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ON ENTIRE RATIONAL MAPS OF REAL SURFACES

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ABSTRACT. In this paper, we define for a component X_0 of a nonsingular compact real algebraic surface X the complex genus of X_0, denoted by $g_{\mathbb{C}}(X_0)$, and use this to prove the nonexistence of nonzero degree entire rational maps $f : X_0 \to Y$ provided that $g_{\mathbb{C}}(Y) > g_{\mathbb{C}}(X_0)$, analogously to the topological category. We construct connected real surfaces of arbitrary topological genus with zero complex genus.

1. Introduction and results

It follows from Poincaré duality that there exists a continuous map of nonzero degree $f : F_1 \to F_2$, between closed connected orientable surfaces F_1 and F_2, if and only if $g(F_1) \geq g(F_2)$, where $g(F_i)$, $i = 1, 2$ denotes the genus of F_i (see [11, Theorem 14.1 (6)]). However, this is not the case in the category of real algebraic surfaces and entire rational maps: In [17] Loday showed that any entire rational map $f : S^1 \times S^1 \to S^2$ is homotopic to a constant map, where S^n denotes the standard sphere in $\mathbb{R}^{n+1}$, using algebraic K-theory.

Bochnak and Kucharz extensively studied entire rational maps from algebraic varieties into standard spheres and Grassmann varieties making use of algebraic K-theory and the group $H^*_{\mathbb{C}-Alg}(X, \mathbb{Z})$ introduced by them ([4, 6, 7, 8, 9, 10]). The author gave another proof of Loday's result observing that $S^1 \times S^1$ bounds in its complexification whereas S^2 does not ([18]).

For an orientable connected component, X_0, of a nonsingular compact real algebraic surface X define the complex genus of X_0, denoted by $g_{\mathbb{C}}(X_0)$, as the greatest integer g such that there exits a continuous

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map of nonzero degree, $\phi : X_0 \to F_g$, to an orientable closed connected surface of genus g, where the kernel of the induced homomorphism

$$\phi_\sharp : \pi_1(X_0, p) \to \pi_1(F_g, \phi(p))$$

contains that of

$$i_\sharp : \pi_1(X_0, p) \to \pi_1(X_{\mathbb{C}}, i(p))$$

where $p \in X_0$ and $i : X \to X_{\mathbb{C}}$ is any complexification. Obviously, the topological genus of X_0, $g(X_0)$, is an upper bound for $g_{\mathbb{C}}(X_0)$.

For real algebraic varieties $X \subseteq \mathbb{R}^r$ and $Y \subseteq \mathbb{R}^s$ a map $F : X \to Y$ is said to be entire rational if there exist $f_i, g_i \in \mathbb{R}[x_1, \dots, x_r]$, $i = 1, \dots, s$, such that each g_i vanishes nowhere on X and $F = (f_1/g_1, \dots, f_s/g_s)$. We say X and Y are (entire rationally) isomorphic if there are entire rational maps $F : X \to Y$ and $G : Y \to X$ such that $F \circ G = id_Y$ and $G \circ F = id_X$. Isomorphic algebraic varieties will be regarded the same.

REMARK 1.1. Let X_0 be a connected component of a compact non-singular real algebraic variety X.

(i) Since birationally isomorphic complex projective varieties have isomorphic fundamental groups (cf. see [14, p. 494]) the homomorphism

$$i_\sharp : \pi_1(X_0) \to \pi_1(X_{\mathbb{C}})$$

is determined only by X up to an (entire rational) isomorphism. Hence, if X is a surface then $g_{\mathbb{C}}(X_0)$ is a well defined invariant of X. In other words, if $f : X \to Y$ is an (entire rational) isomorphism then $f_\sharp(\ker(i_\sharp)) = \ker(j_\sharp)$, where $i_\sharp : X \to X_{\mathbb{C}}$ and $j_\sharp : Y \to Y_{\mathbb{C}}$ are some complexifications.

(ii) Just like in the case of π_1, the kernel of the homomorphism

$$i_\sharp : \pi_2(X_0) \to \pi_2(X_{\mathbb{C}})$$

is an (entire rational) isomorphism invariant of X.

To see this let $i_1 : X \to Z_1$ and $i_2 : X \to Z_2$ be two complexifications. Then Z_1 and Z_2 are birationally isomorphic by some map $\phi : Z_1 \to Z_2$, which is regular on X and defined off a complex codimension two subvariety. Now if a homotopy class, represented by a smooth map $\alpha : S^2 \to X_0$, is trivial in Z_1, then we can move the 3-disk bounding α off the real codimension four indeterminacy set of ϕ, so that α bounds the 3-disk in Z_2 also.

(iii) Let X be a surface. Suppose that $g_{\mathbb{C}}(X_0) > 1$ and let $\phi : X_0 \to F_g$ be a continuous map of nonzero degree to an orientable closed connected surface of genus $g = g_{\mathbb{C}}(X_0)$, where the kernel of the induced homomorphism on π_1 contains that of $i_\sharp : \pi_1(X_0) \to$

$\pi_1(X_{\mathbb{C}})$, and $i : X \to X_{\mathbb{C}}$ is any complexification. Now, if $g > 1$ then $\phi_\sharp : \pi_1(X_0) \to \pi_1(F_g)$ is onto because ϕ has nonzero degree and $g = g_{\mathbb{C}}(X_0)$ is maximal with this property. For, since ϕ has nonzero degree the index $[\pi_1(F_g) : \phi_\sharp(\pi_1(X_0))]$ is finite. On the other hand, if this index larger than one then ϕ lifts to a finite covering $F_h \to F_g$ with $h > g$ (indeed $h - 1 = [\pi_1(F_g) : \phi_\sharp(\pi_1(X_0))]\ (g-1)$) corresponding to the subgroup $\phi_\sharp(\pi_1(X_0)) < \pi_1(F_g)$. However, this contradicts the maximality of g.

If $g = 1$ then $\phi_\sharp : \pi_1(X_0) \to \pi_1(F_g)$ is not necessarily onto. However, again the index $[\pi_1(F_g) : \phi_\sharp(\pi_1(X_0))]$ is finite and thus passing to a finite cover of the torus we may assume that $\phi_\sharp : \pi_1(X_0) \to \pi_1(F_g)$ is onto.

For the rest of paper we will assume that all connected spaces are pointed spaces and we will write simply $f_\sharp : \pi_1(M) \to \pi_1(N)$ instead of

$$f_\sharp : \pi_1(M, p) \to \pi_1(N, f(p))$$

for any continuous map $f : M \to N$.

Here is an application of this invariant.

THEOREM 1.2. *Let X and Y be nonsingular compact real algebraic surfaces, where the latter is assumed to be connected and orientable. Then for any orientable connected component X_0 of X with $g_{\mathbb{C}}(Y) > g_{\mathbb{C}}(X_0)$ and any entire rational map $f : X \to Y$, the restriction map $f_{|X_0} : X_0 \to Y$ has degree zero. Indeed the same holds for any rational map $f : X \to Y$ which is entire on X_0.*

Part (i) of Remark 1.1 enables us to define another invariant of X_0: The homomorphism

$$i_\sharp : \pi_1(X_0) \to \pi_1(X_{\mathbb{C}})$$

is determined only by X up to an isomorphism and hence so does the image of the homomorphism

$$i^* : H^1(X_{\mathbb{C}}, \mathbb{Z}) \to H^1(X_0, \mathbb{Z})$$

which we will denote by $ImH^1(X_0, \mathbb{Z})$. Consider the restriction of the cup product pairing on $H^1(X_0, \mathbb{Z})$ to the subgroup $ImH^1(X_0, \mathbb{Z})$. Then the rank of the restricted pairing, denoted by $r_{\mathbb{C}}(X)$, is also an isomorphism invariant of X_0. $r_{\mathbb{C}}(X)$ is always an even integer since the form is skew symmetric. (It is known that $ImH^i(X, \mathbb{Z})$ is an isomorphism invariant of any orientable nonsingular real algebraic variety, [19].)

If $f : F_1 \to F_2$ is a continuous map of nonzero degree between closed orientable surfaces then the induced map on cohomology rings is injective. Hence, we obtain:

THEOREM 1.3. *Let X and Y be nonsingular compact real algebraic surfaces, where the latter is assumed to be connected and orientable. Then for any orientable connected component X_0 of X with $r_{\mathbb{C}}(Y) > r_{\mathbb{C}}(X_0)$ and any entire rational map $f : X \to Y$, the restriction map $f_{|X_0} : X_0 \to Y$ has degree zero. Indeed the same holds for any rational map $f : X \to Y$ which is entire on X_0.*

REMARK 1.4. The converses of the above theorems do not hold: Let α and β be any two positive real numbers so that the product $\alpha\beta$ is irrational. Then by Remark 13.3.15 of [5] there exist nonsingular real connected elliptic curves D_α and D_β in $\mathbb{R}P^2$ such that any entire rational map from $D_\alpha \times D_\beta$ to the standard sphere S^2 is null homotopic. However, since a connected nonsingular real elliptic curve cannot be nullhomologous in its complexification the inclusion map

$$i : D_\alpha \times D_\beta \to D_{\alpha\mathbb{C}} \times D_{\beta_{\mathbb{C}}} = (D_\alpha \times D_\beta)_{\mathbb{C}}$$

induces an injection on fundamental groups and therefore both $g_{\mathbb{C}}(D_\alpha \times D_\beta)$ and $r_{\mathbb{C}}(D_\alpha \times D_\beta)$ are equal one.

The result below is a partial converse to Theorem 1.3.

THEOREM 1.5. *Let X be a nonsingular compact real algebraic surface. Then X admits an entire rational map $f : X \to A$ to some complex abelian variety A (regarded as a real variety), such that the induced homomorphism $f_* : H_2(X, \mathbb{Z}) \to H_2(A, \mathbb{Z})$ is nontrivial if and only if X has an orientable connected component X_0 with $r_{\mathbb{C}}(X_0) \geq 2$.*

By the method of proof of the above theorem, in general A has dimension larger than one.

For a product of two nonsingular real algebraic curves we get the following immediate corollary of the above theorem. (Compare with [6].)

COROLLARY 1.6. *Let X be the product, $X_1 \times X_2$, of two connected compact nonsingular real algebraic curves. Then the following conditions are equivalent:*

(i) *There exists an entire rational map $f : X \to A$ to some complex abelian variety (regarded as a real variety), such that the homology class $f_*([X_1 \times X_2]) \neq 0$;*
(ii) $r_{\mathbb{C}}(X_1 \times X_2) = 2$;
(iii) *Both X_1 and X_2 are homologously nontrivial in their complexifications.*

COROLLARY 1.7. *Let X_0 be an oriented connected component of a nonsingular compact real algebraic surface X. Then, if $f : F \to X_{\mathbb{C}}$ is a continuous map from an oriented closed surface to some complexification $X_{\mathbb{C}}$ of X so that $f_*([F]) = [X_0]$ then*

$$g(F) \geq \frac{1}{2} r_{\mathbb{C}}(X_0),$$

where $g(F)$ denotes the genus of F.

The proof of the above corollary is just standard algebraic topology and is given in the next section for the sake of completeness. The point is that the inequality

$$g(F) \geq \frac{1}{2} r_{\mathbb{C}}(X_0)$$

holds no matter what the complexification $X_{\mathbb{C}}$ is.

The two invariants $r_{\mathbb{C}}(X_0)$ and $g_{\mathbb{C}}(X_0)$ are related as follows.

PROPOSITION 1.8. *Let X_0 be an oriented connected component of a nonsingular compact real algebraic surface X. If $r_{\mathbb{C}}(X_0) > 0$ then $g_{\mathbb{C}}(X_0) > 0$.*

The converse of the above proposition is not correct as the example below shows:

EXAMPLE 1.9. First we would like to construct a compact connected nonsingular real algebraic curve Y such that $Y_{\mathbb{C}} - Y$ is disconnected but none of the components is a disc. To do this let S be a closed orientable surface of genus 2 and $C \subseteq S$ is a simple closed curve such that $S - C$ is the disjoint union of two tori with boundary.

Consider the reflection map $r : S \to S$ across the circle C. The fixed point set of r is C. Now, put a Riemannian metric g on S such that r is an isometry. If w is the volume form associated to the metric g then $r^*(w) = -w$, because r is orientation reversing. The metric and

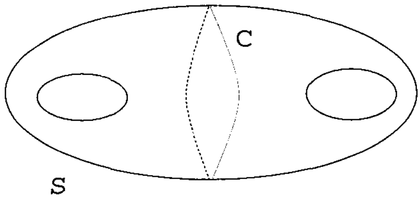

Figure 1.

the volume form defines an almost complex structure J on the tangent bundle T_*S by the formula

$$g(u,v) = w(u, J(v))$$

for any $u, v \in T_p(S)$, and $p \in S$. Moreover, we have $dr \circ J = -J \circ dr$, where dr is the differential of r. Since any almost complex structure on a closed orientable two manifold is integrable we can regard S as a Riemann surface, and hence as a complex algebraic curve. Moreover, r becomes an anti holomorphic involution of S (see [13, p. 43]).

We can embed S into a complex projective space such that r becomes the restriction of the complex conjugation and C its fixed point set; i.e., the real part (see [16, Sections 1 and 2]). Now choose Y as the image of C under this embedding.

Using different methods (perturbation of real curves), one may obtain a concrete example of such a curve blowing up of the singular point, which is the origin, of the plane curve given by the equation $x^4 + y^4 + 3x^2y^2 - 3x^2 - y^2 = 0$.

Let $X = Y \times Y$. Then $X_{\mathbb{C}}$ may be taken as $Y_{\mathbb{C}} \times Y_{\mathbb{C}}$. Now the inclusion map $i : X \to X_{\mathbb{C}}$ induces an injection $i_\sharp : \pi_1(X) \to \pi_1(X_{\mathbb{C}})$ on fundamental groups and the trivial homomorphism $i^* : H^1(X_{\mathbb{C}}, \mathbb{Z}) \to H^1(X, \mathbb{Z})$ on first cohomology. Hence, $r_{\mathbb{C}}(X) = 0$ whereas $g_{\mathbb{C}}(X) = 1$.

REMARK 1.10. If V is a compact nonsingular complex algebraic variety, then we can view V as a real algebraic variety which we will denote by $V_{\mathbb{R}}$. Indeed, $V_{\mathbb{R}}$ is just the fixed point set of the anti holomorphic involution $\sigma : V \times \bar{V} \to V \times \bar{V}$ given by $\sigma(x,y) = (\bar{y}, \bar{x})$, where $\bar{V}$ is the complex conjugate of V. It is well known that there is a complex algebraic subvariety Z of some projective space $\mathbb{C}P^N$ defined by real polynomials which is biregularly isomorphic to $V \times \bar{V}$. Moreover, the real part $Z \cap \mathbb{R}P^N$ is isomorphic to $V_{\mathbb{R}}$. However, any projective real algebraic variety is affine ([5, Proposition 3.4.4]) and hence $V_{\mathbb{R}}$ can be viewed as an affine real algebraic variety. For more details, we refer the reader to Sections 1 and 2 of [16].

If C is a nonsingular complex projective curve regarded as a real algebraic surface then by the above discussion we may take $i : V \to V \times \bar{V}$, $x \mapsto (x, \bar{x})$ as a complexification. If $\pi : V \times \bar{V} \to V$ is the projection onto the first factor then $\pi \circ i : V \to V$ is the identity map and hence the i induces an injective map on fundamental groups. Hence, $g_{\mathbb{C}}(C) = g(C)$ and $r_{\mathbb{C}}(C) = 2g(C)$, where $g(C)$ denotes the topological genus of C. In other words, if a surface is already complex then its complex genus equals its topological genus. This implies the following corollary.

COROLLARY 1.11. *Let X_0 be an oriented connected component of a nonsingular compact real algebraic surface X and $f : X \to C$ an entire rational map to a complex curve C of genus g. If $f_*([X_0]) \in H_2(C, \mathbb{Z})$ is not zero then $g_{\mathbb{C}}(X_0) \geq g$ and $r_{\mathbb{C}}(X_0) \geq 2g(C)$.*

The following examples provide real surfaces with $g(X) > g_{\mathbb{C}}(X)$, $r_{\mathbb{C}}(X)$.

EXAMPLE 1.12. Suppose X is a nonsingular compact real algebraic variety of dimension $n \geq 2$. Further suppose that X has a complete intersection complexification $X_{\mathbb{C}}$; i.e., $X_{\mathbb{C}}$ is a complete intersection in some complex projective space. So this complexification is simply connected and thus $g_{\mathbb{C}}(X_0) = r_{\mathbb{C}}(X_0) = 0$ for any component X_0.

Combining the previous corollary with the above example we arrive at the following corollary.

COROLLARY 1.13. *Suppose that X_0 is a connected component of a real surface X, which has a complete intersection complexification, and $f : X \to C$ is an entire rational map to some smooth complex projective curve (regarded as a real variety) such that the class $f_*([X_0]) \in H_2(C, \mathbb{Z})$ is not zero. Then, $C = \mathbb{C}P^1$.*

The above corollary can be proved by the results of Bochnak and Kucharz making use the group $H^*_{\mathbb{C}-alg}(X, \mathbb{Z})$, the cohomology subgroup of X generated by the pull backs of the complex algebraic cycles of its complexification ([4, 5, 8]).

EXAMPLE 1.14. The following theorem which is a direct consequence of Theorem 2.8.4 of [1], whose weaker form is originally proved by Benedetti and Tognoli ([2]), also provides surfaces with arbitrary topological genus but trivial $g_{\mathbb{C}}$ and $r_{\mathbb{C}}$

THEOREM 1.15. [1] *Let $L \subseteq M \subseteq \mathbb{R}^k$ where L is a nonsingular real algebraic variety and M an embedded closed smooth manifold. Then there is a smooth embedding $g : M \to \mathbb{R}^k \times \mathbb{R}^l$ such that $X = g(M)$ is a nonsingular real algebraic variety with $g(x) = x$, for all $x \in L$, if and only if the normal bundle $N_M(L)$ of L in M has a strongly algebraic structure.*

Let g be any positive integer and $F \subseteq \mathbb{R}^3$ a closed orientable smooth surface of genus g such that F contains g embedded disjoint circles $a_1, \dots, a_g$ with the following properties:

(i) Each a_i is a nonsingular real algebraic curve which is homologously trivial in its complexification;
(ii) The set $\{a_1, \dots, a_g\}$ is part of a basis for $H_1(X, \mathbb{Z})$.

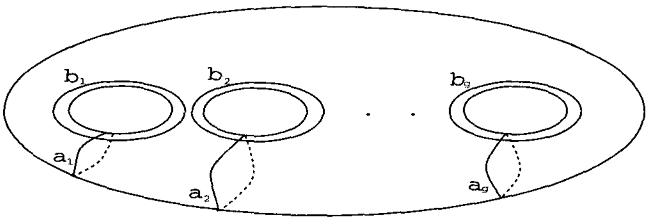

Figure 2.

Using the above theorem we can isotop F to an algebraic surface X, maybe in some larger Euclidean space, keeping each a_i fixed. (Note that the normal bundle of each a_i is trivial and thus has a strongly algebraic structure.)

Since a_i homologously trivial in its complexification $ImH^1(X_0, \mathbb{Z})$ does not contain a pair with nontrivial cup product. Hence, $r_{\mathbb{C}}(X) = 0$.

Now assume further that each a_i is entire rationally isomorphic to the standard circle $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. Since, S^1 bounds (a 2-disc) in its complexification $S^1_{\mathbb{C}} = S^2$ by the above argument $r_{\mathbb{C}}(X) = 0$. Below, we will show that $g_{\mathbb{C}}(X)$ is also zero.

Let $\phi : X \to F_{g'}$ be a continuous map of nonzero degree to an orientable closed connected surface of genus $g' = g_{\mathbb{C}}(X)$, where the kernel of the induced homomorphism on π_1 contains that of $i_\sharp : \pi_1(X) \to \pi_1(X_\mathbb{C})$, $i : X \to X_\mathbb{C}$ being any complexification. S^1 bounds a 2-disc in its complexification $S^1_\mathbb{C} = S^2$ and therefore the map ϕ factors through the 2-complex, say K, obtained by gluing a 2-disc to X along each a_i (for example we may take one of the disc components of $S^1_\mathbb{C} = S^2 - S^1$):

$$X \to K \to F_{g'}.$$

If g' were positive then there would be cohomology classes $a, b \in H^1(F_{g'}, \mathbb{Z})$ with nonzero cup product, and since ϕ has nonzero degree, pull backs of these classes would have nonzero cup product in X and thus in K. However, cohomology of K is generated by the duals (in the sense of Universal Coefficient Theorem) of b_i's and their cup products are clearly trivial.

In the last section we will make use of group homology to prove that $g_\mathbb{C}(X) = 0$.

2. Proofs

All real algebraic varieties under consideration in this report are compact and nonsingular. It is well known that real projective varieties are affine ([1, Proposition 2.4.1] or [5, Theorem 3.4.4]). Moreover, compact affine real algebraic varieties are projective ([1, Corollary 2.5.14]) and therefore, we will not distinguish between compact real affine varieties and real projective varieties.

A complexification $X_\mathbb{C} \subseteq \mathbb{C}P^N$ of X will mean that X is embedded into some projective space $\mathbb{R}P^N$ and $X_\mathbb{C} \subseteq \mathbb{C}P^N$ is the complexification of the pair $X \subseteq \mathbb{R}P^N$. We also require the complexification to be nonsingular (blow up $X_\mathbb{C}$ along smooth centers away from X defined over reals if necessary, [15, 3]). We refer the reader for the basic definitions and facts about real algebraic geometry to [1, 5].

Proof of Theorem 1.2. Let $f : X \to Y$ be an entire rational map and g_1 and g_2 denote $g_\mathbb{C}(X_0)$ and $g_\mathbb{C}(Y)$ respectively. Suppose that $\phi : X_0 \to F_1$ and $\psi : Y \to F_2$ are some continuous maps of nonzero degree to orientable closed connected surfaces of genus g_i, $i = 1, 2$, where the kernel of the induced homomorphisms

$$\phi_\sharp : \pi_1(X_0) \to \pi_1(F_1) \quad \text{and} \quad \psi_\sharp : \pi_1(Y) \to \pi_1(F_2)$$

contains those of

$$i_\sharp : \pi_1(X_0) \to \pi_1(X_\mathbb{C}) \quad \text{and} \quad j_\sharp : \pi_1(Y) \to \pi_1(Y_\mathbb{C})$$

respectively, where $p \in X_0$ is any point, $q = f(p)$ and $i : X \to X_{\mathbb{C}}$ and $j : Y \to Y_{\mathbb{C}}$ are some complexifications such that $f : X \to Y$ extends to $f_{\mathbb{C}} : X_{\mathbb{C}} \to Y_{\mathbb{C}}$.

Consider the composition map $\psi \circ f : X_0 \to F_2$ and the induced homomorphism $\psi_\sharp \circ f_\sharp : \pi_1(X_0) \to \pi_1(F_2)$. Since $f_{\mathbb{C}} : X_{\mathbb{C}} \to Y_{\mathbb{C}}$ extends $f : X \to Y$ the homomorphism $\psi_\sharp \circ f_\sharp$ factors through the image of $i_\sharp : \pi_1(X_0) \to \pi_1(X_{\mathbb{C}})$. Hence, the kernel of $\psi_\sharp \circ f_\sharp$ contains that of $i_\sharp$. By the hypothesis we have $g_1 < g_2$ and therefore the degree of $\psi \circ f$ must be zero. This finishes the proof because the degree of ψ is nonzero.

For the second statement the above proof works fine except we may have to blow up also some real points of $X - X_0$ to get the regular map $f_{\mathbb{C}} : X_{\mathbb{C}} \to Y_{\mathbb{C}}$. □

Proof of Theorem 1.5. Suppose there is a component X_0 with $r_{\mathbb{C}}(X_0) \geq 2$. Then there are cohomology classes $a, b \in H^1(X_{\mathbb{C}}, \mathbb{Z})$ with $i^*(a) \cup i^*(b) \in H^2(X_0, \mathbb{Z})$ is nontrivial, where $i : X \to X_{\mathbb{C}}$ is the inclusion map. Let $\alpha : X_{\mathbb{C}} \to A$ be the Albanese map of $X_{\mathbb{C}}$ and $f = \alpha \circ i$. The induced homomorphism $\alpha^* : H^1(A, \mathbb{Z}) \to H^1(X_{\mathbb{C}}, \mathbb{Z})$ is an isomorphism and thus there are classes $a', b' \in H^1(A, \mathbb{Z})$ with $a = \alpha^*(a')$ and $b = \alpha^*(b')$ which implies that $f_*([X_0]) = (\alpha \circ i)_*[X_0] \neq 0$.

Conversely, assume that there is an entire rational map $f : X \to A$ into some complex abelian variety such that the induced homomorphism $f_* : H_2(X, \mathbb{Z}) \to H_2(A, \mathbb{Z})$ is nontrivial. So, there is an orientable component X_0 of X with $f_*([X_0]) \neq 0$. Since A is a torus there are cohomology classes $a, b \in H^1(A, \mathbb{Z})$ such that $(f^*(a) \cup f^*(b))([X_0]) \neq 0$. By Example 1.10, the map $j : A \to A \times \bar{A}$, $j(p) = (p, \bar{p})$, is a complexification. Clearly, $j^* : H^1(A \times \bar{A}, \mathbb{Z}) \to H^1(A, \mathbb{Z})$ is onto and thus there are classes a', b' in $H^1(A \times \bar{A}, \mathbb{Z})$ with $j^*(a') = a$ and $j^*(b') = b$. Now by Hironaka's theorem ([15, 3]) we can extend $f : X \to A$ to some complexification $f_{\mathbb{C}} : X_{\mathbb{C}} \to A \times \bar{A}$ and obtain the following commutative diagrams:

$$\begin{array}{ccc} X & \xrightarrow{\;f\;} & A \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ X_{\mathbb{C}} & \xrightarrow[f_{\mathbb{C}}]{} & A \times \bar{A} \end{array} \qquad \begin{array}{ccc} H^1(X, \mathbb{Z}) & \xleftarrow{\;f^*\;} & H^1(A, \mathbb{Z}) \\ {\scriptstyle i^*}\uparrow & & \uparrow{\scriptstyle j^*} \\ H^1(X_{\mathbb{C}}, \mathbb{Z}) & \xleftarrow[f_{\mathbb{C}}^*]{} & H^1(A \times \bar{A}, \mathbb{Z}) \end{array}$$

Figure 3.

Now by the commutativity of the second diagram $f^*(a)$ and $f^*(b)$ are in $ImH^1(X_0, \mathbb{Z})$. This finishes the proof. $\square$

Proof of Corollary 1.7. Choose a set of linearly independent, over $\mathbb{Q}$, elements $a_1, b_1, \ldots, a_k, b_k \in H^1(X_{\mathbb{C}}, \mathbb{Q})$, where $2k = r_{\mathbb{C}}(X_0)$, such that $(a_i \cup a_j)([X_0]) = (b_i \cup b_j)([X_0]) = 0$ and $(a_i \cup b_j)([X_0]) = \delta_{ij}$ for all $i, j = 1, \ldots, k$. Now, if $f : F \longrightarrow X_{\mathbb{C}}$ is a continuous map such that $f_*([F]) = [X_0]$ then

$$f_|^* :< a_i, b_i \mid i = 1, \ldots, k > \longrightarrow H^1(F, \mathbb{Q})$$

is injective, yielding that $g(F) \geq k = \dfrac{r_{\mathbb{C}}(X_0)}{2}$. $\square$

Proof of Proposition 1.8. Let $r_{\mathbb{C}}(X_0) > 0$. Then there are cohomology classes $a_0, b_0 \in ImH^1(X_0, \mathbb{Z})$ with $a_0 \cup b_0 \neq 0$. Let $a_0 = i^*(a)$ and $b_0 = i^*(b)$ for some classes $a, b \in H^1(X_{\mathbb{C}}, \mathbb{Z})$. Since we can view $H^1(X_{\mathbb{C}}, \mathbb{Z})$ as homotopy classes of continuous maps from $X_{\mathbb{C}}$ to S^1 there are continuous maps $f_a : X_{\mathbb{C}} \longrightarrow S^1$ and $f_b : X_{\mathbb{C}} \longrightarrow S^1$ so that $f_a^*(c) = a$ and $f_b^*(c) = b$ where $c \in H^1(S^1, \mathbb{Z})$ is a generator. Let $\phi : X_0 \longrightarrow S^1 \times S^1 = T^2$ be given by the composition $\phi(x) = (f_a(i(x)), f_b(i(x)))$. Then, clearly ϕ has nonzero degree and the kernel of $\phi_\sharp : \pi_1(X_0) \longrightarrow \pi_1(T^2)$ contains that of $i_\sharp : \pi_1(X_0) \longrightarrow \pi_1(X_{\mathbb{C}})$. So $g_{\mathbb{C}}(X_0) > 0$. $\square$

3. Example 1.14 revisited

For a connected CW-complex X we have the well known exact sequence

$$\pi_2(X) \to H_2(X, \mathbb{Z}) \to H_2(\pi_1(X), \mathbb{Z}) \to 0$$

giving $H_2(\pi_1(X), \mathbb{Z})$ as the cokernel of the Hurewicz homomorphism ([12]).

Let X be the surface constructed in Example 1.14 and $i : X \longrightarrow X_{\mathbb{C}}$ any complexification. Suppose that $g_{\mathbb{C}}(X) > 0$ and $\phi : X \longrightarrow F$ is a nonzero degree map to a surface with $g(F) = g_{\mathbb{C}}(X)$ such that the kernel of the induced map on the fundamental groups, $\phi_\sharp : \pi_1(X) \longrightarrow \pi_1(F)$, contains that of $i_\sharp : \pi_1(X) \longrightarrow \pi_1(X_{\mathbb{C}})$. Consider the commutative diagram below:

$$\begin{array}{ccccccc}
\pi_2(X) & \longrightarrow & H_2(X,\mathbb{Z}) & \xrightarrow{h} & H_2(\pi_1(X),\mathbb{Z}) & \longrightarrow & 0 \\
\downarrow{\scriptstyle \phi_\sharp} & & \downarrow{\scriptstyle \phi_*} & & \downarrow{\scriptstyle \phi_\sharp} & & \\
\pi_2(F) & \longrightarrow & H_2(F,\mathbb{Z}) & \xrightarrow{h'} & H_2(\pi_1(F),\mathbb{Z}) & \longrightarrow & 0
\end{array}$$

Figure 4.

where all the vertical maps are induced by the map $\phi : X \to F$.

Since the fundamental group of X is

$$\langle a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] \rangle$$

where the bracket $[a, b]$ denotes the commutator of a and b, and each a_i bounds a 2-disc in $X_{\mathbb{C}}$ all a_i's are in the kernel of the homomorphism $i_\sharp : \pi_1(X) \to \pi_1(X_{\mathbb{C}})$ and thus of the homomorphism $\phi_\sharp : \pi_1(X) \to \pi_1(F)$. Hence $\phi_\sharp : \pi_1(X) \to \pi_1(F)$ factors through the free group on g letters, $Fr_g = \langle b_1, \ldots, b_g \rangle$. However, second homology group of a free group is trivial, because free groups admit bouquet of circles as classifying spaces. Hence $\phi_\sharp : H_2(\pi_1(X), \mathbb{Z}) \to H_2(\pi_1(F), \mathbb{Z})$ is trivial. Now, by the above diagram the induced homomorphism $\phi_* : H_2(X, \mathbb{Z}) \to H_2(F, \mathbb{Z})$ is trivial because $\pi_2(X) = \pi_2(F) = 0$. However, this contradicts the fact that ϕ has nonzero degree and thus $g_{\mathbb{C}}(X)$ is zero as claimed.

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