QUIVERS OF FINITE MUTATION TYPE AND SKEW-SYMMETRIC MATRICES

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ABSTRACT. Quivers of finite mutation type are certain directed graphs that first arised in Fomin-Zelevinsky's theory of cluster algebras. It has been observed that these quivers are also closely related with different areas of mathematics. In fact, main examples of finite mutation type quivers are the quivers associated with triangulations of surfaces. In this paper, we study structural properties of finite mutation type quivers in relation with the corresponding skew-symmetric matrices. We obtain a characterization of finite mutation type quivers that are associated with triangulations of surfaces and give a new numerical invariant for their mutation classes.

1. INTRODUCTION

Quivers of finite mutation type are certain directed graphs that first arised in Fomin-Zelevinsky's theory of cluster algebras. It has been observed that these quivers are also closely related with different areas of mathematics. In fact, main examples of finite mutation type quivers are the quivers associated with triangulations of surfaces as introduced in [5]. They also provide interesting classes of non-commutative algebras [1]. A classification of finite mutation type quivers has been obtained recently in [4]. In this paper, we study structural properties of finite mutation type quivers in relation with the corresponding skew-symmetric matrices. We determine a class of subquivers, which we call basic quivers, and show that they have a natural linear-algebraic interpretation. In particular, we obtain a characterization of finite mutation type quivers that are associated with triangulations of surfaces and give a new numerical invariant for their mutation classes. We also give a theoretical proof of the classification of finite mutation type quivers that are not associated with triangulations of a surface (Lemma 3.3), which was obtained in [4] partly using a computer program.

To state our results, we need some terminology. Formally, a quiver is a pair $Q = (Q_0, Q_1)$ where Q_0 is a finite set of vertices and Q_1 is a set of arrows between them. It is represented as a directed graph with the set of vertices Q_0 and a directed edge for each arrow. In this paper, we are more concerned with the number of arrows between the vertices rather than the arrows themselves, so by a quiver we mean a directed graph Q, with no loops or 2-cycles, whose edges are weighted with positive integers. If the weight of an edge is 1, we do not specify it in the picture and call it a single edge; if an edge has weight 2 we call it a double edge for convenience. If all edges of Q are single edges, we call Q "simply-laced". By a subquiver of

Q, we always mean a quiver obtained from Q by taking an induced (full) directed subgraph on a subset of vertices and keeping all its edge weights the same as in Q.

For a quiver Q with vertices $1, ..., n$, there is the uniquely associated skew-symmetric matrix $B = B^Q$ defined as follows: for each edge $\{i, j\}$ directed from i to j, the entry $B_{i,j}$ is the corresponding weight; if i and j are not connected to each other by an edge then $B_{i,j} = 0$. Recall from [6] that, for each vertex k, the mutation of the quiver Q at a vertex k transforms Q to the quiver $Q' = \mu_k(Q)$ whose corresponding skew-symmetric matrix $B' = B^{Q'}$ is the following: $B'_{i,j} = -B_{i,j}$ if $i = k$ or $j = k$; otherwise $B'_{i,j} = B_{i,j} + sgn(B_{i,k})[B_{i,k}B_{k,j}]_+$ (where we use the notation $[x]_+ = max\{x, 0\}$ and $sgn(x) = x/|x|$ with $sgn(0) = 0$). The operation μ_k is involutive, so it defines a mutation-equivalence relation on quivers (or equivalently on skew-symmetric matrices). A quiver Q is said to be of "finite mutation type" if its mutation-equivalence class is finite. It is well known that, in a finite mutation type quiver with at least three vertices, any edge is a single edge or a double edge; any subquiver is also of finite mutation type. The most basic examples of finite mutation type quivers are Dynkin quivers (Figure 2), which correspond to skew-symmetric cluster algebras of finite type [6].

Another important class of finite mutation type quivers has been obtained in [5] using a construction that associates quivers to certain triangulations of surfaces. In this paper, we will not use this construction, so we do not recall it here (we will only use some of their well-known properties). We call these quivers *quivers that come from the triangulation of a surface*. More recently, it has been shown that these are almost all of the finite mutation type quivers:

Theorem 1.1. [4, Theorem 6.1] *A connected quiver Q with at least three vertices is of* finite mutation type *if and only if it comes from the triangulation of a surface or it is mutation-equivalent to one of the exceptional types E_6, E_7, E_8, $E_6^{(1)}$, $E_7^{(1)}$, $E_8^{(1)}$, $E_7^{(1,1)}$, $E_8^{(1,1)}$, X_6, X_7 (Figures 2, 3).*

The main tool in proving this classification theorem is a purely combinatorial characterization of quivers that come from triangulations of surfaces as quivers that can be composed by matching quivers from a small set of simple quivers. We will not use this construction either, so we do not recall it here. The proof is obtained by determining minimal quivers that are indecomposable, i.e. can not be composed from those simple quivers [4, Theorem 5.11].

In this paper, to understand the structure of finite mutation type quivers, we identify another class of subquivers that we call "basic (sub)quivers" and use them give an algebraic/combinatorial characterization of the finite mutation type quivers that come from triangulations of surfaces. More explicitly, we define a basic quiver as one of the following: a Dynkin tree D_4, two adjacent oriented simply-laced triangles, an oriented simply-laced cycle with at least four vertices (see Figure 1). Here by a cycle we mean a subquiver whose vertices can be labeled by elements of $\mathbb{Z}/m\mathbb{Z}$ so that the edges betweeen them are precisely $\{i, i+1\}$ for $i \in \mathbb{Z}/m\mathbb{Z}$. To proceed, we need a little bit more terminology. For each vertex i in a quiver Q with vertex set $\{1, 2, ..., n\}$, we denote by e_i the i-th standard basis vector of $\mathbb{Z}^n$. For any vector u in $\mathbb{Z}^n$, we define $supp_Q(u)$ to be the subquiver of Q on the vertices which correspond to the non-zero coordinates of u and call it the support of u in Q. Now we can state our first main result:

Theorem 1.2. *Suppose that Q is a finite mutation type quiver with at least three vertices. Then Q comes from the triangulation of a surface if and only if the following holds for any basic subquiver S:*

(i) *S contains a subquiver of the form $supp_Q(u)$ where u is a non-zero radical vector of B^Q such that each non-zero coordinate of u is either 1 or -1 (here u is radical if $B^Q u = 0$). The subquiver $supp_Q(u)$ has exactly two vertices or it is a cycle.*

(ii) *furthermore if S is an oriented cycle of length at least 5, then the vector u whose coordinates corresponding to the vertices of S is 1 and 0 in the remaining vertices is a radical vector for B_Q (in particular $S = supp_Q(u)$).*

Thus we have, in particular, obtained an algebraic interpretation of basic subquivers in quivers that come from the triangulation of a surface. We will also give a numerical invariant for their mutation classes which is related to this interpretation, involving another common class of subquivers as well: double edges and non-oriented cycles. For this purpose, it turns out to be convenient to work in $\bar{V} := \mathbb{Z}^n/2\mathbb{Z}^n$, which is a vector space over $\mathbb{Z}/2\mathbb{Z}$ (which is the field with two elements). To be more precise, for a finite mutation type quiver Q, we denote by $\bar{B}^Q$ the skew-symmetric matrix whose entries are the corresponding entries of B^Q modulo $2\mathbb{Z}$. We denote by $\bar{V}_0^Q$ the space of radical vectors of $\bar{B}^Q$ (over $\mathbb{Z}/2\mathbb{Z}$); we call a vector u in $\bar{V}_0^Q$ a "basic radical vector" if $supp_Q(u)$ has exactly two vertices or it is a cycle (oriented or not). We denote by $\bar{V}_{00}^Q$ the subspace spanned by the basic radical vectors of $\bar{B}^Q$ over $\mathbb{Z}/2\mathbb{Z}$; if there are no basic radical vectors, then we take $\bar{V}_{00}^Q$ as the zero subspace. Let us also note that the radical vectors given by Therem 1.2 are basic radical vectors for $\bar{B}^Q$. Our next result relates these vectors to subquivers:

Theorem 1.3. *Suppose that Q is a connected finite mutation type quiver with at least three vertices. Suppose also that S is a subquiver which is a double edge or a non-oriented cycle. Let u be the vector whose coordinates corresponding to the vertices of S is 1 and 0 in the remaining vertices. Then u is a radical vector for $\bar{B}_Q$.*

Furthermore, if Q comes from the triangulation of a surface or it is mutation-equivalent to one of X_6, X_7, then we have the following:

(i) *$dim(\bar{V}_0^Q/\bar{V}_{00}^Q) \leq 1$.*

(ii) *if Q and Q' are mutation-equivalent, then $dim(\bar{V}_{00}^Q) = dim(\bar{V}_{00}^{Q'})$.*

Let us note, in particular, that $dim(\bar{V}_{00}^Q)$ is a numerical invariant for the mutation classes of quivers that come from triangulation of a surface. In view of Theorem 1.2, it can be considered as a count of subquivers S such that S is a double edge or a non-oriented cycle or a basic quiver, modulo those which overlap in a way that the supports of the corresponding basic radical vectors coincide. Let us also note that the first part of the theorem holds for any finite mutation type quiver. However, the second part may not be true for a quiver which belongs to one of the types E in Theorem 1.1. Also, in part (i), the equality may hold; e.g. it holds for a Dynkin quiver Q which is of type A_{2n+1}, $n \geq 2$.

In the classification theorem [4, Theorem 6.1], which is Theorem 1.1 above, the classification of quivers that do *not* come from the triangulation of a surface was done, in part, using a computer program [4, Proof of Theorem 6.1]. Here, using

our approach, we suggest an algebraic/combinatorial proof (Lemma 3.3). More precisely, we show the following:

Theorem 1.4. *Let Q be a connected quiver of finite mutation type. Suppose also that Q has a subquiver which is mutation-equivalent to E_6 (resp. X_6). Then any quiver which is mutation-equivalent to Q also contains a subquiver which is mutation-equivalent to E_6 (resp. X_6). Furthermore Q is mutation-equivalent to a quiver which is one of the types E (resp. X) given in Theorem 1.1.*

We prove our results in Section 3 after some preparation in Section 2.
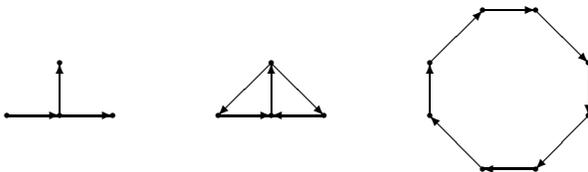

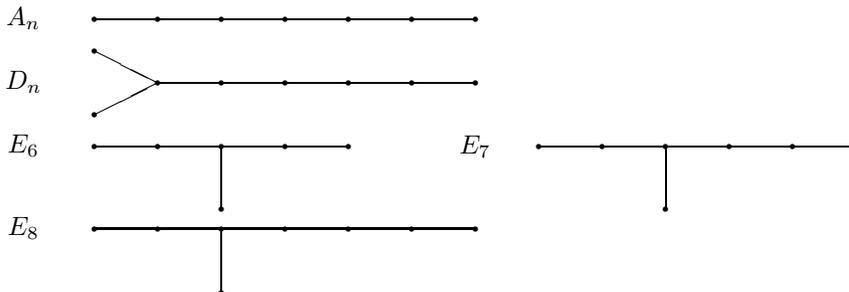FIGURE 1. Basic quivers (the cycle has at least four vertices)

FIGURE 2. Dynkin quivers: each edge is assumed to be arbitrarily oriented

2. PRELIMINARY RESULTS

In this section, we will recall some more terminology and prove some statements that we will use to prove our results. As we discussed in Section 1, we mainly study quivers of finite mutation type. There is also a stronger notion of *finite (cluster) type*: a quiver Q is said to be of finite type if any edge in any quiver Q' which is mutation-equivalent to Q is a single edge. Quivers of finite type were classified by Fomin and Zelevinsky in [6]. Their classification is identical to the Cartan-Killing classification. More precisely, a quiver Q is of finite type if and only if Q is mutation-equivalent to an orientation of a Dynkin quiver (Figure 2). Another related definition is the following: a quiver Q is said to be of *minimal infinite* type if it is of infinite type and any proper subquiver of Q is of finite type. A list of minimal infinite type quivers has been obtained in [8]. In particular, any minimal infinite type quiver with at least three vertices is mutation-equivalent to an extended Dynkin quiver [8, Theorem 3.2].
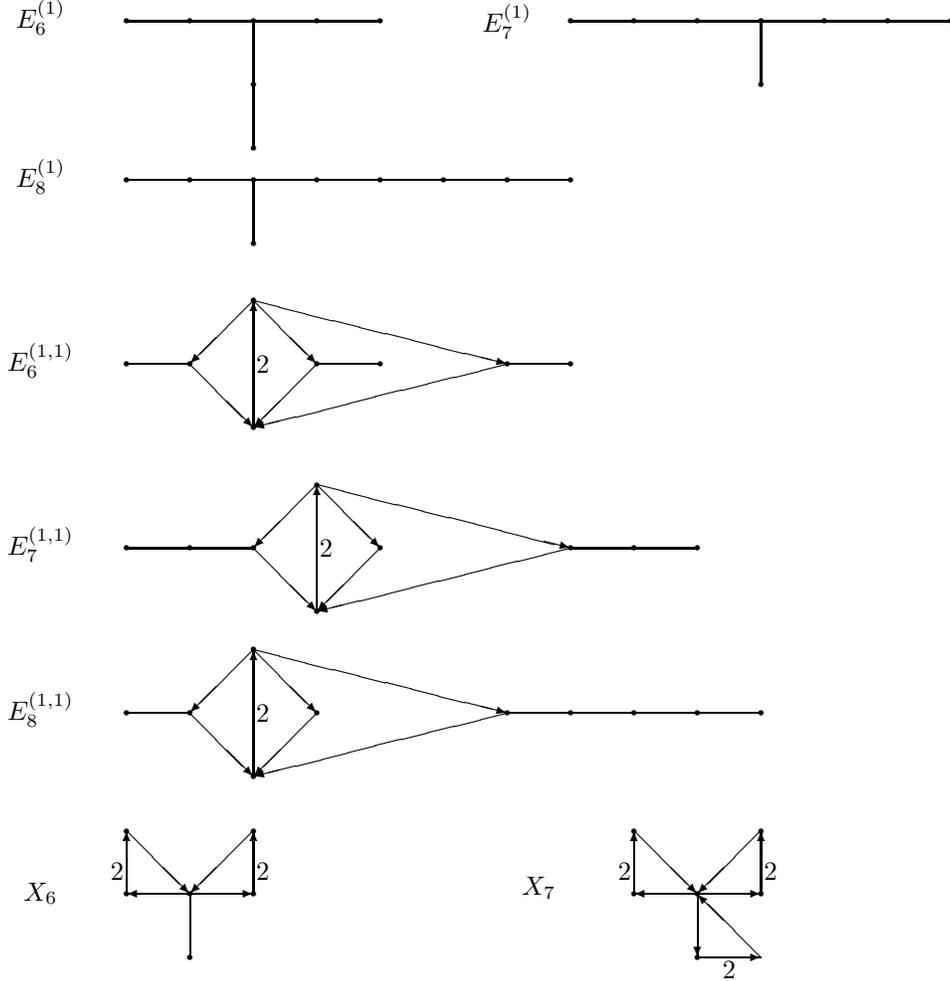
FIGURE 3. exceptional quivers of finite mutation type which are not Dynkin: edges with unspecified orientation are assumed to be arbitrarily oriented

Let us also recall that the mutation operation can be viewed as a base change transformation for a skew-symmetric bilinear form [4, Section 2]. To be more specific, let Q be a quiver with vertices $1, ..., n$. Let Ω be the bilinear form on $\mathbb{Z}^n$ defined as follows: $\Omega(e_i, e_j) = B_{i,j}^Q$; here e_i denotes the i-th standard basis vector. Then $\mu_k(B^Q)$ is the matrix that represents Ω with respect to the basis $\{e_j'\}$: $e_k' = -e_k$, $e_j' = e_j$ if $\Omega(e_k, e_j) > 0$, $e_j' = e_j - \Omega(e_k, e_j)e_k$ if $\Omega(e_k, e_j) < 0$. The coordinates of a vector $u = (u_1, ..., u_n)$ in $\mathbb{Z}^n$ with respect to this new basis will be the same except in the $k-$th coordinate, which becomes $-u_k + \sum u_j$ over all j such that $\Omega(e_k, e_j) < 0$. In particular, the rank of B^Q is invariant under the mutation operation.

Finally, let us recall that a vertex i in a quiver is called a source (rsp. sink) if all adjacent edges are oriented away (resp. towards) i. A quiver is called acyclic if it has no oriented cycles at all. It is well known that an acyclic quiver has a source and a sink. Also acyclic quivers of finite mutation type with at least three vertices are the Dynkin and extended Dynkin quivers [2].

Let us now give some basic examples of quivers which are of infinite mutation type (i.e. not of finite mutation type):

Proposition 2.1. *Let Q be a connected quiver which has at least three vertices.*

  (i) *If Q is of finite mutation type, then any edge of Q is a single edge or a double edge.*

 (ii) *Suppose that Q has exactly three vertices and has a double edge. Then Q is of finite mutation type if and only if it is an oriented triangle with edge weights $2, 1, 1$ or $2, 2, 2$.*

(iii) *Any non-simply-laced, non-oriented cycle is of infinite mutation type.*

(iv) *Suppose that Q is a simply-laced quiver. If Q contains a non-oriented cycle C such that there is a vertex k which is connected to exactly an odd number of vertices in C, then Q is of infinite mutation type. If C is an oriented cycle in Q and k is connected to exactly an odd number greater than or equal to 3 vertices in C, then Q is also of infinite mutation type.*

 (v) *Suppose that Q is a quiver which has no oriented cycles but has at least two non-oriented cycles. Then Q is of infinite mutation type.*

Proof. Statements (i),(ii),(iii) are obtained easily from the definitions by observing that if the conclusions do not hold then, by an iterative process, the mutation class of the quiver contains edges of arbitrarily large weights. Let us now prove (iv) for a non-oriented cycle C. (The second part for an oriented cycle follows by similar arguments). By part (iii), we can assume that C is simply-laced. First we consider the case where k is connected to exactly one vertex, say c, in C. Let us suppose first that C is a triangle. Applying a mutation at a source or sink of C if necessary, we can assume that c is a source or sink; mutating at the vertex which is neither a source or sink, we obtain a quiver which contains a three-vertex tree which has a double edge; then part (ii) applies. Let us now suppose that C has more than 3 vertices. Then, applying a mutation at a source or sink of C if necessary, we can assume that there is a vertex $c' \neq c$ in C which is neither a source nor a sink in C. Then, in $\mu_{c'}(Q)$, the subquiver C' obtained from C by removing c' is a non-oriented cycle and k is connected to exactly one vertex in C'. Then the statement (iv) follows by induction.

Let us now consider the case where k is connected to exactly three vertices in C. Then there are three cycles, say C_1, C_2, C_3, that contain k; one of them, say C_1, is necessarily non-oriented. If one of the cycles C_2 or C_3 has more than three vertices, then there is a vertex in that cycle connected to exactly one vertex in C_1, which is the case we considered above. Thus we can further assume that C_2 and C_3 are triangles. Given all this, we proceed as follows. If C has exactly three vertices, then the statement follows from a direct check. If C has more than three vertices, then one of the cycles C_1, C_2, C_3 also has more than three vertices; since C_2 and C_3 are triangles, the cycle C_1 must have at least four vertices. If any of C_2 or C_3 is non-oriented, then there is a vertex in C_1 which is connected to exactly one vertex in that cycle, which is the case we considered above. Then the only subcase

left to consider is the case where both C_2 and C_3 are oriented. Then, in $\mu_k(Q)$, the subquiver $C \cup \{k\}$ consists of a non-oriented cycle C' that contains k and an additional vertex which is connected to exactly one vertex in C', which is again the case we considered above. To consider the case where k is connected to at least five vertices in C, we note that in this case there is a non-oriented cycle C' which contains k and there is a vertex in C connected to exactly one vertex in C', which is also the case we considered.

To prove part (v), we can assume that any cycle in Q is simply-laced by part(iii). Let us now suppose that C is a cycle in Q with minimal number of vertices. There is a vertex k which is not in C but connected to C. If k is connected to C by a double edge e, then there is a three-vertex acyclic subquiver that contains e, so part (ii) applies. Thus we can also assume that any edge connecting k to C is a single edge. If k is connected to an odd number of vertices in C, then part (iv) applies. If k is connected to an even number of vertices and C is a triangle or a square, then the statement follows from a direct check; if C has at least five vertices, then there is a non-oriented cycle C' containing k such that there is a vertex $r \neq k$ which is connected to exactly an odd number of vertices in C', so part (iv) applies. This completes the proof of the proposition. $\qquad\square$

The following statement follows from the definitions:

Lemma 2.2. *Suppose that Q is a quiver and $S = \{s_1, ..., s_r\}$ is a subquiver of Q. Let $u = e_{s_1} + ... + e_{s_r}$. Then we have the following:*

   (i) *u is a radical vector for B^Q if and only if, for any vertex j and for the edges connecting j to (a vertex in) S, the following holds: the number of such edges entering j is equal to the number of the ones leaving, each edge being multipled by its weight.*
   (ii) *u is a radical vector for $\bar{B}^Q$ if and only if, for any vertex j, the sum of the weights of the edges connecting j to S is even.*

We also give several different characterizations of the mutation class of the Dynkin quiver E_6:

Proposition 2.3. *Suppose that Q is a simply-laced connected quiver with six vertices such that Q does not contain any non-oriented cycle. Then the following statements (a),(b),(c),(d) are equivalent.*

   (a) *Q is mutation-equivalent to E_6.*
   (b) *Q contains a basic subquiver and B^Q has corank 0.*
   (c) *Q contains a basic subquiver and $\bar{B}^Q$ has corank 0.*
   (d) *The following (i),(ii),(iii) hold:*
       (i) *Q contains a basic subquiver,*
       (ii) *for each cycle C in Q, there is a vertex which is connected to exactly one vertex in C,*
       (iii) *for each pair of vertices which are not connected to each other, there is a vertex which is connected to exactly one of them.*

We will first prove that (a) and (b) are equivalent. To show that (a) implies (b), let us suppose that Q is mutation-equivalent to E_6. If Q does not contain any basic quiver, then it is mutation-equivalent to the Dynkin quiver A_6 [8, Corollary 5.15]. Therefore Q contains a basic subquiver. It also follows from a direct computation that B^Q has corank 0 (it is enough to compute it for $Q = E_6$ because the rank of

B^Q is invariant under the mutation operation). For the converse, let us suppose that B^Q has corank 0 and Q contains a basic subquiver S. Let us first assume that Q is of finite (cluster) type. Then, by the classification of finite type quivers, Q is mutation-equivalent to a Dynkin quiver which is of type A_6 or D_6 or E_6. Since Q contains a basic subquiver, it is not mutation-equivalent to A_6; since B^Q has corank 0, the quiver Q is not mutation-equivalent to D_6 either. Therefore Q is mutation-equivalent to E_6. Let us now assume that Q is not of finite type. Then Q contains a minimal infinite type subquiver M. Then M is mutation-equivalent to the extended Dynkin tree $D_4^{(1)}$ [8, Theorem 3.2]. The skew-symmetric matrix B^M has corank 3, so the corank of B^Q is at least 2, contradicting our assumption. Thus Q is necessarily mutation-equivalent to E_6. This completes the proof of the equivalence of (a) and (b). Using similar arguments, the equivalence of (a) and (c) can be proved easily. The equivalence of the other statements follow by Lemma 2.2 and Proposition 2.1(iv).

Let us also record the following statement which follows immediately from the previous proposition and Proposition 2.1(iv):

Proposition 2.4. *Suppose that Q is a simply-laced quiver which contains a non-oriented cycle. If the underlying (undirected) graph satisfies (i),(ii,(iii) of Proposition 2.3, then it is of infinite mutation type.*

In particular, if the underlying (undirected) graph of Q is equal to the underlying graph of a quiver which is mutation-equivalent to E_6, then Q is of infinite mutation type.

3. Proofs of Main Results

3.1. **Proof of Theorem 1.2.** We first show the "only if part", i.e. if Q comes from the triangulation of a surface then it satisfies (i),(ii). For this it is enough to establish the theorem for Q which does not contain any subquiver which is mutation-equivalent to E_6 or X_6 (because quivers that come from the triangulation of a surface have this property [4, Corollary 5.13]). We show this by induction on the number of vertices of S. The basis of the induction is for S with exactly four vertices. There are three types of such basic quivers: Dynkin tree D_4, two adjacent oriented triangles or oriented square (all are simply-laced). For convenience, we will first prove for S which is an oriented square.

Let us now assume that $S = \{s_1, s_2, s_3, s_4\}$ is oriented cyclically (so $s_1 \to s_2 \to s_3 \to s_4 \to s_1$). We will show that one of the vectors $e_{s_1} + e_{s_2} + e_{s_3} + e_{s_4}$ or $e_{s_1} + e_{s_3}$ or $e_{s_2} + e_{s_4}$ is a radical vector using Lemma 2.2(i). This trivially follows from Lemma 2.2 if there is no vertex which is connected to S. Thus we can assume that there is a vertex k, which is not in S, connected to (at least one vertex in) S. For any such k, we denote the subquiver $\{S, k\}$ by Sk for convenience. If k is connected to S by a double edge, then there is necessarily a three-vertex subquiver which is not as in Proposition 2.1(ii), so we assume that *any edge connecting a vertex k to S is a single edge.* Below we will establish the theorem considering possible cases for k to connect to S. During the analysis, if we do not specify an orientation on a subquiver, we assume that it is oriented as required by Proposition 2.1 to be of finite mutation type.

We first show the theorem in the case that (*) for any k which is connected to S, the quiver Sk does not contain any non-oriented cycle. Then there are the following two subcases: (i) k is connected to exactly one vertex in S or (ii) k is connected to

exactly two vertices s_i, s_j and the subquiver $\{k, s_i, s_j\}$ is an oriented triangle. For the subcase (i) let us assume without loss of generality that k is connected to s_1. We show that $e_{s_2} + e_{s_4}$ is a radical vector. This is true if no vertex (outside S) is connected to s_2 or s_4. Similarly, it is also true if any vertex which is connected to s_2 is connected to s_4 with the opposite orientation (recall that any edge incident to any s_i is a single-edge). We now consider the remaining two possibilities: (a) There is a vertex r which is connected to exactly one of s_2, s_4, say connected to s_2 (by a single edge). Note that by asumption (*), the subquiver Sr does not contain any non-oriented cycles either (in particular, r is connected to at most one of s_1, s_3). If r is connected to k by a double edge then, the subquiver $\{k, s_2, r\}$ is a three-vertex tree, so contradiction by Proposition 2.1(iv). Thus we can assume now that Skr is simply-laced. Then we have the following: If r is not connected to both of k and s_3, then the subquiver Skr is mutation-equivalent to E_6 (recall our convention that the edges adjacent to k or r are oriented as required by Proposition 2.1); if r is connected to both of k and s_3, then the subquiver $\{k, r, s_3, s_4, s_1\}$ is a cycle and s_2 is connected to exactly three vertices there so contradiction by Proposition 2.1(iv). (b) There is a vertex r which is connected to both of s_2, s_4 with the same orientation. Then the subquiver Sr contains a non-oriented cycle, contradicting (*).

For the subcase (ii), let us assume without loss of generality that k is connected to s_1 and s_2 (such that the triangle $\{k, s_1, s_2\}$ is oriented). We show that $e_{s_1} + e_{s_2} + e_{s_3} + e_{s_4}$ is a radical vector. If this is not true then, by Lemma 2.2, there is a vertex r as in the following two subcases: (a) r is connected to exactly an odd number of vertices in S. Then, by Proposition 2.1(iv) or (*), the vertex r is connected to exactly one vertex in S. This is the same situation as we considered in the previous subcase (exchanging r and k), which implies that $e_{s_1} + e_{s_3}$ or $e_{s_2} + e_{s_4}$ is radical, however this is not true in this case (Lemma 2.2). (b) r is connected to exactly an even number of vertices in S such that the number of corresponding edges which enter r is different from the ones which leave. Then the subquiver Sr contains a non-oriented cycle, contradicting (*).

Thus for the rest of the proof, we can assume that *there is a vertex k which is contained in a non-oriented cycle $C \subset Sk$.* Then k is connected to at least two vertices in S. If k is connected to exactly two vertices s_i, s_j in S and s_i, s_j are connected, then C is the triangle $\{k, s_i, s_j\}$ and one of the remaining vertices in S is connected to exactly one vertex in C, which implies that Q is not of finite mutation type (Proposition 2.1(iv)), contradicting our assumption. We have the same contradiction if k is connected to exactly three vertices in S. Therefore, we can assume that k is connected to exactly two vertices in S which are not connected to each other or k is connected to all four vertices. We proceed considering possible (sub)cases:

Case 1. *k is connected to exactly two vertices, say s_1, s_3, in S, which are not connected to each other.*

Subcase 1.1. *k is a source or sink in C.* Then Sk has two non-oriented cycles $C = \{k, s_1, s_2, s_3\}$ and $C' = \{k, s_1, s_3, s_4\}$. We will show that $e_{s_2} + e_{s_4}$ is radical. Suppose that this is not true. Then there is a vertex r which is (i) connected to exactly one of s_2, s_4 or is (ii) connected to both of s_2, s_4 with the same orientation (Lemma 2.2).

In the former case (i), assume without loss of generality that r is connected to s_2 (by a single edge) and not connected to s_4 (note that r may be connected to

s_1 or s_3). If r is connected to k by a double edge, then the subquiver $\{r, k, s_2\}$ is a three-vertex tree that contains this double edge, contradicting the assumption that Q is of finite mutation type by Proposition 2.1(ii). If r is connected to k by a single edge, then the vertex r is connected to exactly an odd number of vertices in C or C' (more explicitly, if r is connected to an even number of vertices in C' then it is connected to exactly one of s_1, s_3, because r is not connected to s_4, then r is connected to exactly three vertices in C), contradiction by Proposition 2.1(iv). Similarly, if r is not connected to k then the vertex r is connected to exactly an odd number of vertices in C or C' (more explicitly, if r is not connected to any of s_1, s_3, then it is connected to exactly one vertex, which is s_2, in C; otherwise r is connected to exactly one of s_1, s_3, then r is connected to exactly one vertex in C), contradiction.

In the latter case (ii), first we note that r is connected to k because otherwise there is a non-oriented cycle C'' that contains r (because S is oriented) such that k is connected to exactly one vertex in C'', contradiction by Proposition 2.1(iv). If r is connected to k by a double edge, then there is the three vertex tree $\{k, r, s_2\}$, contradiction by Proposition 2.1(ii). If r is connected to k by a single edge and not connected to any of s_1, s_3, then the subquiver Skr is not of finite mutation type (Proposition 2.1(v)) or it is mutation-equivalent to X_6; if r is connected to k (by a single edge) and connected to both of s_1, s_3, then Skr is mutation-equivalent to X_6, contradiction. If r is connected to exactly one of s_1, s_3, then it is connected to to exactly three vertices in S, so Proposition 2.1(iv) applies.

Subcase 1.2. *k is a not a source or sink in C.* Then Sk consists exactly of an oriented cycle, say $C = \{s_1, s_2, s_3, k\}$, and a non-oriented cycle $C' = \{k, s_1, s_3, s_4\}$ (both containing k). Note then that any edge incident to k is a single edge by Proposition 2.1(ii).

To proceed let us first note the following:

Claim: If a vertex r which is not in Sk is connected to k (by a single edge), then r is connected to exactly one of s_1, s_3.

Proof: if r is not connected to any of them, then it is connected to s_4 (by Proposition 2.1(iv) because C' is non-oriented), which implies that both cycles $\{k, s_1, s_4, r\}$, $\{k, r, s_4, s_3\}$ are non-oriented (because there s_1 or s_3 is a sink or source respectively), then Proposition 2.1(v) applies to their union to give a contradiction. If r is connected to both of s_1, s_3, then r is connected to s_4 as well (to connect to an even number of vertices in C'), so there are four triangles in $C'r$. By Proposition 2.1(v), we can assume that exactly two of them, say T, T', are non-oriented and they are not adjacent (it is not possible that all four of these triangles are oriented because S is oriented). We may assume, without loss of generality, that T does not contain k. Then we have the following: if r is not connected to s_2, then s_2 is connected to exactly one vertex in T, so Proposition 2.1(iv) applies to give a contradiction; if r is connected to s_2, then each of the triangles $T_1 = \{r, s_1, s_2\}$ and $T_2 = \{r, s_3, s_2\}$ is non-oriented (more explicitly, e.g., if the triangle T_1 is oriented, then the triangles $\{r, s_1, k\}$ and $\{r, s_1, s_4\}$ are both non-oriented and adjacent, so Proposition 2.1(v) applies), furthermore one of T_1, T_2 is adjacent to T or T', so Proposition 2.1(v) applies to give a contradiction.

Subsubcase 1.2.1. *There is a vertex r which is not in Sk such that r is connected to k.* Then r is connected to exactly one of s_1, s_3 by the Claim above. Thus we can assume that r is connected to only k and s_3 in C' (if r is connected to s_4, then

Proposition 2.1(iv) applies) and that the triangle $\{k, r, s_3\}$ is oriented (otherwise, since it is adjacent to the non-oriented C', Proposition 2.1(v) applies). Note then that r is not connected to s_2 or s_4 because then it is connected to exactly three vertices in C or C'. We will show that $e_{s_2} + e_{s_4}$ is a radical vector. Suppose that this not true, i.e. there is a vertex t which is not in Sk such that (i) t is connected to exactly one of s_2, s_4 or (ii) t is connected to both of s_2, s_4 such that the edges $\{t, s_2\}$ and $\{t, s_4\}$ have the same orientations (Lemma 2.2). We will show that this contradicts our assumptions. Note that since the triangle $\{k, r, s_3\}$ is oriented, any edge incident to r is a single edge (otherwise Proposition 2.1(ii) applies to give a contradiction).

We consider the subcases for (i). Let us first suppose that (a) t is connected to s_4 and not connected to s_2. If t is connected to k as well, then t is connected to exactly one of s_1, s_3 by the Claim, then it is connected to exactly three vertices in C', which gives a contradiction. Thus we can assume that t is *not* connected to k. Then t is connected to exactly one of s_1, s_3 (otherwise t is connected to an odd number of vertices in the non-oriented cycle C').

Under all these assumptions, suppose that (a1) t is connected to s_3 (and not connected to s_1). Then note that the triangle $\{t, s_3, s_4\}$ is oriented (otherwise, since it is adjacent to C', Proposition 2.1(v) applies). If t is connected to r then the cycle $C'' = \{s_4, t, r, k, s_1\}$ is non-oriented (where s_1 is a sink) and s_2 is connected to exactly one vertex (which is s_1) in C'', contradiction by Proposition 2.1(iv). If t is not connected to r, then the subquiver $\{r, k, s_3, s_4, t, s_2\}$ is mutation-equivalent to E_6, which is a contradiction. Suppose now that (a2) t is connected to s_1 (and not connected to s_3). Similarly the triangle $\{t, s_1, s_4\}$ is oriented. If t is connected to r then the cycle $C''' = \{t, r, k, s_1\}$ is non-oriented (where k is a source) and s_2 is connected to exactly one vertex (which is s_1) in C''', contradiction. If t is not connected to r, then the subquiver $\{r, k, s_1, s_4, t, s_2\}$ is mutation-equivalent to E_6, contradiction.

Let us now suppose that (b) t is connected to s_2 and not connected to s_4. If t is connected to k as well, then t is connected to exactly one of s_1, s_3 by the Claim, so it is connected to exactly three vertices in C contradiction (Proposition 2.1(iv)). Thus assume that t is not connected to k. (b1) If t is connected to r, then we have the following: if t is not connected to s_1, then the cycle $C'''' = \{k, s_1, s_2, t, r\}$ is a non-oriented cycle (where k is a source) and s_4 is connected to exactly one vertex (which is s_1) in C'''', contradiction; if t is connected to s_1, then the cycle $C''''' = \{k, s_1, t, r\}$ is a non-oriented cycle (where k is a source) and s_4 is connected to exactly one vertex (which is s_1) in C''''', contradiction. (b2) If t is not connected to r, then the subquiver $\{t, s_2, s_3, r, k, s_4\}$ is mutation-equivalent to E_6, contradiction.

We consider the subcases for (ii), so suppose that t is connected to both s_2, s_4 such that the edges $\{t, s_2\}$ and $\{t, s_4\}$ have the same orientations. If t is connected to k as well, then t is connected to exactly one of s_1, s_3 by the Claim, so it is connected to exactly three vertices in C contradiction (Proposition 2.1(iv)). Thus assume that t is not connected to k. Then again t is connected to exactly one of s_1, s_3 (otherwise t is connected to an odd number of vertices in the non-oriented cycle C'), however then t is connected to exactly three vertices in S, so again Proposition 2.1(iv) applies to give a contradiction.

Subsubcase 1.2.2. *No vertex r which is not in Sk is connected to k.* First suppose that there is a vertex r which is not in Sk such that r is connected to s_4. Then

(**) r is connected to exactly one of s_1, s_3, say connected to s_1 (otherwise r is connected to exactly an odd number of vertices in C' because r is not connected to k, so Proposition 2.1(iv) applies). Then r is not connected to s_2 (because then it is connected to exactly three vertices in S). This implies, in particular, that the vectors $e_{s_1} + e_{s_3}$ and $e_{s_2} + e_{s_4}$ are both not radical. We claim that $e_{s_1} + e_{s_2} + e_{s_3} + e_{s_4}$ is radical. Suppose that this is not true. Then there is a vertex $t \neq k, r$ connected to S such that, for the edges connecting t to S, the number of the ones going away from t is different from the ones going in (Lemma 2.2). Then either (i) t is connected to exactly one vertex in S or (ii) t is connected exactly two vertices s_i, s_j and the orientations of the corresponding edges are the same (so t is a source or sink in St) (here note that, by (**), the vertex t is not connected to all vertices in S, so it is connected to at most two vertices in S by Proposition 2.1(iv)). In the latter case (ii), the vertices s_i and s_j are not connected (because otherwise the triangle $T = \{t, s_i, s_j\}$ is non-oriented and any of the remaining vertices of S is connected to exactly one vertex in T), so this case is the same as Subcase 1.1 above replacing k by t, which implies that $e_{s_1} + e_{s_3}$ or $e_{s_2} + e_{s_4}$ is radical, however this is not true in this case, so contradiction. Thus here we only need to consider the case (i), where t is connected to exactly one vertex, say v, in S. Then $v = s_2$ because otherwise t is connected to exactly one vertex in the non-oriented cycle C' (note that t is not connected to k by the definition of this case). If t is connected to r, then the subquiver $\{t, r, s_1, s_2\}$ is a non-oriented cycle that contains t and r and k is connected to exactly one vertex there (note that the triangle $\{r, s_1, s_4\}$ is oriented because otherwise, since it is adjacent to the non-oriented C', Proposition 2.1(v) applies), contradiction. If t is not connected to r, then the subquiver Str is mutation-equivalent to E_6, which also contradicts an assumption.

Now suppose that no vertex r which is not in Sk is connected to s_4. Note that if such a vertex r is connected to s_1 or s_3, then it is connected to the other one as well (otherwise t is connected to exactly one vertex in C') and it is not connected to s_2 (otherwise it is connected to exactly three vertices in S and Proposition 2.1(iv) applies). Also if r is connected to both s_1 and s_3 with the same orientations, then the cycles $\{r, s_1, s_4, s_3\}$ and $\{r, s_1, s_3, k\}$ are non-oriented and their union has no oriented cycles, so contradiction by Proposition 2.1(v). Thus if a vertex is connected s_1 or s_3, it is connected to both of them with opposite orientations. This implies that $e_{s_1} + e_{s_3}$ is radical.

Case 2. *k is connected to all four vertices in S.* Then Sk has four triangles that contain k; two of them are oriented and the other two are non-oriented and the non-oriented ones are not adjacent (Proposition 2.1(v)). We will show that the vector $e_{s_1} + e_{s_2} + e_{s_3} + e_{s_4}$ is radical. This trivially follows from Lemma 2.2 if there is no other vertex which is connected to S. Let us now denote by r an arbitrary vertex which is connected to S. If r is connected to k by a double edge, then it is connected to an oriented triangle, so Proposition 2.1(ii) applies necessarily; if r is connected to k by a single edge, then it is connected to both of s_1, s_3 or both s_2, s_4, but not all four of them (otherwise k is connected to exactly three vertices in one of the non-oriented triangles of Sk containing k, then Proposition 2.1(iv) applies) so Skr is mutation-equivalent to X_6, contradiction. Let us now assume that r is not connected to k and the subquiver Skr is simply-laced. If r is connected to a vertex in one of the non-oriented triangles, then it is connected to the other vertex as well

with the opposite orientation by Proposition 2.1(iv,v) (note that every vertex of S lies in exactly one of the non-oriented triangles). Then for the edges connecting r to S, the number of the ones leaving r is the same as the ones entering r. Thus $e_{s_1} + e_{s_2} + e_{s_3} + e_{s_4}$ is radical by Lemma 2.2.

We have completed the proof that the basic quiver which is oriented square contains the support of a non-zero radical vector u as in Theorem 1.2. To show this for the other basic subquivers with 4 vertices, let us first note that they are mutation-equivalent to the oriented square. Furthermore, mutation of a basic quiver with four vertices is also a basic quiver. Let us now suppose that S is a basic subquiver in Q and u is a radical vector as in the theorem. Let k be a vertex in S and let u' be the vector that represents u in the basis which corresponds to $B^{Q'} = \mu_k(B^Q)$ (Section 2). Then $supp_{Q'}(u')$ lies in the basic subquiver $S' = \mu_k(S) \subset \mu_k(Q) = Q'$. Also Q' does not contain any subquiver which is mutation-equivalent to E_6 or X_6 (by [4, Corollary 5.13] or by Theorem 1.4, which we will prove without using the current theorem). Thus we can conclude that any basic subquiver S with four vertices contains the support of a non-zero radical vector as in Theorem 1.2. For a basic quiver S with $m > 4$ vertices, applying the mutation at a vertex k of S gives a basic subquiver S' with $m - 1$ vertices, then the existence of the non-zero radical vector u follows by induction and the base change formula as we discussed.

Conversely, to show the if part of the theorem, suppose that Q is a finite mutation type quiver which does not come from the triangulation of a surface. Then Q contains a subquiver which is mutation-equivalent to E_6 or X_6 [4, Corollary 5.13]. In fact, it is enough to show it for E_6 and X_6 (see also Theorem 1.4). For E_6 it follows from Proposition 2.3, for X_6 it follows from a direct check using Lemma 2.2.

3.2. **Proof of Theorem 1.3.** The first part of the theorem, where Q contains a double edge or a non-oriented cycle, follows from Proposition 2.1(ii,iv) and Lemma 2.2. Let us now prove the second part. For this, let us assume that Q is a quiver which comes from the triangulation of a surface or Q is mutation equivalent to one of X_6, X_7. Then Q does not contain any subquiver which is mutation-equivalent to E_6 [4, Corollary 5.13]. We first show (ii) for convenience. Let $Q' = \mu_k(Q)$. Suppose that u be a basic radical vector for $\bar{B}^Q$ and let u' be the vector that represents u in the basis that corresponds to $\bar{B}^{Q'}$ (see Section 2 for the base change formula corresponding to μ_k). We will show the following:

Claim: u' is in the span of basic radical vectors for $\bar{B}^{Q'}$.

Proof: If k is in $supp_Q(u)$, then it follows from a direct check that u' is a basic radical vector for $\bar{B}^{Q'}$. We consider the case where k is not in $supp_Q(u)$ (but connected to it). It is easy to check the claim if $supp_Q(u)$ has exactly two vertices. Thus we assume that $C = supp_Q(u)$ is a cycle. We denote the subquiver $C \cup \{k\}$ by Ck for convenience. Note that if u lies in the span of basic radical vectors whose support contain k, then we are done. First we consider the subcase that k is connected to a vertex, say c, in C by a double edge. Let c', c'' be the vertices which are adjacent to c in C. Then, by Proposition 2.1(ii), the vertex k is connected to both of c' and c'' and it is not connected to any other vertex in C. Then the subquiver C' obtained from Ck by removing c is a non-oriented cycle (k is a source or sink there). Thus $u = y - x$ where y the vector such that $supp_Q(y) = C'$ and x is the vector such that $supp_Q(x) = \{k, c\}$ (recall that we work modulo $2\mathbb{Z}$). the vectors x, y are basic radical vectors for $\bar{B}^Q$ by the first part of the theorem. Since

the claim is true for x, y (their supports contain k), it is also true for u. Similar arguments, in view of Theorem 1.2, also show the Claim if Ck has a subquiver S which is a non-oriented cycle or a basic subquiver such that S contains k. Then the remaining subcase is where the vertex k is connected to exactly two vertices c_1, c_2 in C and that the triangle $\{k, c_1, c_2\}$ is oriented. Then u' is a basic radical vector for $\bar{B}^{Q'}$ with $supp_{Q'}(u) = \mu_k(Ck)$, which is a cycle in $Q' = \mu_k(Q)$.

Thus, by the Claim, we have $dim(\bar{V}_{00}^Q) \leq dim(\bar{V}_{00}^{Q'})$. Since μ_k is involutive, it is also true that $dim(\bar{V}_{00}^{Q'}) \leq dim(\bar{V}_{00}^Q)$, so $dim(\bar{V}_{00}^Q) = dim(\bar{V}_{00}^{Q'})$.

To prove part (i), let us first note that the statement is true if Q is (mutation-equivalent to) the Dynkin quiver A_n. For arbitrary Q, we will reduce the claim to the A_n case, which will prove the statement. Let us now assume that u, v are two non-zero radical vectors which are not in $\bar{V}_{00}^Q$. Since we could replace u by $u - w$, we can assume without loss of generality that:

(****) the union of $supp_Q(u)$ and $supp_Q(v)$ (in particular each of them) does not contain the support of a basic radical vector w of $\bar{B}^Q$.

Then, by Theorem 1.2, any connected component of $supp_Q(u)$ (or $supp_Q(v)$) is a single vertex or a simply-laced oriented triangle. In view of part (ii), applying some mutations if necessary, we can assume that each connected component of $supp_Q(u)$ and $supp_Q(v)$ is a single vertex (if mutations are applied the conditions of the statement will also be satisfied for the resulting quiver because of Theorem 1.4). Let us now note that (****) implies the following: a minimal connected subquiver M that contains $supp_Q(u)$ and $supp_Q(v)$ does not contain any basic subquiver or a non-oriented cycle or a double edge. This is because if M contains such a subquiver S then, by Theorem 1.2 and part (ii) of the current theorem, there is a basic radical vector w such that $supp_Q(w)$ lies in S; furthermore, by the assumption (****), there is a vertex in $supp_Q(w)$ which is not contained in any of $supp_Q(u)$ or $supp_Q(v)$; removing this vertex gives a connected subquiver (because if a vertex is connected to $supp_Q(w)$ it is connected to at least two vertices there). This contradicts minimality of M. Thus M is mutation-equivalent to A_n. Since u, v belong to V_{00}^M, whose dimension is at most 1, they are linearly dependent, so $\bar{V}_0^Q / \bar{V}_{00}^Q$ has dimension at most 1. This completes the proof of the theorem.

3.3. **Proof of Theorem 1.4.** We prove the theorem in three lemmas:

Lemma 3.1. *Let Q be a quiver of finite mutation type and let k be a vertex in Q. Suppose that Q has a subquiver which is mutation-equivalent to X_6. Then $\mu_k(Q)$ also contains a subquiver which is mutation-equivalent to X_6.*

Proof. Let us denote by X the subquiver which is mutation-equivalent to X_6. The lemma is obvious if k is in X, so we can assume that k is not in X. We then denote the subquiver $\{X, k\}$ by Xk for convenience. For quivers which are mutation-equivalent to X_7 the lemma follows from a direct check (these quivers are given in [3]). Then, to complete the proof of the lemma for a general Q, it is enough to show that Xk is mutation-equivalent to X_7. For this purpose, applying mutations if necessary, we can assume that $X = X_6$. Let us then denote the double edges of X by $\{i_1, i_2\}$ and $\{j_1, j_2\}$, the center vertex by c and the remaining vertex by d. By Proposition 2.1(i,ii), we can assume without loss of generality that k is connected to $\{i_1, i_2\}$ such that the triangle $\{i_1, i_2, k\}$ is oriented. If k is not connected to any other vertex in X_6, then the subquiver obtained from Xk by removing i_1 (or i_2) is

$m \geq 1$

$m \geq 1$

$m \geq 1,\ r = 3, 4, 5,\ 1 \leq i \leq r$
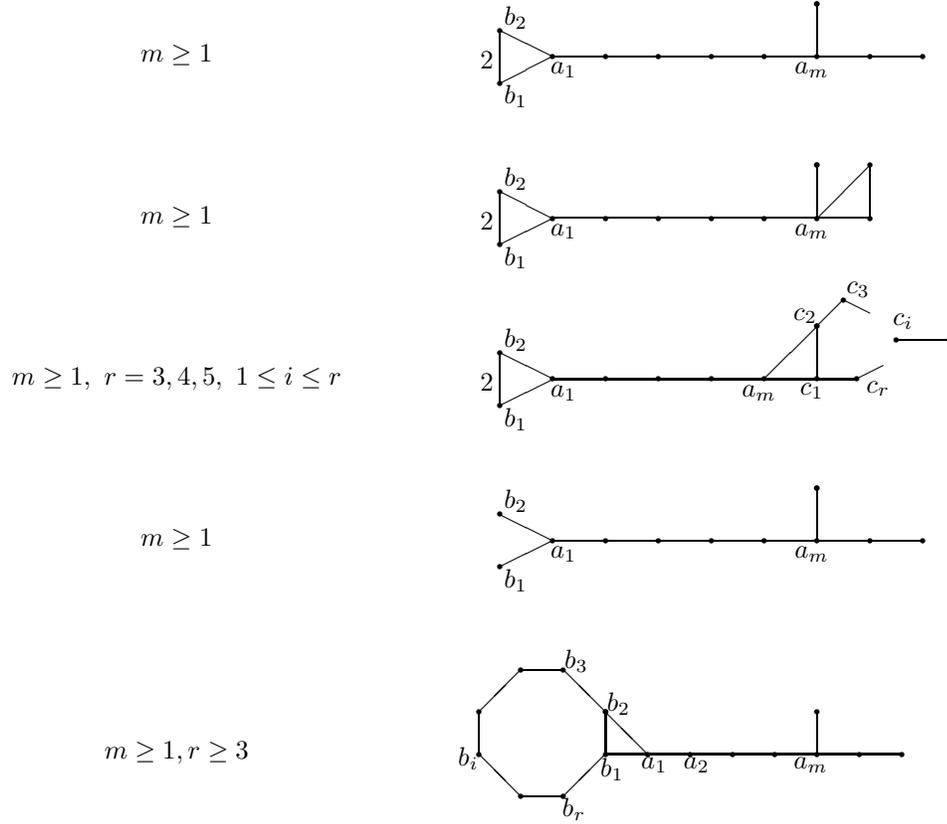
$m \geq 1$

$m \geq 1, r \geq 3$

FIGURE 4. Some quivers of infinite mutation type; each edge can be taken to be arbitrarily oriented (see also Proposition 2.1).

of infinite mutation type (it belongs to Figure 4), contradicting that Q is of finite mutation type. If k is connected to a vertex which is different from i_1, i_2, then it follows from a direct check that there is a non-oriented cycle that contains k such that Proposition 2.1(iv,v) applies. Thus Xk is mutation-equivalent to X_7. This completes the proof of the lemma. □

Lemma 3.2. *Suppose that Q is a quiver of finite mutation type and let k be a vertex in Q. Suppose also that Q has a subquiver which is mutation-equivalent to E_6. Then Q does not contain any subquiver which is mutation equivalent to X_6. Furthermore, the quiver $\mu_k(Q) = Q'$ also contains a subquiver which is mutation-equivalent to E_6.*

Proof.   For the first part of the lemma, let us assume to the contrary that Q contains a subquiver which is mutation-equivalent to X_6. Then, by Lemma 3.1, the quiver Q is mutation-equivalent to X_6 or X_7. However, it follows from a direct check on the quivers which are mutation equivalent to X_6 or X_7 as given in [3], that

Q does not contain any subquiver which is mutation-equivalent to E_6, contradicting our assumption.

For the second part of the lemma, we first note that, by [4, Theorem 5.11], the quiver Q, so Q', does not come from the triangulation of a surface. Therefore Q' also contains a subquiver which is mutation eqivalent to E_6 or X_6 (which are minimal quivers that do not come from the triangulation of a surface). By the first part of the lemma, the quiver Q' does not contain any subquiver which is mutation-equivalent to X_6, thus Q' contains a subquiver which is mutation-equivalent to E_6. This completes the proof of the lemma.   □

Lemma 3.3. *Let Q be a connected quiver of finite mutation type. Suppose also that Q has a subquiver which is mutation-equivalent to E_6. Then Q is mutation-equivalent to a quiver which is one of the (exceptional) types E_6, E_7, E_8, $E_6^{(1)}$, $E_7^{(1)}$, $E_8^{(1)}$, $E_7^{(1,1)}$, $E_8^{(1,1)}$ [4, Figure 6.1].*

Proof. If Q is of finite type, then the lemma follows from the classification of finite type quivers under the mutation operation [6]. Thus we can assume that Q is not of finite type. Then it is mutation-equivalent to a quiver Q' which has a double edge $e = \{u_1, u_2\}$. Note that the quiver Q' also contains a subquiver which is mutation-equivalent to E_6 by Lemma 3.2. Below we consider cases depending on the number of vertices connected to e. In our analysis, if we do not specify an orientation on a subquiver, it is assumed to be oriented as required by Proposition 2.1 to be of finite mutation type.

Case 1. *There is exactly one vertex, say v_1, connected to e.* In this case it can be checked easily, using Proposition 2.3(iii), that Q' contains a subquiver M' as in Figure 4. (More explicitly M' is a minimal connected subquiver which contains e and a subquiver E' which is mutation-equivalent to E_6). Since the quivers in Figure 4 are of infinite mutation type, the quiver Q' is also of infinite mutation type, contradicting our assumption.

Case 2. *There are exactly two vertices, say v_1, v_2, connected to e.* By Proposition 2.1(ii,iii), we can assume that the subquiver $\{e, v_i\}$ is an oriented triangle for $i = 1, 2$. Let us also note that if there is a subquiver E' which is mutation-equivalent to E_6 such that E' contains at most one of v_1 and v_2, then Q' contains a subquiver as in Case 1 (it is a minimal connected subquiver that contains E' together with the edge e and one of the vertices v_1 or v_2), which we already considered. Thus here we only need to consider the case when any subquiver E' which is mutation-equivalent to E_6 contains both v_1 and v_2. Let us note that, by Proposition 2.3(iii), there are vertices v_1' and v_2' in E' which are connected to v_1 and v_2 respectively. Let us denote $e_1 = \{v_1, v_1'\}$, $e_2 = \{v_2, v_2'\}$

We first consider the subcase where E' contains a vertex which is adjacent to e (i.e. E' contains u_1 or u_2). Then any path connecting any of v_1, v_1' to v_2 or v_2' in E' contains u_1 or u_2 because otherwise there is a non-oriented cycle in E' (note that, since the subquiver $\{e, v_i\}$ is an oriented triangle, the edges that connect v_1 and v_2 to $u_j, j = 1, 2$ have the same orientation). In particular, we have $v_1' \neq v_2'$. Furthermore, the remaining vertex of E' is connected to exactly one of e_1 and e_2. However, then E' is either the Dynkin tree D_6 or it is mutation-equivalent to A_6, contradiction.

Let us now consider the subcase where E' does not contain any vertex which is in e (i.e. E' does not contain any of u_1, u_2). By Proposition 2.3, E' contains a basic

subquiver S. Since S has at least four vertices, it contains at least two vertices from the set $\{v_1, v_1', v_2, v_2'\}$. Let us first assume that S does not contain any vertex from one of the edges e_1, e_2, say it does not contain any vertex from e_2. Then S has exacly four vertices (because E' has six vertices and it already contains e_1 and e_2). Also the vertices in e_2 are connected to S as required Proposition 2.3(ii,iii). Then, it follows from an easy check that Q' contains a non-oriented cycle C such that there is a vertex r (in E') which is connected (by singles edges) to an odd number of vertices in C, contradiction by Proposition 2.1(iv).

To complete the treatment of this case, we now assume that S contains a vertex from each of e_1 and e_2 and consider the subcases:

Subcase 2.1. *S is the Dynkin tree D_4.* Let $S = \{a, b, c, d\}$ such that a is in e_1; b is in e_2; c is connected to both a, b and d is connected to only c in S (so c is the "center" of S). Let a', b' be the remaining vertices in e_1 and e_2 respectively (so $\{a, a'\} = \{v_1, v_1'\}$ and $\{b, b'\} = \{v_2, v_2'\}$).

Let us first assume that no vertex in e_1 is connected to any vertex in e_2. Then the vertex d is not connected to a' nor to b', by Proposition 2.3(iii) (applied to the pairs of vertices d, a and d, b). Let us note that, since the triangles $\{e, v_1\}$ and $\{e, v_2\}$ are oriented, there are two non-oriented cycles in Q' which contain v_1, v_2 and the vertex c. The vertex d is connected to exactly one vertex (which is c) in these cycles, contradiction by Proposition 2.1(iv). If there is a vertex in e_1 which is connected to a vertex in e_2, the subcase follows by similar arguments.

Subcase 2.2. *S is formed by two adjacent triangles.* As in the previous subcase, let $S = \{a, b, c, d\}$ such that a is in e_1; b is in e_2; c, d are connected to both a, b and to each other (so the triangles of S are $\{a, c, d\}$ and $\{b, c, d\}$). Let a', b' be the remaining vertices in e_1, e_2 respectively (so $\{a, a'\} = \{v_1, v_1'\}$ and $\{b, b'\} = \{v_2, v_2'\}$). If there is a vertex in e_1 which is connected to a vertex in e_2, then there is a non-oriented cycle in E', which is not the case, so we can assume that no vertex in e_1 is connected to any vertex in e_2. Then, by Proposition 2.3(ii), the vertices c, d are not connected to a' or b'. Since the triangles $\{e, v_1\}$ and $\{e, v_2\}$ are oriented, there are two non-oriented cycles say C_1, C_2 in Q' which contain v_1, v_2 together with the vertex c or d respectively. Then, e.g., the vertex c is connected to exactly three vertices in C_2, contradiction by Proposition 2.1(iv).

Subcase 2.3. *S is a square.* Let $S = \{a, b, c, d\}$ such that a is in e_1, b is in e_2, the vertices c, d are connected to both a, b and not connected to each other (so $\{a, a'\} = \{v_1, v_1'\}$ and $\{b, b'\} = \{v_2, v_2'\}$). As in the previous subcase, if there is a vertex in e_1 which is connected to a vertex in e_2, then there is a non-oriented cycle in E', which is not the case, so we can assume that no vertex in e_1 is connected to any vertex in e_2. By Proposition 2.3(iii), one of the vertices a', b', say a', is connected to exactly one of c or d, say connected to c. As in the previous subcases, there are two non-oriented cycles say C_1, C_2 in Q' which contain v_1, v_2 together with the vertex c or d. Then a' or c is connected to exactly an odd number of vertices in one of these non-oriented cycles, contradiction by Proposition 2.1(iv).

In the current set-up of this case, it is not possible that S is a cycle with five vertices, so we have completed our analysis for this case.

Case 3. *There are exactly three vertices, say v_1, v_2, v_3, connected to e.* Let us first note that if there is a subquiver E' which is mutation-equivalent to E_6 such that E' contains at most two of v_1, v_2, then we are in Case 2. Thus here we only need to consider the case when any subquiver E' which is mutation-equivalent to

E_6 contains all v_1, v_2, v_3. For any v_i, $i = 1, 2, 3$ we denote by Pv_i the subquiver
on the vertices which are connected to v_i by a path that does not contain any
vertex adjacent to e (v_i is included in Pv_i). We first show that for any $v_i \neq v_j$
connected to e, the subquivers Pv_i and Pv_j are disjoint. Suppose this is not true
and assume without loss of generality that $i = 1$, $j = 2$. Then there is a path
$P' = \{v_1 = w_1, w_2, ..., w_r = v_2\}$, $r \geq 2$, that connects v_1 and v_2 such that P'
does not contain any of u_1, u_2 (which are the vertices adjacent to e). We can
assume without loss of generality that P' a shortest path connecting two vertices
which are connected to e, implying that v_3 is not connected to any vertex in P'
except possibly to v_1 or v_2. If v_3 is not connected any of v_1, v_2, then the cycle
$C = \{P', u_1\}$ is non-oriented (because u_1 is connected to v_1 and v_2 by the same
orientation) and v_3 is connected to exactly one vertex (which is u_1) in C, which is a
contradiction by Proposition 2.1(iv). If v_3 is connected any of v_1, v_2 then similarly
Proposition 2.1(iv) or (v) applies to give a contradiction. Thus for the rest of this
case, we can assume that the subquivers Pv_i and Pv_j are disjoint.

Let us now note that by Proposition 2.3(iii) at least two of Pv_i's, say Pv_1 and
Pv_2, have at least two vertices. This implies that each Pv_i does not contain any
subquiver which is one of the following: a basic subquiver, a non-oriented cycle or
a double edge, because otherwise Q' contains a subquiver as in Figure 4. Thus each
Pv_i is mutation-equivalent to the Dynkin quiver A_n, applying some mutations if
necessary we can assume that each Pv_i is of type A_n such that v_i is an end vertex
of Pv_i (otherwise Q' also contains a subquiver as in Figure 4). Now we can proceed
to establish the lemma:

(i) Suppose that each Pv_i has at at least two vertices. If each of them has exactly
two vertices, then Q' is mutation-equivalent to $E_6^{(1,1)}$; if one of them has more than
two vertices, then Q' contains a tree which is extended Dynkin (it contains $E_6^{(1)}$ as
a proper subquiver), so it is of infinite mutation type [2].

Thus for the rest of the proof we can assume that Pv_3 has exactly one vertex.
Then we have the following subcases:

(ii) Suppose that each of Pv_1 and Pv_2 has at least three vertices. If both have
exactly two vertices, then Q' is mutation-equivalent to $E_7^{(1,1)}$; otherwise Q' contains
a tree which is not extended Dynkin (it contains $E_7^{(1)}$ as a proper subquiver), so it
is of infinite mutation type, which is a contradiction.

(iii) Suppose now, without loss of generality, that Pv_2 has exactly two vertices.
If Pv_1 has exactly two vertices then Q' is mutation-equivalent to $E_6^{(1)}$; if Pv_1 has
exactly three vertices, then Q' is mutation-equivalent to $E_7^{(1)}$; if Pv_1 has exactly
four vertices, then Q' is mutation-equivalent to $E_8^{(1)}$; if Pv_1 has exactly five vertices,
then Q' is mutation-equivalent to $E_8^{(1,1)}$; if Pv_1 has more than five vertices, then
Q' contains a tree which is not extended Dynkin (it contains $E_8^{(1)}$ as a proper
subquiver), so it is of infinite mutation type, which is a contradiction.

Case 4. There are at least four vertices connected to e. Let us assume that
v_1, v_2, v_3, v_4 are connected to e. Then the subquiver $S = \{u_1, v_1, v_2, v_3, v_4\}$ is the
extended Dynkin tree $D_4^{(1)}$. Since Q' contains a subquiver which is mutation-
equivalent to E_6, there is a vertex which is connected to e or S. Then there is
necessarily a tree that contains S as a proper subquiver, so it is of infinite mutation
type, which is a contradiction.

This completes the proof of the lemma.   □

Given these lemmas, let us now show how Theorem 1.4 follows. Let us first assume that Q is a finite mutation type quiver which contains a subquiver which is mutation equivalent to E_6. Then, by Lemma 3.2, any quiver which is mutation-equivalent to Q contains a subquiver which is mutation-equivalent to E_6; furthermore Q is mutation equivalent to a quiver which is one of the (exceptional) types E_6, E_7, E_8, $E_6^{(1)}$, $E_7^{(1)}$, $E_8^{(1)}$, $E_7^{(1,1)}$, $E_8^{(1,1)}$ by Lemma 3.3. Let us now assume that Q is a finite mutation type quiver which contains a subquiver which is mutation-equivalent to X_6. Then by Lemma 3.1 and its proof, any quiver which is mutation-equivalent to Q contains a subquiver which is mutation-equivalent to X_6 and Q is in fact mutation-equivalent to the quiver X_6 or X_7. This completes the proof of the theorem.

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