

Integral laminations on non-orientable surfaces

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Abstract

We describe triangle coordinates for integral laminations on a non-orientable surface $N_{k,n}$ of genus k with n punctures and one boundary component, and give an explicit bijection from the set of integral laminations on $N_{k,n}$ to $(\mathbb{Z}^{2(n+k-2)} \times \mathbb{Z}^k) \setminus \{0\}$.

Keywords: non-orientable surfaces, triangle coordinates, Dynnikov coordinates

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1 Introduction

Let $N_{k,n}$ be a non-orientable surface of genus k with n punctures and one boundary component. In this paper we shall describe the generalized Dynnikov coordinate system for the set of integral laminations $\mathcal{L}_{k,n}$, and give an explicit bijection between $\mathcal{L}_{k,n}$ and $(\mathbb{Z}^{2(n+k-2)} \times \mathbb{Z}^k) \setminus \{0\}$. To be more specific, we shall first take a particular collection of $3n + 2k - 4$ arcs and k curves embedded in $N_{k,n}$, and describe each integral lamination by an element of $\mathbb{Z}_{\geq 0}^{3n+2k-4} \times \mathbb{Z}^k$, its geometric intersection numbers with these arcs and curves. *Generalized Dynnikov coordinates* are certain linear combinations of these integers that provide a one-to-one correspondence between $\mathcal{L}_{k,n}$ and $(\mathbb{Z}^{2(n+k-2)} \times \mathbb{Z}^k) \setminus \{0\}$.

The motivation for this paper comes from a recent work of Papadopoulos and Penner [7] where they provide analogues for non-orientable surfaces of several results from Thurston theory of surfaces which were studied only for orientable

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surfaces before [4, 8]. Here we shall give the analogy of the Dynnikov Coordinate System [2] on the finitely punctured disk which has several useful applications such as giving an efficient method for the solution of the word problem of the n -braid group [1], computing the geometric intersection number of integral laminations [9], and counting the number of components they contain [11].

Throughout the text we shall work on a standard model of $N_{k,n}$ as illustrated in Figure 1 where a disc with a cross drawn within it represents a crosscap, that is the interior of the disc is removed and the antipodal points on the resulting boundary component are identified (i.e. the boundary component bounds a Möbius band).

The structure of the paper is as follows. In Section 1.1 we give the necessary terminology and background. In Section 2 we describe and study the triangle coordinates for integral laminations on $N_{k,n}$, and construct the generalized Dynnikov Coordinate System giving the bijection $\rho: \mathcal{L}_{k,n} \rightarrow (\mathbb{Z}^{2(n+k-2)} \times \mathbb{Z}^k) \setminus \{0\}$. An explicit formula for the inverse of this bijection is given in Theorem 2.14.

1.1 Basic terminology and background

A simple closed curve in $N_{k,n}$ is inessential if it bounds an unpunctured disk, once punctured disk, or an unpunctured annulus. It is called essential otherwise. A simple closed curve is called *2-sided* (respectively *1-sided*) if a regular neighborhood of the curve is an annulus (respectively Möbius band). We say that a 2-sided curve is *non-primitive* if it bounds a Möbius band [7], and a 1-sided curve is *non-primitive* if it is a core curve of a Möbius band. They are called *primitive* otherwise.

An integral lamination \mathcal{L} on $N_{k,n}$ is a disjoint union of finitely many essential simple closed curves in $N_{k,n}$ modulo isotopy. Let $\mathcal{A}_{k,n}$ be the set of arcs α_i ($1 \leq i \leq 2n-2$), β_i ($1 \leq i \leq n+k-1$), γ_i ($1 \leq i \leq k-1$) which have each endpoint either on the boundary or at a puncture, and the curves c_i ($1 \leq i \leq k$) which are the core curves of Möbius bands in $N_{k,n}$ as illustrated in Figure 1: the arcs α_{2i-3} and α_{2i-2} for $2 \leq i \leq n$ join the i -th puncture to $\partial N_{k,n}$, the arc β_i has both end points on $\partial N_{k,n}$ and passes between the i -th and $(i+1)$ -st punctures for $1 \leq i \leq n-1$, the n -th puncture and the first crosscap for $i = n$, and the $(i-n)$ -th and $(i+1-n)$ -th crosscaps for $n+1 \leq i \leq n+k-1$. The arc γ_i ($1 \leq i \leq k-1$) has both endpoints on $\partial N_{k,n}$ and surrounds the i -th crosscap.

The surface is divided by these arcs into $2n+2k-2$ regions, $2n+k-3$ of these are triangular since each Δ_i ($1 \leq i \leq 2n-2$) and Σ_i ($1 \leq i \leq k-1$) is bounded by three arcs when the boundary of the surface is identified to a point. The two triangles Δ_{2i-3} and Δ_{2i-2} on the left and right hand side of the i -th puncture are defined by the arcs $\alpha_{2i-3}, \alpha_{2i-2}, \beta_{i-1}$ and $\alpha_{2i-3}, \alpha_{2i-2}, \beta_i$ respectively. The triangle Σ_i is defined by the arcs $\gamma_i, \beta_{n+i-1}, \beta_{n+i}$. Each Δ'_i ($1 \leq i \leq k-1$) is bounded by γ_i , and the two end regions Δ_0 and Δ'_k are bounded by β_1 and β_{n+k-1} respectively.

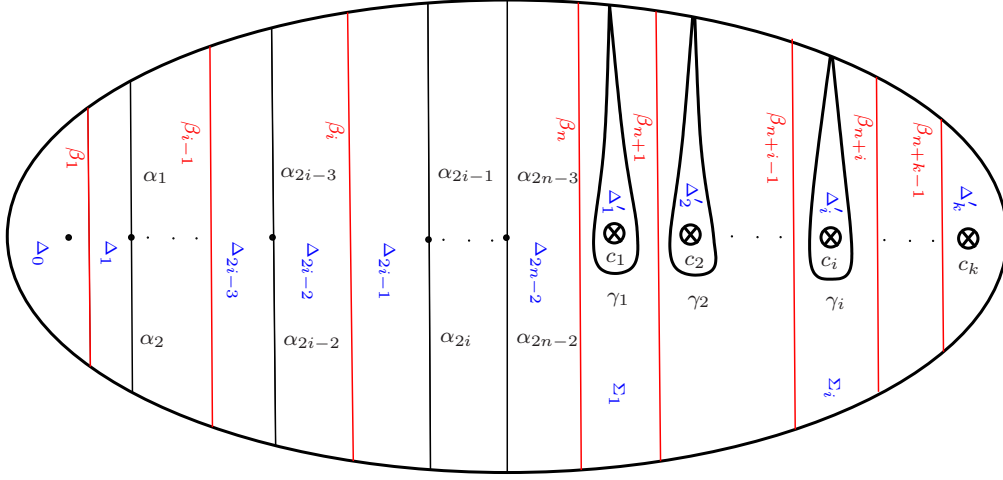


Figure 1: The arcs α_i , β_i , γ_i , the 1-sided curves c_1, c_2, \dots, c_k and the regions Δ_i and Σ_i .

Given $\mathcal{L} \in \mathcal{L}_{k,n}$, let L be a taut representative of \mathcal{L} with respect to the elements of $\mathcal{A}_{k,n}$. That is, L intersects each of the arcs and curves in $\mathcal{A}_{k,n}$ minimally.

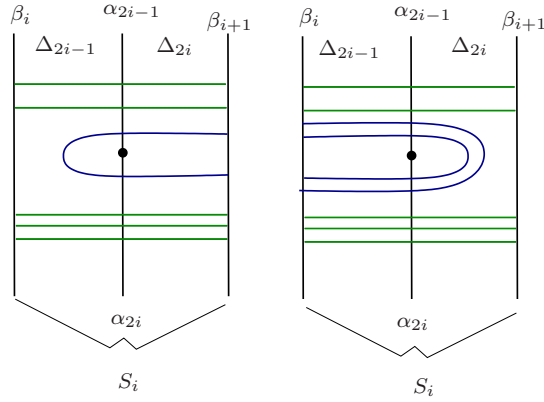


Figure 2: There is 1 left loop component in the first case and 2 right loop components in the second case. There are 2 above and 3 below components in each case.

Definition 1.1. Set $S_i = \Delta_{2i-1} \cup \Delta_{2i}$ for each i with $1 \leq i \leq n-1$. A *path component* of L in S_i is a component of $L \cap S_i$. There are four types of path components in S_i as depicted in Figure 2:

- An *above component* has end points on β_i and β_{i+1} , passing across α_{2i-1} ,
- A *below component* has end points on β_i and β_{i+1} , passing across α_{2i} ,
- A *left loop component* has both end points on β_{i+1} ,
- A *right loop component* has both end points on β_i .

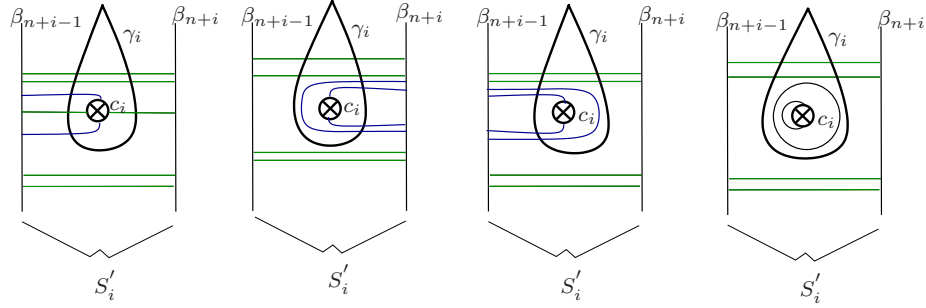


Figure 3: There is 1 right core loop and 1 straight core component in the first case; 1 left loop and 1 left core loop component in the second case; 1 right non-core loop and 1 right core loop component in the third case and 1 1-sided and 1 2-sided non-primitive curves in the fourth case. There are 2 above and 2 below components in each case.

Definition 1.2. Set $S'_i = \Delta'_i \cup \Sigma_i$ for each $1 \leq i \leq k-1$. A *path component* of L in S'_i is a component of $L \cap S'_i$. There are 7 types of path components in S'_i as depicted in Figure 3.

- An *above component* has end points on β_{n+i-1} and β_{n+i} , and passes across γ_i without intersecting c_i ,
- A *below component* has end points on β_{n+i-1} and β_{n+i} , and doesn't pass across γ_i ,
- A *left loop component* has both end points on β_{n+i} ,
- A *right loop component* has both end points on β_{n+i-1} ,

If a loop component intersects c_i , it is called *core loop component* otherwise it is called non-core loop component.

- A *straight core component* has end points on β_{n+i-1} and β_{n+i} , and intersects c_i ,

- A non-primitive 1-sided curve,

If L contains a non-primitive 1-sided curve c_i we depict it with a ring with end points on the i -th crosscap as shown in the fourth case in Figure 3.

- A non-primitive 2-sided curve.

2 Triangle coordinates

Let L be a taut representative of \mathcal{L} . Write $\alpha_i, \beta_i, \gamma_i$ and c_i for the geometric intersection number of L with the arc $\alpha_i, \beta_i, \gamma_i$ and the core curve c_i respectively. It will always be clear from the context whether we mean the arc or the geometric intersection number assigned on the arc.

Definition 2.1. The triangle coordinate function $\tau: \mathcal{L}_{k,n} \rightarrow (\mathbb{Z}_{\geq 0}^{3n+2k-4} \times \mathbb{Z}^k) \setminus \{0\}$ is defined by

$$\tau(\mathcal{L}) = (\alpha_1, \dots, \alpha_{2n-2}; \beta_1, \dots, \beta_{n+k-1}; \gamma_1, \dots, \gamma_{k-1}; c_1, \dots, c_k).$$

where $c_i = -1$ if L contains the i -th core curve; $c_i = -2m$ if it contains $m \in \mathbb{Z}^+$ disjoint copies 2-sided non-primitive curves around the i -th crosscap, and $c_i = -2m - 1$ if it contains m disjoint copies of 2-sided non-primitive curves around the i -th crosscap plus the i -th core curve.

Remark 2.2. Let $b_i = \frac{\beta_i - \beta_{i+1}}{2}$ for $1 \leq i \leq n+k-2$. Then in each S_i ($1 \leq i \leq n-1$) and S'_i ($n \leq i \leq n+k-2$) there are $|b_i|$ loop components. Furthermore, if $b_i < 0$ these loop components are left, and if $b_i > 0$ they are right.

The proof of the next lemma is obvious from Figure 2.

Lemma 2.3. Let $1 \leq i \leq n-1$. The number of above and below components in S_i are given by $a_{S_i} = \alpha_{2i-1} - |b_i|$ and $b_{S_i} = \alpha_{2i} - |b_i|$ respectively.

Let λ_i and λ_{c_i} denote the number of non-core and core loop components, ψ_i the number of straight core components, and $a_{S'_i}$ and $b_{S'_i}$ the number of above and below components in S'_i .

Lemma 2.4. Let L be a taut representative of $\mathcal{L} \in \mathcal{L}_{k,n}$, and set $c_i^+ = \max(c_i, 0)$. Then for each $1 \leq i \leq k-1$ we have

$$\begin{aligned} \lambda_i &= \max(|b_{n+i-1}| - c_i^+, 0), & \lambda_{c_i} &= \min(|b_{n+i-1}|, c_i^+), \\ \psi_i &= \max(c_i^+ - |b_{n+i-1}|, 0). \end{aligned}$$

Proof. Assume that L doesn't contain any non-primitive curve in S'_i . Since c_i gives the sum of straight core and core loop components and $|b_{n+i-1}|$ gives the sum of non-core loop and core loop components in S'_i (see Figure 3) we have

$$c_i = \psi_i + \lambda_{c_i} \quad \text{and} \quad |b_{n+i-1}| = \lambda_i + \lambda_{c_i}. \quad (1)$$

If $c_i > |b_{n+i-1}|$, then clearly there exists a straight core component in S'_i , and hence no non-core loop component in S'_i that is $\lambda_i = 0$. Therefore in this case, $\lambda_{c_i} = |b_{n+i-1}|$ and hence $\psi_i = c_i - |b_{n+i-1}|$ by Equation 1.

If $c_i < |b_{n+i-1}|$, there exists a non-core loop component in S'_i , and hence no straight core components in S'_i that is $\psi_i = 0$. Therefore $c_i = \lambda_{c_i}$ and hence $\lambda_i = |b_{n+i-1}| - c_i$ by Equation 1. We get

$$\begin{aligned} \lambda_i &= \max(|b_{n+i-1}| - c_i, 0) \\ \psi_i &= \max(c_i - |b_{n+i-1}|, 0). \end{aligned}$$

Also if $|b_{n+i-1}| < c_i$, $\lambda_i = 0$ and hence $\lambda_{c_i} = |b_{n+i-1}|$, if $|b_{n+i-1}| > c_i$, $\psi_i = 0$ and hence $\lambda_{c_i} = c_i$ by Equation 1. Therefore we get, $\lambda_{c_i} = \min(|b_{n+i-1}|, c_i)$.

Finally, if L contains a non-primitive curve in S'_i , there can be no straight core and core loop component in S'_i that is $\psi_i = \lambda_{c_i} = 0$, hence $\lambda_i = |b_{n+i-1}|$. Since $c_i < 0$ by definition, setting $c_i^+ = \max(c_i, 0)$ we can write

$$\begin{aligned} \lambda_i &= \max(|b_{n+i-1}| - c_i^+, 0), & \lambda_{c_i} &= \min(|b_{n+i-1}|, c_i^+), \\ \psi_i &= \max(c_i^+ - |b_{n+i-1}|, 0). \end{aligned}$$

□

Lemma 2.5. *Let L be a taut representative of $\mathcal{L} \in \mathcal{L}_{k,n}$. For each $1 \leq i \leq k-1$ we have*

$$\begin{aligned} a_{S'_i} &= \frac{\gamma_i}{2} - |b_{n+i-1}| - \psi_i \\ b_{S'_i} &= \max(\beta_{n+i-1}, \beta_{n+i}) - |b_{n+i-1}| - \frac{\gamma_i}{2}. \end{aligned}$$

Proof. To compute the number of above and below components in S'_i we observe that each path component other than a below component in S'_i intersects γ_i twice, that is $\gamma_i = 2(a_{S'_i} + |b_{n+i-1}| + \psi_i)$. Therefore we get,

$$a_{S'_i} = \frac{\gamma_i}{2} - |b_{n+i-1}| - \psi_i.$$

To compute the number of below components, we note that the sum of all path components in S'_i is given by $\beta = \max(\beta_{n+i-1}, \beta_{n+i})$. Then $b_{S'_i}$ is β minus the number of above, straight core components and twice the number loop components in S'_i (each loop component intersects β twice). We get

$$\begin{aligned} b_{S'_i} &= \max(\beta_{n+i-1}, \beta_{n+i}) - a_{S'_i} - 2|b_{n+i-1}| - \psi_i \\ &= \max(\beta_{n+i-1}, \beta_{n+i}) - |b_{n+i-1}| - \frac{\gamma_i}{2} \end{aligned}$$

□

Another way of expressing $a_{S'_i}$ and $b_{S'_i}$ is given in item P4. in Properties 2.12.

Remark 2.6. Observe that the loop components in Δ_0 are always left and the number of them is given by $\frac{\beta_1}{2}$. Similarly, the loop components in Δ'_k are always right and the number of core and non-core loop components in Δ'_k are given by c_k and $\lambda_k = \frac{\beta_{n+k-1}}{2} - c_k$.

Lemma 2.7 and Lemma 2.8 are obvious from Figure 2 and Figure 3.

Lemma 2.7. *There are equalities for each S_i :*

- When there are left loop components ($b_i < 0$),

$$\begin{aligned} \alpha_{2i} + \alpha_{2i-1} &= \beta_{i+1} \\ \alpha_{2i} + \alpha_{2i-1} - \beta_i &= 2|b_i|, \end{aligned}$$

- When there are right loop components ($b_i > 0$),

$$\begin{aligned} \alpha_{2i} + \alpha_{2i-1} &= \beta_i \\ \alpha_{2i} + \alpha_{2i-1} - \beta_{i+1} &= 2|b_i|, \end{aligned}$$

- When there are no loop components ($b_i = 0$),

$$\alpha_{2i} + \alpha_{2i-1} = \beta_i = \beta_{i+1}.$$

Lemma 2.8. *There are equalities for each S'_i :*

- When there are left loop components ($b_{n+i-1} < 0$),

$$\begin{aligned} a_{S'_i} + b_{S'_i} + \psi_i + 2|b_{n+i-1}| &= \beta_{n+i} \\ a_{S'_i} + b_{S'_i} + \psi_i &= \beta_{n+i-1} \end{aligned}$$

- When there are right loop components ($b_{n+i-1} > 0$)

$$\begin{aligned} a_{S'_i} + b_{S'_i} + \psi_i + 2|b_{n+i-1}| &= \beta_{n+i-1}. \\ a_{S'_i} + b_{S'_i} + \psi_i &= \beta_{n+i} \end{aligned}$$

- When there are no loop components $b_{n+i-1} = 0$

$$a_{S'_i} + b_{S'_i} + \psi_i = \beta_{n+i} = \beta_{n+i-1}.$$

Example 2.9. Let $\tau(\mathcal{L}) = (4, 2, 2, 6; 2, 6, 8, 4; 8; 1, 1)$ be the triangle coordinates of an integral lamination $\mathcal{L} \in \mathcal{L}_{2,3}$. We shall show how we draw \mathcal{L} from its given triangle coordinates. First, we compute the loop components in the two end regions Δ_0 and Δ'_2 using Remark 2.6. Since $\beta_1 = 2$ there is one loop component in Δ_0 . Similarly, since $\beta_4 = 4$ and $c_2 = 1$, we get $\lambda_2 = \frac{\beta_4}{2} - c_2 = 1$.

Next, we compute loop components in S_1 , S_2 and S'_1 . Since $b_i = \frac{\beta_i - \beta_{i+1}}{2}$ for each $1 \leq i \leq 3$ we have $b_1 = -2, b_2 = -1$. Hence there are two left loop components in S_1 , and one left component in S_2 . Similarly since $b_3 = 2$ there are 2 right loop components in S'_1 , and by Lemma 2.4, $\lambda_1 = \max(|b_3| - c_1, 0) = 1$ (hence $\psi_1 = 0$) and $\lambda_{c_1} = \min(|b_3|, c_1) = 1$. Using Lemma 2.3 and Lemma 2.5 we compute the number of above and below components. We get $a_{S_1} = \alpha_1 - |b_1| = 2$, $b_{S_1} = \alpha_2 - |b_1| = 0$, $a_{S_2} = \alpha_3 - |b_2| = 1$, $b_{S_2} = \alpha_4 - |b_2| = 5$, and

$$\begin{aligned} a_{S'_1} &= \frac{\gamma_1}{2} - |b_3| - \psi_1 = 2 \\ b_{S'_1} &= \max(\beta_3, \beta_4) - |b_3| - \frac{\gamma_1}{2} = 2. \end{aligned}$$

Connecting the path components in each Δ_0 , Δ'_2 , S_1 , S_2 and S'_1 we draw the integral lamination as shown in Figure 4.

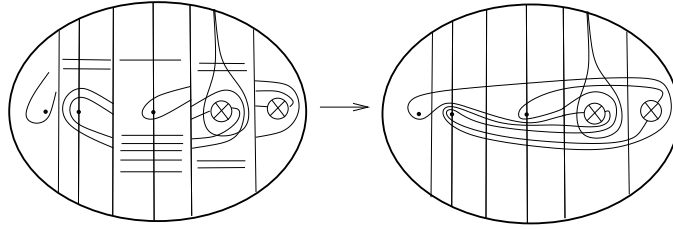


Figure 4: $\tau(L) = (4, 2, 2, 6; 2, 6, 8, 4; 8; 1, 1)$

Lemma 2.10. *The triangle coordinate function $\tau: \mathcal{L}_{k,n} \rightarrow (\mathbb{Z}_{\geq 0}^{3n+2k-4} \times \mathbb{Z}^k) \setminus \{0\}$ is injective.*

Proof. We can determine the number of loop, above and below components in each S_i by Remark 2.2 and Lemma 2.3; core and non-core loop, straight core, above and below components in each S'_i by Lemma 2.4 and Lemma 2.5 as illustrated in Example 2.9. The components in each S_i and S'_i are glued together in a unique way up to isotopy, and hence \mathcal{L} is constructed uniquely. \square

Remark 2.11. The triangle coordinate function $\tau: \mathcal{L}_{k,n} \rightarrow (\mathbb{Z}_{\geq 0}^{3n+2k-4} \times \mathbb{Z}^k) \setminus \{0\}$ is not surjective: an integral lamination must satisfy the triangle inequality in each S_i and S'_i , and some additional conditions such as the equalities in Lemma 2.7 and Lemma 2.8.

Next, we give a list of properties an integral lamination $\mathcal{L} \in \mathcal{L}_{k,n}$ satisfies in terms of its triangle coordinates as in [9], and then construct a new coordinate system from the triangle coordinates which describes integral laminations in a unique way. In particular, we shall generalize the Dynnikov coordinate system [1–3, 5, 9–11] for $N_{k,n}$.

Properties 2.12. *Let L be a taut representative of $\mathcal{L} \in \mathcal{L}_{k,n}$.*

- P1. *Every component of L intersects each β_i an even number of times. Recall from Remark 2.2 that the number of loop components is given by $|b_i|$ where $b_i = \frac{\beta_i - \beta_{i+1}}{2}$.*
- P2. *Set $x_i = |\alpha_{2i} - \alpha_{2i-1}|$ and $t_i = |a_{S'_i} - b_{S'_i}|$. Then x_i and t_i gives the difference between the number of above and below components in S_i and S'_i respectively. Set $m_i = \min \{|\alpha_{2i} - \beta_i|, |\alpha_{2i-1} - \beta_i|\}$; $1 \leq i \leq n-1$ and $n_i = \min \{a_{S'_i}, b_{S'_i}\}$; $1 \leq i \leq k-1$. See Figure 5. Note that x_i is even since L intersects $\alpha_{2i} \cup \alpha_{2i-1}$ an even number of times. Clearly, this may not hold for t_i since when ψ_i is odd the sum of above and below components (and hence their difference) is odd. See Lemma 2.8.*
- P3. *Set $2a_i = \alpha_{2i} - \alpha_{2i-1}$ ($|a_i| = x_i/2$). Then, by Lemma 2.7 we get*

- *If $b_i \geq 0$, then $\beta_i = \alpha_{2i} + \alpha_{2i-1}$ and hence*

$$\alpha_{2i} = a_i + \frac{\beta_i}{2} \text{ and } \alpha_{2i-1} = -a_i + \frac{\beta_i}{2}.$$

- *If $b_i \leq 0$, then $\beta_{i+1} = \alpha_{2i} + \alpha_{2i-1}$ and hence*

$$\alpha_{2i} = a_i + \frac{\beta_{i+1}}{2} \text{ and } \alpha_{2i-1} = -a_i + \frac{\beta_{i+1}}{2}.$$

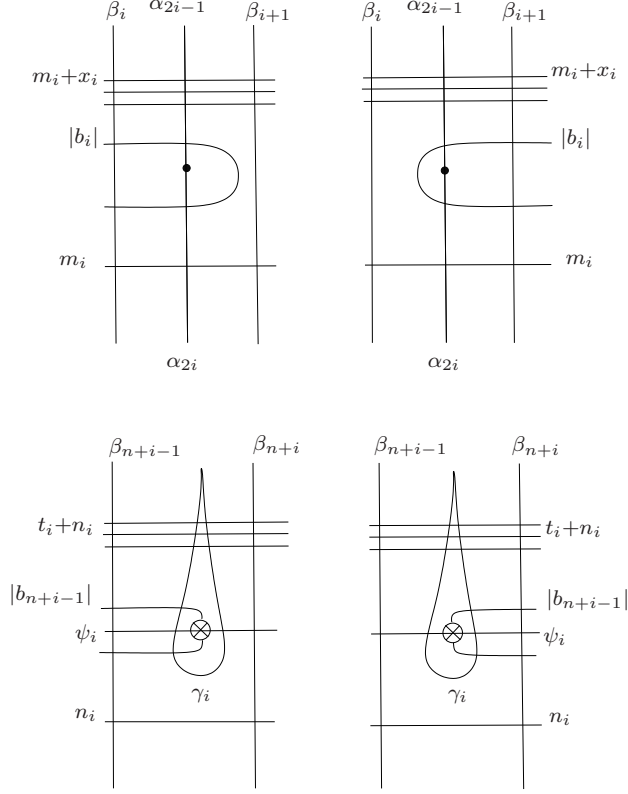


Figure 5: m_i and n_i denote the smaller of above and below components in S_i and S'_i respectively

That is,

$$\alpha_i = \begin{cases} (-1)^i a_{\lceil i/2 \rceil} + \frac{\beta_{\lceil i/2 \rceil}}{2} & \text{if } b_{\lceil i/2 \rceil} \geq 0, \\ (-1)^i a_{\lceil i/2 \rceil} + \frac{\beta_{1+\lceil i/2 \rceil}}{2} & \text{if } b_{\lceil i/2 \rceil} \leq 0. \end{cases}$$

where $\lceil i/2 \rceil$ denotes the smallest integer that is not less than $i/2$.

P4. Since $t_i = a_{S'_i} - b_{S'_i}$ for $1 \leq i \leq k-1$, from Lemma 2.8 we get

- If $b_{n+i-1} \geq 0$ then $a_{S'_i} + b_{S'_i} + \psi_i + 2b_{n+i-1} = \beta_{n+i-1}$, and

$$a_{S'_i} = \frac{t_i - \psi_i + \beta_{n+i-1} - 2b_{n+i-1}}{2}$$

- If $b_{n+i-1} \leq 0$ then $a_{S'_i} + b_{S'_i} + \psi_i - 2b_{n+i-1} = \beta_{n+i}$, and

$$a_{S'_i} = \frac{t_i - \psi_i + \beta_{n+i} + 2b_{n+i-1}}{2}$$

And hence

$$a_{S'_i} = \frac{t_i - \psi_i + \max(\beta_{n+i}, \beta_{n+i-1}) - 2|b_{n+i-1}|}{2}$$

Similarly we compute

$$b_{S'_i} = \frac{-t_i - \psi_i + \max(\beta_{n+i}, \beta_{n+i-1}) - 2|b_{n+i-1}|}{2}$$

P5. It is easy to observe from Figure 5 that

$$\begin{aligned} \beta_i &= 2[|a_i| + \max(b_i, 0) + m_i] \quad \text{for } 1 \leq i \leq n-1 \\ \beta_{n+i} &= |t_i| + 2\max(b_{n+i-1}, 0) + \psi_i + 2n_i \quad \text{for } 1 \leq i \leq k-1. \end{aligned}$$

Therefore, since $b_i = \frac{\beta_i - \beta_{i+1}}{2}$; $1 \leq i \leq n+k-2$ we can compute β_1 using one of the two equations below:

$$\begin{aligned} \beta_1 &= 2 \left[|a_i| + \max(b_i, 0) + m_i + \sum_{j=1}^{i-1} b_j \right] \quad \text{for } 1 \leq i \leq n-1, \\ \beta_1 &= |t_i| + 2\max(b_{n+i-1}, 0) + \psi_i + 2n_i + 2 \sum_{j=1}^{n+i-2} b_j \quad \text{for } 1 \leq i \leq k-1. \end{aligned}$$

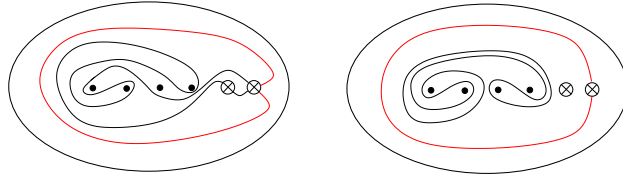


Figure 6: L^* is a simple closed curve on the right but it is not on the left.

P6. *Some integral laminations contain R-components: an R-component of L has geometric intersection numbers $i(R, \alpha_j) = 1$ for each $1 \leq j \leq 2n - 2$, $i(R, \beta_j) = 2$ for each $1 \leq j \leq n+k-1$ and $i(R, \gamma_j) = 2$ for each $1 \leq j \leq k-1$, which has its end points on the k -th crosscap (denoted red in Figure 6). Set $L^* = L \setminus R$. Note that L^* is a component of L which isn't necessarily a simple closed curve (the two possible cases are depicted in Figure 6). Let α_i^*, β_i^* and γ_i^* denote the number of intersections of L^* with the arcs α_i, β_i and γ_i respectively. Define a_i^*, b_i^*, t_i^* and $\lambda_i^*, \lambda_{c_i}^*, a_{S'}^*, b_{S'}^*$ and ψ_i^* similarly as above. We therefore have*

$$\beta_1^* = 2 \left[|a_i^*| + \max(b_i^*, 0) + m_i^* + \sum_{j=1}^{i-1} b_j^* \right] \quad \text{for } 1 \leq i \leq n-1,$$

$$\beta_1^* = |t_i^*| + 2 \max(b_{n+i-1}^*, 0) + \psi_i^* + 2n_i^* + 2 \sum_{j=1}^{n+i-2} b_j^* \quad \text{for } 1 \leq i \leq k-1.$$

where $m_i^* = \min \{ \alpha_{2i}^* - |b_i^*|, \alpha_{2i-1}^* - |b_i^*| \}; 1 \leq i \leq n-1$ and $n_i^* = \min \{ a_{S'_i}^*, b_{S'_i}^* \}; 1 \leq i \leq k-1$. Furthermore, there is some $m_i^* = 0$, or some $n_i^* = 0$ since otherwise L^* would have above and below components in each S_i and S'_i which would yield curves parallel to $\partial N_{k,n}$, or L^* would contain R-components which is impossible by definition. Write $a_i^* = a_i, b_i^* = b_i, t_i^* = t_i$ since deleting R-components doesn't change the a, b, t values. Set

$$X_i = 2 \left[|a_i| + \max(b_i, 0) + \sum_{j=1}^{i-1} b_j \right] \quad \text{for } 1 \leq i \leq n-1,$$

$$Y_i = |t_i| + 2 \max(b_{n+i-1}, 0) + \psi_i + 2 \sum_{j=1}^{n+i-2} b_j \quad \text{for } 1 \leq i \leq k-1.$$

Then one of the three following cases hold for L^* :

- I. If $m_i^* > 0$ for all $1 \leq i \leq n-1$, then there is some j with $1 \leq j \leq k-1$ such that $n_j^* = 0$. Therefore, $\beta_1^* > X_i$ and $\beta_1^* = Y_j$.
- II. If $n_i^* > 0$ for all $1 \leq i \leq k-1$, then there is some j with $1 \leq j \leq n-1$ such that $m_j^* = 0$. Therefore, $\beta_1^* > Y_i$ and $\beta_1^* = X_j$.

III. There is some i with $1 \leq i \leq n-1$ such that $m_i^* = 0$ and some j with $1 \leq j \leq k-1$ such that $n_j^* = 0$. Therefore, $\beta_1^* = X_i = Y_j$.

We therefore have

$$\beta_i^* = \max(X, Y) - 2 \sum_{j=1}^{i-1} b_j$$

where

$$X = 2 \max_{1 \leq r \leq n-1} \left\{ |a_r| + \max(b_r, 0) + \sum_{j=1}^{r-1} b_j \right\}$$

and

$$Y = \max_{1 \leq s \leq k-1} \left\{ |t_s| + 2 \max(b_{n+s-1}, 0) + \psi_s + 2 \sum_{j=1}^{n+s-2} b_j \right\}.$$

P7. If L doesn't have an R -component, that is if $L^* = L$ then $2c_k \leq \beta_{n+k-1}^* = \beta_{n+k-1}$ since $\beta_{n+k-1} = 2c_k + 2\lambda_k$. If L has an R -component then $2c_k > \beta_{n+k-1}^*$ and $\lambda_k = 0$. See Figure 6. Hence the number of R -components of L is given by

$$R = \max(0, 2c_k - \beta_{n+k-1}^*)/2.$$

For example, the integral laminations in Figure 6 (from left to right) has $c_1 = 2, \beta_5^* = 2$, and hence $R = 1$; and $c_1 = 1, \beta_5^* = 0$, and hence $R = 1$. Then L is constructed by identifying the two end points of an R component with the pieces of L^* on the k -th crosscap. Since R -components intersect each β_i twice we get

$$\beta_i = \beta_i^* + 2R; 1 \leq i \leq n+k-1.$$

Then

$$\beta_i = \max(X, Y) - 2 \sum_{j=1}^{i-1} b_j + 2R$$

Also, from item P3. we have

$$\alpha_i = \begin{cases} (-1)^i a_{\lceil i/2 \rceil} + \frac{\beta_{\lceil i/2 \rceil}}{2} & \text{if } b_{\lceil i/2 \rceil} \geq 0, \\ (-1)^i a_{\lceil i/2 \rceil} + \frac{\beta_{1+\lceil i/2 \rceil}}{2} & \text{if } b_{\lceil i/2 \rceil} \leq 0, \end{cases}$$

Finally, it is easy to observe from Figure 3 that

$$\gamma_i = 2(a_{S'_i} + |b_{n+i-1}| + \psi_i)$$

Making use of the properties above, we shall define the generalized Dynnikov coordinate system which coordinatizes $\mathcal{L}_{k,n}$ bijectively and with the least number of coordinates.

Definition 2.13. The generalized Dynnikov coordinate function

$$\rho: \mathcal{L}_{k,n} \rightarrow (\mathbb{Z}^{2(n+k-2)} \times \mathbb{Z}^k) \setminus \{0\}$$

is defined by

$$\rho(\mathcal{L}) = (a; b; t; c) := (a_1, \dots, a_{n-1}; b_1, \dots, b_{n+k-2}; t_1, \dots, t_{k-1}; c_1, \dots, c_k)$$

where

$$\begin{aligned} a_i &= \frac{\alpha_{2i} - \alpha_{2i-1}}{2} & \text{for } 1 \leq i \leq n-1, \\ b_i &= \frac{\beta_i - \beta_{i+1}}{2} & \text{for } 1 \leq i \leq n+k-2, \\ t_i &= a_{S'_i} - b_{S'_i} & \text{for } 1 \leq i \leq k-1, \end{aligned}$$

where $a_{S'_i}$ and $b_{S'_i}$ are as given in Lemma 2.5.

Theorem 2.14 gives the inverse of $\rho: \mathcal{L}_{k,n} \rightarrow (\mathbb{Z}^{2(n+k-2)} \times \mathbb{Z}^k) \setminus \{0\}$.

Theorem 2.14. Let $(a; b; t; c) \in (\mathbb{Z}^{2(n+k-2)} \times \mathbb{Z}^k) \setminus \{0\}$. Set

$$\begin{aligned} X &= 2 \max_{1 \leq r \leq n-1} \left\{ |a_r| + \max(b_r, 0) + \sum_{j=1}^{r-1} b_j \right\} \\ Y &= \max_{1 \leq s \leq k-1} \left\{ |t_s| + 2 \max(b_{n+s-1}, 0) + \psi_s + 2 \sum_{j=1}^{n+s-2} b_j \right\}. \end{aligned}$$

Then $(a; b; t; c)$ is the Dynnikov coordinate of exactly one element $\mathcal{L} \in \mathcal{L}_{k,n}$ which has

$$\beta_i = \max(X, Y) - 2 \sum_{j=1}^{i-1} b_j + 2R, \quad (2)$$

$$\alpha_i = \begin{cases} (-1)^i a_{\lceil i/2 \rceil} + \frac{\beta_{\lceil i/2 \rceil}}{2} & \text{if } b_{\lceil i/2 \rceil} \geq 0, \\ (-1)^i a_{\lceil i/2 \rceil} + \frac{\beta_{1+\lceil i/2 \rceil}}{2} & \text{if } b_{\lceil i/2 \rceil} \leq 0, \end{cases} \quad (3)$$

$$\gamma_i = 2(a_{S'_i} + |b_{n+i-1}| + \psi_i) \quad (4)$$

where $a_{S'_i}$ is defined as in item P4. in Properties 2.12.

Proof. Given $L \in \mathcal{L}_{k,n}$ with $\tau(L) = (\alpha, \beta, \gamma, c)$ and $\rho(L) = (a, b, t, c)$, Properties 2.12 show that α, β and γ must be given by (2), (3) and (4) respectively, and hence L is unique by Lemma 2.10. Therefore ρ is injective. By Properties 2.12 we can draw non-intersecting path components in each S_i ($1 \leq i \leq n-1$), S'_i ($1 \leq i \leq k-1$), Δ_0 and Δ'_k which intersect each element of $\mathcal{A}_{k,n}$ the number of times given by $(\alpha, \beta, \gamma, c)$. Gluing together these path components gives a disjoint union of simple closed curves in $N_{k,n}$. There are no curves that bound a puncture or parallel to the boundary by construction, and hence $(\alpha, \beta, \gamma, c)$ where α, β and γ are defined by (2), (3) and (4) respectively, correspond to some L with $\rho(L) = (a, b, t, c)$. Therefore, ρ is surjective. \square

Example 2.15. Let $\rho(\mathcal{L}) = (a_1; b_1, b_2; t_1; c_1, c_2) = (-1; 2, 0; 1; 1, 0)$ be the generalized Dynnikov coordinates of an integral lamination \mathcal{L} on $N_{2,2}$. We shall use Theorem 2.14 to compute the triangle coordinates of \mathcal{L} from which we determine the number of path components in S_1 and S'_1 , and hence draw \mathcal{L} as illustrated in Example 2.9. By Lemma 2.4, $\psi_1 = \max(c_1^+ - |b_2|, 0) = 1$ so we have

$$X = 2(|a_1| + \max(b_1, 0)) = 6 \quad \text{and} \quad Y = |t_1| + 2\max(b_2, 0) + \psi_1 + 2b_1 = 6.$$

Therefore

$$\begin{aligned} \beta_1 &= \max(6, 6) = 6, \quad \beta_2 = \max(6, 6) - 2b_1 = 2, \quad \beta_3 = \max(6, 6) - 2(b_1 + b_2) = 2, \\ \alpha_1 &= -a_1 + \frac{\beta_1}{2} = 4, \quad \alpha_2 = a_1 + \frac{\beta_1}{2} = 2. \end{aligned}$$

Since $0 = 2c_2 < \beta_3^* = 2$, there are no R -components by item P8. of Properties 2.12. Since $\beta_1 = 6$ there are 3 loop components in Δ_0 , and since $\beta_3 = 2$ and $c_2 = 0$, there is one non-core loop component in Δ'_2 that is $\lambda_2 = 1$. By Remarks 2.2, $b_1 = 2$ and $b_2 = 0$, and hence there are 2 right loop components in S_1 and no

loop components in S'_1 . By Lemma 2.3 we compute that $a_{S_1} = \alpha_1 - |b_1| = 2$ and $b_{S_1} = \alpha_2 - |b_1| = 0$. Finally by item P4. of Properties 2.12,

$$a_{S'_1} = \frac{t_1 - \psi_1 + \max(\beta_2, \beta_3) - 2|b_2|}{2} = 1$$

$$b_{S'_1} = \frac{-t_1 - \psi_1 + \max(\beta_2, \beta_3) - 2|b_2|}{2} = 0$$

Gluing together the path components in S_1 and S'_1 we construct the integral lamination depicted in Figure 7.

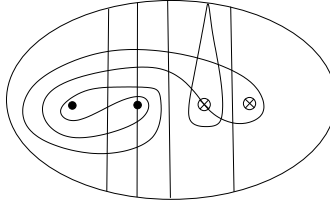


Figure 7: $\rho(L) = (-1; 2, 0; 1; 1, 0)$

Remark 2.16. Generalized Dynnikov coordinates for integral laminations can be extended in a natural way to generalized Dynnikov coordinates of measured foliations [5]: the transverse measure on the foliation [4, 7, 8] assigns to each element in $\mathcal{A}_{k,n}$ a non-negative real number, and hence each measured foliation is described by an element of $(\mathbb{R}_{\geq 0}^{3n+2k-4} \times \mathbb{R}^k) \setminus \{0\}$, the associated measures of the arcs and curves of $\mathcal{A}_{k,n}$. Therefore, the *Generalized Dynnikov coordinate system* for measured foliations is defined similarly (see Definition 2.13), and provides a one-to-one correspondence between the set of measured foliations (up to isotopy and Whitehead equivalence) on $N_{k,n}$ and $(\mathbb{R}^{2(n+k-2)} \times \mathbb{R}^k) \setminus \{0\}$.

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