

TIGHT CONTACT STRUCTURES ON HYPERBOLIC THREE-MANIFOLDS

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ABSTRACT. We show the existence of tight contact structures on infinitely many hyperbolic three-manifolds obtained via Dehn surgeries along sections of hyperbolic surface bundles over circle.

1. INTRODUCTION

A *contact three-manifold* is a pair (M, ξ) where M is a smooth 3-manifold and $\xi \subset TM$ is a totally non-integrable 2-plane field distribution on M . Here we always assume that ξ is a *co-oriented positive* contact structure, that is, $\xi = \text{Ker}(\alpha)$ for a *contact* 1-form α satisfying $\alpha \wedge d\alpha > 0$ with respect to a pre-given orientation on M . A disk D in a contact 3-manifold (M, ξ) is called *overtwisted* if the boundary circle ∂D is tangent to ξ everywhere. A contact structure ξ is called *overtwisted* if there is an *overtwisted* disk in (M, ξ) , otherwise it is called *tight*. It is known that every closed oriented 3-manifold admits an overtwisted contact structure ([7], [19]). On the other hand, there are 3-manifolds which do not admit a tight contact structure [10].

There are some classification results on tight contact structures with respect to the geometric speciality of 3-manifolds. Lisca and Stipsicz in [18] proved that a closed oriented Seifert fibered 3-manifold admits a tight contact structure if and only if it is not gotten $(2q-1)$ -surgery along the $(2, 2q+1)$ torus knot in S^3 for $q \geq 1$. In two independent work ([2], [16]), they showed the existence of tight contact structures on toroidal 3-manifolds. It is known that every irreducible 3-manifold that is neither toroidal nor Seifert fibered is hyperbolic. Kaloti and Tosun in [17] find infinitely many hyperbolic rational homology spheres admitting tight contact structures. Etgü in [9] also explored that infinitely many hyperbolic 3-manifolds that carry tight contact structures. His construction uses Dehn surgeries along sections of hyperbolic torus bundles over S^1 . Here we'll follow similar ideas for surface bundles over S^1 with fiber genus at least two.

Let Σ_g be a closed connected orientable surface with genus g . In this paper assume that g is always greater than 1. We will denote $MCG(\Sigma_g)$ by the *mapping class group* of Σ_g , i.e, the group of isotopy classes of orientation preserving homeomorphisms of Σ_g . Let t_a be the positive Dehn twist along a simple closed curve a .

Let $\phi \in MCG(\Sigma_g)$ be the mapping class representing the homeomorphism

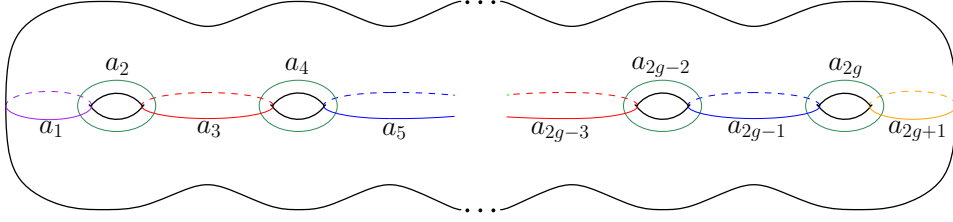
$$(1) \quad t_{a_1}^m t_{a_2} \cdots t_{a_{2g}} t_{a_{2g+1}}^n$$

where a_i 's are simple closed curves on Σ_g as indicated in Figure 1.

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FIGURE 1. Simple closed curves on the surface Σ_g .

Denote by M_ϕ the *mapping torus* with fibers Σ_g and monodromy ϕ . Let $M_\phi(r)$ be the surgered manifold obtained by performing rational r -surgery along a section of M_ϕ . Clearly, ϕ has a fixed point, so such a section exists. The following theorems give examples required:

Theorem 1.1. *Suppose $g \geq 2$, $m, n \in \mathbb{Z}$, $r \in \mathbb{Q}$ and ϕ as indicated in (1). Then $M_\phi(r)$ is hyperbolic for all but finitely many m and r .*

Theorem 1.2. *Suppose $g \geq 1$, $r \in \mathbb{Q}$ and ϕ as indicated in (1). Then $M_\phi(r)$ admits a tight contact structure ξ for any $m, n \in \mathbb{Z}^+$ and for all $r \neq 2g - 1$.*

As a consequence of the theorems we have:

Corollary 1.3. *Suppose $g \geq 2$, $m, n \in \mathbb{Z}^+$, $r \in \mathbb{Q}$ and ϕ as indicated in (1). Then $M_\phi(r)$ is a hyperbolic manifold admitting a tight contact structure for all $r \neq 2g - 1$ and all but finitely many $m \in \mathbb{Z}^+$. \square*

The proof of Theorem 1.1 and Theorem 1.2 will be given in Section 2 and Section 3.

2. PROOF OF THEOREM 1.1

In order to prove the theorem, we'll focus on pseudo-Anosov homeomorphisms and construct infinitely many hyperbolic 3-manifolds via pseudo-Anosov monodromies. A hyperbolic 3-manifold is a 3-manifold which admits a complete finite-volume hyperbolic structure. Thurston [22] demonstrated that an orientable surface bundle over circle whose fiber is a compact surface of negative Euler characteristic is hyperbolic if and only if the monodromy of the surface bundle is a pseudo-Anosov homeomorphism. Another deep result of Thurston is hyperbolic Dehn surgery theorem which states that a hyperbolic 3-manifold remains hyperbolic after Dehn filling along a link for all slopes except finitely many of them (For details see [23]). In order to apply these results, we need a lemma where we construct infinitely many pseudo-Anosov diffeomorphisms as products of certain Dehn twists:

Lemma 2.1. *Let ϕ be the class in $MCG(\Sigma_g)$ as described in (1) above. Then ϕ is pseudo-Anosov for any integer n and for all but at most 7 consecutive values of m .*

Denote by $\iota(\alpha, \beta)$ geometric intersection number of the curves α and β . We say a set of simple closed curves $\{\gamma_1, \gamma_2, \dots, \gamma_k\}$ fills Σ_g if $\Sigma_g \setminus \{\gamma_1, \gamma_2, \dots, \gamma_k\}$ is a disjoint union of topological disks. In order to prove Lemma 2.1, we use the following theorem of Fathi:

Theorem 2.2. ([12]) *Let f be the class in $MCG(\Sigma_g)$ and let γ be a simple closed curve in Σ_g . If the orbit of γ under f fills Σ_g , then $t_\gamma^m f$ is a pseudo-Anosov class except for at most 7 consecutive values of m .*

Proof of Lemma 2.1. Let γ represents the curve a_1 and let f be the product of Dehn twists $t_{a_1} t_{a_2} \cdots t_{a_{2g}} t_{a_{2g+1}}^n$. Then conclude that

$$f(\gamma) = t_{a_1} t_{a_2}(a_1) = a_2, \quad f^2(\gamma) = t_{a_1} t_{a_2} t_{a_3}(a_2) = a_3,$$

and inductively,

$$f^i(\gamma) = a_{i+1} \text{ for all } i \in 1, 2, \dots, 2g - 1.$$

Since the complement $\Sigma_g \setminus \{a_1, \dots, a_{2g}\}$ is a topological disk, we can say the orbit of γ under f fills Σ_g . As a result of Theorem 2.2, ϕ is pseudo-Anosov except for at most 7 consecutive m values. \square

Now we have a family of pseudo-Anosov monodromies. Using [22] we can say that the surface bundles M_ϕ are all hyperbolic. By hyperbolic Dehn surgery theorem the surgered manifolds $M_\phi(r)$ are hyperbolic for all $m, n \in \mathbb{Z}$ and $r \in \mathbb{Q}$ except 7 values of m and finitely many “bad” slopes r . This finishes the proof of Theorem 1.1. \square

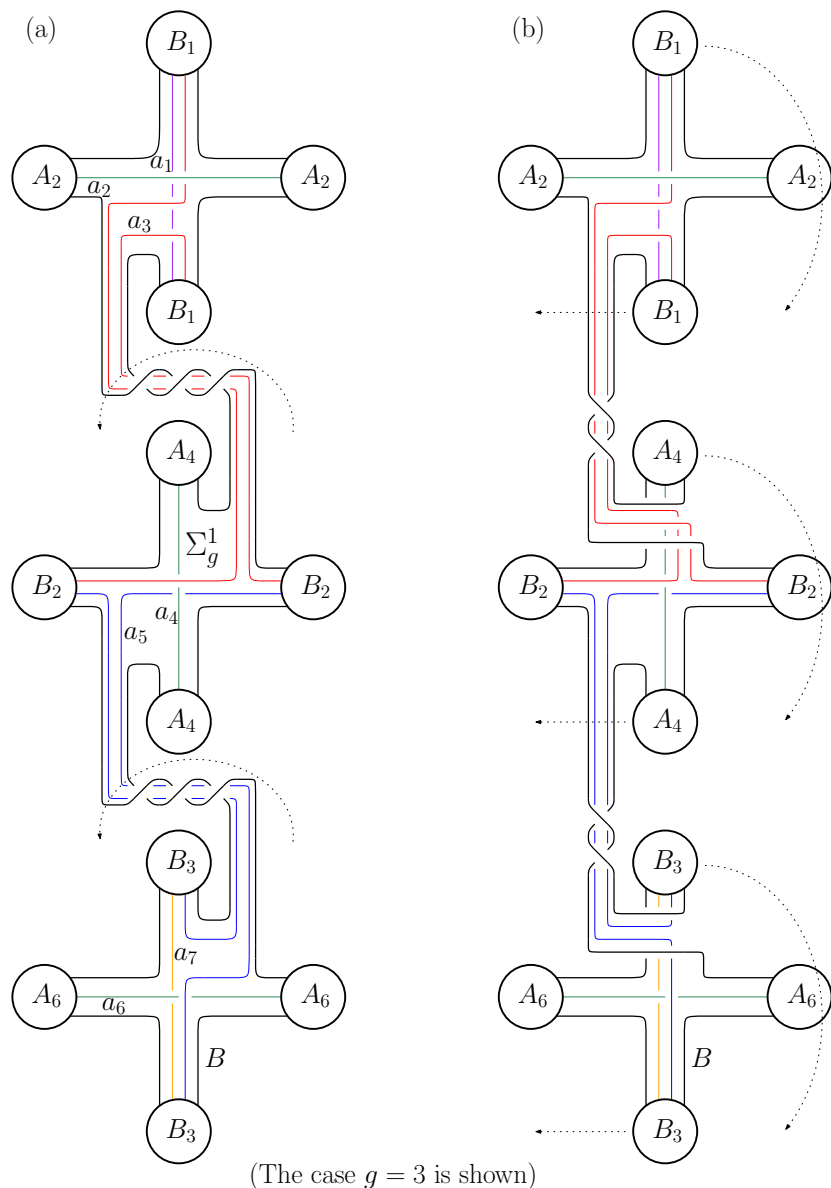


FIGURE 2.

3. PROOF OF THEOREM 1.2

We will analyze the proof with respect to the parity of the genus g of the fiber Σ_g . First assume $g \geq 3$ odd. Note that conjugation of the monodromy by any class of $MCG(\Sigma_g)$ does not change the mapping torus up to diffeomorphism. Since

$$t_{a_2} \cdots t_{a_{2g}} t_{a_{2g+1}}^n t_{a_1}^m = t_{a_1}^{-m} \phi t_{a_1}^m$$

we may replace ϕ in (1) with the mapping class $t_{a_2} \cdots t_{a_{2g}} t_{a_{2g+1}}^n t_{a_1}^m$. Also observe that $M_\phi(r)$ can be also obtained from a Dehn surgery on the binding of an open book decomposition whose page is Σ_g^1 (punctured Σ_g) and monodromy can be still assumed to be $\phi \in MCG(\Sigma_g^1)$. We will construct the required contact structure ξ on $M_\phi(r)$ via Dehn surgery on the open book decomposition (Σ_g^1, ϕ) along its binding.

It is known (see [1], [14]) that the contact structure, say ξ_0 , (before the surgery along binding) supported by (Σ_g^1, ϕ) is Stein fillable. More precisely, consider the handlebody diagram of the smooth 4-manifold X_ϕ given in Figure 2-(a) (in the case of genus 3) with “ $2g$ ” 1-handles and “ $m + n + 2g - 1$ ” 2-handles. Note that Figure 2-(a) describes a Lefschetz fibration structure on X_ϕ with a regular fiber Σ_g^1 and the vanishing cycles $a_1, a_2, \dots, a_{2g+1}$. There are n copies for a_{2g+1} and m copies for a_1 (not drawn for simplicity). All coefficients (except on B) are -1 with respect to the framing given by the page Σ_g^1 . We remark that no handle is attached along the binding of the induced open book (Σ_g^1, ϕ) on the boundary ∂X_ϕ which is realized as B in the figure.

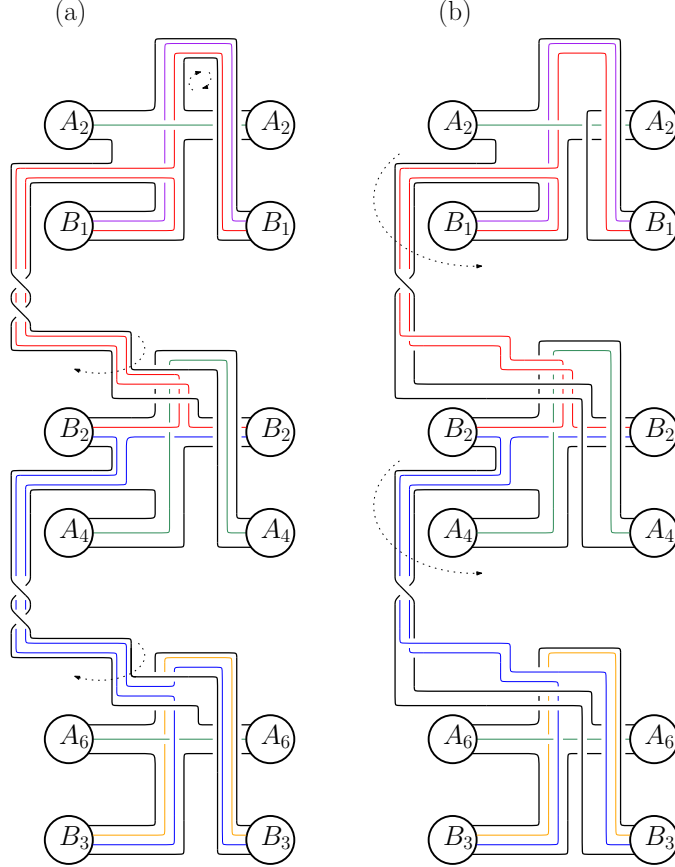


FIGURE 3.

Next starting from the topological description in Figure 2-(a) of X_ϕ , we'll get a diagram describing a Stein structure on X_ϕ inducing ξ_0 as follows: First we flip the twisted bands over the 1–handles as pointed out in Figure 2-(a) and get Figure 2-(b). Figure 3-(a) gives another handle description of X_ϕ obtained by moving the feet of 1–handles as indicated by the dotted arrows in Figure 2-(b). Then flip the bands as shown in Figure 3-(a) to get rid of one more left half twist for each band (see Figure 3-(b)), and obtain Figure 4-(a) by flipping the connecting bands over the feet of 1–handles suggested by the dotted arrows in Figure 3-(b). Figure 4-(b) defines a Stein structure on X_ϕ obtained by putting the attaching circles in part (a) into Legendrian positions, where a Legendrian realization L_0 of B in the tight contact boundary ∂X_ϕ is also provided. All coefficients (except on L_0) are -1 with respect to Thurston-Bennequin (contact) framing in ∂X_ϕ and no handle is attached along L_0 . Note that $tb(L_0) = 2$ (the case $g = 3$ is shown). In the general case, $tb(L_0) = g - 1$. Finally, we use the trick (“Move 6”) in Figure 20 of [15] to obtain a Legendrian representation L of B with $tb(L) = 2g - 1$ (see Figure 5). Note that Figure 5 describes the same Stein structure on X_ϕ as in Figure 4-(b).

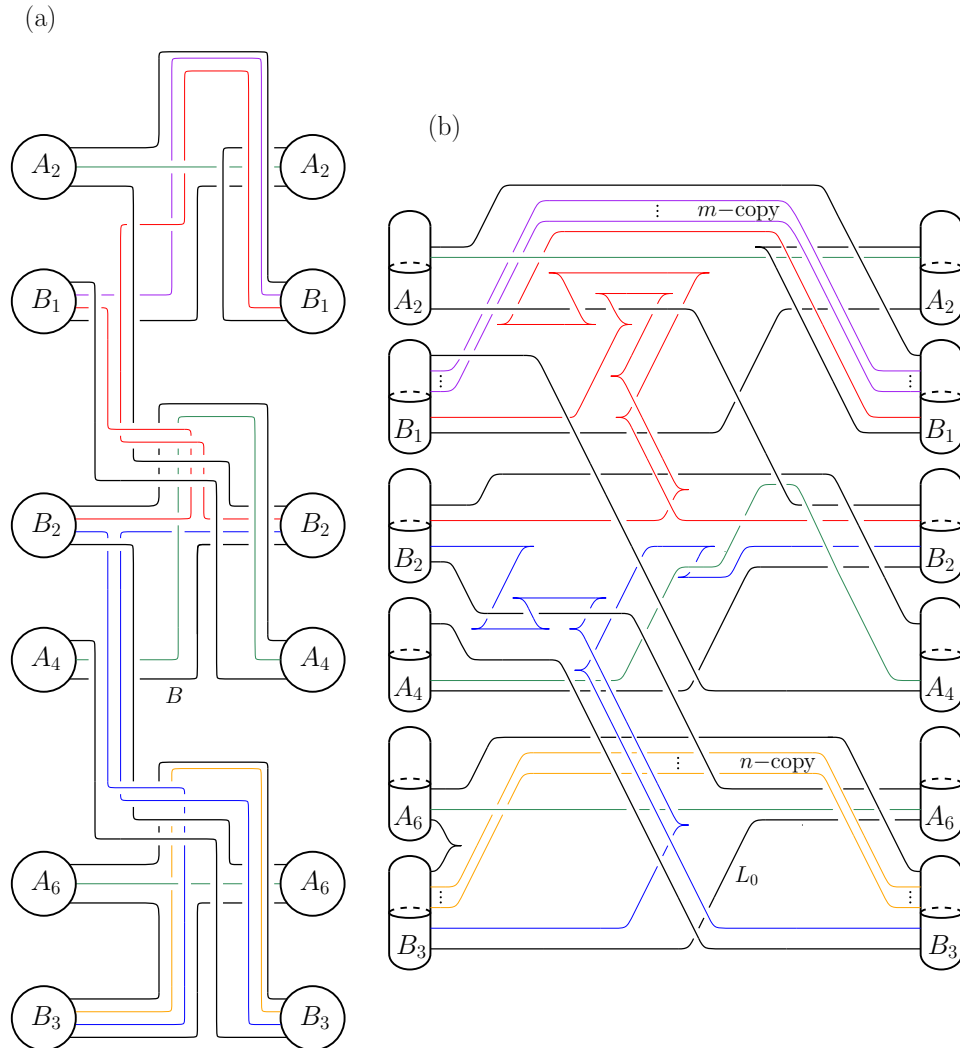


FIGURE 4.

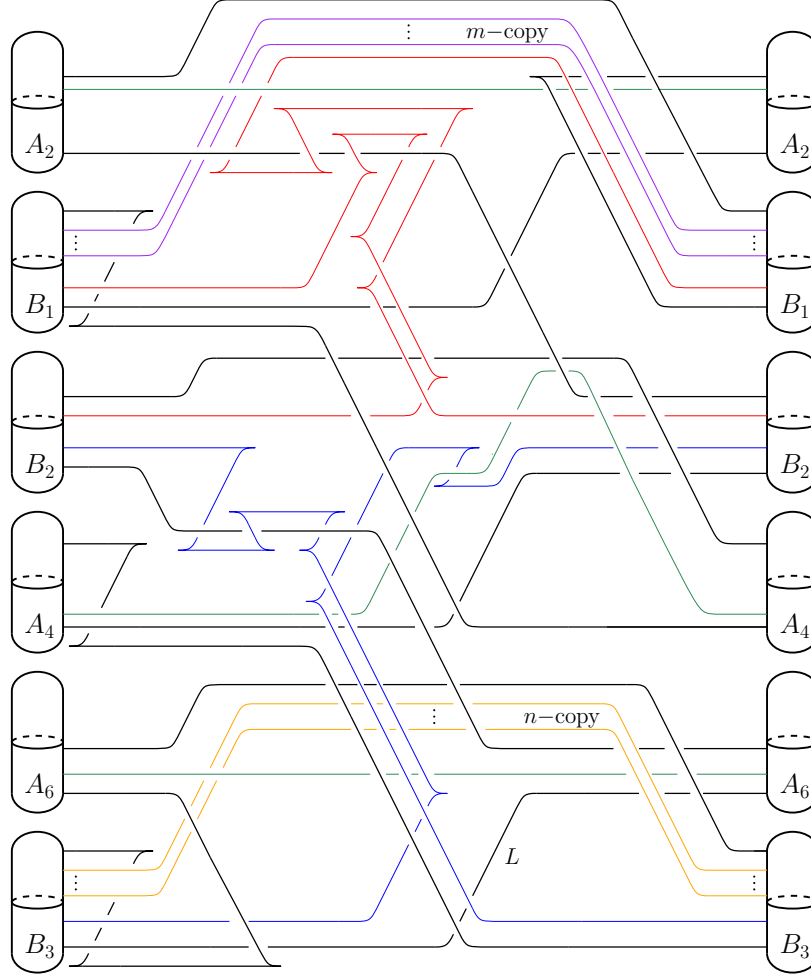


FIGURE 5. The same Stein structure on X_ϕ as in Figure 4-(b), and another Legendrian realization L of the binding B in the tight contact boundary ∂X_ϕ . L is obtained from L_0 by applying “Move 6” (smooth but non-Legendrian isotopy of L_0) g times using the left foot of the corresponding 1-handles (when $g = 3$, handles are B_1, A_4, B_3). All coefficients (except on L) are -1 with respect to Thurston-Bennequin (contact) framing in ∂X_ϕ . No handle attached along L . Note that $tb(L) = 5$ (the case $g = 3$ is shown). In the general case, $tb(L) = 2g - 1$.

Now if $g \geq 2$ is even, we replace the monodromy ϕ with $t_{a_{2g+1}}^n t_{a_2} \cdots t_{a_{2g}} t_{a_1}^m$ since

$$t_{a_{2g+1}}^n t_{a_1}^{-m} \phi t_{a_{2g+1}}^{-n} t_{a_1}^m = t_{a_{2g+1}}^n t_{a_2} \cdots t_{a_{2g}} t_{a_1}^m.$$

Then starting from the handlebody diagram given in Figure 6-(a) (where the case $g = 4$ is shown) and following the moves as in the case of odd genus, one can get Figure 6-(b) describing a Stein structure realizing a Legendrian representation L with $tb(L) = 2g - 1$ as in Figure 5. One should note that we need to consider different monodromies (but still giving the same mapping torus) depending on the parity of g to make the contact and the page framing on any attaching circle coincide.

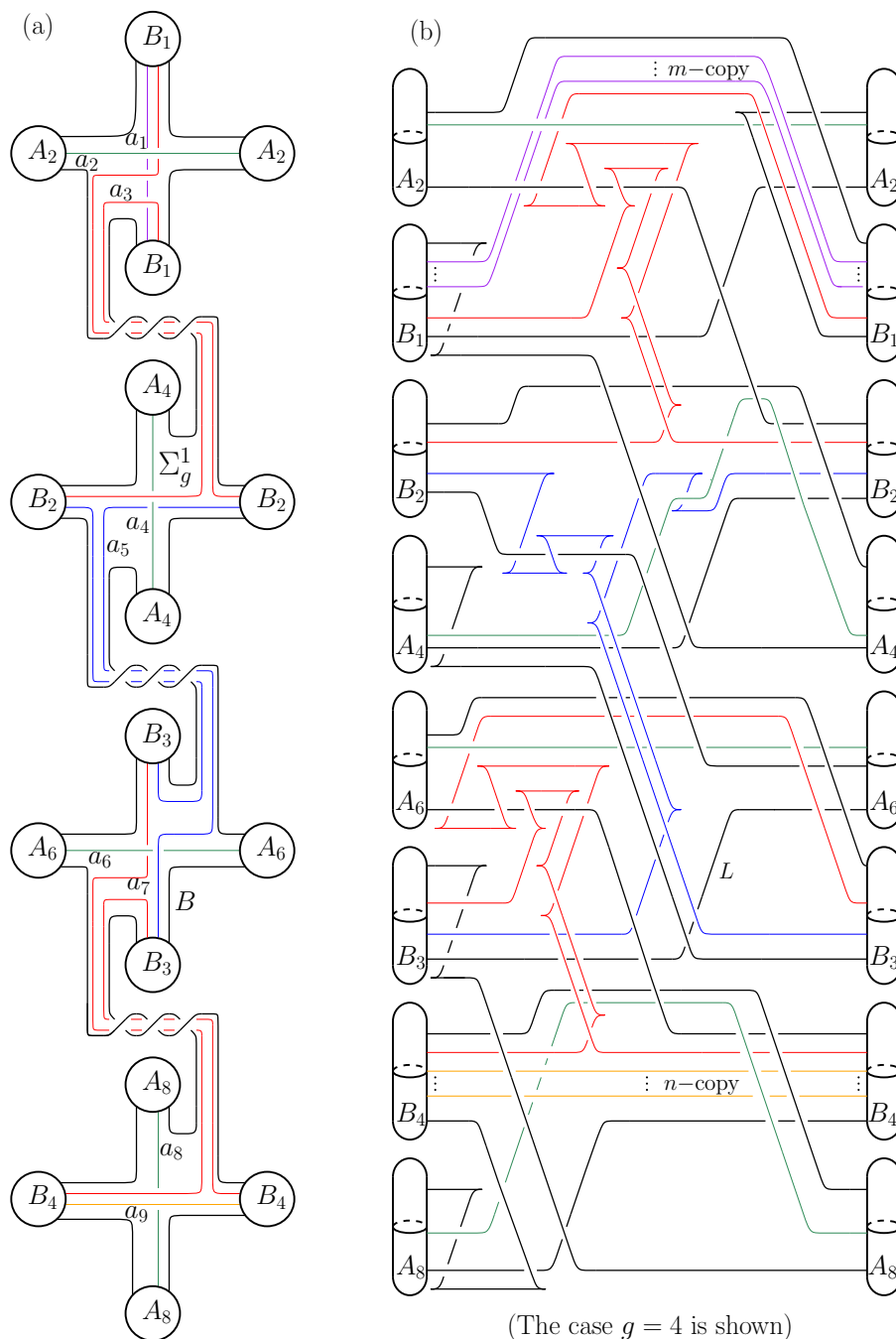


FIGURE 6.

Now (in any case of g) we first (Legendrian) slide (Stein) 2–handle corresponding a_3 over the ones represented by the curves $a_1, a_5, a_7, \dots, a_{2g+1}$, and then cancel the 2–handles represented by $a_5, a_7, \dots, a_{2g-1}$ with the corresponding 1–handles. Second, we (Legendrian) slide 2–handles represented by the curves a_1 and a_{2g+1} over a fixed one (chosen from each family in Figure 5 / Figure 6-(b)), and then cancel 1–handles B_1 and B_g with the chosen 2–handles corresponding a_1 and a_{2g+1} respectively. Also we cancel each 1–handle A_i with the 2–handle corresponding

the curve a_i for each i even. As a result, we obtain another (but equivalent) Stein structure on X_ϕ which can be also considered as the contact surgery diagram for ξ_0 on ∂X_ϕ . Finally, we set $r' = r - 2g + 1$ and perform r' -contact surgery along $L \subset (\partial X_\phi, \xi_0)$ to get a contact structure ξ on $M_\phi(r)$ whose diagram is given in Figure 7 (where we use continued fractions).

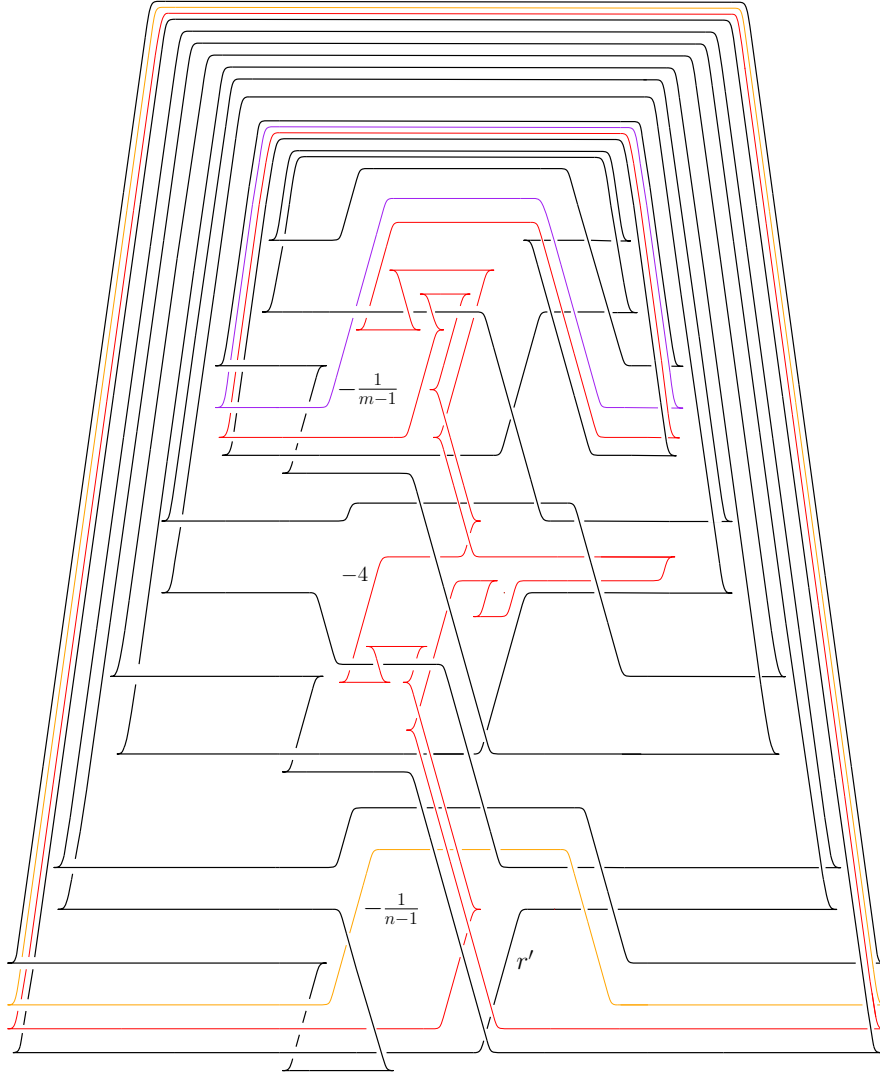


FIGURE 7. The contact 3-manifold $(M_\phi(r), \xi)$. (The case $g = 3$ is shown.)

First suppose $r' = r - 2g + 1 < 0$. We know any contact surgery with negative contact framing can be converted to a sequence of contact (-1) -surgeries and (-1) -surgeries preserve Stein fillability ([5], [6], [14]). Thus $(M_\phi(r), \xi)$ is Stein fillable (hence tight).

Now let $r' = r - 2g + 1 > 0$. By Thurston-Winkelnkemper construction ([24]), it is known that the binding B is transverse to the contact structure supported by the open book decomposition. Also since ∂X_ϕ is Stein fillable, ξ_0 has nonzero contact invariant [21]. As a result of Conway's work (see [3], Theorem 1.6) if K is a fibered transverse knot in a contact 3-manifold (M, η) where η has nonvanishing contact class, then r -surgery along K preserves the non-vanishing of

the contact class if $r > 2g - 1$ where g is the genus of K . Hence we conclude that $(M_\phi(r), \xi)$ has nonzero contact invariant (hence tight) through Conway's result. This finishes the proof of Theorem 1.2. \square

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