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Analytical solutions to orthotropic variable thickness disk problems

Ortotropik değişken kalınlıklı disk problemlerinin analitik çözümleri

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Abstract

An analytical model is developed to estimate the mechanical response of nonisothermal, orthotropic, variable thickness disks under a variety of boundary conditions. Combining basic mechanical equations of disk geometry with the equations of orthotropic material, the elastic equation of the disk is obtained. This equation is transformed into a standard hypergeometric differential equation by means of a suitable transformation. An analytical solution is then obtained in terms of hypergeometric functions. The boundary conditions used to complete the solutions simulate rotating annular disks with two free surfaces, stationary annular disks with pressurized inner and free outer surfaces, and free inner and pressurized outer surfaces. The results of the solutions to each of these cases are presented in graphical forms. It is observed that, for the three cases investigated the elastic orthotropy parameter turns out to be an important parameter affecting the elastic behavior

Keywords: Orthotropic disk, Variable thickness, Thermoelasticity, Hypergeometric equation

Öz

Bu çalışmada eş ısıl olmayan, ortotropik, değişken kesitli disklerin, farklı sınır koşulları altında mekanik davranışlarını tahmin edebilmek için analitik bir model geliştirilmiştir. Disk geometrisi için temel mekanik denklemleri, ortotropik malzeme denklemleri ile birleştirilerek elastik denklem elde edilmiştir. Bu denklem uygun bir dönüşüm tekniği ile standart hipergeometrik diferansiyel denkleme dönüştürülmüş ve bunun analitik çözümü hipergeometrik fonksiyonlar cinsinden bulunmuştur. Çözümü tamamlayan sınır koşulları, iki ucu serbest dönen, iç veya dış yüzeyden basınçlandırılmış durağan değişken kesitli disklerin benzetişimini sağlayacak şekilde seçilmiştir. Elde edilen sonuçlar grafiksel olarak sunulmuştur. Sonuçlar göstermiştir ki makalede incelenen her üç problem için de elastik ortotropi parametresi diskin elastik davranışını etkileyen en önemli parametre olarak ortaya çıkmıştır.

Anahtar kelimeler: Ortotropik diskler, Değişken kesit, Termo elastisite, Hipergeometrik denklem

1 Introduction

The mechanical response of rotating and stationary disks has been treated extensively by researchers because of the importance of these structures in various branches of engineering [1]-[14]. It appears that most of these investigations involve isotropic or functionally graded disk materials.

An orthotropic disk is the one in which the modulus of elasticity, E, and the Poisson's ratio, ν, differ in radial and circumferential directions. The ratio of the modulus of elasticity in one direction to the other is considered as the measure of material orthotropy [15]. Wood is an example of a natural orthotropic material in which material properties in radial and circumferential directions are different [16],[17]. Graphite-epoxy, glass-epoxy and plywood disks are among artificial orthotropic ones [15].

Although to a lesser extent than isotropic disks, there appear numerable research articles on the subject of disk orthotropy in the literature. These are shortly mentioned here in chronological order. Dumir and Mehta [15] have numerically investigated the stresses in orthotropic, uniform thickness, annular disks under external or internal pressure. The stress response of rotating orthotropic uniform thickness circular plates has been studied by Tutuncu [18] using laminated plate theory. Jain et al. [19] have proposed a calculation procedure to design uniform strength orthotropic constant thickness disks by adjusting the elastic orthotropy parameter in the radial direction. In a similar work, Guven et al. [20] have determined transverse vibrations of an orthotropic, variable thickness, solid disk. The degree of orthotropy has been adjusted radially so that the corresponding stress component remained constant. The stresses in rotating, orthotropic, constant thickness, annular disks have been determined analytically by Callioglu [21] in existence of a prescribed radial temperature gradient. In a later work, Callioglu et al. [22] have derived an analytical solution to determine the stress response in uniform thickness, isothermal, annular, rotating disks in the partially plastic state of stress. In a more recent work, Nie et al. [23] have described material tailoring in the radial direction to design orthotropic, rotating, uniform thickness annular disks with either constant radial stress or hoop stress or in-plane shear stress. Creep analysis based on Hill's yield criterion and Sherby's law in orthotropic, variable thickness, rotating annular disks has been the subject of the investigation carried out by Gupta and Singh [24]. Recently, a general formulation has been realized by Lubarda [25] to investigate the elastic behavior of pressurized, orthotropic, annular disks, hollow cylinders and spherical shells. In the most recent work that appear in the literature, Eraslan et al. [26] have developed a computational model to analyze partially plastic stresses in an orthotropic variable thickness disk under external pressure. Using Hill's quadratic yield criterion and a Swift type hardening law a nonlinear hardening material behavior has been simulated.

This work deals with the analysis of stress and deformation in orthotropic disks under different boundary conditions. An analytical model is developed for this purpose. The fact that material properties vary in different coordinate directions in an orthotropic disk brings additional difficulty in the analytical treatment. A general derivation which takes into account orthotropy, thickness variability, and the existence of a radial temperature gradient is carried out. The variation of the disk thickness along the radial direction is described by the thickness function h given by

$$h(r) = h_0\left[1 - n\left(\frac{r}{b}\right)^k\right] \tag{1}$$

in which h_0 is the thickness at the center, r the radial coordinate, b the radius of the disk, n and k the thickness parameters. Considering this thickness profile and using the equation of equilibrium, the compatibility relation, an orthotropic form of Hooke's law and the strain-displacement relations, the governing differential equation describing the elastic response of the disk is obtained in terms of a predefined stress function. The elastic equation turns into a familiar hypergeometric type by performing an appropriate transformation. The analytical solution is then obtained in terms of hypergeometric functions. Three different boundary conditions to model realistic loading conditions are handled.

2 Formulation and solution

2.1 Basic equations

The equation of motion

$$\frac{d}{dr}(hr\sigma_r) - h\sigma_\theta + h\rho\omega^2 r^2 = 0, \tag{2}$$

the equations of the generalized Hooke's Law

$$\varepsilon_r = \frac{du}{dr} = \frac{\sigma_r}{E_r} - \frac{\upsilon_{\theta r}}{E_\theta}\sigma_\theta, \tag{3}$$

$$\varepsilon_\theta = \frac{u}{r} = \frac{\sigma_\theta}{E_\theta} - \frac{\upsilon_{r\theta}}{E_r}\sigma_r, \tag{4}$$

the compatibility equation

$$\frac{d}{dr}(r\varepsilon_\theta) - \varepsilon_r = 0, \tag{5}$$

and the Maxwell relation

$$\frac{\upsilon_{\theta r}}{E_\theta} = \frac{\upsilon_{r\theta}}{E_r}, \tag{6}$$

form the basic equations of the problem [1]-[4]. In these equations σ_r and σ_θ represent the radial and circumferential stress components, ρ the mass density of the disk material, ω the angular speed, ε_r and ε_θ the radial and circumferential strains, u the displacement in the radial direction, E_r and E_θ the elasticity moduli in coordinate directions and $\upsilon_{r\theta}$ and $\upsilon_{\theta r}$ the Poisson's ratios in coordinate directions. Introducing a ratio which will be referred to as the elastic orthotropy parameter

$$R = \frac{E_r}{E_\theta}, \tag{7}$$

the Maxwell relation takes the form

$$\upsilon_{r\theta} = R\upsilon_{\theta r}. \tag{8}$$

Accordingly, the total elastic strains can be written as

$$\varepsilon_r = \tfrac{1}{E}(\sigma_r - \upsilon\sigma_\theta) + \alpha\Delta T,$$

$$\varepsilon_r = \frac{1}{E}(\sigma_r - \upsilon\sigma_\theta) + \alpha\Delta T, \tag{9}$$

$$\varepsilon_\theta = \frac{1}{E}(R\sigma_\theta - \upsilon\sigma_r) + \alpha\Delta T, \tag{10}$$

in which α represents the coefficient of thermal expansion, ΔT the temperature gradient in the radial direction, and $E = E_r$, $\upsilon = \upsilon_{r\theta}$. At this stage, basic equations are put into their nondimensional and normalized forms for convenience. For this purpose, we use the following variables: the radial coordinate $\overline{r} = r/b$, the thickness function $\overline{h} = h/h_0$, the stress $\overline{\sigma} = \sigma/\sigma_0$, the angular speed $\Omega = \omega b\sqrt{\rho/\sigma_0}$, the strain $\overline{\varepsilon} = \varepsilon E/\sigma_0$, the displacement $\overline{u} = uE/b\sigma_0$, the coefficient of thermal expansion $\overline{\alpha} = \alpha E/\sigma_0$, where σ_0 is the yield strength of the disk material. From here on nondimensional variables are used without overbars for convenience. The equation of motion and the equations of Hooke's law respectively take the forms

$$\frac{d}{dr}(hr\sigma_r) - h\sigma_\theta + h\Omega^2 r^2 = 0, \tag{11}$$

$$\varepsilon_r = \frac{du}{dr} = \sigma_r - \upsilon\sigma_\theta + \alpha\Delta T, \tag{12}$$

$$\varepsilon_\theta = \frac{u}{r} = R\sigma_\theta - \upsilon\sigma_r + \alpha\Delta T. \tag{13}$$

The nondimensional compatibility relation has a form similar to Eq. (5) as the overbars are not used.

2.2 The elastic equation

Introducing the stress function

$$Y(r) = hr\sigma_r, \tag{14}$$

and using the equation equilibrium, Eq. (2), the stresses take the forms

$$\sigma_r = \frac{Y}{h\,r}\ ,\quad \sigma_\theta = r^2\Omega^2 + \frac{1}{h}\frac{dY}{dr}. \tag{15}$$

With stresses expressed in Y and with the nondimensional thickness function

$$h(r) = 1 - nr^k, \tag{16}$$

the elastic strains read

$$\varepsilon_r = \frac{Y}{r(1-nr^k)} - \nu\left(r^2\Omega^2 + \frac{1}{1-nr^k}\frac{dY}{dr}\right) + \alpha\Delta T, \tag{17}$$

$$\varepsilon_\theta = -\frac{\nu Y}{r(1-nr^k)} + R\left(r^2\Omega^2 + \frac{1}{1-nr^k}\frac{dY}{dr}\right) + \alpha\Delta T. \tag{18}$$

The elastic equation is obtained by substitution of the strains into the compatibility relation, Eq. (5). The result is

$$\begin{aligned} r^2(1-nr^k)\frac{d^2Y}{dr^2} + r[1-(1-k)nr^k]\frac{dY}{dr} - \left[\frac{1-(1-k\upsilon)nr^k}{R}\right]Y \\ = -\frac{(1-nr^k)^2(\upsilon+3R)\Omega^2 r^3}{R} \\ + \frac{(1-nr^k)^2 r^2\alpha}{R}\frac{dT}{dr}. \end{aligned} \tag{19}$$

2.3 Analytical solution

The homogeneous part of the elastic equation is

$$r^2(1-nr^k)\frac{d^2Y}{dr^2}+r[1-(1-k)nr^k]\frac{dY}{dr}-\left[\frac{1-(1-k\nu)nr^k}{R}\right]Y=0. \quad (20)$$

Using the transformation $Y(r)=\phi(z)$ with $z=nr^k$ we first derive

$$\frac{dY}{dr}=knr^{k-1}\frac{d\phi}{dz}, \quad (21)$$

$$\frac{d^2Y}{dr^2}=kn(k-1)r^{k-2}\frac{d\phi}{dz}+k^2n^2r^{2(k-1)}\frac{d^2\phi}{dz^2}, \quad (22)$$

and replace r with $(z/n)^{1/k}$ which is substituted into Eq. (20). After tedious simplifications we arrive at

$$z(1-z)\frac{d^2\phi}{dz^2}+\frac{d\phi}{dz}-\left[\frac{1-z(1-k\nu)}{k^2Rz}\right]\phi=0. \quad (23)$$

This is a hypergeometric differential equation which assumes the exact solution

$$\phi(z)=A\,\phi_1(z)+B\,\phi_2(z), \quad (24)$$

where A and B are arbitrary constants and

$$\phi_1(z)=z^{-\frac{M}{k}}F(\alpha,\beta,\delta;z), \quad (25)$$

$$\phi_2(z)=z^{\frac{M}{k}}F(\alpha-\delta+1,\beta-\delta+1,2-\delta;z). \quad (26)$$

In these equations $M=1/\sqrt{R}$ and $F(\alpha,\beta,\delta;z)$ is the hypergeometric function with the following arguments:

$$F(\alpha,\beta,\delta;z)=1+\frac{\alpha\beta}{\delta 1!}z+\frac{\alpha(\alpha+1)\beta(\beta+1)}{\delta(\delta+1)2!}z^2+\frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{\delta(\delta+1)(\delta+2)3!}z^3+\ldots, \quad (27)$$

$$\alpha=-\frac{1}{2}-\frac{M}{k}-\frac{M\sqrt{4(1-k\nu)+k^2R}}{2k} \quad (28)$$

$$\beta=-\frac{1}{2}-\frac{M}{k}-\frac{M\sqrt{4(1-k\nu)+k^2R}}{2k} \quad (29)$$

$$\delta=1-\frac{2M}{k} \quad (30)$$

The general solution to the elastic equation, Eq (19), can now be written as

$$Y(r)=C_1Y_1(r)+C_2Y_2(r)+Y_P(r), \quad (31)$$

in which C_1 and C_2 are constants, $Y_1(r)$, $Y_2(r)$ and $Y_P(r)$ are two homogeneous and particular solutions, respectively. The homogeneous solutions take the form

$$Y_1(r)=r^{-M}F(\alpha,\beta,\delta;nr^k), \quad (32)$$

$$Y_2(r)=r^{M}F(\alpha-\delta+1,\beta-\delta+1,2-\delta;nr^k). \quad (33)$$

The method of variation of parameters is used to calculate $Y_P(r)$ as

$$Y_P(r)=U_1Y_1+U_2Y_2, \quad (34)$$

where

$$U_1(r)=\int_a^r G_1(\lambda)d\lambda;\ U_2(r)=\int_a^r G_2(\lambda)d\lambda, \quad (35)$$

with a being the dimensionless inner radius and

$$G_1(r)=-\frac{Y_2(r)\,f(r)}{W(r)}\text{ and }G_2(r)=-\frac{Y_1(r)\,f(r)}{W(r)}, \quad (36)$$

$$f(r)=-\frac{(1-nr^k)(\upsilon+3R)\Omega^2r}{R}-\frac{\alpha}{R}\frac{dT(r)}{dr}, \quad (37)$$

$$W(r)=Y_1(r)\frac{dY_2}{dr}-Y_2(r)\frac{dY_1}{dr}. \quad (38)$$

As U_1 and U_2 have polynomial integrands, the integrals in Eq. (35) can be evaluated exactly by expanding them in series at Gaussian points:

$$U_1(r)=\frac{r-a}{2}\sum_{i=1}^{N}\Phi_i\times G_1\left(\frac{(r-a)X_i+r+a}{2}\right), \quad (39)$$

$$U_2(r)=\frac{r-a}{2}\sum_{i=1}^{N}\Phi_i\times G_2\left(\frac{(r-a)X_i+r+a}{2}\right), \quad (40)$$

where Φ_i's are the weights and X_i's the roots. Note that $U_1(a)=U_2(a)=0$, and as a result $Y_P(a)=0$.

The stresses and displacement are then determined from

$$\sigma_r(r)=\frac{1}{h\,r}[C_1Y_1(r)+C_2Y_2(r)+Y_P(r)], \quad (41)$$

$$\sigma_\theta(r)=r^2\Omega^2+\frac{1}{h}\left[C_1\frac{dY_1}{dr}+C_2\frac{dY_2}{dr}+\frac{dY_P}{dr}\right], \quad (42)$$

$$u(r)=\frac{C_1}{h}\left[rR\frac{dY_1}{dr}-\nu Y_1\right]+\frac{C_2}{h}\left[rR\frac{dY_2}{dr}-\nu Y_2\right]+\frac{1}{h}\left[rR\frac{dY_P}{dr}-\nu Y_P\right]+R\Omega^2r^3+r\alpha\Delta T. \quad (43)$$

It should be noted that according to the Hill's quadratic yield condition, the yield stress σ_Y is obtained from [27].

$$\sigma_Y=\sqrt{\sigma_r^2-\frac{2R^*}{1+R^*}\sigma_r\sigma_\theta+\sigma_\theta^2}, \quad (44)$$

where R^* is another orthotropy parameter. When $R^*=1$, this criterion reduces to the well-known von Mises's yield criterion. The elastic equation is valid as long as $\sigma_Y\le 1$, and the elastic limit load corresponds to $\sigma_Y=1$.

2.4 Evaluation of integration constants

2.4.1 Rotating annular disk

In the case of the rotating annular disk, the boundary conditions read $\sigma_r(a)=0$ and $\sigma_r(1)=0$. Accordingly, integrating constants are found to be

$$C_1=-\frac{Y_P(1)Y_2(a)}{Y_1(a)Y_2(1)-Y_1(1)Y_2(a)}, \quad (45)$$

$$C_2=-\frac{Y_P(1)Y_1(\alpha)}{Y_1(1)Y_2(\alpha)-Y_1(\alpha)Y_2(1)}. \quad (46)$$

2.4.2 Disk subjected to internal pressure

The boundary conditions in this case are $\sigma_r(a) = -P_{in}$, and $\sigma_r(1) = 0$ where P_{in} is the nondimensional internal pressure. Thus, the integration constants become

$$C_1 = \frac{aP_{in}h(a)Y_2(1) - Y_2(a)Y_P(1)}{Y_1(1)Y_2(a) - Y_1(a)Y_2(1)}, \tag{47}$$

$$C_2 = \frac{aP_{in}h(a)Y_1(1) - Y_1(a)Y_P(1)}{Y_1(a)Y_2(1) - Y_1(1)Y_2(a)}. \tag{48}$$

2.4.3 Disk subjected to external pressure

The conditions take the form $\sigma_r(a) = 0$ and $\sigma_r(1) = -P_{ex}$ so that we determine

$$C_1 = \frac{Y_2(a)[P_{ex}h(1) + Y_P(1)]}{Y_1(a)Y_2(1) - Y_1(1)Y_2(a)}, \tag{49}$$

$$C_2 = -\frac{Y_1(a)[P_{ex}h(1) + Y_P(1)]}{Y_1(a)Y_2(1) - Y_1(1)Y_2(a)}, \tag{50}$$

where P_{ex} represents the nondimensional external pressure.

3 Results and discussion

In the following calculations, Poisson's ratio $\nu = \upsilon_{r\theta} = 0.3$ and in the stress-displacement diagrams the solid lines represent the results of the orthotropic, variable thickness disk, whereas the dashed-lines represent the isotropic uniform thickness disk.

The rotating annular disk yields at the inner surface $r = a$. Since at this location $\sigma_r(a) = 0$, the yield condition, Eq. (44) reduces to

$$\sigma_Y = \sigma_\theta. \tag{51}$$

Hence, the elastic limit turns out independent of the plastic orthotropy parameter R^* and corresponds to $\sigma_\theta = 1$. For a uniform thickness isotropic annular disk of inner radius $a = 0.4$ the elastic limit angular speed is determined as $\Omega = 1.08274$. The corresponding integration constants are calculated as $C_1 = -0.065$ and $C_2 = 0.406213$. Taking the parameter values $R = 1.4$, $n = 0.4$, $k = 0.8$, the elastic limit angular speed for the orthotropic variable thickness rotating disk of the same inner radius is determined as $\Omega = 1.11906$ (The shape of the corresponding disk profile can be examined in App. A-(a)). The integration constants at this limit are $C_1 = -0.0828$ and $C_2 = 0.257816$. The stresses and displacement at this limiting load are then calculated and plotted in Fig. 1 in comparison to those in the isotropic uniform thickness disk at its elastic limit. As seen in this figure, although the stresses are not affected to a great extent by orthotropy and thickness variability, the effect on the displacement is obvious.

To investigate the effect of the elastic orthotropy parameter $R = E_r/E_\theta$ on the elastic limit rotating speed, a parametric analysis is carried out. An orthotropic disk of inner radius $a = 0.25$ accompanied by the parameter values $n = 0.5$ and $k = 1.2$ is taken into consideration. The elastic limit rotation speeds are calculated for different values of R in the range $0.5 < R < 1.5$. The results of these calculations are plotted in Fig. 2. As seen in this figure, the elastic limit angular speed decreases with the increasing value of the parameter R. However, the change in the limiting speed is not more than 7% in the range considered.

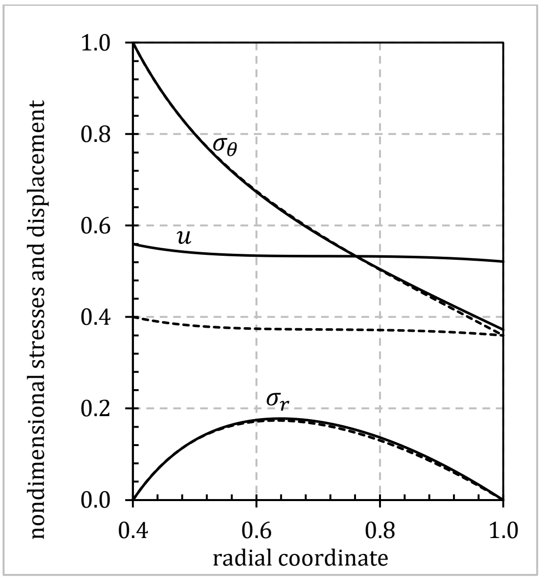

Figure 1: The states of stress and displacement in rotating annular disks at corresponding elastic limits.

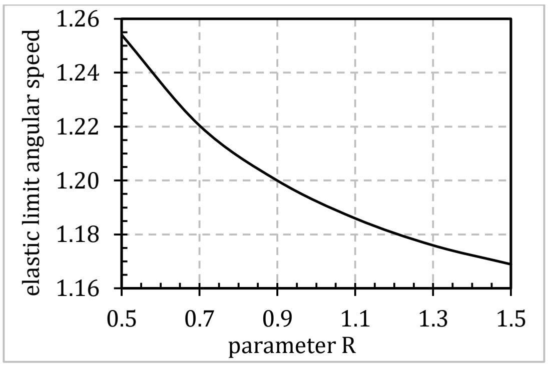

Figure 2: Variation of the elastic limit angular speed with the parameter $R = E_r/E_\theta$.

Like the rotating annular disk, the stationary annular disk subjected to internal pressure yields at the inner surface. Since at this location $\sigma_r(a) = -P_{in} \neq 0$, yielding takes place according to the Hill's quadratic yield condition given by Eq. (44). Hence, the plastic orthotropy parameter R^* is effective in this case. Note that $R^* = 1$ for the isotropic disk. The elastic limit internal pressure for a uniform thickness isotropic annular disk of inner radius $a = 0.4$ is determined as $P_{in} = 0.482918$. The corresponding integration constants are $C_1 = -0.092$ and $C_2 = 0.092$. The orthotropic variable thickness disk of the same inner radius possessing the parameters $R = 0.5$, $R^* = 1.1$, $n = 0.4$, $k = 0.6$ reaches the elastic limit when $P_{in} = 0.405606$. Under this load, the integration constants take the values $C_1 = -0.0468$ and $C_2 = 0.039$. The corresponding states of stress and displacement are compared in Fig. 3. Again, the difference in the displacement is apparent. The effects of both orthotropy parameters R and R^* can be visualized in Fig. 4. The parameters for this disk are chosen to be $a = 0.25, n = 0.5$ and $k = 1.2$ (The corresponding disk profile can be seen in App. A-(b)). As seen in Fig. 4, the increase in R increases the elastic limit internal pressure, and conversely the increase in R^* reduces this pressure.

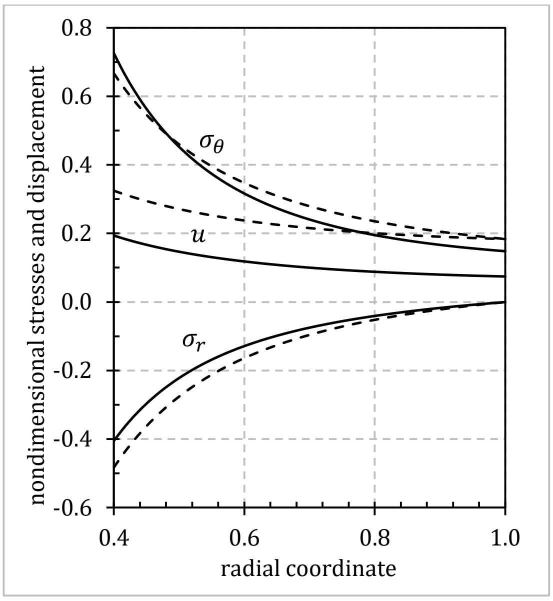

Figure 3: The states of stress and displacement in stationary annular disks subjected to internal pressure at corresponding elastic limits.

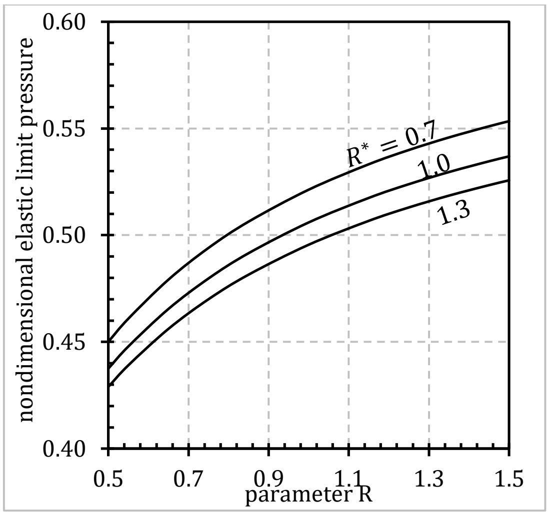

Figure 4: Variation of the elastic limit internal pressure with the parameter $R = E_r/E_\theta$ using R^* (see Eq. (44)) as another parameter.

Finally, we consider stationary disks under external pressure. The inner surface is the location of the maximum principal stresses. The yield criterion is the one in Eq. (51) as $\sigma_r = 0$ at this location. The elastic limit external pressure is independent of R^*. For the stationary constant thickness isotropic disk of $a = 0.4$ the elastic limit external pressure is determined to be $P_{ex} = 0.42$ which corresponds to the constants $C_1 = 0.08$ and $C_2 = -0.5$. On the other hand, this limit is $P_{ex} = 0.570813$ for an orthotropic variable thickness disk of $a = 0.4$, $R = 0.7$, $n = 0.4$, $k = 1.2$. The corresponding integration constants for the orthotropic disk are calculated as $C_1 = 0.0543$ and $C_2 = -0.881925$. The stresses and displacement in this loading are plotted in Fig. 5. As can be seen in this figure, the stresses differ considerably at the outer surface, i.e. at the pressurized surface in accordance with the different P_{ex} values for isotropic and orthotropic disks.

For an orthotropic disk having the parameters $a = 0.25$, $n = 0.5$ and $k = 1.2$ the effect of the elastic orthotropy parameter R on the elastic limit external pressure can be visualized in Fig. 6. As seen in this figure, the elastic limit pressure decreases notably with the increasing value of the parameter R.

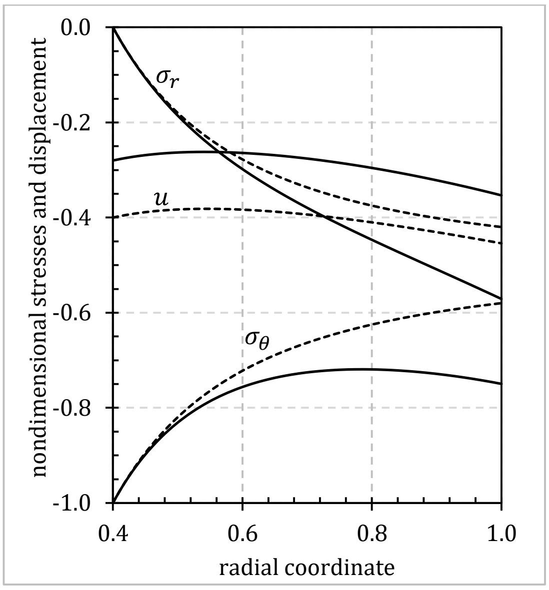

Figure 5: The states of stress and displacement in stationary annular disks subjected to external pressure at corresponding elastic limits.

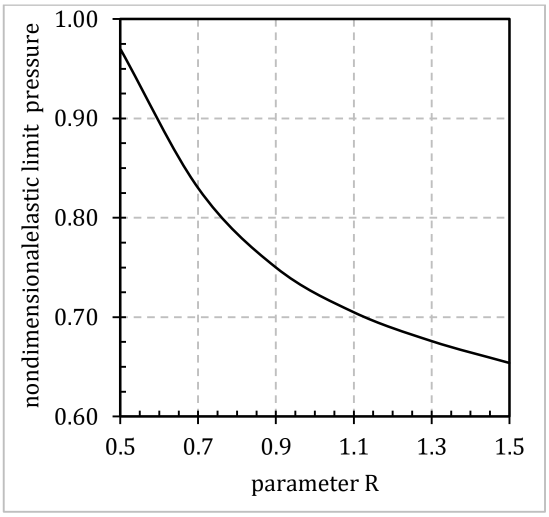

Figure 6: Variation of the elastic limit external pressure with the parameter $R = E_r/E_\theta$.

4 Conclusion

In this work concise analytical treatments of orthotropic variable thickness disk problems in the elastic state of stress are presented. In the formulation the ratio of the modulus of elasticity in radial direction to the one in circumferential direction, i.e. E_r/E_θ, is considered as the elastic material orthotropy parameter and is denoted by R. The analytical solution derived is applied to simulate rotating annular disks with two free surfaces and stationary annular disks with pressurized and free surfaces. Elastic limit loads are determined by the use of Hill's quadratic yield criterion described by Eq. (44), which contains another orthotropy parameter shown by R^*.

The stress state diagrams indicate that, for the three cases investigated, the inner surface of the disk is critical in the sense that the difference between the principal stresses takes its maximum value at that location. In case of rotating orthotropic variable thickness annular disks with two free surfaces, it is observed that, the elastic limit angular speed depends only on the elastic orthotropy parameter R. The elastic limit angular speed decreases with increasing value of R. The stationary annular orthotropic disk subjected to internal pressure yields at the inner surface and the plastic orthotropy parameter R^* in Hill's yield criterion is effective in this case. A parametric study reveals that the increase in R increases the elastic limit internal pressure, and conversely the increase in R^* reduces this pressure. In case of the stationary annular orthotropic disk subjected to external pressure the elastic limit pressure turns out to be independent of R^* like in the rotating disk. However, the limit depends strongly on the elastic orthotropy parameter R such that it decreases notably with the increasing value of the parameter R.

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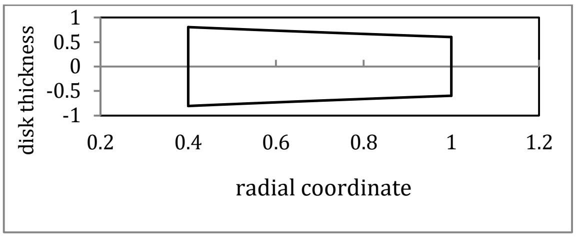
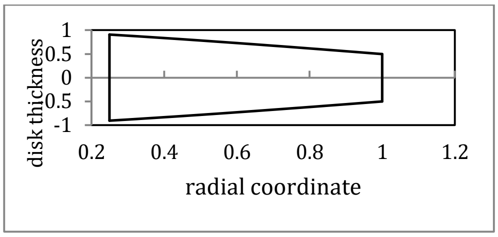
Appendix A

Disk profiles for different parameters are given in the following figures in which dimensionless disk profile is plotted against dimensionless radial coordinate for each.



a: Disk profile for the parameters *n*=0.4, *k*=0.8.



b: Disk profile for the parameters *n*=0.5, *k*=1