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Gauge Theories with Fuzzy Extra Dimensions and Noncommutative Vortices and Fluxons

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Abstract. A $U(2)$ Yang-Mills theory on the space $\mathcal{M} \times S_F^2$ is considered, where it is assumed that \mathcal{M} is an arbitrary noncommutative space and S_F^2 is a fuzzy sphere spontaneously generated from a noncommutative $U(\mathcal{N})$ Yang-Mills theory on \mathcal{M} coupled to a triplet of scalars in the adjoint of $U(\mathcal{N})$. $SU(2)$ -equivariant reduction of this theory leads to a noncommutative $U(1)$ gauge theory coupled adjointly to a set of scalar fields. The emergent model is studied on the Groenewald-Moyal plane \mathbb{R}_θ^2 and it is found that, in certain limits, it admits noncommutative, non-BPS vortex as well as fluxon solutions.

1. Introduction

Dimensional reduction of Yang-Mills theories over coset spaces of the form G/H has long been an interesting theme in modern particle physics. It was first formulated in a systematic manner by Forgacs and Manton [1]. The essential idea in this context can be clearly illustrated by considering a Yang-Mills theory over $\mathcal{M} \times G/H$, where \mathcal{M} is a given manifold. G has a natural action on its coset, and requiring the Yang-Mills gauge fields to be invariant under the G action up to a gauge transformation leads to the dimensional reduction of the theory after integrating over the coset space G/H . This technique is usually called “coset space dimensional reduction” (CSDR), and it has been widely used as a method in attempts to obtain the standard model on the Minkowski space M^4 starting from a Yang-Mills-Dirac theory on the higher dimensional space $M^4 \times G/H$; for a review on this topic see [2]. Recently, CSDR techniques have also been applied to Yang-Mills theories over $\mathbb{R}_\theta^{2d} \times S^2$ [3], where \mathbb{R}_θ^{2d} is the $2d$ dimensional Groenewald-Moyal space; a prime example of a noncommutative space. In this framework, Donaldson-Uhlenbeck-Yau (DUY) equations of a $U(2k)$ Yang-Mills theory have been reduced to a set of equations on \mathbb{R}_θ^{2d} whose solutions are given by BPS vortices on \mathbb{R}_θ^{2d} and the properties of the latter have been elaborated.

On another front, there have been significant advances in understanding the structure of gauge theories possessing fuzzy extra dimensions (for a review on fuzzy spaces see [4]). Gauge theories with fuzzy extra dimensions using CSDR scheme were first studied in [5]. Later on this was followed by [6], where it was shown that an $SU(\mathcal{N})$ Yang-Mills theory on the four dimensional Minkowski space M^4 coupled to an appropriate set of scalar fields develops fuzzy extra dimensions in the form of fuzzy spheres S_F^2 via spontaneous symmetry breaking. The vacuum expectation values (VEV) for the scalar fields form the fuzzy sphere, while the fluctuations around this vacuum are interpreted as gauge fields over S_F^2 . The resulting theory

can therefore be viewed as a gauge theory over $M^4 \times S_F^2$ with a smaller gauge group; which is further corroborated by the expansion of a tower of Kaluza-Klein modes of the gauge fields over $M^4 \times S_F^2$. Inclusion of the fermions into this theory was considered in [7]. Gauge theory on $M^4 \times S_F^2 \times S_F^2$ has recently been investigated in [8]. For a review on these and related results [9] can be consulted. It is also worthwhile to mention that, in [10] starting from a suitable matrix gauge theory, noncommutative gauge theories on \mathbb{R}_θ^4 possessing extra dimensions have been proposed. Extra dimensions are interpreted as scalars on \mathbb{R}_θ^4 coupled to the gauge fields, and it was shown that scalars could take vacuum expectation values leading to their identification as fuzzy spheres. Consequently, spontaneous symmetry breaking in a particular gauge theory has been investigated, and its content is compared with that of the standard model.

In a recent article together with D. Harland [11], the question of dimensional reduction of gauge theories over fuzzy coset spaces has been addressed. For this purpose, a $U(2)$ Yang-Mills theory over $\mathcal{M} \times S_F^2$ is considered, where \mathcal{M} is taken as a Riemannian manifold and S_F^2 is as the fuzzy sphere. The equivariant reduction of this model over S_F^2 has been performed by finding the most general $SU(2)$ -equivariant gauge field over $\mathcal{M} \times S_F^2$ and applying the well-known CSDR techniques. It was shown that for $\mathcal{M} = \mathbb{R}^2$ the emergent theory has non-BPS vortex solutions depending on the parameters in the model, which correspond to instantons in the original theory.

In this paper, we report on some of the results of a recent work in which we consider a $U(2)$ Yang-Mills theory $\mathcal{M} \times S_F^2$ where \mathcal{M} is an arbitrary noncommutative manifold. Performing $SU(2)$ -equivariant dimensional reduction of this theory leads to a noncommutative $U(1)$ gauge model coupled adjointly to a set of scalar fields [12]. We find that on the Groenewald-Moyal plane $\mathcal{M} = \mathbb{R}_\theta^2$ the emergent models admit noncommutative, non-BPS vortex as well as fluxon solutions. A broader discussion and further details can be found in [12].

2. Yang-Mills Theory on $\mathcal{M} \times S_F^2$

In this section, we collect the essential features of gauge theory on $\mathcal{M} \times S_F^2$. We start with considering the following $U(\mathcal{N})$ Yang-Mills theory over a suitable noncommutative space \mathcal{M} which we leave unspecified for the time being:

$$S = \int_{\mathcal{M}} \text{Tr}_{\mathcal{N}} \left(\frac{1}{4g^2} F_{\mu\nu}^\dagger F_{\mu\nu} + (D_\mu \phi_a)^\dagger (D_\mu \phi_a) \right) + \frac{1}{\tilde{g}^2} V_1(\phi) + a^2 V_2(\phi). \quad (1)$$

Here, ϕ_a ($a = 1, 2, 3$) are anti-Hermitian scalars, transforming in the adjoint of $SU(\mathcal{N})$, $D_\mu \phi_a = \partial_\mu \phi_a + [A_\mu, \phi_a]$ are the covariant derivatives and A_μ are the $u(\mathcal{N})$ valued anti-Hermitian gauge fields associated to the curvature $F_{\mu\nu}$. We take the potentials of the form

$$V_1(\phi) = \text{Tr}_{\mathcal{N}}(F_{ab}^\dagger F_{ab}), \quad V_2(\phi) = \text{Tr}_{\mathcal{N}}((\phi_a \phi_a + \tilde{b})^2). \quad (2)$$

where in $V_1(\phi)$ we have defined

$$F_{ab} := [\phi_a, \phi_b] - \varepsilon_{abc} \phi_c, \quad (3)$$

whose purpose will become evident shortly.

In the expressions above a , \tilde{b} , g and \tilde{g} are constants and $\text{Tr}_{\mathcal{N}} = \mathcal{N}^{-1} \text{Tr}$ denotes a normalized trace. We further note that ϕ_a transform in the vector representation of an additional global $SO(3)$ symmetry and that V_1 and V_2 are invariant under this symmetry.

As its commutative counterpart [6], this theory spontaneously develops extra dimensions in the form of fuzzy spheres. Following [6], let us very briefly see how this actually comes about. We observe that the potential $\tilde{g}^{-2} V_1 + a^2 V_2$ is positive definite, and that solutions of

$$F_{ab} = [\phi_a, \phi_b] - \varepsilon_{abc} \phi_c = 0, \quad -\phi_a \phi_a = \tilde{b} \quad (4)$$

are evidently a global minima. Most general solution to this equation is not known. However depending on the values taken by the parameter \tilde{b} , a large class of solutions has been found in [6]. Here we restrict ourselves to the simplest situation and refer the reader to [6] for a general discussion and its physical consequences.

Taking the value of \tilde{b} as the quadratic Casimir of an irreducible representation of $SU(2)$ labeled by ℓ , $\tilde{b} = \ell(\ell + 1)$ with $2\ell \in \mathbb{Z}$ and assuming further that the dimension \mathcal{N} of the matrices ϕ_a is $(2\ell + 1)n$, (4) is solved by the configurations of the form

$$\phi_a = X_a^{(2\ell+1)} \otimes \mathbf{1}_n, \quad (5)$$

where $X_a^{(2\ell+1)}$ are the (anti-Hermitian) generators of $SU(2)$ in the irreducible representation ℓ , which has dimension $2\ell + 1$. We observe that this vacuum configuration spontaneously breaks the $U(\mathcal{N})$ down to $U(n)$ which is the commutant of ϕ_a in (5).

Fluctuations about the vacuum (5) may be written as

$$\phi_a = X_a + A_a, \quad (6)$$

where $A_a \in u(2\ell + 1) \otimes u(n)$ and we have used the short-hand notation $X_a^{(2\ell+1)} \otimes \mathbf{1}_n =: X_a$. Then A_a ($a = 1, 2, 3$) may be interpreted as three components of a $U(n)$ gauge field on the fuzzy sphere S_F^2 .

A short definition of the fuzzy sphere and some of its properties are in order here. The fuzzy sphere at level ℓ is defined to be the algebra of $(2\ell + 1) \times (2\ell + 1)$ matrices $\text{Mat}(2\ell + 1)$. The three Hermitian ‘‘coordinate functions’’

$$\hat{x}_a := \frac{i}{\sqrt{\ell(\ell + 1)}} X_a^{(2\ell+1)} \quad (7)$$

satisfy

$$[\hat{x}_a, \hat{x}_b] = \frac{i}{\sqrt{\ell(\ell + 1)}} \varepsilon_{abc} \hat{x}_c, \quad \hat{x}_a \hat{x}_a = 1, \quad (8)$$

and generate the full matrix algebra $\text{Mat}(2\ell + 1)$. Derivations of functions can be given by the adjoint action of $su(2)$ on S_F^2 :

$$f \rightarrow adX_a^{(2\ell+1)} f := [X_a^{(2\ell+1)}, f], \quad f \in \text{Mat}(2\ell + 1). \quad (9)$$

In the limit $\ell \rightarrow \infty$, the functions \hat{x}_a are identified with the standard coordinates x_a on \mathbb{R}^3 , restricted to the unit sphere, and the infinite-dimensional algebra $\mathcal{C}^\infty(S^2)$ of functions on the sphere is recovered. Also in this limit, the derivations $[X_a^{(2\ell+1)}, \cdot]$ become the vector fields $-i\mathcal{L}_a = \varepsilon_{abc} x_a \partial_b$, induced by the usual action of $SO(3)$.

Tracking back to our main discussion in this section, we observe that ϕ_a are the ‘‘covariant coordinates’’ on S_F^2 and (3) defines the associated curvature F_{ab} . The latter may be expressed in terms of the gauge fields A_a as:

$$F_{ab} = [X_a, A_b] - [X_b, A_a] + [A_a, A_b] - \varepsilon_{abc} A_c. \quad (10)$$

Obviously, the term V_1 corresponds to the Yang-Mills action on S_F^2 . However, with this term alone, gauge theory on the sphere is not recovered in the commutative limit, since the fuzzy gauge field has three components rather than two. Rather, one obtains gauge theory with an additional scalar; the scalar is more precisely the component of the gauge field pointing in the radial direction when S^2 is embedded in \mathbb{R}^3 . The purpose of the term V_2 in the action is to suppress this scalar.

To summarize, with (6) the action in (1) takes the form of a $U(n)$ gauge theory on $\mathcal{M} \times S_F^2(2\ell + 1)$ with the gauge field components $A_M(\hat{y}) = (A_\mu(\hat{y}), A_a(\hat{y})) \in u(n) \otimes u(2\ell + 1)$ and field strength tensor (\hat{y} are a set of coordinates for the noncommutative manifold \mathcal{M})

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \\ F_{\mu a} &= D_\mu \phi_a = \partial_\mu \phi_a + [A_\mu, \phi_a] \\ F_{ab} &= [\phi_a, \phi_b] - \epsilon_{abc} \phi_c. \end{aligned} \quad (11)$$

3. The $SU(2)$ -Equivariant Gauge Field

Let us focus on the case of a $U(2)$ gauge theory on $\mathcal{M} \times S_F^2$. The construction of the most general $SU(2)$ -equivariant gauge field on S_F^2 was given in a recent article by the present author with D. Harland [11]. This construction uses essentially the representation theory of $SU(2)$. Here we give a brief account and refer the reader to [11] for further details.

We begin by selecting

$$\omega_a = X_a^{(2\ell+1)} \otimes \mathbf{1}_2 - \mathbf{1}_{2\ell+1} \otimes \frac{i\sigma^a}{2}, \quad \omega_a \in u(2) \otimes u(2\ell + 1), \text{ for } a = 1, 2, 3 \quad (12)$$

These ω_a are the generators of the representation $\frac{1}{2} \otimes \underline{\ell}$ of $SU(2)$, where by \underline{m} we denote the spin m representation of $SU(2)$ of dimension $2m + 1$. The two terms which make up ω_a generate rotations and gauge transformations, therefore imposing ω -equivariance amounts to requiring that rotations can be compensated by gauge transformations. There are certainly more possible choices for ω_a ; for example, $\omega_a = X_a^{(2\ell+1)} \otimes \mathbf{1}_2$ was studied in [5].

$SU(2)$ -equivariance of the theory requires the fulfillment of the symmetry constraints,

$$[\omega_a, A_\mu] = 0, \quad [\omega_a, \phi_b] = \epsilon_{abc} \phi_c, \quad (13)$$

on the gauge field. A consistency condition on these constraints is $[\omega_a, \omega_b] = \epsilon_{abc} \omega_c$ and it is readily satisfied by our choice of ω_a .

The solutions to these constraints are obtained using the representation theory of $SU(2)$ and are presented in [11]. They are conveniently parametrized as

$$A_\mu = \frac{1}{2} Q a_\mu(\hat{y}) + \frac{1}{2} i b_\mu(\hat{y}), \quad (14)$$

$$A_a = \frac{1}{2} \varphi_1(\hat{y}) [X_a, Q] + \frac{1}{2} (\varphi_2(\hat{y}) - 1) Q [X_a, Q] + i \frac{1}{2} \varphi_3(\hat{y}) \frac{1}{2} \{\hat{X}_a, Q\} + \frac{1}{2} \varphi_4(\hat{y}) \hat{\omega}_a, \quad (15)$$

with $\phi_a = X_a + A_a$ and a_μ, b_μ are Hermitian $U(1)$ gauge fields, φ_i are Hermitian scalar fields over \mathcal{M} , the curly brackets denote anti-commutators throughout, and we have used

$$\hat{X}_a := \frac{1}{\ell + 1/2} X_a, \quad \hat{\omega}_a := \frac{1}{\ell + 1/2} \omega_a. \quad (16)$$

We have further introduced the anti-Hermitian matrix

$$Q := \frac{X_a \otimes \sigma^a - i/2}{\ell + 1/2}, \quad Q^\dagger = -Q, \quad Q^2 = -\mathbf{1}_{2(2\ell+1)}. \quad (17)$$

Indeed, Q is the fuzzy version of $q := i\sigma \cdot \mathbf{x}$ and converges to it in the $\ell \rightarrow \infty$ limit.

4. Reduced Action

Using the $SU(2)$ -equivariant gauge field in the noncommutative $U(2)$ Yang-Mills theory on $\mathcal{M} \otimes S_F^2$, we can explicitly trace it over the fuzzy sphere and reduce it to a theory on \mathcal{M} . The reduced action has the general form

$$S = \int_{\mathcal{M}} \mathcal{L}_F + \mathcal{L}_G + \frac{1}{\tilde{g}^2} \tilde{V}_1 + a^2 \tilde{V}_2 \quad (18)$$

where \mathcal{L}_F stands for the curvature term, \mathcal{L}_G for the gradient term and \tilde{V}_1, \tilde{V}_2 for the reduced forms of V_1, V_2 , respectively. The explicit form of these terms and the related results are omitted here, they can be found in [12].

It turns out that the presence of extra degrees of freedom, namely φ_3, φ_4 , in the $SU(2)$ -equivariant gauge field on S_F^2 leads to a further symmetry breaking in the reduced action and the reduced action is invariant only under a noncommutative $U(1)$ gauge group and it has the form [12],

$$S = \int_{\mathcal{M}} \frac{1}{4g^2} |F_{\mu\nu}|^2 + \frac{1}{2} \frac{\ell^2 + \ell}{(\ell + 1/2)^2} D_\mu \varphi D_\mu \varphi^\dagger + \frac{1}{8} \frac{\ell^2 + \ell}{(\ell + 1/2)^2} \left(\frac{(\ell + \frac{3}{2})(\ell - \frac{1}{2})}{(\ell + \frac{1}{2})^2} + 1 \right) (D_\mu \varphi_3)^2 + \frac{\ell^2 + \ell + \frac{3}{4}}{4(\ell + \frac{1}{2})^2} (D_\mu \varphi_4)^2 + \frac{\ell^2 + \ell}{4(\ell + \frac{1}{2})^3} \{D_\mu \varphi_3, D_\mu \varphi_4\} + \frac{1}{\tilde{g}^2} \tilde{V}_1 + a^2 \tilde{V}_2. \quad (19)$$

where we have

$$D_\mu \cdot = \partial_\mu \cdot + [c_\mu, \cdot], \quad F_{\mu\nu} = \partial_\mu c_\nu - \partial_\nu c_\mu + i[c_\mu, c_\nu], \quad c_\mu = \frac{1}{2} b_\mu. \quad (20)$$

5. Solutions of the Reduced Theory on \mathbb{R}_θ^2

We now wish to study the classical solutions of the system governed by the action given in (19) on the Groenewald-Moyal plane \mathbb{R}_θ^2 . As emphasized in [11] there is no canonical choice for the coefficient a^2 of the fuzzy constraint term; we will consider the two extreme cases of $a^2 = \infty$ and $a^2 = 0$ corresponding, respectively, to imposing the constraint $\phi_a \phi_a + \ell(\ell + 1) = 0$ in full (i.e. “by hand”) and imposing no constraint at all. In both cases, we consider the large ℓ limit; in the $a = \infty$ theory, we include only terms appearing at $O(\ell^{-2})$, whereas for the case $a = 0$, we assume $\ell = \infty$.

5.1. Definitions and Conventions for the Groenewald-Moyal Plane \mathbb{R}_θ^2

Using the operator formalism, \mathbb{R}_θ^2 is defined by two operators \hat{y}_1, \hat{y}_2 acting on the standard Harmonic oscillator Fock space \mathcal{H} . They fulfill the Heisenberg algebra commutation relation

$$[\hat{y}_1, \hat{y}_2] = i\theta, \quad (21)$$

where θ is the noncommutativity parameter.

It is often useful to switch to the complex basis which we take as

$$z = \frac{1}{\sqrt{2}}(y_1 + iy_2), \quad \bar{z} = \frac{1}{\sqrt{2}}(y_1 - iy_2), \quad (22)$$

fulfilling

$$[z, \bar{z}] = \theta. \quad (23)$$

The derivatives on \mathbb{R}_θ^2 maybe expressed as

$$\partial_\mu \cdot = -\frac{i}{\theta} \varepsilon_{\mu\nu} [\hat{y}_\nu, \cdot], \quad \partial_z \cdot = -\frac{1}{\theta} [\bar{z}, \cdot], \quad \partial_{\bar{z}} \cdot = \frac{1}{\theta} [z, \cdot]. \quad (24)$$

The integration over \mathbb{R}^2 becomes a trace over the Fock space \mathcal{H} on \mathbb{R}_θ^2 :

$$\int_{\mathbb{R}^2} d^2 y \longrightarrow 2\pi\theta \text{Tr}_{\mathcal{H}}. \quad (25)$$

5.2. Case 1: The constraint fully imposed

After performing the dimensional reduction, the fuzzy constraint $\phi_a \phi_a + \ell(\ell + 1) = 0$ can be solved order by order in powers of the parameter $\frac{1}{\ell}$ to obtain φ_3 and φ_4 in terms of φ_1 and φ_2 . Substituting back into the action yields an action involving only the scalar $\varphi = \varphi_1 + i\varphi_2$.

For large but finite ℓ , one can solve the constraint approximately by expanding it to leading order in powers of ℓ^{-1} around the $\ell = \infty$. Performing this to order $O(\ell^{-3})$, we find [12]

$$\varphi_3 = -i\frac{4}{\ell}[\varphi_1, \varphi_2] + \frac{1}{2\ell^2}(\varphi_1^2 + \varphi_2^2 - 1) + O(\ell^{-3}), \quad (26)$$

$$\varphi_4 = -\frac{1}{2\ell}(\varphi_1^2 + \varphi_2^2 - 1) + i\frac{3}{\ell^2}[\varphi_1, \varphi_2] + O(\ell^{-3}). \quad (27)$$

Note that these indeed preserve the gauge symmetry since both sides transform covariantly under the action of the gauge group.

Using these in (19), we find

$$S = 2\pi\theta \text{Tr}_{\mathcal{H}} \left[\frac{1}{4g^2} |F_{\mu\nu}|^2 + \frac{1}{2} \left(1 - \frac{1}{4\ell^2} \right) D_\mu \varphi D_\mu \varphi^\dagger + \frac{1}{\ell^2} \left(D_\mu [\varphi, \varphi^\dagger] \right)^2 + \frac{1}{32\ell^2} \left(D_\mu \{ \varphi, \varphi^\dagger \} \right)^2 + \frac{1}{\tilde{g}^2} \left(\left(\frac{1}{2} + \frac{1}{4\ell^2} \right) \left(\frac{1}{2} \{ \varphi, \varphi^\dagger \} - 1 \right)^2 + \frac{1}{8} \left(1 - \frac{1}{\ell} - \frac{3}{4\ell^2} \right) [\varphi, \varphi^\dagger]^2 \right) + O(\ell^{-3}) \right]. \quad (28)$$

To obtain this result we have also used the cyclicity property of the trace $\text{Tr}_{\mathcal{H}}$, under which the terms proportional to $[\varphi, \varphi^\dagger]$ and $\{ \{ \varphi, \varphi^\dagger \}, [\varphi, \varphi^\dagger] \}$ vanish. The expression (28) is clearly invariant under the noncommutative $U(1)$ gauge symmetry, as it should be.

It is possible to employ the solution generating techniques introduced in [13, 14] to find noncommutative vortex type solutions of (28). To this end we proceed as follows. Let us first define the covariant coordinates

$$X = -\frac{1}{\theta} \bar{z} + ic_z, \quad X^\dagger = -\frac{1}{\theta} z - ic_{\bar{z}}, \quad (29)$$

where we have used the complex combinations $c_z = \frac{1}{\sqrt{2}}(c_1 - ic_2)$, $c_{\bar{z}} = \frac{1}{\sqrt{2}}(c_1 + ic_2)$. The covariant derivatives and the field strength may be expressed as

$$D_z \varphi = [X, \varphi], \quad D_{\bar{z}} \varphi = -[X^\dagger, \varphi]. \quad (30)$$

$$\begin{aligned} F_{z\bar{z}} &= \partial_z c_{\bar{z}} - \partial_{\bar{z}} c_z + i[c_z, c_{\bar{z}}], \\ &= i[X, X^\dagger] + \frac{i}{\theta}, \end{aligned} \quad (31)$$

All the basic constituents of the action (28) transform covariantly under the gauge symmetry

$$X \longrightarrow UXU^\dagger, \quad \varphi \longrightarrow U\varphi U^\dagger, \quad D_z\varphi \longrightarrow UD_z\varphi U^\dagger, \quad F_{z\bar{z}} \longrightarrow UF_{z\bar{z}}U^\dagger. \quad (32)$$

It follows that the equations of motion will transform covariantly, that is,

$$\frac{\delta S}{\delta X} \longrightarrow U \frac{\delta S}{\delta X} U^\dagger, \quad \frac{\delta S}{\delta \varphi} \longrightarrow U \frac{\delta S}{\delta \varphi} U^\dagger, \quad (33)$$

under a partial isometry U satisfying

$$U^\dagger U = 1, \quad UU^\dagger = P, \quad (34)$$

where P is a projection operator [13]. Thus, the partial isometries (34) generate solutions from a known solution.

A trivial solution to the equations of motion of (28) may easily found to be $X = -\frac{1}{\theta}\bar{z}, \varphi = 1$. Taking $U = S^m$, where S is the usual shift operator $S = \sum_{k=0}^{\infty} |k+1\rangle\langle k|$, we can write a set of non-trivial solutions for the theory governed by (28) as

$$\begin{aligned} \varphi &= S^m S^{\dagger m} = 1 - P_m, \\ X &= -\frac{1}{\theta} S^m \bar{z} S^{\dagger m}, \end{aligned} \quad (35)$$

where

$$P_n = \sum_{k=0}^{n-1} |k\rangle\langle k|, \quad (36)$$

is the projection operator of rank m . The corresponding field strength is $F_{12} = -iF_{z\bar{z}} = \frac{1}{\theta}P_m$. We can view these solutions as noncommutative vortices [13, 14, 15, 16, 17] carrying m units of flux:

$$2\pi\theta \text{Tr}F_{12} = 2\pi m. \quad (37)$$

It is useful to evaluate the value of the action (28) on these solutions; we find

$$S = \pi\theta m \left(\frac{1}{g^2\theta^2} + \frac{1}{\tilde{g}^2} \left(1 + \frac{1}{2\ell^2} \right) \right) + O(\ell^{-3}). \quad (38)$$

This corresponds to the energy of the static vortices in 2 + 1 dimensions, $\mathbb{R}_\theta^2 \times \mathbb{R}^1$ with \mathbb{R}^1 standing for time. We observe that to leading order in ℓ^{-1} there is a ℓ^{-2} contribution adding to the energy, which is a residue of the fact that the present model has descended from a model with a fuzzy sphere of order ℓ , $S_F^2(\ell)$ as extra dimensions.

Two limiting cases may also be easily recorded from (38). For $\tilde{g} \rightarrow \infty$, our solutions collapse to the fluxon solutions discussed in [18, 19]; whereas, for $\theta \rightarrow \infty$, the action gets a contribution only from the potential term, and our vortex solution collapses to a noncommutative soliton solution of the type first discussed in [20].

5.3. Case 2: No constraint

With $a = 0$ and $\ell = \infty$, the action reduces to

$$S = 2\pi\theta \text{Tr}_{\mathcal{H}} \left(\frac{1}{4g^2} |F_{\mu\nu}|^2 + \frac{1}{2} D_\mu\varphi D_\mu\varphi^\dagger + \frac{1}{4} (D_\mu\varphi_3)^2 + \frac{1}{4} (D_\mu\varphi_4)^2 + \frac{1}{\tilde{g}^2} \tilde{V}_1 \right). \quad (39)$$

where \tilde{V}_1 is

$$\tilde{V}_1 \underset{\ell \rightarrow \infty}{=} \frac{1}{4} \left((\varphi\varphi^\dagger)^2 + (\varphi^\dagger\varphi)^2 + \{\varphi\varphi_3, \varphi^\dagger\varphi_3\} + \{\varphi, \varphi^\dagger\} (\varphi_3^2 + 2(\varphi_3 - 1)) + 2(\varphi_3 - 1)^2 \right. \\ \left. \{[\varphi, \varphi^\dagger], \varphi_4\} + 2\varphi_4^2 \right). \quad (40)$$

We see that there are linear terms in the potential (40) in φ_3 , which will prevent us from applying the solution generating technique used in the previous subsection since these lead to terms in the equations of motion proportional to identity, and, therefore, they do not transform adjointly under the solution generating transformations [13, 14]. However, in the present model this situation can be remedied by defining a new field $\varphi'_3 = \varphi_3 - 1$. In this manner, all the terms in the potential are quadratic or higher order or a constant, while the gradient term involving φ_3 is unaffected by this substitution. The equations of motion are

$$(D_z D_{\bar{z}} + D_{\bar{z}} D_z)\phi - \frac{\partial V'_1}{\partial \phi} = 0, \quad \text{for } \phi : \varphi, \varphi'_3, \varphi_4, \\ \frac{1}{g^2} D_z F_{z\bar{z}} + i(\varphi D_{\bar{z}} \varphi^\dagger - D_{\bar{z}} \varphi \varphi^\dagger) + i[\varphi'_3, D_{\bar{z}} \varphi'_3] + i[\varphi_4, D_{\bar{z}} \varphi_4] = 0. \quad (41)$$

We observe that a trivial solution to these equations is given by $\varphi = 1, \varphi'_3 = -1, \varphi_4 = 0, X = -\frac{1}{\theta} \bar{z}$. Applying the solution generating technique with $U = S^m$, we find

$$\varphi = S^m S^{\dagger m} = 1 - P_m, \quad \varphi'_3 = -S^m S^{\dagger m} = P_m - 1, \quad \varphi_4 = 0, \quad X = -\frac{1}{\theta} S^m \bar{z} S^{\dagger m}, \quad (42)$$

where S and X are defined as in the previous subsection.

Evaluating the value of the action (39) on these solutions we find

$$S = \frac{\pi m}{g^2 \theta}, \quad (43)$$

and the flux carried by these solutions is again

$$2\pi\theta \text{Tr} F_{12} = 2\pi m. \quad (44)$$

As it turns out, there is in fact no contribution to (43) from the potential term. Thus, we can interpret (42) as flux-tube solutions carrying m units of flux [18]. It is easy to see that (42) satisfies the equations of motion (41), by noting that

$$D_z \phi = [X, \phi] = 0, \quad D_{\bar{z}} \phi = -[X^\dagger, \phi] = 0, \quad (45)$$

where ϕ are the solutions for $\varphi, \varphi'_3, \varphi_4$ given in (42).

We also wish to remark that the field redefinition for φ_3 used above works only in the infinite ℓ limit. In fact, there does not appear to be a field redefinition at finite ℓ or at leading order around $\ell = \infty$ which will allow the use of solution generating transformations to construct non-trivial solutions.

Results of this section are generalized to \mathbb{R}^{2d}_θ in a rather straightforward manner, for details we refer the reader to [12].

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