NILPOTENT LENGTH OF A FINITE SOLVABLE GROUP WITH A COPRIME FROBENIUS GROUP OF AUTOMORPHISMS

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Abstract We prove that a finite solvable group G admitting a Frobenius group FH of automorphisms of coprime order with kernel F and complement H so that $[G,F]=G$ and $C_{C_G(F)}(h)=1$ for every $1\neq h\in H$, is of nilpotent length equal to the nilpotent length of the subgroup of fixed points of H.

1. Introduction

Let A be a finite group that acts on the finite solvable group G by automorphisms. There have been a lot of research to obtain information about certain group theoretical invariants of G in terms of the action of A on G. A particular major problem is to bound the nilpotent length of G in terms of information about the structure of A alone when $C_G(A)=1$, that is, the action of A is fixed point free. One of the recent results in this framework is [1] in which Khukhro handled the case where $A=FH$ is a Frobenius group with kernel F and complement H. He proved that the nilpotent lengths of G and $C_G(H)$ are the same if $C_G(F)=1$ and $(|G|,|H|)=1$ and later in [2], he removed the coprimeness assumption of the theorem in [1]. In the present paper, we keep the coprimeness condition but weaken the fixed point freeness of F on G slightly, and obtain the same conclusion about the nilpotent length of G. More precisely, we prove the following:

Theorem 2.1 *Let G be a finite solvable group admitting a Frobenius group of automorphisms FH of coprime order with kernel F and complement H so that $C_{C_G(F)}(h)=1$ for every $1\neq h\in H$. Then $f([G,F])=f(C_{[G,F]}(H))$ and $f(G)\leq f([G,F])+1$.*

We obtained Proposition 2.2. below as a key result in proving Theorem 2.1.

Proposition 2.2 *Let Q be a normal q-subgroup of a group having a complement FH which is a Frobenius group with kernel F and complement H so that $C_{C_Q(F)}(h)=1$ for every $1\neq h\in H$. Assume further that $|FH|$ is not divisible by q and Q is of class at most 2. Let V be a $kQFH$-module where k is a field with characteristic not dividing $|QFH|$. Then we have*

$$Ker(C_{[Q,F]}(H) \text{ on } C_V(H)) = Ker(C_{[Q,F]}(H) \text{ on } V).$$

Throughout the article all groups are finite. The notation and terminology are mostly standard with the exception that $f(G)$ denotes the nilpotent length of the group G.

2. Main Results

In this section we establish Theorem 2.1. We also prove Proposition 2.2 which is crucial in proving Theorem 2.1. and of independent interest, too.

Theorem 2.1. *Let G be a finite solvable group admitting a Frobenius group of automorphisms FH of coprime order with kernel F and complement H so that $C_{C_G(F)}(h) = 1$ for every $1 \neq h \in H$. Then $f([G,F]) = f(C_{[G,F]}(H))$ and $f(G) \leq f([G,F]) + 1$.*

Proof. $C_G(F)$ is nilpotent by a well known result due to Thompson and $G/[G,F]$ is covered by the image of $C_G(F)$. Then $f(G) \leq f([G,F]) + 1$. Hence we may assume $G = [G,F]$ and prove that $f(G) = f(C_G(H))$. Let $f(G) = n$. We proceed by induction on the order of G. As $G = [G,F]$ and $(|G|, |FH|) = 1$, there exists an irreducible FH-tower $\hat{P}_1, ..., \hat{P}_n$ in the sense of [3] where

(a) $\hat{P}_i$ is an FH-invariant p_i-subgroup, p_i is a prime, $p_i \neq p_{i+1}$, for $i = 1, ..., n-1$;
(b) $\hat{P}_i \leq N_G(\hat{P}_j)$ whenever $i \leq j$;
(c) $P_n = \hat{P}_n$ and $P_i = \hat{P}_i / C_{\hat{P}_i}(P_{i+1})$ for $i = 1, ..., n-1$ and $P_i \neq 1$ for $i = 1, ..., n$;
(d) $\Phi(\Phi(P_i)) = 1$, $\Phi(P_i) \leq Z(P_i)$, and $exp(P_i) = p_i$ when p_i is odd for $i = 1, ..., n$;
(e) $[\Phi(P_{i+1}), P_i] = 1$ and $[P_{i+1}, P_i] = P_{i+1}$ for $i = 1, ..., n-1$;
(f) $(\Pi_{j<i} \hat{P}_j)FH$ acts irreducibly on $P_i/\Phi(P_i)$ for $i = 1, ..., n$;
(g) $P_1 = [P_1, F]$.

Set now $X = \hat{P}_1 .... \hat{P}_n$. As $P_1 = [P_1, F]$ by *(g)*, we observe that $X = [X,F]$ and so F is not contained in $Ker(FH \text{ on } X)$. Therefore $FH/Ker(FH \text{ on } X)$ is a Frobenius group of automorphisms of the group X. If X is proper in G, by induction we have $f(X) = f(C_X(H)) = n$. So $f(C_G(H)) = n$ and the theorem follows. Hence we can assume that $X = G$. We set next $\overline{G} = G/F(G)$. The group $\overline{G}$ is nontrivial, because otherwise $G = F(G)$ implying that the conclusion of the theorem is true since $C_G(H) \neq 1$ by Lemma 1.3 in [1]. As $\overline{G} = [\overline{G}, F]$, it follows by induction that $f(\overline{G}) = n-1 = f(C_{\overline{G}}(H))$. That is, $Y = [C_{\hat{P}_{n-1}}(H), ..., C_{\hat{P}_1}(H)] \nleq F(G) \cap \hat{P}_{n-1} = C_{\hat{P}_{n-1}}(\hat{P}_n)$. Notice that Y centralizes $C_{\hat{P}_n}(H)$ because otherwise $f(C_G(H)) = n$ which is not the case. Then, as $Y \leq Ker(C_{\hat{P}_{n-1}}(H) \text{ on } C_{\hat{P}_n}(H))$, we have

$$Ker(C_{\hat{P}_{n-1}}(H) \text{ on } C_{\hat{P}_n/\Phi(\hat{P}_n)}(H)) \neq Ker(C_{\hat{P}_{n-1}}(H) \text{ on } \hat{P}_n/\Phi(\hat{P}_n)) \leq C_{\hat{P}_{n-1}}(\hat{P}_n)$$

Note that $C_{[\hat{P}_{n-1},F]}(H) = C_{\hat{P}_{n-1}}(H)$ as $C_{\hat{P}_{n-1}}(FH) = 1$. Also $[\hat{P}_{n-1}, F] \neq 1$ as we have $[G,F] = G$. Now the action of the group $\hat{P}_{n-1}FH$ on $\hat{P}_n/\Phi(\hat{P}_n)$ satisfies the hypothesis of the proposition below and we get a contradiction completing the proof.

□

Proposition 2.2. *Let Q be a normal q-subgroup of a group having a complement FH which is a Frobenius group with kernel F and complement H so that $C_{C_Q(F)}(h) = 1$ for every $1 \neq h \in H$. Assume further that $|FH|$ is not divisible by q and Q is of class at most 2. Let V be a $kQFH$-module where k is a field with characteristic not dividing $|QFH|$. Then we have*

$$Ker(C_{[Q,F]}(H) \text{ on } C_V(H)) = Ker(C_{[Q,F]}(H) \text{ on } V).$$

Proof. We proceed by induction on $dim(V) + |QFH|$ over a sequence of steps. We may assume that k is a splitting field for all subgroups of QFH. To simplify the notation we set $K = Ker(C_Q(H) \text{ on } C_V(H))$.

Claim 1. We have $Q = [Q, F]$ and hence $C_Q(F) \leq Q' \leq Z(Q)$.

Proof. We may assume that $[Q, F]$ acts nontrivially on V. If $[Q, F]$ is a proper subgroup of Q, we get the conclusion of the theorem by applying induction to the action of the group $[Q, F]FH$ on V. This contradiction shows that $[Q, F] = Q$ and hence $C_Q(F) \leq Q'$. □

Claim 2. V is an irreducible QFH-module on which Q acts faithfully.

Proof. For each irreducible QFH-component W of V on which Q acts nontrivially, if $W \neq V$ then $Ker(C_Q(H) \text{ on } C_W(H)) = Ker(C_Q(H) \text{ on } W)$ holds by induction. Using the fact that V is completely reducible as a QFH-module, in case V is not irreducible, we have

$$Ker(C_Q(H) \text{ on } V) = \bigcap_W Ker(C_Q(H) \text{ on } W) = \bigcap_W Ker(C_Q(H) \text{ on } C_W(H)) = K$$

which is nothing but the claim of the theorem. Therefore we can regard V as an irreducible QFH-module. We set next $\overline{Q} = Q/Ker(Q \text{ on } V)$ and consider the action of the group $\overline{Q}FH$ on V. An induction argument gives

$$Ker(C_{\overline{Q}}(H) \text{ on } C_V(H)) = Ker(C_{\overline{Q}}(H) \text{ on } V)$$

which leads to a contradiction. Thus we may assume that Q acts faithfully on V. □

It should be noted that we need only to prove $K = 1$ due to the faithful action of Q on V. So we assume this to be false.

Claim 3. FH acts faithfully on V.

Proof. We have $C_{FH}(V) = C_F(V)C_H(V)$. Suppose first that $C_H(V) \neq 1$. Then the group $FC_H(V)$ is Frobenius with kernel F and complement $C_H(V)$, and hence $[F, C_H(V)] = F$. It follows by the three-subgroup lemma that $[V, F] = 1$ and so $[V, Q] = 1$ which is not the case. Thus we have $C_H(V) = 1$. Notice that $C_F(V)$ centralizes Q, and hence we can consider the action of the group $(QFH)/C_F(V)$ on V. This yields the conclusion of the theorem by induction in case $C_F(V) \neq 1$, and so the claim is established. □

Claim 4. Let Ω denote the set of Q-homogeneous components of V, and let Ω_1 be an F-orbit on Ω. Set $H_1 = Stab_H(\Omega_1)$. Then H_1 is a nontrivial subgroup of H stabilizing exactly one element W of Ω_1 and all the remaining orbits of H_1 on Ω_1 are of length $|H_1|$. Furthermore K acts trivially on each member of Ω_1 except W.

Proof. Suppose that $H_1 = 1$. Pick an element W from Ω_1. Clearly, we have $Stab_H(W) \leq H_1 = 1$ and hence $X = \sum_{h \in H} W^h = \bigoplus_{h \in H} W^h$. It is straightforward to verify that $\left\{\sum_{h \in H} v^h \;\middle|\; v \in W\right\} \leq C_X(H)$. Since K normalizes each W^h and acts trivially on $C_X(H)$, K acts trivially on X. Notice that the action of

H on the set of F-orbits on Ω is transitive, and hence K acts trivially on the whole of V. Thus $H_1 \neq 1$.

Let now $S = Stab_{FH_1}(W)$ and $F_1 = F \cap S$. Then $|F : F_1| = |\Omega_1| = |FH_1 : S|$. Notice next that $(|S : F_1|, |F_1|) = 1$ as $(|F|, |H_1|) = 1$. Let S_1 be a complement of F_1 in S. Then we have $|F : F_1| = |F| |H_1| / |F_1| |S_1|$ which implies that $|H_1| = |S_1|$. Therefore we may assume that $S = F_1 H_1$, that is W is H_1-invariant.

It remains to show that W is the only member of Ω_1 which is stabilized by H_1, and all the remaining orbits are of length $|H_1|$: Let $x \in F$ and $1 \neq h \in H_1$ such that $(W^x)^h = W^x$ holds. Then $[h, x^{-1}] \in F_1$ and so $F_1 x = F_1 x^h = (F_1 x)^h$ implying the existence of an element $g \in F_1 x \cap C_F(h)$. Now the Frobenius action of H on F gives that $x \in F_1$. That is, for each $x \in F - F_1$, $Stab_{H_1}(W^x) = 1$. Then, as a consequence of the argument in the first paragraph, K acts trivially on W^x for every $x \in F - F_1$. □

Claim 5. F acts transitively on Ω and hence we have $H = H_1$.

Proof. The group H acts transitively on $\{\Omega_i | i = 1, 2, ..., s\}$, the collection of F-orbits on Ω. Let now $V_i = \bigoplus_{W \in \Omega_i} W$ for $i = 1, 2, ..., s$. Suppose that $s > 1$. Then $H_1 = Stab_H(\Omega_1)$ is a proper subgroup of H. Applying induction to the action of QFH_1 on V_1 we obtain

$$Ker(C_Q(H_1) \text{ on } C_{V_1}(H_1)) = Ker(C_Q(H_1) \text{ on } V_1).$$

It follows that $Ker(C_Q(H) \text{ on } C_{V_1}(H_1)) = Ker(C_Q(H) \text{ on } V_1)$ holds as $C_Q(H) \leq C_Q(H_1)$. On the other hand we have $C_V(H) = \{u^{x_1} + u^{x_2} + ... + u^{x_s} | u \in C_{V_1}(H_1)\}$ where $x_1, ..., x_s$ is a complete set of right coset representatives of H_1 in H. K acts trivially on $C_V(H)$ and normalizes each V_i. Then K is trivial on $C_{V_1}(H_1)$ and hence on V_1. As K is normalized by H we see that K acts trivially on each V_i and hence on V. This contradiction proves the claim. □

Claim 6. $C_Q(F) = 1$.

Proof. We observe that $C_Q(F) = [C_Q(F), H] \leq [Z(Q), H] \leq C_Q(W)$ as $Z(Q)$ acts by scalars on W. Then

$$C_Q(F) \leq \bigcap_{x \in F} C_Q(W)^x = C_Q(V) = 1,$$

□

Claim 7. Final Contradiction.

Proof. Since $1 \neq K \trianglelefteq C_Q(H)$, the group $L = K \cap Z(C_Q(H))$ is nontrivial. Let now $1 \neq z \in L$ and consider the group $\langle z^F \rangle$. As $C_Q(F) = 1$ we have $[\langle z^F \rangle, F] = \langle z^F \rangle$. An induction argument applied to the action of the group $\langle z^F \rangle FH$ on V shows that $\langle z^F \rangle = Q$. Note that $Q = [Q, H] C_Q(H)$ as $(|Q|, |H|) = 1$. We have $[Q, L, H] \leq [Q', H] \leq [Z(Q), H] \leq C_Q(W)$ and also $[L, H, Q] = 1$. It follows by the three subgroup lemma that $[H, Q, L] \leq C_Q(W)$. Clearly $[C_Q(H), L] = 1$ by the definition of L. Thus $LC_Q(W)/C_Q(W) \leq Z(Q/C_Q(W))$ and hence $z^f \in zC_Q(W)$ for any $f \in F_1$ due to the scalar action of $Z(Q/C_Q(W))$ on W. On the other hand we know that K acts trivially on $W^{g^{-1}}$ and hence $z^g \in C_Q(W)$ for any $g \in F - F_1$. So we have $\langle z^F \rangle = Q = \langle z \rangle C_Q(W)$ implying that $Q' \leq C_Q(W)$. This forces that $Q' = 1$, that is, Q is abelian. We consider now $\prod_{f \in F} z^f$. It is a well defined element of Q which lies in $C_Q(F) = 1$. Thus we have

$$1 = \prod_{f \in F} z^f = (\prod_{f \in F_1} z^f)(\prod_{f \in F - F_1} z^f) \in (\prod_{f \in F_1} z^f) C_Q(W) = z^{|F_1|} C_Q(W)$$

leading to the contradiction $z \in C_Q(W)$. This completes the proof of Proposition 2.2. □

References

[1] E.I.Khukhro, "Nilpotent length of a finite group admitting a Frobenius group of automorphisms with fixed-point-free kernel", Algebra and Logic, 49, 551-560, 2011

[2] E.I.Khukhro, "Fitting heigth of a finite group with a Frobenius group of automorphisms", Journal of Algebra 366, 1-11, 2012

[3] A.Turull, "Fitting Height of Groups and of Fixed Points", Journal of Algebra 86, 555-556 (1984)

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