Along the horizontal the flow is unbounded which means that periodicity is implicitly assumed in lateral directions. The basic flow is perturbed on a large scale by a weak perturbation field and the resulting interaction between the basic and the perturbation fields is examined.

From the normal mode solutions for the perturbation field, it is found out that the initial basic baroclinic flow is hydrodynamically unstable. This result reveals that sufficently long perturbation wavelengths will amplify in time. These normal modes also include the decaying perturbation waves.

A measure of the growth rates of the growing waves can be displayed by a plot of $\chi\sqrt{-f(\chi)}$ versus χ shown in Fig.3.3. This curve has a maximum value of 0.309 at $\chi = 1.61$ stating that the wave that will be observed dominating the spectrum is that of the wavelength corresponding to maximum growth rate. From Table 3.1. which exhibits the calculated wavelengths of some of the amplifying waves, the wave of maximum instability has a wavelength 3902km. The e-folding time for this wave $T_e \simeq 1 day$. As

χ increases, the wavelength of the unstable waves decreases, indicating a short- wave cut-off at about $\chi \simeq 2.40$. A further point of importance that should be noted is that the waves are stationary in the coordinate system used. This means that they move at the speed of the fluid at the mid-level. The calculations presented in this work differ slightly from those given in Eady's paper. This is because of the slight differences between the characteristic length scales used for some of the flow parameters. In this study, the typical values selected are $U \simeq 15m/sec$, $L \simeq 10^6m$, $N \simeq 10^{-2}sec^{-1}$ etc...

Furthermore a structural analysis for the most unstable wave is made. From the vertical phase variations given in Fig.3.4, it is observed that the pressure wave tilts westward with height whereas the thermal wave tilts eastward with height. This is consistent with the energetic theory being valid for baroclinic flows suggesting northward and upward transports of heat by these growing perturbation waves. For the same wave the amplitudes of the stream function and vertical velocity as a function of height are also shown in Fig.3.5. The stream function, which is proportional to the pressure wave, is largest on the boundaries but the vertical velocity is largest at the mid-level.

LIST OF REFERENCES

[1] Charney, J.G., Journal of Meteorology, 4, 135 (1947)

[2] Eady, E.T., Tellus, 1, 33 (1949)

[3] Kuo, H.L., Journal of Meteorology, 10, 235 (1953)

[4] Green, J.S.A., Quart.J.Roy.Meteor.Soc., 86, 237 (1960)

[5] Pedlosky, J., Tellus, 15, 20 (1963)

[6] Wiin-Nielsen, A., Tellus, 15, 1 (1963)

[7] Brown, A.John Jr., J.Atmos.Sci., 26, 352 (1969)

[8] Simmons, A.J. and Hoskins, B.J., J.Atmos.Sci., 33, 1454 (1976)

[9] Blumen,W., J.Atmos.Sci., 36, 1925 (1979)

[10] Blumen,W., J.Atmos.Sci., 37, 1984 (1980)

[11] Mechoso, C.R., J.Atmos.Sci., 37, 1393 (1980)

[12] Farrell, F., J.Atmos.Sci., 39, 1663 (1982)

[13] Branscome, L.E., J.Atmos.Sci., 40, 1393 (1983)

[14] Simmons, A.J. and Hoskins, B.J., Quart.J.Roy.Meteor.Soc.,101, 551 (1975)

[15] Bannon, P.R., J.Atmos.Sci., 46, 402 (1989)

[16] Nakamura, N. and Held, I.M., J.Atmos.Sci., 46, 3055 (1989)

[17] Holton, J.R., An Introduction to Dynamic Meteorology, London, Academic Press(1979), p.92.

[18] Holton, J.R., An Introduction to Dynamic Meteorology, London, Academic Press(1972), p.305.

[19] Hoskins, B.J. and Bretherton, F.B., J.Atmos.Sci., 29, 11 (1972)

Appendix A

Quasi-Geostrophic Vorticity Equation

The quasi-geostrophic vorticity equation for an inviscid flow is given by [17]

$$\frac{\partial}{\partial t}(\zeta + f) + \vec{V}_h \cdot \vec{\nabla}_h(\zeta + f) + f_0\vec{\nabla}_h \cdot \vec{V}_h = 0 \tag{A.1}$$

Since $f = f_0 \equiv constant$ then (A.1) reduces to the following form

$$(\frac{\partial}{\partial t} + \vec{V}_h \cdot \vec{\nabla}_h)\zeta + f_o\vec{\nabla}_h \cdot \vec{V}_h = 0 \tag{A.2}$$

Any horizontal velocity vector can be partitioned as [18]

$$\vec{V}_h = \vec{V}_{h\psi} + \vec{V}_{he} \tag{A.3}$$

such that

$$\vec{\nabla} \cdot \vec{V}_{h\psi} = 0 \qquad \vec{\nabla} \times \vec{V}_{he} = 0$$

Thus

$$\vec{\nabla}_h \cdot \vec{V}_h = \vec{\nabla}_h \cdot \vec{V}_{h\psi} + \vec{\nabla}_h \cdot \vec{V}_{he} \tag{A.4}$$

Now (A.2) becomes

$$(\frac{\partial}{\partial t} + \vec{V}_h \cdot \vec{\nabla}_h)\zeta + f_o\vec{\nabla}_h \cdot \vec{V}_{he} = 0 \tag{A.5}$$

Also since the flow is assumed to be nearly incompressible, the continuity equation

$$\frac{1}{\rho}\frac{d\rho}{dt} + \vec{\nabla} \cdot \vec{V} = 0$$

simplifies to the form

$$\vec{\nabla} \cdot \vec{V} = \vec{\nabla}_h \cdot \vec{V}_{he} + \frac{\partial w}{\partial z} \tag{A.6}$$

This means that (A.5) may now be expressed as

$$\frac{\partial \zeta}{\partial t} + \vec{V}_h \cdot \vec{\nabla}_h \zeta = f_o \frac{\partial w}{\partial z} \tag{A.7}$$

Appendix B

Relation Between Geostrophic Wind And Potential Temperature

Using geostrophic wind, ideal gas law, hydrostatic equations and the definitions of Brunt- Vaisalla frequency and potential temperature we can obtain the desired equations. By taking the derivative of the x-component of the geostrophic wind equation with respect to z one gets

$$\frac{\partial u}{\partial z} = -\frac{R}{f_0}[\frac{\partial T}{\partial z}\frac{\partial lnp}{\partial y} + T\frac{\partial}{\partial y}(\frac{\partial lnp}{\partial z})] \tag{B.1}$$

Substituting for $\partial lnp/\partial z$ in (B.1) from hydrostatic and ideal gas equations (B.1) becomes

$$\frac{\partial u}{\partial z} = -\frac{R}{f_0}[\frac{\partial T}{\partial z}\frac{\partial lnp}{\partial y} + \frac{g}{RT}\frac{\partial T}{\partial y}] \tag{B.2}$$

One can find $\partial T/\partial z$ and $\partial lnT/\partial y$, by taking the derivative of the potential temperature relation with respect to z and y respectively and by substituting them in (B.2)

$$\frac{\partial u}{\partial z} = -\frac{g}{f_0}\frac{\partial ln\theta}{\partial y} + \frac{N^2 u}{g} \tag{B.3}$$

As a result of scaling analysis using midlatitudinal observed charecteristic values of the field variables in (B.3), the second term on the right can be shown to be one order of magnitude smaller than the other two terms, so that it can be neglected. This means (B.3) now becomes

$$\frac{\partial u}{\partial z} = -\frac{g}{f_0}\frac{\partial ln\theta}{\partial y} \tag{B.4}$$

Similarly by taking the derivative of y-component of the geostrophic wind equation with respect to z one can get

$$\frac{\partial v}{\partial z} = \frac{R}{f_0}[\frac{\partial T}{\partial z}\frac{\partial lnp}{\partial x} + T\frac{\partial}{\partial x}(\frac{\partial lnp}{\partial z})] \quad (B.5)$$

Substituting for $\partial lnp/\partial z$ in (B.5) and by using hydrostatic and ideal gas equations (B.5) takes the form

$$\frac{\partial v}{\partial z} = \frac{R}{f_0}[\frac{\partial T}{\partial z}\frac{\partial lnp}{\partial x} + \frac{g}{RT}\frac{\partial T}{\partial x}] \quad (B.6)$$

To find $\partial T/\partial z$ and $\partial lnT/\partial x$, one takes the derivative of the potential temperature relation with respect to z and x respectively and substitutes them in (B.6)

$$\frac{\partial v}{\partial z} = \frac{g}{f_0}\frac{\partial ln\theta}{\partial x} + \frac{N^2 v}{g} \quad (B.7)$$

Again the result of a scaling analysis using midlatitudinal observed charecteristic values of the field variables in (B.7) the second term on the right is seen to be one order of magnitude smaller than the other terms, so it can also be neglected. Thus (B.7) takes the following form

$$\frac{\partial v}{\partial z} = \frac{g}{f_0}\frac{\partial ln\theta}{\partial x} \quad (B.8)$$

Appendix C

Derivation of Equation 3.22

Using the definitions of the G_1, G_2, G_3 and G_4, (3.21)

$$G_2G_3 = G_1G_4 \tag{C.1}$$

can be shown to be written explicitly by

$$\begin{aligned}&[(\overline{U}-c)\nu c' + \frac{\Delta U}{2}\nu c' - \frac{\Delta U}{H}s'] \ [\ -(\overline{U}-c)\nu s' + \frac{\Delta U}{2}\nu s' - \frac{\Delta U}{H}c'] \\ &= [(\overline{U}-c)\nu s' + \frac{\Delta U}{2}\nu s' - \frac{\Delta U}{H}c'] \ [\ (\overline{U}-c)\nu c' - \frac{\Delta U}{2}\nu c' + \frac{\Delta U}{H}s']\end{aligned} \tag{C.2}$$

Expanding the terms and rearranging them, one obtains

$$\begin{aligned}&-(\overline{U}-c)^2\nu^2 s'c' + (\overline{U}-c)\frac{\Delta U}{2}\nu^2 s'c' - (\overline{U}-c)\frac{\Delta U}{H}\nu c'^2 \\ &-(\overline{U}-c)\frac{\Delta U}{2}\nu^2 s'c' + (\frac{\Delta U}{2})^2\nu^2 s'c' - \frac{(\Delta U)^2}{2H}\nu c'^2 \\ &+(\overline{U}-c)\frac{\Delta U}{H}\nu s'^2 - \frac{(\Delta U)^2}{2H}\nu s'^2 + (\frac{\Delta U}{H})^2 s'c'\end{aligned}$$

$$\begin{aligned}&= (\overline{U}-c)^2\nu^2 s'c' - (\overline{U}-c)\frac{\Delta U}{2}\nu^2 s'c' + (\overline{U}-c)\frac{\Delta U}{H}\nu s'^2 \\ &+(\overline{U}-c)\frac{\Delta U}{2}\nu^2 s'c' - (\frac{\Delta U}{2})^2\nu^2 s'c' + \frac{(\Delta U)^2}{2H}\nu s'^2 \\ &-(\overline{U}-c)\frac{\Delta U}{H}\nu c'^2 + \frac{(\Delta U)^2}{2H}\nu c'^2 - (\frac{\Delta U}{H})^2 s'c'\end{aligned}$$

Cancelling out the crossing terms and collecting the remaining ones on the left hand side, one gets

$$(\overline{U}-c)^2\nu^2 s'c' - (\frac{\Delta U}{2})^2\nu^2 s'c' + \frac{(\Delta U)^2}{2H}\nu(s'^2 + c'^2) - (\frac{\Delta U}{H})^2 s'c' = 0 \tag{C.3}$$

Also, it can be shown that

$$\begin{aligned} c'^2 + s'^2 &= (c' + s')^2 - 2s'c' \\ (c' + s')^2 &= \exp \nu H \\ s'c' &= \frac{\exp \nu H - \exp - \nu H}{4} \end{aligned} \quad (C.4)$$

Thus after substituting (C.4) in (C.3) and dividing by $\nu^2\ s'c'$ and letting $\chi = \nu H$ one can finally obtain

$$(\overline{U} - c)^2 = (\Delta U)^2 [\frac{1}{\chi^2} - \frac{\coth \chi}{\chi} + \frac{1}{4}] \quad (C.5)$$