BASICS OF MODEL THEORY

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ABSTRACT

BASICS OF MODEL THEORY

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This thesis is based on mainly the first two chapters of Chang and Keisler's monumental work 'ModelTheory.' As a preperation to this material, a section on a general disscussion of formal languages and formal systems is added. In this section several basic notions of formal languages and systems as well as some set theoretic notions such as, legitimate strings of a language, deducibility of one string from another, axiomatic system, finiteness, countability and uncountability are defined with a few examples. The main result of this section is that, the set of finite strings over a countable alphabet is countable.

After this general introduction, the language and system of propositional logic is introduced. This is followed by the section on the model theory for propositional logic. In this section, symbols and legitimate strings of the language of propositional logic are interpreted by means of the definition of

the model. Then, model theoretic properties, some relations between model theoretic and proof theoretic properties of propositional logic and relations between models of the language of propositional logic are established.

In section 2.4, a more sophisticated formal language and formal system, the language and system of quantificational logic, is introduced, syntactic properties of terms and formulas of quantificational logic and proof theoretic properties of quantificational logic are given. In the last section of chapter 2, the basic model theory for quantificational logic is given.

Chapter 3 is devoted to the completeness theorem for quantificational logic: if T is a set of sentences of the language of quantificational logic, if T is consistent then T has a model. The proof does not only show the existence of a model of T. but also shows the way of constructing a model of T.

Keywords: Logic, formal language, formal system, theory, model. model theory, completeness theorem

ÖZ

MODEL TEORİSİNİN TEMELLERİ

Taşdelen, İskender

Yüksek Lisans, Felsefe Bölümü

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Bu tez başlıca olarak Chang ve Keisler'in anıtsal eseri 'Model Theory' nin ilk iki bölümü üzerine temellenmiştir. Bu malzemeye hazırlık olarak, formel diller ve formel sistemler ile ilgili genel bir tartışma kısmı eklenmiştir. Bu kısımda, bir dildeki düzgün diziler, bir dizinin bir başka diziden çıkarılabilirliği, aksiyomatik sistem gibi formel dillere ve sistemlere ait temel nosyonların yanısıra sonluluk, sayılabilirlik ve sayılamazlık gibi set teorisine dair bazı nosyonlar da birkaç örnekle birlikte tanımlanmıştır.

Bu genel girişten sonra, önermeler mantığının dili ve sistemi tanıtılmaktadır. Daha sonra gelen kısım önermeler mantığı için model teorisi üzerinedir. Bu kısımda, önermeler mantığının dilinin sembolleri ve düzgün dizileri model tanımı aracılığıyla yorumlanmıştır. Daha sonra, model teorisine dair özellikler ile önermeler mantığının model teorisiyle ve ispat teorisiyle ilgili özellikleri arasındaki bazı ilişkiler ve önermeler mantığının dilinin modelleri arasındaki ilişkiler kurulmuştur.

Kısım 2.4'te daha karmaşık bir formel dil ve formel sistem olan niceleme mantığının dili ve sistemi tanıtılmış, terimlerin sentaktik özellikleri, niceleme mantığının formülleri ve niceleme mantığının ispat teorisiyle ilgili özellikleri verilmiştir. 2. bölümün son kısmında niceleme mantığı için temel model teorisi verilmiştir.

Bölüm 3, niceleme mantığı için olan eksiksizlik teoremine ayrılmıştır: Eğer T niceleme mantığının dilinin bir cümleler kümesi ise, o zaman eğer T tutarlı ise, T'nin bir modeli vardır. Bu ispat yalnızca T'nin bir modelinin varolduğunu göstermez, aynı zamanda T'nin bir modelini kurma yolunu da gösterir.

Anahtar Kelimeler: Mantık, formel dil, formel sistem, theory, model, model teorisi, eksiksizlik teoremi

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CHAPTER 1

INTRODUCTION

There are several ways to study mathematical structures in a general setting. Universal Algebra studies algebraic structures independently of the domain of a given algebraic structure. Thus, the basic notion of Universal Algebra is that of 'algebras' and not specific algebras. Algebras then become examples and means of justification for the general claims of Universal Algebra. In Category Theory. a type of mathematical structures and morphisms among the members of the given type constitutes the object of the study.

Model Theory, the first results of which constitutes the subject of this study, is another way of studying mathematical structures in general. First, a 'language' which is defined end structures in which symbols of the given language gets a meaning is called a model of the language. Model Theory is the study of the relationships between a language and its models.

The aim of this work is to give the most basic results of this rapidly growing field of mathematics s much as accessible to everyone with little knowledge in logic and set theory. With this thesis I aimed at making a modest contribution to its being a subject of a graduate logic course in a department of philosophy of a Turkish University. To this end, this work requires much to be added. I tried to take a few steps for my part as much as a thesis format allows its being a contribution in that direction. To achieve this aim to add an appendix on set theory and logic is among my plans. Moreover, the number of examples is not sufficient to provide a concrete basis for the theory and to provide a justification of the claim that model theory is really a strong tool of doing and understanding mathematics.

First two chapters of the Chang and Keisler's monumental work 'Model Theory' is the basis of the results and many of the proofs of this thesis. When a result or proof is taken from another author, this is indicated by the name of the author in parentheses, The same material is used for a serious course on Model Theory at the graduate level. I followed their plan to introduce the ideas with minor changes and my work must be seen as an attempt to contribute to a reading of Chang and Keisler. I tried to complete parts of proofs that has been left to the reader, or give proofs for the propositions stated without proofs. Moreover, exercises in the book are such that results in many of them with proof can safely be integrated into the main text of the book without disturbing the unity. Moreover, some of the important notions of Model Theory is defined in the exercises. I tried to make use of this fact several times.

I would like to choose a different manner of presentation of some notions in the part on the Propositional Logic. Chang and Keisler deviates from the tradition in that they intermixed proof theoretic and semantic notions. For example, the sign '⊢' is used for tautologies and tautologyhood is introduced as a syntactic notion while '⊢' is commonly used for theoremhood. However, I have choosen to follow their presentation and notation and just to warn the reader about this point. To give a list of syntactic and semantic notions is, then necessary. Theoremhood, consistency and deducibility are syntactic notions while validity, satisfiability and being a consequence of a set of sentences are semantic notions.

I hope that the section on the formal languages will be useful. Before introducing the formal languages and formal systems of propositional logic and quantificational logic, a general discussion of formal languages and formal systems gives us the chance to see the motivation behind the way of construction of specific formal languages and systems. Main result of this section is that, the set of finite strings over a countable alphabet is countable. This result is extremely useful since the languages we will actually encounter will be languages with countable alphabets and we will mainly deal with finite expressions of these languages. Moreover, knowledge of the fact that the set of expressions under consideration is countable proves to be used in proofs and model theoretic constructions.

Moreover, I have given formal definitions of free and bound variables and substitution in the first chapter. This was necessary since the notion of satisfaction in a model is one of the main concepts in model theory and understanding and usage of this concept requires a sound knowledge of these notions.

CHAPTER 2

BASICS

2.1 Formal Languages and Formal Systems

Definition 2.1.1. *A* ***formal language*** *is a structure $\mathcal{F}$ consisting of a set $\mathcal{A}$ of symbols which is called the* alphabet *of $\mathcal{F}$ and a list of prescriptions $\mathcal{F}$ determining whether a sequence of symbols from $\mathcal{A}$ is an acceptable sequence or not such that:*

- *Elements of $\mathcal{A}$ are not symbols of anything, all that matters is that they are just distinct members of $\mathcal{A}$.*
- *Our list of prescriptions on sequences of elements of $\mathcal{A}$ must determine the set of acceptable sequences independently of any interpretation.*

Hereafter we denote arbitrary formal languages by $\mathcal{L}, \mathcal{L}', \mathcal{L}''$ .... When the alphabet of the language in question is $\mathcal{A}$, $\mathcal{A}^*$ denotes the set of all finite strings of symbols of $\mathcal{A}$.

Definition 2.1.2. *The* ***length*** *of a string $\zeta \in \mathcal{A}^*$ is the number of symbols, counting repetitions, occurring in ζ.*

Definition 2.1.3. *The string of length zero is denoted by* $\circ$ *and is called* ***the empty string.***

This definition will not be used in this work and we will use only nonempty strings of symbols of the languages that we will actually work with. However, we include it since it is an important feature of formal languages that they may allow empty string. .

The size or cardinality of the alphabet is essential to the study of the formal language. Here is a list of definitions concerning the cardinality of a set:

Definition 2.1.4. *A set is said to be* ***finite*** *iff it can be put into one to one correspondance with a natural number.*

Definition 2.1.5. *A set is said to be* ***countable*** *iff it is not finite and there is a surjective map of the set* $\mathbb{N}$ *of natural number onto it. A set is said to be* ***uncountable*** *iff it is not countable and not finite.*

Definition 2.1.6. *A set is said to be* ***at most countable*** *iff it is either finite or countable.*

Example 2.1.1. Let $\mathcal{L}$ be such that $\mathcal{A}_1 = \{0, 1\}$ and every finite sequence of 1's and 0's is an acceptable sequence. It is easy to say that $\mathcal{L}$ is a formal language once you have some working knowledge of finiteness.

Note that neither 0 nor 1 as members of $\mathcal{A}_1$ has its usual meaning, they are just distinct symbols of $\mathcal{A}$. If you have any doubt, please reread the definition of the alphabet of a formal system.

Example 2.1.2. Let $\mathfrak{L}$ be such that; $\mathcal{A} = \{A, B, \ldots X, Y, Z, a, b, \ldots x, y, z\}$ and our rule is such that; any finite sequence of symbols from $\mathcal{A}$ which is a meaningful word in English is a legitimate expression.

Clearly $\mathfrak{L}$ is not a formal language since this time we are forced to interprete symbols of the alphabet (as letters of the English alphabet) in order to decide whether an expression is a legitimate one or not.

Definition 2.1.7. *A **formal system** is a structure consisting of a formal language $\mathfrak{L}$ together with a set of rules on the set of legitimate expressions of $\mathfrak{L}$ which determines whether a legitimate expresion of $\mathfrak{L}$ immediately follows from another one without regard to any interpretation.*

Remark 2.1.1. We may restate the definition as follows: Since we defined the rules of formal systems as relations, in fact as binary relations, such a rule must give a yes or no answer to the question: Is "$\langle X, Y\rangle \in \mathbf{R}$?" where X and Y are legitimate expressions of $\mathfrak{L}$ and $\mathbf{R}$ is a transformation rule of $\mathfrak{L}$ stated as a relation.

Remark 2.1.2. Note that a formal system is an extension of a formal language; we take a formal language and we define a set of relations on the set $\mathcal{A}^*$ to obtain a formal language. Expansion of a formal language in this way is not unique. In fact, if cardinality of $\mathcal{A}^*$ is η, the number of possible formal systems definable on $\mathcal{L}$ with one rule is $2^{\eta \times \eta}$. However, with a formal language of suitably large cardinality, this number gets very large if we introduce more than one rule of transformation. Complexity increases with the possibility of defining rules of transformation with more than one arguments. Such a rule of transformation takes, for example, a couple $\langle X, Y \rangle$ of legitimate $\mathcal{L}$-expressions and sends them to another one (or, to one of them).

Example 2.1.3. Let our language $\mathcal{L}$ be the language of example 2.1.1 and let our rules of transformation be:

(i). X is an immediate consequence of Y *iff* X can be obtained from Y by adding a finite number of 1's to the end of Y.

(ii). Z is the immediate consequence of X and Y *iff* Z is obtained by adding all symbols of Y to the end X.

Thus, by (i)., X is an immediate consequence of X; we obtain X by adding a finite number, 0, of 1's to X. By (ii). 011101001 is the immediate consequence of 01 and 1101001, of 0111 and 01001 ...

Formal systems enable us to work with a formal language without any reference to the meanings of the symbols of that formal language. The above definition of immediate consequence is the first step in that direction. Using the notion of immediate consequence and going further we obtain another important notion;

Definition 2.1.8. *We say that a string X is* ***deducible from*** *another string Y in a formal system* $\mathfrak{F}$*, in symbols* $X \vdash_{\mathfrak{F}} Y$ *iff there is a finite sequence of strings of* $\mathfrak{F}$ $X_0, X_1, \ldots, X_n$ *such that* $X_0 = X$*,* $X_n = Y$ *and each string* X_i *where* $0 < i \leq n$ *is an immediate consequence of a string preceeding it.*

For simplicity we assumed in this definition that all rules of transformations of the formal system under consideration are binary relations that is, a rule that takes one string and yields another string from that one. However, the reader is invited to enlarge the definition to the general case; take an n-ary transformation rule i.e., a rule that takes n strings and yield another.

A special type of formal systems is an ***axiomatic system***. An axiomatic system is a formal system with a set of initial legitimate expressions of its formal language. Elements of this set of initial strings are called the ***axioms*** of the formal system. In the case of axiomatic systems we modify definition 2.1.8 as; X is deducible from Y *iff* there is a finite sequence of legitimate expressions $X_0, X_1, \ldots, X_n$ such that $X_0 = X$. $X_n = Y$ and each string X_i where $0 < i \leq n$ is either an axiom, or an immediate consequence of a string preceeding it.

We said that the cardinality of the alphabet of a formal language plays an essential role in the study of formal languages and formal systems. We give and prove a result to be used in this direction.

Definition 2.1.9. *We will call two strings in a formal language* ***equivalent*** *if they have the same length and their corresponding elements are the same symbols from the alphabet.*

Lemma 2.1.1. If $\mathcal{A}$ is an at most countable alphabet, then the set $\mathcal{A}^*$ of strings over $\mathcal{A}$ is countable.

Proof. Without proof we will use the following result; a set $\mathbf{S}$ is at most countable iff there is an injective map $\alpha : \mathbf{S} \longrightarrow \mathbb{N}$. To prove the lemma it suffices to show that $\mathcal{A}^*$ is at most countable since it is easy to see that it is not finite. Using the mentioned result, if we can define an injective map $\alpha : \mathcal{A}^* \longrightarrow \mathbb{N}$ the proof is complete.

Let us denote the nth prime number by p_n If $\mathcal{A}$ is finite we may write $\mathcal{A} = \{a_0, a_1, a_2, \ldots, a_n\}$. If it is countable, $\mathcal{A} = \{a_0, a_1, a_2, \ldots\}$ In any case the map $\alpha : \mathcal{A}^* \longrightarrow \mathbb{N}$ defined by

$$\alpha(\circ) = 1, \qquad \alpha(a_{i_0} \ldots a_{i_r}) = {p_0}^{i_0+1} \cdot \ldots \cdot {p_r}^{i_r+1}.$$

This map α is injective and $\mathcal{A}^*$ is at most countable. To see that α is injective, choose a string $(a_{j_0} \ldots a_{j_s})$ different from $(a_{i_0} \ldots a_{i_r})$ and see that they take different values under α. For a string to be different from $(a_{i_0} \ldots a_{i_r})$ we have the following possibilities (i). $(a_{j_0} \dots a_{j_s})$ have the same length as $(a_{i_0} \dots a_{i_r})$. In that case again we have two possibilities:

- It has the same symbols as $(a_{i_0} \dots a_{i_r})$ but the order of symbols is different.
- It has at least one symbol different from$(a_{i_0} \dots a_{i_r})$.

(ii). $(a_{j_0} \dots a_{j_s})$ and $(a_{i_0} \dots a_{i_r})$ have different lengths. Then the length of $(a_{i_0} \dots a_{i_r})$ is either greater or less than the length of $(a_{j_0} \dots a_{j_s})$.

Using these cases, the reader may easily verify that α yields different values for different strings thus showing that α is an injection. □

Thus, if we have an at most countable language, the set of expressions, moreover, the set of legitimate expressions is countable. The importance of this result lies in the fact that, we may list elements of a countable set and then base very useful constructions on that counting since counting is firstly an ordering and ordering enables us 'to choose the next.' The reader is invited to a quick reading of the statement and proof of lemma 2.3.4 if he desires an example at this moment.

2.2 A System of Propositional Logic

After examples of section 2.1, we are now ready to present a special formal system; system of propositional logic $\mathcal{P}$. The alphabet $\mathcal{A}(\mathcal{P})$ of the language of $\mathcal{P}$, consists of:

(i). A set $\{p_0, p_1, \ldots\}$;

(ii). A unary sentential connective $\neg$;

(iii). A binary sentential connective $\wedge$;

(iv). Symbols of grouping (.).

Symbols of (i). are called *propositional variables*; $\neg$ is called *the negation*; $\wedge$ is called *and*; (,) are called *the left parenthesis* and *the right parenthesis* respectively.

Definition 2.2.1. *Legitimate expressions of $\mathcal{P}$ constitute the smallest set $Prop(\mathcal{P})$ such that:*

(i). Propositional variables are in $Prop(\mathcal{P})$;

(ii). If $\varphi \in Prop(\mathcal{P})$ then $\neg\varphi \in Prop(\mathcal{P})$;

(iii). If $\varphi \in Prop(\mathcal{P})$ and $\vartheta \in Prop(\mathcal{P})$ then $(\varphi \wedge \vartheta) \in Prop(\mathcal{P})$.

Remark 2.2.1. To guarantee that $Prop(\mathcal{P})$ is the smallest such set, we may, instead of saying that it is such a set, add the following clause to the definition of $Prop(\mathcal{P})$:

(iv). A string is in $Prop(\mathcal{P})$ *iff* it can be generated by one of (i).-(iii).

We call the elements of $Prop(\mathcal{P})$ $\mathcal{P}$*-sentences* or simply *sentences.*

We may construct the set $Prop(\mathcal{P})$ by recursion on the length of strings belonging to $\mathcal{A}^*(\mathcal{P})$ as follows

(i). A string of length 1 is in $Prop(\mathcal{P})$ *iff* it is a propositional variable;

(ii). A string α of length ≥ 1 is in $Prop(\mathcal{P})$ *iff* there is a member ς in $Prop(\mathcal{P})$ such that $\alpha = \neg\varsigma$ or there are members φ and ϑ in $Prop(\mathcal{P})$ such that $\alpha = (\varphi \wedge \vartheta)$

Just another way of constructing $Prop(\mathcal{P})$ as a set just as we want is this; we say that a set $\mathbf{S}$ is *inductive on* the language of $\mathcal{P}$, $\mathcal{L}(\mathcal{P})$, *iff*

(i). Every propositional variable of $\mathcal{L}(\mathcal{P})$ is in $\mathbf{S}$;

(ii). The string $\neg\alpha \in \mathbf{S}$ whenever $\alpha \in \mathbf{S}$;

(iii). The string $(\alpha \wedge \beta) \in \mathbf{S}$ whenever both $\alpha \in \mathbf{S}$ and $\beta \in \mathbf{S}$.

Thus, we see that

$$Prop(\mathcal{P}) = \bigcap\{\mathbf{S} : \mathbf{S} \text{ is inductive on } \mathcal{L}(\mathcal{P})\}$$

Particularly important is the inductive definition of $Prop(\mathcal{P})$ based on 2.2.1. Many important properties of $\mathcal{P}$-sentences can be proved to hold for every $\mathcal{P}$-sentence based on the definition 2.2.1 of $Prop(\mathcal{P})$. Let P be a property. To prove that every $\mathcal{P}$ sentence has the property P, we show that

(i). Every propositional variable has the property P;

(ii). If α has the property P, then $\neg\alpha$ has the property P;

(iii). If α and β both have the property P, so does $(\alpha \wedge \beta)$.

We call this method of proof as *proof by induction on the complexity of formulas.*

We depend on the structure of $\mathcal{P}$ also for defining a function f of the sentences of $\mathcal{P}$ as follows: First we define f for every propositional variable, second, we define $f(\neg\varphi)$ as a function of $f(\varphi)$, and third. we define $f(\varphi \wedge \vartheta)$ as a function of $f(\varphi)$ and $f(\vartheta)$. Then, it is guaranteed that, the function f is defined for all sentences of $\mathcal{P}$. This method of definition is called *definition by recursion on the complexity of sentences.* After a few examples we will be ready to give the idea of recursive definition in a general setting.

Example 2.2.1. The number of parentheses $p(\varphi)$ of a sentence is:

(i). 0 if φ is a propositional variable;

(ii). $p(\vartheta)$ if $\varphi = \neg\vartheta$;

(iii). $p(\vartheta) + p(\varsigma) + 2$ if $\varphi = (\vartheta \wedge \varsigma)$.

Thus we may calculate the number of parentheses of a sentence once we calculate this for its subformulas.

Example 2.2.2. *The rank of a sentence $r(\varphi)$ is defined as follows*

(i). $r(\varphi) = 0$ if φ is a propositional variable;

(ii). $r(\varphi) = r(\vartheta)$ if $\varphi = \neg\vartheta$;

(iii). $r(\varphi) = max\{r(\vartheta), r(\varsigma)\} + 1$ if $\varphi = (\vartheta \wedge \varsigma)$.

Informally, the rank of a sentence is the number of pairs of parentheses to be erased to reach at the level of propositional variables.To illustrate this, take the sentence

$$((\neg p_0 \wedge (p_0 \wedge p_1)) \wedge (\neg p_9 \wedge \neg p_0)) \tag{2.1}$$

Erasing the outermost parentheses we obtain

$$(\neg p_0 \wedge (p_0 \wedge p_1)) \tag{2.2}$$

and

$$(\neg p_9 \wedge \neg p_0) \tag{2.3}$$

Now, to find rank of 2.2, by erasing its outermost parentheses we further divide it into

$$\neg p_0 \tag{2.4}$$

and

$$(p_0 \wedge p_1) \tag{2.5}$$

Rank of 2.1 is, by 2.2.2-(iii)., the maximum of ranks of 2.2 and 2.3 plus 1. To calculate this value, we find the rank of 2.3 which is the maximum of the ranks of 2.4 and 2.5 plus 1. Thus, $r(((\neg p_0 \wedge (p_0 \wedge p_1)) \wedge (\neg p_9 \wedge \neg p_0)))$ is $max\{max\{0, max\{0,0\}+1\}+1, max\{0,0\}+1\}+1 = 3$

We will employ three more symbols as abbreviations for composite functions of $\neg$ and $\wedge$. These are called ***the disjunction, the conditional*** and **the biconditional** respectively.

2.2.1.

$$(\varphi \vee \vartheta) \quad =_{df} \quad \neg(\neg\varphi \wedge \neg\vartheta) \tag{2.6}$$

$$(\varphi \rightarrow \vartheta) \quad =_{df} \quad (\neg\varphi \vee \vartheta) \tag{2.7}$$

$$(\varphi \leftrightarrow \vartheta) \quad =_{df} \quad ((\varphi \rightarrow \vartheta) \wedge (\vartheta \rightarrow \varphi)) \tag{2.8}$$

Here we shall also adopt common rules of dropping parentheses. Thus we'll drop the outermost parentheses. Since we keep our definition of sentence unchanged and regard abbreviated sentences not sentences in the official sense but just their informal counterparts, the definition of rank is not affected. Moreover, we will accept that, $\neg$ binds more strongly than $\wedge$ and $\vee$, these two binding more strongly than $\rightarrow$ and $\leftrightarrow$. For example, instead of $((\varphi \wedge \vartheta) \rightarrow (\neg\varphi \wedge \varsigma))$ we will write $\varphi \wedge \vartheta \rightarrow \neg\varphi \wedge \varsigma$ and instead of $((\varphi \rightarrow (\varphi \vee \vartheta)) \leftrightarrow (\varphi \wedge \vartheta))$ we will write $(\varphi \rightarrow \varphi \vee \vartheta) \leftrightarrow \varphi \wedge \vartheta$.

2.3 Model Theory for Propositional Logic

Formal systems are extensions of formal languages; we impose a set of transformation rules on the set of legitimate expressions of the language. *Proof theory* is the study of formal systems with particular attention to their properties related to their transformation rules.

Another way of working with formal languages arises from a naive question: what are these symbols for? When we answer that question we are said to interpret the formal language. A formal language plus an interpretation of it is called a *structure* (for the formal definition of this notion see 2.4). *Model theory* is the study of structures with particular attention to the interpretation inherent in the structure.

Symbols of the language of propositional logic which are to be interpreted are propositional variables. Other symbols do not need any interpretation other than they usually have (this is why they are called *logical constants*). The reader is assumed to know the standard interpretation. Let $\mathcal{L}(\mathcal{P})$ denote the language of propositional logic. An interpretation of $\mathcal{L}(\mathcal{P})$ consists of assigning meanings to members of propositional variables. As their name indicates, they are thought to stand for propositions – sentences which are either true or false. Since we are only interested in their truth functional properties, we do not have to know what particular proposition a propositional variable stands for. In that sense, a propositional variable that is interpreted as a true proposition, stands for every true proposition. Thus, instead of assigning a particular true proposition to a propositional variable, we simply assign the truth value "true" to that variable. The same remarks hold for "false". Thus we may as well do with only two propositions; one true and one false.

Definition 2.3.1. *An* ***assignment*** *is a function from the set of propositonal variables to the set* $\{t, f\}$. *We will denote assignments by* $< a >, < a_0 >, < a_1 > \ldots$. *The* ***value*** $\varphi_{<a>}$ ***of a sentence*** φ ***under an assignment*** $< a >$ *is defined inductively as follows;*

(i). $p_{i_{<a>}} =< a > (p_i)$;

(ii).

$$\neg\varphi_{<a>} = \begin{cases} t & \text{if } \varphi_{<a>} = f \\ f & \text{if } \varphi_{<a>} = t \end{cases}$$

(iii).

$$(\varphi \wedge \vartheta)_{<a>} = \begin{cases} t & \text{if } \varphi_{<a>} = \vartheta_{<a>} = t \\ f & \text{otherwise} \end{cases}$$

Using this definition, it is easy to see that every sentence has a unique value under an assignment. Note that, although an assignment is defined as a total function from the set of propositional variables, we do not take care of the value of every propositional variable under that assignment in order to find the value of a given sentence. Just considering values of propositional variables occuring in that sentence suffices. We define the ***truth table*** of a sentence φ as the listing of possible assignments to its propositional variables together with the values that φ takes under these assignments. We may immediately give the following definition.

Definition 2.3.2. *A sentence φ is a **tautology**, in symbols $\vdash \varphi$, if it takes the value t under any assignment or, (in the view of the preceding remark) equivalently, under any assignment to its propositional variables.*

We will now define *immediate consequence, deducibility* and related notions for the formal system of propositional logic. We pointed out in section 2.1 that, using these definitions we may work within a formal system without any reference to meanings of symbols and expressions of the underlying formal language. For the system of propositional logic $\mathcal{P}$ we give only one rule of transformation:

$$\vartheta \text{ is the immediate consequence of } \varphi \text{ and } (\varphi \to \vartheta)$$

We call this rule of transformation of $\mathcal{P}$ as *Modus Ponens.* As we pointed out earlier, we may also define it as a ternary (3-ary) relation that is, as a set;

$$\mathcal{MP} = \{< x, y, z >: x,\ y,\ z \text{ are sentences, } x = \varphi,\ y = (\varphi \to \vartheta),\ z = \vartheta\}$$

Definition 2.3.3. *Let Σ be a set of sentences. We say that φ is* ***deducible from*** *Σ, in symbols $\Sigma \vdash \varphi$, iff there is a finite sequence of sentences $\varphi_0, \varphi_1, \ldots, \varphi_n$ such that $\varphi_n = \varphi$ and each φ_i, where $i < n$, is such that, φ_i is either a tautology or an element of Σ or there are sentences φ_j φ_k where $j, k < i$ from which φ_i follows by the rule of Modus Ponens. We call such a sequence as* ***the deduction*** *of φ from Σ. φ is deducible from ϑ, in symbols, $\vartheta \vdash \varphi$ if $\{\vartheta\} \vdash \varphi$. Particularly important are sentences φ such that $\{\} \vdash \varphi$. If φ is such a sentence, we will call it as a* ***theorem*** *of $\mathcal{P}$ and write $\vdash_{\mathcal{P}} \varphi$. Note that $\vdash$ does not refer to any transformation rule while $\vdash_{\mathcal{P}}$ does. Thus, being a tautology is guaranteed against rules of transformation as long as we accept the standard definition of being true under an assignment.*

Definition 2.3.4. *Let Σ be a set of sentences. We say that Σ is* ***inconsistent*** *if for every sentence φ, $\Sigma \vdash \varphi$. Σ is* ***consistent*** *if it is not inconsistent i.e., if there is at least one sentence ϑ such that it is not the case that $\Sigma \vdash \vartheta$.*

Definition 2.3.5. *A set of sentences Σ is said to be* ***maximally consistent*** *if Σ is the only consistent set containing Σ. That is, if $\Sigma \subset \Gamma$ then Γ is inconsistent.*

We are now ready to introduce model theoretic properties of $\mathcal{L}(\mathcal{P})$.

Definition 2.3.6. *By a* ***model*** *of $\mathcal{L}(\mathcal{P})$ we understand a subset of the set of propositional variables of $\mathcal{L}(\mathcal{P})$.*

Remark 2.3.1. Another standart way of interpretation of $\mathcal{L}(\mathcal{P})$ is to define a function from the set of propositional variables to the set $\{t, f\}$. There is no essential difference between these two, since given one way of interpretation, we may easily define the other. Let our model be A. We may define the corresponding interpretation function f by

$$f(p_i) = \begin{cases} \mathbf{t} & if \quad p_i \in A \\ \mathbf{f} & \text{otherwise.} \end{cases}$$

Remark 2.3.2. The number of models of $\mathcal{L}(\mathcal{P})$ is $2^{|Var(\mathcal{L}(\mathcal{P}))|}$ by the definition 2.3.6. where $Var(\mathcal{L}(\mathcal{P}))$ denotes the set of propositional variables of $\mathcal{L}(\mathcal{P})$).

Above description of model theory can be restated as follows: model theory studies the relation between formal languages on the one hand and their interpretation, structures on the other. Since the interpretation of the language of propositional logic $\mathcal{L}(\mathcal{P})$ is quite standard, instead of giving a structure as an interpretation of it, we will be just contented with choosing a subset of the set of propositional variables as models of $\mathcal{L}(\mathcal{P})$. The following definition gives us the relation between $\mathcal{L}(\mathcal{P})$ and its models;

Definition 2.3.7. *Let A be a model of $\mathcal{L}(\mathcal{P})$, and let φ be a $\mathcal{L}(\mathcal{P})$-sentence, we define the relation φ* ***is true in*** *A or A is* ***a model of*** *φ, in symbols*

$$A \models \varphi$$

by the following inductive definition;

2.3.1.

(i). If φ is a propositional variable p_i, $A \models \varphi$ iff $p_i \in A$;

(ii). If $\varphi = \neg\vartheta$, $A \models \varphi$ iff it is not the case that $A \models \vartheta$;

(iii). If $\varphi = (\vartheta \wedge \varsigma)$, $A \models \varphi$ iff $A \models \vartheta$ and $A \models \varsigma$.

Here is a basic list of model theoretical properties of sentences related to definition 2.3.7:

Definition 2.3.8. *A sentence φ is called* ***valid****, in symbols $\models \varphi$, iff $A \models \varphi$ for every model A of $\mathcal{P}$.*

Definition 2.3.9. *A sentence φ is called **satisfiable** iff it has at least one model.*

Definition 2.3.10. *A set of sentences Σ is satisfiable iff there is a model A such that $A \models \varphi$ for every $\varphi \in \Sigma$.*

Definition 2.3.11. *A sentence φ is called **a consequence of** another sentence ϑ. in symbols $\vartheta \models \varphi$, iff every model of ϑ is a model of φ.*

Definition 2.3.12. *A sentence φ is a consequence of a set of formulas Σ iff every model of Σ is also a model of φ.*

Definition 2.3.13. *Two sentences φ and ϑ are said to be **semantically equivalent** iff they have exactly the same models.*

Definition 2.3.14. $\Sigma \models \varphi$ *iff $A \models \varphi$ for every model A of Σ.*

Definition 2.3.15. $A \models \Sigma$ *iff $A \models \varphi$ for every $\varphi \in \Sigma$.*

We will now state a result which is used to establish many others as the reader will see immediately. We will not give a proof of this proof theoretical result. For the proof, the reader may refer to Hunter.

Theorem 2.3.2 (Deduction Theorem). *If $\Sigma \cup \{\varphi\} \vdash \vartheta$. then*

$\Sigma \vdash (\varphi \to \vartheta)$

Proposition 2.3.3.

(i). If Σ is consistent and $\Gamma = \{\varphi : \Sigma \vdash \varphi\}$, then Γ is consistent.

(ii). If Σ is maximal consistent and $\Sigma \vdash \varphi$ then $\varphi \in \Sigma$.

(iii). Σ is inconsistent *iff* $\Sigma \vdash (\varphi \wedge \neg\varphi)$ for any sentence φ.

Proof. (i). Assume that Γ is inconsistent. Then for every sentence ϑ, $\Gamma \vdash \vartheta$ in particular $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg\varphi$ for some sentence φ. Then there are deductions;

(1) $$\varphi_0, \varphi_1, \ldots, \varphi_n \varphi,$$

(2) $$\varphi'_0, \varphi'_1, \ldots, \varphi'_m \neg\varphi,$$

in Γ.

Now. each φ_i and φ'_j is an element of Γ. Thus, by definition of Γ, each φ_i and each φ'_j is deducible from Σ. That is. there are deductions;

(3) $$\varphi_{i_0} \varphi_{i_1}, \ldots, \varphi_{i_p} \varphi_i,$$

(4) $$\varphi'_{j_0} \varphi'_{j_1}, \ldots, \varphi'_{j_s} \varphi'_j$$

for every φ_i and φ'_j. Substituting these deductions in deductions of φ and $\neg\varphi$, we obtain deductions of φ and $\neg\varphi$ this time in Σ. Note that, since the sentence φ is chosen arbitrarily, the same result holds for every sentence. Thus, $\Sigma \vdash (\varphi \wedge \neg\varphi)$ for any sentence φ. By this result and (iii)., we conclude that Σ is inconsistent which is a contradiction.

(ii). Assume that $\varphi \notin \Sigma$, then $\Sigma \cup \{\varphi\}$ is inconsistent. Thus, every sentence is deducible from Σ. In particular; $\Sigma \cup \{\varphi\} \vdash \varphi$ and $\Sigma \cup \{\varphi\} \vdash$

$\neg\varphi$. Then, by Deduction Theorem, $\Sigma \vdash (\varphi \to \varphi)$ and $\Sigma \vdash (\varphi \to \neg\varphi)$. Thus, $\Sigma \vdash ((\varphi \to \varphi) \wedge (\varphi \to \neg\varphi))$ and, since $(((\varphi \to \varphi) \wedge (\varphi \to \neg\varphi)) \to \neg\varphi)$ is a tautology, by Modus Ponens, we have; $\Sigma \vdash \neg\varphi$ leading to a contradiction.

(iii) It is obvious that if Σ is inconsistent, then $\Sigma \vdash \varphi \wedge \neg\varphi$ since every sentence is deducible from an inconsistent set Σ. We prove the other direction; Let Σ be such that, $\Sigma \vdash (\varphi \wedge \neg\varphi)$ for some arbitrary sentence φ. Then, since $((\varphi \wedge \neg\varphi) \to \vartheta)$ is a tautology, $\Sigma \vdash \vartheta$ for any sentence ϑ making Σ inconsistent. □

Lemma 2.3.4 (Lindenbaum's Theorem). *Every consistent set of sentences Σ is a subset of a maximally consistent set Γ of sentences.*

proof. Using the fact that the set of $\mathcal{L}(\mathcal{P})$)-sentences is countable, let us arrange all the $\mathcal{L}(\mathcal{P})$)-sentences in a list, $\varphi_0, \varphi_1, \ldots, \varphi_\alpha, \ldots$. We shall construct a chain of consistent sets of sentences as follows;

(i). $\Sigma_0 = \Sigma$

(ii). $\Sigma_{n+1} = \begin{cases} \Sigma_n \cup \{\varphi_n\} & if \quad \Sigma_n \cup \{\varphi_n\} \text{ is consistent} \\ \Sigma_n & if \quad \Sigma_n \cup \{\varphi_n\} \text{ is inconsistent.} \end{cases}$

(iii). $\Sigma_\alpha = \bigcup_{\beta<\alpha} \Sigma_\beta$ where α is a limit ordinal.

Note that;

$$\Sigma = \Sigma_0 \subseteq \Sigma_1 \subseteq \Sigma_2 \subseteq \ldots \subseteq \Sigma_\alpha \subseteq \ldots$$

Let $\Gamma = \bigcup \Sigma_\alpha$. We will first show that Γ is consistent. Assume that it is not. Then, by proposition 2.3.3 there is a deduction of the sentence $(p_i \wedge \neg p_i)$ from Γ. That is, there is a finite sequence $\psi_0, \psi_1, \ldots, \psi_n$ where $\psi_n = (p_i \wedge \neg p_i)$. Let $\psi_{i_0}, \psi_{i_1}, \ldots, \psi_{i_m}$ be the list of sentences from Γ that are used in this deduction. We may choose Σ_α such that $\psi_{i_0}, \psi_{i_1}, \ldots, \psi_{i_m} \in \Sigma_\alpha$. Then, $\Sigma_\alpha \vdash (p_i \wedge \neg p_i)$ making Σ_α inconsistent but each Σ_β in Γ were consistent.

It remains to show that Γ is maximally consistent. Suppose that there is a consistent set of sentences Δ such that Δ contains Γ: $\Gamma \subseteq \Delta$. Let $\varphi_\alpha \in \Delta$. Since Δ is consistent and $\Sigma_\alpha \subseteq \Gamma \subseteq \Delta$, $\Sigma_\alpha \cup \{\varphi_\alpha\}$ is consistent hence $\Sigma_{\alpha+1}$ is consistent. Since $\varphi_\alpha \in \Sigma \subseteq \Gamma$, $\varphi_\alpha \in \Gamma$. Thus $\Delta \subseteq \Gamma$ showing that Γ is the only consistent set of sentences containing Γ. □

Lemma 2.3.5. *Let Γ be a maximally consistent set of sentences of $\mathcal{P}$. Then:*

(i). For each sentence φ, exactly one of φ or $\neg\varphi$ is an element of Γ:

(ii). For each pair of sentences φ and ϑ. $(\varphi \wedge \vartheta) \in \Gamma$ *iff* both $\varphi \in \Gamma$ and $\vartheta \in \Gamma$.

Proof. (i). It is clear that both can not be elements of Γ. Thus, assume that $\varphi \notin \Gamma$ and $\neg\varphi \notin \Gamma$. Then both $\Gamma \cup \{\varphi\}$ and $\Gamma \cup \{\neg\varphi\}$ are inconsistent by maximality of Γ. Then, $\Gamma \vdash \neg\varphi$ and $\Gamma \vdash \varphi$. Since, if $\Gamma \cup \{\varphi\}$ is inconsistent,

$$\Gamma \cup \{\varphi\} \vdash \neg\varphi \text{ and } \Gamma \cup \{\varphi\} \vdash \varphi$$

Then, since $(\varphi \to \varphi)$ is a tautology,

$$\Gamma \vdash (\varphi \to \varphi)$$

and, by Deduction Theorem,

$$\Gamma \vdash (\varphi \to \neg\varphi)$$

Thus,

$$\Gamma \vdash (\varphi \to \varphi) \wedge (\varphi \to \neg\varphi)$$

Since, $((\varphi \to \varphi) \wedge (\varphi \to \neg\varphi) \to \neg\varphi)$ is a tautology,

$$\Gamma \vdash \neg\varphi$$

Similarly, one can obtain;

$$\Gamma \vdash \varphi$$

from the assumption that $\Gamma \cup \{\neg\varphi\}$ is inconsistent.

Thus, $\Gamma \vdash (\varphi \wedge \neg\varphi)$ making Γ inconsistent. But, since Γ is maximally consistent, *a fortiori* it is consistent. Thus, the assumption that $\varphi \notin \Gamma$ and $\neg\varphi \notin \Gamma$ leads to a contradiction.

(ii). $(\varphi \wedge \vartheta) \in \Gamma$ *iff* $\Gamma \vdash (\varphi \wedge \vartheta)$ *iff* $\Gamma \vdash \varphi$ and $\Gamma \vdash \vartheta$ *iff* $\varphi \in \Gamma$ and $\vartheta \in \Gamma$. □

Proposition 2.3.6 (Completeness Theorem). *A sentence is a tautology if and only if it is valid. In symbols* $\vdash \varphi$ *if and only if* $\models \varphi$

proof. Let φ be a sentence. For every model A an assignment for φ can be found such that $p_{i<a>} = t$ if and only if $p_i \in A$: call the assignment corresponding to A as a. Then it is easily proved by induction that,

$$(1) \qquad A \models \varphi \text{ if and only if } \varphi_{<a>} = t$$

Thus, if φ is valid, then it is true under any assignment since every assignment can be obtained from a model and φ is true in all models. Similarly, if φ is true under every assignment, then it must be valid since for every model, we have a corresponding assignment for φ under which φ is true and by (1), φ is true in every model. □

The reader must note the importance of the proof of the following theorem for further reference; the idea used in the proof of the if part is applied in the proof of the completeness theorem for Quantificational Logic.

Theorem 2.3.7 (Extended Completeness Theorem). *Σ is consistent iff Σ is satisfiable.*

proof. Assuming that Σ is satisfiable, we will show that it is consistent. Since Σ is satisfiable, it has a model. Let $A \models \Sigma$ and let φ_n be deducible from Σ. Then there is a sequence $\varphi_0, \varphi_1, \ldots, \varphi_n$ which is a deduction of φ_n from Σ. We will prove by induction on the place m of sentences occuring in the deduction that each $m \leq n$ holds in A.

(i). If φ_m is a tautology, by completeness theorem it will trivially hold in A;

(ii). If $\varphi_m \in \Sigma$ it will hold in A by definition 2.3.15;

(iii). If φ_m is inferred from two previous elements of the sequence φ_j and $(\varphi_j \rightarrow \varphi_m)$ such that $A \models \varphi_j$ and $A \models (\varphi_j \rightarrow \varphi_m)$ then it is easy to see that $A \models \varphi_m$.

We have shown that φ_m holds in A for every $m \leq n$ in particular for n. Thus every φ deducible from Σ holds in every model of Σ. Since $(\varphi \wedge \neg\varphi)$ does not hold in A, $(\varphi \wedge \neg\varphi)$ is not deducible from A. Since we have found a sentence not deducible from Σ, Σ is consistent.

We will now prove the other direction. Assume that Σ is consistent. By Lindenbaum Theorem, there is a maximally consistent set Γ such that $\Sigma \subseteq \Gamma$. Our strategy is to construct a model for Γ. Since $\Sigma \subseteq \Gamma$ this model will also be a model of Σ and this shows that Σ is satisfiable.

Let $\mathcal{A} = \{p : p \in \Gamma\}$. We want to show that $\mathcal{A} \models \varphi$ for all φ in Γ. The result follows by induction. Let $\varphi \in \Gamma$:

(i). If φ is a propositional variable, the result follows by the definition of A.

(ii). Let $\varphi = \neg\vartheta$ and let $\neg\vartheta \in \Gamma$. By 2.3.5-i, $\vartheta \notin \Gamma$. By induction hypothesis, it is not the case that $A \models \vartheta$. Thus $A \models \neg\vartheta$.

(iii). Let $\varphi = (\vartheta \wedge \varsigma)$ and let $(\vartheta \wedge \varsigma) \in \Gamma$. By Lemma 2.3.5-ii, $\vartheta \in \Gamma$ and $\varsigma \in \Gamma$. By induction hypothesis, $A \models \vartheta$ and $A \models \varsigma$. By definition of satisfaction, $A \models (\vartheta \wedge \varsigma)$.

Thus $A \models \Gamma$ and, since $\Sigma \subseteq \Gamma$, $A \models \Sigma$. □

Definition 2.3.16. *A set of sentences Σ is* ***finitely satisfiable*** *iff every finite subset of Σ is satisfiable.*

The following theorem proves to be very useful in model theory, it gives us a very simple means to see whether a set of sentences has a model or not.

Corollary 2.3.1 (Compactness Theorem). *A set of sentences Σ is satisfiable iff it is finitely satisfiable.*

Proof. Assume that Σ is finitely satisfiable but it is not satisfiable. Then, by the Extended Completeness Theorem, it is inconsistent. By proposition 2.3.3 $\Sigma \vdash (p_i \wedge \neg p_i)$. Thus, since a deduction is defined as a finite sequence, there is a finite $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0 \vdash (p_i \wedge \neg p_i)$. Thus Σ_0 is not satisfiable. However, then Σ is not finitely satisfiable contrary to our assumption. □

Remark 2.3.3. *We note that another formulation of corollary 2.3.1 is useful: a set of sentences Σ is satisfiable iff every finite subset of Σ is satisfiable. Thus, in order to prove that Σ is satisfiable, (or, in the light of the Extended Completeness Theorem, to prove that Σ is consistent), it suffices to show that an arbitrary finite subset of Σ is satisfiable (or consistent).*

Corollary 2.3.2.

(i). $\Sigma \models \varphi$ *iff* $\Sigma \vdash \varphi$.

(ii). If $\Sigma \models \varphi$, there is a finite subset $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0 \models \varphi$.

Proof. (i) Assume that $\Sigma \models \varphi$ and it is not the case that $\Sigma \vdash \varphi$ then, $\Sigma \cup \{\neg\varphi\}$ is not satisfiable. Then, by the Extended Completeness Theorem, $\Sigma \cup \{\neg\varphi\}$ is inconsistent. Thus, $\Sigma \cup \{\neg\varphi\} \vdash \varphi$ and $\Sigma \cup \{\neg\varphi\} \vdash \neg\varphi$. Then, by Deduction Theorem. $\Sigma \vdash (\neg\varphi \to \varphi)$ and $\Sigma \vdash (\neg\varphi \to \neg\varphi)$. Since $(((\neg\varphi \to \varphi) \wedge (\neg\varphi \to \neg\varphi)) \to \varphi)$ is a tautology, we may infer that $\Sigma \vdash \varphi$ contrary to our assumption.

That if $\Sigma \vdash \varphi$ then $\Sigma \models \varphi$ can be proved by the induction on the place of φ_i in the proof of φ from Σ as in the proof of the Extended Completeness Theorem.

(ii). Assume that $\Sigma \models \varphi$. Then $\Sigma \cup \{\neg\varphi\}$ is not satisfiable. Then, $\Sigma \cup \{\neg\varphi\}$ is inconsistent. Then $\Sigma \cup \{\neg\varphi\} \vdash \varphi$ and $\Sigma \cup \{\neg\varphi\} \vdash \neg\varphi$. Thus $\Sigma \vdash (\neg\varphi \to \varphi)$ and $\Sigma \vdash (\neg\varphi \to \neg\varphi)$. By the same reasoning, we have $\Sigma \vdash \varphi$. So there is a finite subset Σ_0 of Σ such that $\Sigma \vdash \varphi$. Thus, by (i)., we obtain the result that a finite subset Σ_0 of Σ is a model of φ. □

Definition 2.3.17. *A set* Γ *of sentences is called a* ***theory****.*

Definition 2.3.18. *A theory is* ***closed*** *if every consequence of* Γ *is an element of* Γ *i.e.,* $\{\varphi : \Gamma \models \varphi\} \subset \Gamma$*. Note that* $\Gamma \subset \{\varphi : \Gamma \models \varphi\}$ *trivially holds.*

The set of consequences of Γ, $\bar{\Gamma} = \{\varphi : \Gamma \models \varphi\}$ *is called the closure of* Γ. *Thus we may say that* Γ *is closed iff* $\Gamma = \bar{\Gamma}$.

Definition 2.3.19. *A set of sentences* Δ *is called a set of* ***axioms*** *for a theory* Γ *iff they have exactly the same set of consequences. That is,*

$$\bar{\Delta} = \{\varphi : \Delta \models \varphi\} = \{\varphi : \Gamma \models \varphi\} = \bar{\Gamma}$$

by the axiom of extensionality this holds if and only if $\Delta \models \varphi$ *iff* $\Gamma \models \varphi$.

Definition 2.3.20. *A theory is said to be* ***finitely axiomatizable*** *iff it has a finite set of axioms.*

In mathematics we encounter both finitely axiomatizable and non-finitely axiomatizable theories and there is no reason to regard non-finitely axiomatizable theories as defective. As examples form group theory , the theory of groups, abelian groups and abelian groups with every element of order $\leq n$ are finitely axiomatizable while the theory of divisible groups and torsion-free groups are not (see, Barwise). Now we state a result concerning a basic model theoretic relation between a theory and a set of axioms for that theory.

Proposition 2.3.8. Δ *is a set of axioms for* Γ *iff* Δ *and* Γ *have exactly the same models.*

Proof. We will first show that, if Δ is a set of axioms for Γ then $A \models \Sigma$ *iff* $A \models \Gamma$ for every model A. Assume that $\mathcal{A}$ is a model of Δ but not a model of Γ. Then there is a sentence $\varphi \in \Gamma$ such that $\mathcal{A} \not\models \varphi$. So $\mathcal{A} \models \neg\varphi$. Since $\varphi \in \Gamma$, $\Gamma \models \varphi$ but then $\Delta \models \varphi$. However $\mathcal{A} \models \varphi$ since $\mathcal{A}$ is a model of Δ. This is a contradiction with the fact that $\mathcal{A} \models \neg\varphi$. Similarly the assumption that $\mathcal{A}'$ is a model of Γ but not a model of Δ leads to a contradiction.

Now assume that Δ and Γ have exactly the same models. We will show that they have the same set of consequences. Assume that there is a sentence φ such that $\Delta \models \varphi$ but not $\Gamma \models \varphi$. Then there is a model $\mathcal{A}$ of Γ such that it is not the case that $\mathcal{A} \models \varphi$. Then $\mathcal{A} \models \neg\varphi$. Since $\mathcal{A}$ is also a model of Δ, $\mathcal{A} \models \varphi$ leading to a contradiction. Similarly if $\Gamma \models \varphi$ then $\Delta \models \varphi$. Thus, Δ is a set of axioms for Γ. □

A group of results in model theory are related to syntactic properties of sentences. Looking at syntactic properties of sentences in a theory, we may say something about its models.

Definition 2.3.21. *A sentence φ is called a* ***conditional sentence*** *if it is of the form $\varphi_1 \wedge \ldots \wedge \varphi_n$ such that, for each φ_i one of the following holds:*

(i). $\varphi_i = p_j$;

(ii). $\varphi_i = \neg p_{j_1} \vee \neg p_{j_2} \vee \ldots \vee \neg p_{j_s}$;

(iii). $\varphi_i = \neg p_{j_1} \vee \neg p_{j_2} \vee \ldots \vee \neg p_{j_s} \vee p_l$.

Definition 2.3.22. *A theory Σ is said to be* ***preserved under finite intersections*** *iff $\mathcal{A} \models \Sigma$ and $\mathcal{B} \models \Sigma$ implies $\mathcal{A} \cap \mathcal{B} \models \Sigma$. Σ is said to be*

preserved under arbitrary intersections *iff for every nonempty set of models* $\{\mathcal{A}_i : \mathcal{A}_i \models \Sigma\}$, $\bigcap_{i \in I} \mathcal{A}_i \models \Sigma$.

In view of the following lemma, we will drop the property 'finite' or 'arbitrary' and just say that Σ is preserved under intersections.

Lemma 2.3.9. Σ *is preserved under finite intersections iff* Σ *is preserved under arbitrary intersections.*

Proposition 2.3.10.

(i). A theory Γ is preserved under intersections *iff* it has a set of conditional axioms.

(ii). A sentence φ is preserved under intersections *iff* φ is equivalent to a conditional sentence.

proof. We will prove only the fact that every conditional sentence is preserved under intersections which is the basis of the proof of the (i). For the remaining part of the proof, the reader may refer to Chang and Keisler. We will establish this fact by induction.

(i). $\varphi = p_j$. Let $\{A_i : i \in I\}$ be such that, each $A_i \models p_j$. Then $p_j \in A_i$ for each $i \in I$. Thus $p_j \in \bigcap \{A_i : i \in I\}$. Thus, $\bigcap\{A_i : i \in I\} \models \varphi$;

(ii). $\varphi = \neg p_{j_1} \vee \neg p_{j_2} \vee \ldots \vee \neg p_{j_s}$. And let $\{A_i : i \in I\}$ be a set of models of φ. Then for each A_i we may say that, at least one of p_{j_k}'s is not in A_i. Thus, clearly, $\bigcap\{A_i : i \in I\}$ can not have all the p_{j_k}'s since $\bigcap\{A_i : i \in I\} \subseteq A_i$ for every $i \in I$. Thus, $\bigcap\{A_i : i \in I\} \models \varphi$;

(iii). $\varphi = \neg p_{j_1} \vee \neg p_{j_2} \vee \ldots \vee \neg p_{j_s} \vee p_l$ and let A_i be a set of models of φ. Then, each A_i is such that, either it lacks one of p_{j_k}'s or it has p_l. If all A_i's have p_l, $\bigcap\{A_i : i \in I\}$ also has it and $\bigcap\{A_i : i \in I\} \models \varphi$. If at least one A_i does not have p_l, then that A_i does not have at least one of p_{j_k}'s. Thus, by the same reasoning as in (a), $\bigcap\{A_i : i \in I\}$ does not have all p_{j_k}'s, and $\bigcap\{A_i : i \in I\} \models \varphi$.

In order to complete the induction we will show that, if φ and ϑ are preserved under intersection, then so is $(\varphi \wedge \vartheta)$. Assume that $\{A_i : i \in I\}$ is a set of models for $(\varphi \wedge \vartheta)$. Then $\{A_i : i \in I\}$ is a set of models for φ and for ϑ. By our assumption, $\bigcap\{A_i : i \in I\} \models \varphi$ and $\bigcap\{A_i : i \in I\} \models \vartheta$. Thus, $\bigcap\{A_i : i \in I\} \models (\varphi \wedge \vartheta)$ □

Definition 2.3.23. *A theory* Γ *is said to be* ***complete*** *iff for every sentence* φ*, exactly one of* $\Gamma \models \varphi$ *or* $\Gamma \models \neg\varphi$ *holds. Complete theories have striking model theoretic properties as the following proposition shows:*

Definition 2.3.24. *Two models* A *and* B *are* ***equivalent*** *if and only if exactly the same sentences hold in* A *and* B*.*

Proposition 2.3.11. *Let* Σ *be a set of sentences. The following statements are equivalent:*

(i). The set of consequences of Σ is maximally consistent.

(ii). Σ is a complete theory.

(iii). Σ has exactly one model up to equivalence.

(iv). There is a model A such that for all φ, $\Sigma \models \varphi$ *iff* $A \models \varphi$.

Proof. (i).$\Rightarrow$(ii). Assume that $\bar{\Sigma} = \{\varphi : \Sigma \models \varphi\}$ is maximally consistent. Then, for every sentence φ, exactly one of φ, $\neg\varphi$ belongs to $\bar{\Sigma}$. That means, for every sentence φ, exactly one of φ, $\neg\varphi$ is a consequence of Σ. Thus Σ is complete.

(ii).$\Rightarrow$(iii). Assume that Σ is complete and A and B are both models of Σ. Without loss of generality, assume that $p_i \in A$ but $p_i \notin B$. Since Σ is complete, exactly one of $\Sigma \models p_i$ and $\Sigma \models \neg p_i$ holds. If $\Sigma \models p_i$, since $A \models \Sigma$, $A \models p_i$. Thus $p_i \in B$ contrary to our assumption. If $\Sigma \models \neg p_i$, then $A \models \neg p_i$. Thus $p_i \notin A$. This also leads to a contradiction. Similarly one can show that, $p_i \in B$ and $p_i \notin A$ is not possible. Thus, any two models of Σ are equivalent.

The existence of at least one model of Σ trivially follows but the reader must not omit to see it. Since Σ is complete, it is consistent; otherwise any sentence would be deducible and thus be a consequence of Σ.

(iii).$\Rightarrow$(iv). Assume that the model of Σ is A. We will show that $\Sigma \models \varphi$ *iff* $A \models \varphi$. If $\Sigma \models \varphi$ then since A is a model of Σ, $A \models \varphi$ obviously. Let

$A \models \varphi$. Then since A is the only model of Σ, φ holds in every model of Σ trivially but this means that $\Sigma \models \varphi$.

(iv).$\Rightarrow$(i). Let there be a model A such that, $\Sigma \models \varphi$ if and only if $A \models \varphi$ and let $\bar{\Sigma} = \{\varphi : \Sigma \models \varphi\}$. Thus $\varphi \in \bar{\Sigma}$ *iff* $\Sigma \models \varphi$ *iff* $A \models \varphi$. We will show that $\bar{\Sigma}$ is maximally consistent. Since $\varphi \in \bar{\Sigma}$ *iff* $A \models \varphi$, A is also a model of $\bar{\Sigma}$. Then $\bar{\Sigma}$ is satisfiable and consistent (another way to see it is that, since Σ is consistent and $\bar{\Sigma}$ is the set of consequences of Σ, $\bar{\Sigma}$ must also be consistent). Let $\bar{\Sigma} \subset \Delta$ where Δ is a consistent set of sentences. Assume that $\varphi \in \Delta$ but $\varphi \notin \bar{\Sigma}$. Then it is not the case that $A \models \varphi$. Then $A \models \neg\varphi$. This shows that $\neg\varphi \in \bar{\Sigma}$. Thus $\neg\varphi \in \Delta$ (since $\bar{\Sigma} \subseteq \Delta$). Then Δ must be inconsistent. □

2.4 Languages and Theories

Propositional logic is suprisingly powerful for several applications and it reveals many interesting model theoretic properties. It is not, however, strong enough to express many mathematical ideas. In this chapter we will introduce a new language, describe its models and state interrelations among these models.

Definition 2.4.1 (First order language). A first order language consists of two sets of symbols;

(i). **Logical symbols** of a first order language consists of;

(a) a set of variables $\{v_0, v_1, \ldots, v_n, \ldots\}$;

(b) truth functional connectives $\neg$ and $\wedge$;

(c) parentheses (,);

(d) equality symbol $\equiv$

(e) quantifier $\forall$.

(ii). **Non-logical symbols** of a first order language consists of;

(a) $\forall n \geq 1$, a (possibly empty) set of n-ary relation symbols $\{P_0, P_1, \ldots, P_n, \ldots\}$;

(b) $\forall n \geq 1$, a (possibly empty) set of n-ary function symbols $\{F_0, F_1, \ldots, F_n, \ldots\}$;

(c) A (possibly empty) set of individual constant symbols $\{c_0, c_1, \ldots, c_n, \ldots\}$.

Hereafter, we will reserve symbols $\mathcal{L}, \mathcal{L}', \mathcal{L}'', \ldots$ for first order languages, and not for arbitary formal languages. Since we accept that, logical symbols are common to all first order languages and their interpretation is quite standard, we will not list them among the symbols of a first order language. However, the reader must keep in mind that, they are always in the picture though not in the front view. Thus, we will define a first order language as;

$$\mathcal{L} = \{P_0, P_1, \ldots P_n, \ldots, F_0, F_1, \ldots, F_m, \ldots, c_0, c_1, \ldots, c_s, \ldots\}$$

Since almost all structures studied in mathematics have finite set of functions and relations, we usually describe a first order language as;

$$\mathcal{L} = \{P_0, P_1 \ldots, P_n, F_0, F_1, \ldots, F_m, c_0, c_1, \ldots, c_s, \ldots\}$$

Let $\mathcal{L}$ be a first order language, we sometimes want to keep working with all symbols of $\mathcal{L}$ and add some new symbols. When $\mathcal{L}'$ is obtained from $\mathcal{L}$ by the addition of new symbols, we say that $\mathcal{L}'$ is an **expansion** of $\mathcal{L}$, in symbols, $\mathcal{L} \subset \mathcal{L}'$. or $\mathcal{L}$ is a **reduction** of $\mathcal{L}'$. When $\mathcal{L}'$ is obtained by the addition of new constant symbols to $\mathcal{L}$, we say that $\mathcal{L}'$ is a **simple expansion** of $\mathcal{L}$. This phenomenon is not rare in mathematics. Sometimes we need to add a new constant symbol to a language in order to describe a structure better. For example. a description of a group requires an explicit mention of the identity element of the group. Moreover, in some other cases, addition is made to obtain an extended mathematical structure for example, if we add a constant symbol which is to be interpreted as the identity to the language of ring theory, we may obtain a language to study rings with identity.

We will now give our rules of formation to obtain a formal system for a first order theory. Here, as we shall see in other syntactic and model theoretic considerations, complexity arises with respect to formal system of propositional logic. First of all we will need two syntactic categories while in propositional logic there were only one; sentences.

Definition 2.4.2 (term). Inductively **$\mathcal{L}$-terms** are defined as follows;

(i). Variables and constant symbols are $\mathcal{L}$-terms;

(ii). If $t_1, t_2, \ldots, t_n$ are terms and F_i^n is an n-ary function symbol, then $F_i^n(t_1.t_2, \ldots, t_n)$ is an $\mathcal{L} - term.$

(iii). A sequence of $\mathcal{L}$-symbols is a term iff that it is a term can be verified on the basis of finite number of applications of (i). and (ii).

Remark 2.4.1. A set theoretic definition for $\mathcal{L}$-terms can be given as follows; The set of $\mathcal{L}$-terms is the smallest set X such that;

(i). Every variable and constant symbol is in X;

(ii). If $t_1, t_2 \ldots, t_n$ are all in X, and $F_n{}^i$ is an n-ary function symbol, then $F_i{}^n(t_1, t_2 \ldots, t_n)$ is also in X.

Definition 2.4.3 (formula). ***$\mathcal{L}$-formulas*** *are defined inductively as follows;*

(i). If t_1 and t_2 are $\mathcal{L}$-terms, $t_1 \equiv t_2$ is an $\mathcal{L}$-formula;

(ii). If P is an n-ary relation symbol and $t_1, t_2, \ldots, t_n$ are $\mathcal{L}$-terms, then $P(t_1.t_2, \ldots, t_n)$ is an $\mathcal{L}$-formula;

(iii). If φ is an $\mathcal{L}$-formula, then so is $\neg\varphi$;

(iv). If φ and ϑ are $\mathcal{L}$-formulas, then so is $(\varphi \wedge \vartheta)$;

(v). If φ is an $\mathcal{L}$-formula and v_i is a variable, then $\forall v_i \varphi$ is an $\mathcal{L}$-formula;

(vi). An $\mathcal{L}$-expression is an $\mathcal{L}$-formula iff that it is a formula can be shown by a finite number of applications of (i).-(v).

We call formulas obtained by (i). and (ii). **atomic formulas**. As an abbreviation in addition to the those of section 2.2, we have;

$$(\exists v \varphi) = \neg \forall v \neg \varphi$$

$\exists$ is called the 'existential quantifier'.

An anologous set theoretic definition can be given for $\mathcal{L}$-formulas. However, any property meaningfully assertible of $\mathcal{L}$-terms or $\mathcal{L}$-formulas can be shown to hold for every $\mathcal{L}$-term or $\mathcal{L}$-formula on the basis of an induction principle based on the inductive definition of $\mathcal{L}$-terms and $\mathcal{L}$-formulas. Let P be such a property of $\mathcal{L}$-terms or $\mathcal{L}$-formulas and let us symbolize 'P holds for t' and 'P holds for φ' as $p(t)$ and $P(\varphi)$ respectively. Then. P holds for every $\mathcal{L}$-term if it can be shown that:

(i). $P(t)$ where t is a variable or t is a constant symbol;

(ii). $P(F_i^n(t_1, t_2, \ldots, t_n))$ wherever $P(t_1)$, $P(t_2)$, ..., $P(t_n)$.

Similarly, P holds for every $\mathcal{L}$-formula if it can be shown that;

(i). $P(\varphi)$ for every atomic $\mathcal{L}$-formula;

(ii). $P(\neg\varphi)$ whenever $P(\varphi)$;

(iii). $P((\varphi \wedge \vartheta))$ whenever $P(\varphi)$ and $P(\vartheta)$;

(iv). $P(\forall v_i \varphi)$ whenever $P(\varphi)$.

Example 2.4.1. We will show that every $\mathcal{L}$-formula have the same number of left parentheses as right parentheses. Let us write $P(\varphi)$ if φ has the same number of left as right parentheses. Then,

(i). For $\varphi = t_1 \equiv t_2$ and $\varphi = P(t_1, t_2, \ldots, t_n)$ is an $\mathcal{L}$ that $P(\varphi)$ is clear;

(ii). If $\varphi = \neg(\vartheta)$ we have $P(\varphi)$ since $\neg$ adds no extra parentheses;

(iii). If $\varphi = (\vartheta \wedge \varsigma)$ and if φ and ϑ are such that φ has the same number n of left and right parentheses and ϑ has m left and right parentheses ϑ then the reader may easily see that, with $\wedge$ the number of left parentheses adds up to m+n+1 which holds also for the number of right parentheses;

(iv). If $\varphi = \forall v_i(\varphi)$ by the same reasoning as in (iii). we have $P(\varphi)$.

As in the case of propositional logic, we will first give the formal system of quantificational logic and state and prove some of its proof theoretical properties and then we will define its models, give truth definition of a legitimate expression of the language $\mathcal{L}$ in a model, relations between $\mathcal{L}$ models and so on. First, we need some syntactic notions. Recall that a formula is defined as a special string or say a sequence of symbols from the alphabet of $\mathcal{L}$.

Definition 2.4.4 (J. Malitz).

(i). Let $S = s_0, s_1, \ldots, s_n$ be a sequence of symbols from the alphabet of $\mathcal{L}$ and $0 \leq i \leq j \leq n$ then the $i, j - subsequence$ of S is $s_i, s_{i+1}, \ldots, s_j$. A sequence S' is a subsequence of S iff S' is an $i, j - subsequence$ of S for some $0 \leq i \leq j \leq n$.

(ii). If S is a formula then a formula φ is an i, j-**subformula of** S iff φ is an $i, j - subsequence$ of S. φ is a $subformula$ of S iff φ is an $i, j - subformula$ of S for some $0 \leq i \leq j \leq n$.

(iii). The symbol s **occurs at** i **in** S iff s is the $i-th$ term of the sequence S.

(iv). The variable v **is bound at** k **in** φ iff v occurs at k in φ and for some

$i < k < j$, the $i, j - subformula$ of φ is of the form $\forall v\vartheta$ or $\exists v\vartheta$. v **is bound in** φ *iff* v occurs bound at k in φ for some k.

(v). The variable v **occurs free at** k in φ *iff* v occurs at k in φ but it does not occur bound at k in φ. v **is free in** φ *iff* v occurs free at k in φ for some k.

(vi). A formula φ is a **sentence** *iff* no variable occurs free in φ.

Definition 2.4.5 (J. Malitz).

(i). The variable v is **free at** k **for the term** t in the formula φ *iff*

(a) v occurs free at k in φ

(b) If u is a variable of t then, if ϑ is the result of replacing v at the k-th place by u, u must be free at k in ϑ.

(ii). The variable v is **free for the term** t **in** φ *iff* whenever v occurs free at k in φ, v is free for t at k in φ.

From now on we will use the notation $t(v_0, \ldots, v_n)$ to show that the set of variables of t form a subset of $\{v_0, \ldots, v_n\}$ and $\varphi(v_0, \ldots, v_n)$ to show that the set of free variables of φ form a subset of $\{v_0, \ldots, v_n\}$. A formula with no free variables is called a sentence.

Remark 2.4.2. We may give the definition of free variable by means of an inductive definition. Let $Var(t)$ and $Var(\varphi)$ denote the set of variables of the term t and the formula φ respectively. Similarly, $Free(\varphi)$ denotes the set of free variables of the formula φ. Then:

(i). $Free(t_1 \equiv t_2) = Var(t_1) \cup Var(t_2)$;

(ii). $Free(P(t_1, t_2, \ldots, t_n) = Free(t_1) \cup Free(t_2) \cup \ldots \cup Free(t_n)$;

(iii). $Free(\neg\varphi) = Free(\varphi)$;

(iv). $Free((\varphi \wedge \vartheta)) = Free(\varphi) \cup Free(\vartheta)$;

(v). $Free(\forall v \varphi) = Free(\varphi) - \{v\}$.

Definition 2.4.6. *Let s and t be terms and v be a variable of s. We define* ***Substitution of the term** t **for the variable** v **in** s*, *in symbols $s[t/v]$, is defined inductively as follows:*

(i). $u[t/v] = \begin{cases} \text{u} & \text{if } u \not\equiv v \\ \text{t} & \text{if } u \equiv v \end{cases}$

(ii). $c[t/v] = c$

(iii). $F_i^n(t_1, t_2, \ldots, t_n)[t/v] = F_i^n(t_1[t/v], t_2[t/v], \ldots, t_n[t/v])$

Definition 2.4.7. *Let φ be a formula, t a term and v a variable. Then,* ***substitution of the term** t **for the variable** v **in** φ *is defined as follows;*

(i). $P(t_1, t_2, \ldots, t_n)[t/v] = P(t_1[t/v], t_2[t/v], \ldots, t_n[t/v])$

(ii). $(t_1 \equiv t_2)[t/v] = (t_1[t/v] \equiv t_2[t/v])$

(iii). $\neg\varphi[t/v] = \neg\{\varphi[t/v]\}$

(iv). $(\varphi \wedge \vartheta)[t/v] = (\varphi[t/v] \wedge \vartheta[t/v])$

(v). $\forall u\varphi[t/v] = \begin{cases} \forall u\varphi & \text{if } v \text{ is not free for } t \text{ in } \forall u\varphi \\ \forall u\{\varphi[t/v]\} & \text{if } v \text{ is free for } t \text{ in } \forall u\varphi \end{cases}$

As an axiomatic system $\mathcal{L}$ consists of the following in addition to the language of quantificational language described above:

(i). Axioms;

(a) Sentential axioms; Let φ be a tautology of propositional logic and let ϑ be a formula of quantificational logic obtained from φ by uniform and simultaneous substitution of $\mathcal{L}$ formulas for propositional variables of φ. Then ϑ is an axiom. To give an example; $\neg(p \wedge \neg p)$ is a tautology. Substituting $P(x)$ for p we obtain an $\mathcal{L}$ formula $\neg(P(x) \wedge \neg P(x))$.

(b) Quantificational Axioms;

i. Let φ and ϑ be formulas and v be a variable which is not free in φ. Then.

$$\forall v(\varphi \rightarrow \vartheta) \rightarrow (\varphi \rightarrow \forall v\vartheta)$$

is an axiom.

ii. Let φ be a formula and let t be a term free for a variable v in φ. If the formula ϑ is obtained from φ by freely substituting t for v in φ then,

$$\forall v(\varphi \to \vartheta)$$

is an axiom.

(c) Identity Axioms; Let x and y be variables, $t(v_0, v_1, \ldots, v_n)$ a term and $\varphi(v_0, v_1, \ldots, v_n)$ a formula. Then the following are axioms;

$$x \equiv y$$

$$x \equiv y \to t(v_0, \ldots, v_{i-1}, x, v_{i+1}, \ldots, v_n) \equiv t(v_0, \ldots, v_{i-1}, y, v_{i+1}, \ldots, v_n)$$

$$x \equiv y \to (\varphi(v_0, \ldots, v_{i-1}, x, v_{i+1}, \ldots, v_n) \to \varphi(v_0, \ldots v_{i-1}, x, v_{i+1}, \ldots, v_n))$$

(ii). Transformation Rules;

(a) Rule of Modus Ponens;

ϑ is the immediate consequence of φ and $(\varphi \to \vartheta)$.

(b) Rule of generalization;

from φ we may infer $\forall\varphi$.

Since we have given our axioms and transformation rules, we are ready to give proof theoretic definitions for $\mathcal{L}$. The notions of consistency, inconsistency and maximal consistency are the same as in the chapter 2.

Definition 2.4.8. *A formula φ of $\mathcal{L}$ is said to be deducible from a set of $\mathcal{L}$-formulas Σ, in symbols $\Sigma \vdash \varphi$, iff there is a finite sequence of $\mathcal{L}$-formulas $\varphi_0, \varphi_1, \ldots, \varphi_n$ such that $\varphi_n = \varphi$ and for each φ_i one of the following holds;*

(i). φ_i is an axiom;

(ii). φ_i is an element of Σ;

(iii). There is an element φ_j $j < i$ of the sequence such that φ_i follows from φ_j by the rule of generalization;

(iv). There are φ_j and φ_k where $j.k < i$ such that φ_i follows from φ_j and φ_k by the rule of Modus Ponens.

Definition 2.4.9. $\varphi \vdash \vartheta$ iff $\{\varphi\} \vdash \vartheta$.

Definition 2.4.10. *φ is a **theorem** of quantificational logic, in symbols $\vdash \varphi$, iff $\{\} \vdash \varphi$.*

Definition 2.4.11. *The **length of a proof** is the number of elements of the sequence forming the proof.*

Note that being a theorem of quantificational logic depends on the set of axioms and the set of rules of transformation chosen. That is, it depends upon the given formal system of quantificational logic. Similar note applies to other proof theoretical notions such as consistency, inconsistency which are defined exactly as in the case of propositional logic. Since we accept a fixed system of quantificational logic, we omit the subscript of $\vdash$ sign. However, it is necessary when several systems are under consideration.

Proposition 2.4.1.

(i). Σ is consistent *iff* every finite subset of Σ is consistent.

(ii). $\Sigma \cup \{\varphi\}$ is inconsistent *iff* $\Sigma \vdash \neg\varphi$ where φ is a sentence.

(iii). Let Σ be maximally consistent and φ and ϑ be sentences, then;

$$\Sigma \vdash \varphi \text{ iff } \varphi \in \Sigma$$

$$\varphi \notin \Sigma \text{ iff } \neg\varphi \in \Sigma$$

$$(\varphi \wedge \vartheta) \in \Sigma \text{ iff } \varphi \in \Sigma \text{ and } \vartheta \in \Sigma$$

proof.

(i). That if Σ is consistent then every finite subset of Σ is consistent trivially holds. Assume that every finite subset of Σ is consistent and Σ is not consistent. Then $\Sigma \vdash \varphi$ and $\Sigma \vdash \neg\varphi$ for any formula φ. By definition of proof in $\mathcal{L}$, both deductions are finite sequences. Let Σ_0 be the finite subset of Σ whose members are used in the deduction of φ and let Σ_1 be the finite subset of Σ such that $\Sigma_1 \vdash \neg\varphi$. Then $\Sigma_0 \cup \Sigma_1 \vdash (\varphi \wedge \neg\varphi)$. But $\Sigma_0 \cup \Sigma_1$ which is a finite subset of Σ is not consistent. This is a contradiction.

(ii). Assume that $\Sigma \cup \{\varphi\}$ is inconsistent. If Σ is inconsistent, then $\Sigma \vdash \neg\varphi$ trivially holds. Hence, assume that Σ is consistent. We have two cases;

(a) $\Sigma \cup \{\neg\varphi\}$ is consistent. Then,

$$\Sigma \cup \{\neg\varphi\} \vdash \neg\varphi \text{ since } \neg\varphi \in \Sigma \cup \{\neg\varphi\}.$$

$$\Sigma \cup \{\neg\varphi\} \vdash \varphi \text{ since } \Sigma \cup \{\neg\varphi\} \text{ is inconsistent.}$$

then $\Sigma \vdash \neg\varphi$.

(b) $\Sigma \cup \{\neg\varphi\}$ is inconsistent. Then,

$$\Sigma \cup \{\neg\varphi\} \vdash \varphi \text{ since } \Sigma \cup \{\neg\varphi\} \text{ is inconsistent.}$$

$$\Sigma \cup \{\neg\varphi\} \vdash \neg\varphi \text{ since } \neg\varphi \in \Sigma \cup \{\neg\varphi\}.$$

then $\Sigma \vdash \neg\varphi$.

Assume that $\Sigma \vdash \neg\varphi$. Then $\Sigma \cup \{\varphi\} \vdash \neg\varphi$. Moreover, $\Sigma \cup \{\varphi\} \vdash \varphi$. Thus, $\Sigma \cup \{\varphi\}$ is inconsistent.

(iii). Assume that Σ is maximally consistent, $\Sigma \vdash \varphi$ but $\varphi \notin \Sigma$. Then, by the maximality of Σ, $\Sigma \cup \{\varphi\}$ is inconsistent. Then $\Sigma \cup \{\varphi\} \vdash \neg\varphi$ but also $\Sigma \cup \{\neg\varphi\} \vdash \neg\varphi$. Therefore, $\Sigma \vdash \neg\varphi$. This contradicts the consistency of Σ. That if $\varphi \in \Sigma$ then $\Sigma \vdash \varphi$ holds trivially.

Assume that $\varphi \notin \Sigma$. Then $\Sigma \cup \{\varphi\}$ is inconsistent by the maximality of Σ. Thus $\Sigma \cup \{\varphi\} \vdash \neg\varphi$ but $\Sigma \cup \{\neg\varphi\} \vdash \neg\varphi$ also. Then $\Sigma \vdash \neg\varphi$ and by (a), $\neg\varphi \in \Sigma$.

Conversely, if $\neg\varphi \in \Sigma$ then $\varphi \notin \Sigma$ since otherwise Σ can not be consistent.

(iv). $(\varphi \wedge \vartheta) \in \Sigma$ *iff* $\Sigma \vdash (\varphi \wedge \vartheta)$ *iff* $\Sigma \vdash \varphi$ and $\Sigma \vdash \vartheta$ *iff* $\varphi \in \Sigma$ and $\vartheta \in \Sigma$ by (a).

□

2.5 Model Theory of First Order Languages

The language of propositional logic is simple so are the models of propositional logic, truth definition of a sentence of propositional logic and properties of its models. In the case of quantificational logic and first order theories based on the language of quantificational logic, we have languages and structures complicated enough to do mathematics. To give meaning to symbols and legitimate expressions (terms, formulas and sentences). establishing properties of models of these languages and establishing properties of these models and relations between them requires much more effort.

In the language of a first order theory, we have relation, function and constant symbols. To see whether a sentence of propositional logic is true or false in a model, we have a simple definition since simple sentences (propositional variables) can be seen to be true or false at once. Truth or falsity of compound sentences then follows by means of truth or falsity of smaller parts in that model.

We will follow the same procedure to give a truth definition for formulas and sentences of first order theories. However, establishing truth or falsity of simple parts in a model is not that simple. While propositional sentences have only propositional variables to be interpreted whose possible interpretation consists of either *truth* or *falsity*, first order formulas may include relation and function symbols whose possible interpretation may be any relation or function. The following definition is due to Tarski:

Definition 2.5.1. *Let $\mathcal{L}$ be a first order language*

$$\{P_0{}^{n_0}, P_1{}^{n_1}, \ldots, F_0{}^{m_0}, F_1{}^{m_1} \ldots, c_0, c_1, \ldots\}$$

where n_i and m_j codes the arity of the relation or function symbol P_i or F_j respectively. A model for the language $\mathcal{L}$ is a pair $\mathfrak{A} =< A,{}^{\mathfrak{A}} >$ such that A is a set, the domain of $\mathfrak{A}$, and ${}^{\mathfrak{A}}$ is ***the interpretation function*** *such that:*

(i). For each n-ary relation symbol P of $\mathcal{L}$. $P^{\mathfrak{A}}$ is an n-ary relation on A.

(ii). For each n-ary function symbol F of $\mathcal{L}$. $F^{\mathfrak{A}}$ is an n-ary function from A^n into A.

(iii). For each constant symbol c of $\mathcal{L}$. $c^{\mathfrak{A}}$ is an element of A.

Given a first order language $\mathcal{L}$ the number of models of $\mathcal{L}$ may be very large. Even when we have a fixed domain A, depending on the number of relation and function symbols, we have many possible interpretation functions. One of the main concerns of model theory is to find out interesting relations between models of a given language and finding out ways of establishing these relations. Our notation gives an appropriate notation so that models of a given language will be comparable.

Let $\mathcal{L} = \{P_0, P_1, \ldots, P_n, F_0, F_1, \ldots, F_m, c_0, c_1, \ldots, c_s, \ldots\}$ and let $< A,^{\mathfrak{A}} >$ and $< B,^{\mathfrak{B}} >$ be models of $\mathcal{L}$. Then $P_i^{\mathfrak{A}}$ and $P_i^{\mathfrak{B}}$, $F_i^{\mathfrak{A}}$ and $F_i^{\mathfrak{B}}$, $c_i^{\mathfrak{A}}$ and $c_i^{\mathfrak{B}}$ are called ***the corresponding relations, functions and constants*** in $\mathfrak{A}$ and $\mathfrak{B}$ respectively.

We said that for a first order language we have two syntactic categories; term and formula. We will now attach meaning to terms and formulas of a first order language $\mathcal{L}$. Recall that $t(v_0, \ldots, v_n)$ says that variables of t form a subset of $\{v_0, \ldots, v_n\}$ and the notation $\varphi(v_0, \ldots, v_n)$ says that free variables of φ form a subset of $\{v_0, \ldots, v_n\}$. We will enlarge the notation for formulas and let $\varphi(v_0, \ldots, v_n)$ denote the assertion that all variables, free or bound, of φ are among $v_0, \ldots, v_n$ so that our truth definition applies not only to sentences but also to formulas.

Definition 2.5.2. ***The value of a term*** *t, $t(v_0, \ldots, v_n)$ at $[x_0, \ldots, x_n]$ denoted by $t[x_0, \ldots, x_n]$ is defined inductively as follows;*

(i). If $t = v_i$, then $t[x_0, \ldots, x_n] = x_i$;

(ii). If $t = c_i$, then $t[x_0, \ldots, x_n] = c_i^{\mathfrak{A}}$;

(iii). If $t = F_i^m(t_0, \ldots, t_m)$ then

$$t[x_0, \ldots, x_n] = F_i^{\mathfrak{A}}(t_0[x_0, \ldots, x_n], \ldots, t_m[x_0, \ldots, x_n])$$

Definition 2.5.3. *The relation $\varphi(v_0, \ldots, v_n)$ is satisfied by the sequence of elements $x_0, \ldots, x_n$ of A in the model $\mathfrak{A}$, where all variables of φ are among $v_0, \ldots, v_n$, in symbols $\mathfrak{A} \models \varphi[x_0, \ldots, x_n]$,is defined inductively as follows;*

(i). If $\varphi = t_1 \equiv t_2$ then $\mathfrak{A} \models \varphi[x_0, \ldots, x_n]$ *iff* $t_1[x_0, \ldots. x_n] = t_2[x_0, \ldots, x_n]$;

(ii). If $\varphi = P_i^m(t_0, \ldots, t_m)$ then $\mathfrak{A} \models \varphi[x_0. \ldots, x_n]$ *iff* $P_i^{\mathfrak{A}}(t_0[x_0, \ldots, x_n], \ldots, t_m[x_0, \ldots, x_n])$;

(iii). If $\varphi = \neg\vartheta$ then $\mathfrak{A} \models \varphi[x_0, \ldots, x_n]$ *iff* it is not the case that $\mathfrak{A} \models \vartheta[x_0, \ldots, x_n]$;

(iv). If $\varphi = (\vartheta \wedge \varsigma)$ then $\mathfrak{A} \models \varphi[x_0, \ldots, x_n]$ *iff* $\mathfrak{A} \models \vartheta[x_0, \ldots, x_n]$ and $\mathfrak{A} \models \varsigma[x_0, \ldots, x_n]$;

(v). If $\varphi = \forall v_i \varphi$ where $i \leq n$ then $\mathfrak{A} \models \varphi[x_0, \ldots, x_n]$ *iff* for every $x \in A$ $\mathfrak{A} \models \varphi[x_0, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n]$.

Proposition 2.5.1. *Let $t(v_0, \ldots, v_n)$ be a term and $\varphi(v_0, \ldots, v_n)$ be a formula. Let $[x_1, \ldots, x_p]$, $[y_0, \ldots, y_q]$ be two sequences of elements of the domain*

A of a model $\mathfrak{A}$ of $\mathcal{L}$ such that $n \leq p$ and $n \leq q$ and $x_i = y_i$ whenever v_i is a variable of t or a free variable of φ. Then,

(i). $t[x_0, \ldots, x_p] = t[y_0, \ldots, y_q]$

(ii). $\mathfrak{A} \models \varphi[x_0, \ldots, x_p]$ *iff* $\mathfrak{A} \models \varphi[y_0, \ldots, y_q]$

proof. For the first part of the proof, we use an induction on the complexity of terms;

(i). If $t = v_i$, $t[x_0, \ldots, x_p] = x_i = y_i = t[y_0, \ldots, y_q]$;

(ii). If $t = c_i$, $t[x_0, \ldots, x_p] = c_i^{\mathfrak{A}} = t[y_0, \ldots, y_q]$;

(iii). If $t = F_i(t_0, \ldots, t_m)$,

$$\begin{aligned} t[x_0, \ldots, x_p] &= F_i^{\mathfrak{A}}(t_0[x_0, \ldots, x_p], \ldots, t_m[x_0, \ldots, x_p]) \\ &= F_i^{\mathfrak{A}}(t[y_0, \ldots, y_q], \ldots, t_m[y_0, \ldots, y_q]) \end{aligned}$$

$= t[y_0, \ldots, y_q]$

Similarly we use induction on the complexity of formulas for the proof of the second part of the proposition;

(i). If $\varphi = t_1 \equiv t_2$,

$$\mathfrak{A} \models \varphi[x_0, \ldots, x_p]$$

$$iff$$

$$t_1[x_0, \ldots, x_p] = t_2[x_0, \ldots, x_p]$$

which is equivalent, by the first part, to;

$$t_1[y_0, \ldots, y_q] = t_2[y_0, \ldots, y_q]$$

$$iff$$

$$\mathfrak{A} \models \varphi[y_0, \ldots, y_q]$$

(ii). If $\varphi = P(t_0, \ldots, t_m)$

$$\mathfrak{A} \models \varphi[x_0, \ldots, x_p]$$

$$iff$$

$$P^{\mathfrak{A}}(t_0[x_0, \ldots, x_p], \ldots . t_m[x_0, \ldots, x_p])$$

$$iff$$

$$P^{\mathfrak{A}}(t_0[y_0, \ldots, y_q], \ldots, t_m[y_0, \ldots, y_q])$$

$$iff$$

$$\mathfrak{A} \models \varphi[y_0, \ldots, y_q]$$

(iii). If $\varphi = \neg\vartheta$

$$\mathfrak{A} \models \varphi[x_0, \ldots, x_p]$$

$$iff$$

$$\text{It is not the case that } \mathfrak{A} \models \vartheta[x_0, \ldots, x_p]$$

$$iff$$

$$\text{It is not the case that } \mathfrak{A} \models \vartheta[y_0, \ldots, y_q]$$

$$iff$$

$$\mathfrak{A} \models \neg\vartheta[y_0, \ldots, y_q]$$

$$iff$$

$$\mathfrak{A} \models \varphi[y_0, \ldots, y_q]$$

(iv). If $\varphi = (\vartheta \wedge \varsigma)$

$$\mathfrak{A} \models \varphi[x_0, \ldots, x_p]$$

$$iff$$

$$\mathfrak{A} \models \vartheta[x_0, \ldots, x_p] \text{ and } \mathfrak{A} \models \varsigma[x_0, \ldots, x_p]$$

$$iff$$

$$\mathfrak{A} \models \vartheta[y_0, \ldots, y_q] \text{ and } \mathfrak{A} \models \varsigma[y_0, \ldots, y_q]$$

$$iff$$

$$\mathfrak{A} \models \varphi[y_0, \ldots, y_q]$$

(v). If $\varphi = \forall v_i \vartheta$ where $i \leq n$

$$\mathfrak{A} \models \varphi[x_0, \ldots, x_p]$$

$$iff$$

$$\text{for every } x \in A \; \mathfrak{A} \models \vartheta[x_0, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_p]$$

$$iff$$

$$\text{for every } y \in A \; \mathfrak{A} \models \vartheta[y_0, \ldots, y_{i-1}, y, y_{i+1}, \ldots, y_q]$$

$$iff$$

$$\mathfrak{A} \models \varphi[y_0, \ldots, y_q]$$

□

This proposition allows us to redefine the notion of satisfaction of a formula by a sequence in a model. We may only look at the free variables of the formula. Sequences which are equivalent as to the free variables of the formula will be considered essentially the same. What if our formula has no free variable, that is, if it is a sentence? In that case, since trivially all sequences are equivalent as to the free variables of the sentence, it suffices to have at least one sequence satisfying the sentence. Therefore, we may say that, given a sentence, either all sequences satisfy it or there is no sequence satisfying it.

Definition 2.5.4. *A sentence φ* ***holds in a model*** *$\mathfrak{A}$, in symbols $\mathfrak{A} \models \varphi$, iff there is at least one sequence of elements (or, equivalently every sequence* *of elements) from the domain of the model satisfying it.* φ *is called* ***valid,*** *in symbols* $\models \varphi$ *iff in every model, for every sequence of elements from the domain of the model, the formula is satisfied by that sequence.*

Proposition 2.5.2. *Let* $\varphi(v_0, \ldots, v_n)$ *be a formula and* $t_i(v_0, \ldots, v_n)$, $0 \leq i \leq n$, *be terms such that no variable of these terms occurs bound in* φ *and let* $\varphi(t_0, \ldots, t_n)$ *be the formula obtained from* φ *by replacing* v_i *by* t_i. *Then, for any sequence* $[x_0, \ldots, x_n]$,

$$\mathfrak{A} \models \varphi(t_0, \ldots, t_n)[x_0, \ldots, x_n] \; iff \; \mathfrak{A} \models \varphi[t_0[x_0, \ldots, x_n], \ldots, t_n[x_0, \ldots, x_n]]$$

Proof. If $\varphi(v_0, \ldots, v_n) = s(v_0, \ldots, v_n) \equiv t(v_0, \ldots, v_n)$ then

$$\mathfrak{A} \models \varphi(t_0, \ldots, t_n)[x_0, \ldots, x_n] \; iff$$

$$\mathfrak{A} \models s(t_0, \ldots, t_n) \equiv t(t_0, \ldots, t_n)[x_0, \ldots, x_n] \; iff$$

$$s[t_0[x_0, \ldots, x_n], \ldots, t_n[x_0, \ldots, x_n]] = t[t_0[x_0, \ldots, x_n], \ldots, t_n[x_0, \ldots, x_n]] \; iff$$

$$\mathfrak{A} \models (s \equiv t)[t_0[x_0, \ldots, x_n], \ldots, t_n[x_0, \ldots, x_n]]$$

If $\varphi = P(v_0, \ldots, v_n)$,

$$\mathfrak{A} \models P(t_0, \ldots, t_p)[x_0, \ldots, x_n] \; iff$$

$$P^{\mathfrak{A}}[t_0[x_0, \ldots, x_n], \ldots, t_n[x_0, \ldots, x_n]] \; iff$$

$$\mathfrak{A} \models \varphi[t_0[x_0, \ldots, x_n], \ldots, t_n[x_0, \ldots, x_n]]$$

If $\varphi = \neg\vartheta(v_0, \ldots, v_n)$,

$$\mathfrak{A} \models \varphi(t_0, \ldots, t_n)[x_0, \ldots, x_n] \ iff$$

$$\text{it is not the case that } \mathfrak{A} \models \vartheta(t_0, \ldots, t_n)[x_0, \ldots, x_n] \ iff$$

$$\text{it is not the case that } \mathfrak{A} \models \vartheta[t_0[x_0, \ldots, x_n], \ldots, t_n[x_0, \ldots, x_n]] \ iff$$

$$\mathfrak{A} \models \neg\vartheta[t_0[x_0, \ldots, x_n], \ldots, t_n[x_0, \ldots, x_n]] \ iff$$

$$\mathfrak{A} \models \varphi[t_0[x_0, \ldots, x_n], \ldots, t_n[x_0, \ldots, x_n]] \ iff$$

If $\varphi = (\vartheta \wedge \varsigma)$,

$$\mathfrak{A} \models \varphi(t_0, \ldots, t_n)[x_0, \ldots, x_n] \ iff$$

$$\mathfrak{A} \models \vartheta(t_0, \ldots, t_n)[x_0, \ldots, x_n] \text{ and } \mathfrak{A} \models \varsigma(t_0, \ldots . t_n)[x_0 . \ldots, x_n] \ iff$$

$$\mathfrak{A} \models \vartheta[t_0[x_0, \ldots, x_n], \ldots, t_n[x_0, \ldots, x_n]] \text{ and}$$

$$\mathfrak{A} \models \varsigma[t_0[x_0, \ldots, x_n], \ldots, t_n[x_0, \ldots, x_n]] \ iff$$

$$\mathfrak{A} \models \varphi[t_0[x_0, \ldots, x_n], \ldots, t_n[x_0, \ldots, x_n]]$$

If $\varphi = \forall v\vartheta$ where $v \neq v_i$ for $0 \leq i \leq n$ since none of the variables of t_i occurs bound in φ,

$$\mathfrak{A} \models \varphi(t_0, \ldots, t_n)[x_0, \ldots, x_n] \ iff$$

$$\mathfrak{A} \models \forall v\vartheta(t_0, \ldots, t_n)[x_0, \ldots, x_n] \ iff$$

$$\mathfrak{A} \models \vartheta(t_0, \ldots, t_n)[x_0, \ldots, x_n] \ iff$$

$$\mathfrak{A} \models \vartheta[t_0[x_0, \ldots, x_n], \ldots, t_n[x_0, \ldots . x_n]]$$

□

Note that in the last part of the proof we used the fact that the quantifier $\forall$ in $\forall v \vartheta(v_0, \ldots, v_n)$ is redundant since v is not equal to any of the variables $v_i : 0 \leq i \leq n$. The reader is invited to see this point himself by using the definition of satisfaction.

Giving the definition of model and the notion of satisfaction in a model we will now state a few important relations between models.

Definition 2.5.5. *Let $\mathcal{L}$ be a first order language, $\mathcal{L}'$ a subset of $\mathcal{L}$ and $\mathfrak{A}$ is a model of $\mathcal{L}$. Then the model $\mathfrak{B}$ with the same domain as $\mathfrak{A}$ such that $\mathfrak{B}$ interprets only $\mathcal{L}'$ symbols and exactly same as $\mathfrak{A}$ is called* ***the $\mathcal{L}'$-reduction of*** *$\mathfrak{A}$.*

Definition 2.5.6. *Let $\mathcal{L}$ be a language and $\mathfrak{A}$ and $\mathfrak{B}$ are $\mathcal{L}$ models. We say that $\mathfrak{A}$* ***is a submodel of*** *$\mathfrak{B}$ if*

(i). $A \subseteq B$;

(ii). For every n-ary relation symbol P of $\mathcal{L}$, $P^{\mathfrak{A}} = P^{\mathfrak{B}} \restriction A^n$;

(iii). For every n-ary function symbol F of $\mathcal{L}$, $F^{\mathfrak{A}} = F^{\mathfrak{B}} \restriction A^n$;

(iv). For each constant symbol c of $\mathcal{L}$, $c^{\mathfrak{A}} = c^{\mathfrak{B}}$

Definition 2.5.7. *Let $\mathcal{L}$ be a language, $\mathfrak{A}$ and $\mathfrak{B}$ are $\mathcal{L}$ models and $f : A \longrightarrow B$ is a mapping. We say that f is a* ***homomorphism*** *between $\mathfrak{A}$ and $\mathfrak{B}$ if* (i). For every n-ary relation symbol P of $\mathcal{L}$, and for every sequence of elements $x_1, \ldots, x_n$ of A, $P^{\mathfrak{A}}(x_1, \ldots, x_n) \rightarrow P^{\mathfrak{B}}(f(x_1), \ldots, f(x_n))$;

(ii). For every n-ary function symbol F of $\mathcal{L}$ and for every sequence of elements $x_1, \ldots, x_n$ of A, $f(F^{\mathfrak{A}}(x_1, \ldots, x_n)) = F^{\mathfrak{B}}(f(x_1), \ldots, f(x_n))$;

(iii). For each constant c in $\mathfrak{A}$, $f(c^{\mathfrak{A}}) = c^{\mathfrak{B}}$.

If a homomorphism f from $\mathfrak{A}$ to $\mathfrak{B}$ is such that there is a homomorphism g from $\mathfrak{B}$ to $\mathfrak{A}$ and the composition of f and g, $g \circ f$, is the identity mapping on $\mathfrak{A}$ and $f \circ g$ is the identity mapping on $\mathfrak{B}$, then f is said to be an **isomorphism** between $\mathfrak{A}$ and $\mathfrak{B}$. If there is an isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$ then $\mathfrak{A}$ is said to be **isomorphic** to $\mathfrak{B}$,in symbols $\mathfrak{A} \cong \mathfrak{B}$. Using the notation of definition 2.5.7 we may define the notion of isomorphism as follows.

2.5.3.

Let f be a mapping as defined in definition 2.5.7. We say that f is an isomorphism if and only if:

(i). For every n-ary relation symbol P of $\mathcal{L}$, and for every sequence of elements $x_1, \ldots, x_n$ of A, $P^{\mathfrak{A}}(x_1, \ldots x_n)\ iff\ P^{\mathfrak{B}}(f(x_1), \ldots, f(x_n))$;

(ii). For every n-ary function symbol F of $\mathcal{L}$ and for every sequence of elements $x_1, \ldots, x_n$ of A, $f(F^{\mathfrak{A}}(x_1, \ldots, x_n)) = F^{\mathfrak{B}}(f(x_1), \ldots, f(x_n))$;

(iii). For each constant c in $\mathfrak{A}$, $f(c^{\mathfrak{A}}) = c^{\mathfrak{B}}$.

Definition 2.5.8. *Let $\mathcal{L}$ be a language and $\mathfrak{A}$ and $\mathfrak{B}$ be $\mathcal{L}$-models. We say that $\mathfrak{A}$ and $\mathfrak{B}$ are* ***elementarily equivalent****, in symbols $\mathfrak{A} \equiv \mathfrak{B}$ if;*

$$\mathfrak{A} \models \sigma \textit{ iff } \mathfrak{B} \models \sigma \textit{ for every sentence } \sigma$$

The following proposition gives the basic relationship between the notions of isomorphism and elementary equivalence.

Proposition 2.5.4. *If $\mathfrak{A} \cong \mathfrak{B}$ then $\mathfrak{A} \equiv \mathfrak{B}$.*

The idea behind this proposition is as follows: if there is an isomorphism f between $\mathfrak{A}$ and $\mathfrak{B}$ then since f is a bijection between A and B, each element a of A corresponds to an element $f(a)$ of B. Moreover, if $(a_1, \ldots, a_n)$ is a sequence of elements from A and $(f(a_1), \ldots, f(a_n))$ is the sequence of corresponding elements of B, then $(a_1, \ldots, a_n)$ satisfies a sentence σ in $\mathfrak{A}$ if and only if $(f(a_1), \ldots, f(a_n))$ satisfies σ in $\mathfrak{B}$ since behaviour of elements of models $\mathfrak{A}$ and $\mathfrak{B}$ are the same under the relations and functions of $\mathcal{L}$. The proof of the proposition follows this line of reasoning. Let f be an isomorphism between $\mathfrak{A}$ and $\mathfrak{A}$. Then, for example, the following statements are equal:

$$\mathfrak{A} \models c \equiv d$$

$$c^{\mathfrak{A}} = d^{\mathfrak{A}}$$

$$f(c^{\mathfrak{A}}) = f(c^{\mathfrak{B}})$$

$$c^{\mathfrak{B}} = d^{\mathfrak{B}}$$

$$\mathfrak{B} \models c \equiv d$$

CHAPTER 3

COMPLETENESS THEOREM

Let T be a consistent set of sentences in a language $\mathcal{L}$. As in the case of propositional logic, we will show that T is satisfiable, that is, it has a model. Since the only information about T is that T is consistent and consistency is a syntactic notion, we have only syntactic tools to construct a model of T. In order to construct a model, we have to choose an appropriate set as the domain, and interpret relation, function and constant symbols of the language $\mathcal{L}$ of T in a suitable way. If we use the terms of the language $\mathcal{L}$ directly, we face a difficulty: let t_1 and t_2 be $\mathcal{L}$ terms. If T is such that $T \vdash f(t_1) \equiv f(t_2)$ while $f(t_1) \neq f(t_2)$ in our model, then our model constructed with its domain all $\mathcal{L}$-terms, would not be a model of T. If we take care of the behaviour of the terms in formulas deducible from T, no such difficulty arises.

Definition 3.0.9. *Let T be a set of sentences in a language $\mathcal{L}$ and C a set of constants of $\mathcal{L}$. C is a set of* **witnesses** *for T if whenever φ is a formula of $\mathcal{L}$ with at most one free variable x, then there exists $c \in C$ such that:*

$$T \models \exists x\varphi \rightarrow \varphi(c)$$

The following proof of the fact that a consistent first order theory in the language $\mathcal{L}$ has at least one model is due to Henkin. The proof is constructive; we really construct a model for the consistent theory T.

We use two facts in the proof;

(i). If C is a set of witnesses for a set of sentences T, and if T' is an extension of T, i.e., $T \subset T'$, then C is also a set of witnesses for T'.

(ii). If $\mathfrak{A}$ is a model of an extension T' of T, then $\mathfrak{A}$ is also a model of T.

The outline of the proof is as follows: we may expand $\mathcal{L}$, the language of T, to a language $\mathcal{L}'$ such that we may find an extension T' of T in $\mathcal{L}'$ and such that T' has a set of witnesses. Then. we may assume that T' is a maximally consistent set of sentences in $\mathcal{L}'$ since, by Lindenbaum's theorem, we may always enlarge T' to a maximally consistent set. We will use the fact that T' is maximally consistent and that T' has witnesses to find a model of T' and use the fact that T' is a satisfiable extension of T to show that T has a model.

Proposition 3.0.5. *Let T be a consistent set of $\mathcal{L}$-sentences. Let $\bar{\mathcal{L}}$ be a simple expansion of $\mathcal{L}$ obtained by the addition of a set of new constant symbols C, where $|C| = ||\mathcal{L}||$, to $\mathcal{L}$ ie., $\bar{\mathcal{L}} = \mathcal{L} \cup C$. Then there is an extension $\bar{T}$ of T in $\bar{\mathcal{L}}$ such that $\bar{T}$ is a consistent set of sentences in $\bar{\mathcal{L}}$ and C is a set of witnesses in $\bar{\mathcal{L}}$ for $\bar{T}$.*

proof. Let $||\mathcal{L}|| = \alpha$ and

$$C = \{c_\beta : \beta \leq \alpha, c_\beta \not\models c_\gamma \text{ if } \beta < \gamma < \alpha\}$$

and let $\bar{\mathcal{L}} = \mathcal{L} \cup C$. We wish to enlarge T to a consistent set $\bar{T}$ in $\bar{\mathcal{L}}$ such that C is a set of witnesses for $\bar{T}$. Note that, $||\bar{\mathcal{L}}|| = \alpha$ and thus we may list all $\bar{\mathcal{L}}$-formulas with at most one variable as a sequence $\varphi_\beta; \beta < \alpha$. Using this fact, we enlarge T by defining an increasing sequence of sets

$$T = T_0 \subset T_1 \subset \ldots \subset T_\beta \ldots, \quad \beta < \alpha$$

and a sequence of constants $d_\beta; \beta < \alpha$ such that;

(i). Each $T_\beta; \beta < \alpha$ is consistent in $\bar{\mathcal{L}}$;

(ii). Assume that T_β has been defined, then

$$T_{\beta+1} = \begin{cases} T_\beta \cup \{\exists x_\beta \varphi_\beta \to \varphi_\beta(d_\beta)\} & \text{if } x_\beta \text{ is the free variable of } \varphi_\beta \\ T_\beta \cup \{\exists v_0 \varphi_\beta \to \varphi_\beta(d_\beta)\} & \text{if } \varphi_\beta \text{ has no free variable} \end{cases}$$

(iii). $T_\gamma = \bigcup_{\beta<\gamma} T_\beta$ if γ is a limit ordinal.

We let $\bar{T} = \bigcup_{\beta<\alpha} T_\beta$. It is obvious that $\bar{T}$ is consistent. Moreover, C is a set of witnesses for $\bar{T}$. Since let φ be an $\bar{\mathcal{L}}$-formula with at most x free in it. Then $\varphi = \varphi_\beta$ and $x = x_\beta$ for some $\beta < \alpha$. By the above construction of T_i's,

$$T_{\beta+1} \vdash \exists x_\beta \varphi_\beta \to \varphi_\beta(d_\beta)$$

$$\bar{T} \vdash \exists x_\beta \varphi_\beta \to \varphi_\beta(d_\beta)$$

□

Thus we have shown, as we promised, that a consistent set of sentences can always be extented to a consistent set with witnesses in the simply expanded language. Now, we will show that, we may construct a model for such a set in the expanded language.

Proposition 3.0.6. *Let T be a consistent set, with the set of witnesses C, in the language $\mathcal{L}$. Then there is a model $\mathfrak{A}$ of T such that every element of the domain of $\mathfrak{A}$ is an interpretation of a constant $c \in C$.*

proof. We may assume that T is maximally consistent. Since by Lindenbaum's theorem a consistent set T can always be extended to a maximal consistent set and any model of this maximal consistent set is also a model of T. Define an equivalence relation $\sim$ on the set C as $c \sim d \; iff \; c \equiv d \in T$. Since T is maximaly consistent, we have

$$c \sim c$$

$$\text{if } c \sim d \text{ and } d \sim e \text{ then } c \sim e$$

$$\text{if } c \sim d \text{ then } d \sim c$$

We will define the domain of $\mathfrak{A}$ as;

$$A = \{\tilde{c} : c \in C\}$$

where $\tilde{c} = \{d \in C : c \sim d\}$ is the equivalence class of c under $\sim$. Now, we will give an interpretation of relation, function and constant symbols of $\mathcal{L}$ in $\mathfrak{A}$.

(i). Let P be an n-ary relation symbol of $\mathcal{L}$. Then,

$$P^{\mathfrak{A}}(\tilde{c}_1, \ldots, \tilde{c}_n) \; iff \; P(c_1, \ldots, c_n) \in T$$

(ii). Let F be an n-ary function symbol of $\mathcal{L}$. Then,

$$F^{\mathfrak{A}}(\tilde{c}_1, \ldots \tilde{c}_n) = \tilde{c} \; iff \; F(c_1, \ldots, c_n) = c \in T$$

(iii). Let c be a constant symbol of $\mathcal{L}$. Then $c^{\mathfrak{A}} = \tilde{d}$ where $c \sim d$.

Using the fact that T is maximally consistent and C is a set of witnesses for T, we may easily show that the interpretation of relation, function and constant symbols of $\mathcal{L}$ in $\mathfrak{A}$ is well defined. We will now show that $\mathfrak{A}$ is a model of T. We will do this by induction on the complexity of sentences in T. We first note that. since C is a set of witnesses for T, every term t is equivalent to a constant c_t. For every closed term, i.e., term with no variables, t and for every $c \in C$, the following are equivalent;

$$\mathfrak{A} \models d \equiv c$$

$$\tilde{d} = \tilde{c}$$

$$d \sim c$$

$$d \equiv c \in T$$

The following are also equivalent;

$$\mathfrak{A} \models F(t_1, \ldots, t_n) \equiv c$$

$$\mathfrak{A} \models F(c_{t_1}, \ldots, c_{t_n}) \equiv c$$

$$F^{\mathfrak{A}}(\tilde{c}_{t_1}, \ldots, \tilde{c}_{t_n}) = \tilde{c}$$

$$F(c_{t_1}, \ldots, c_{t_n}) \equiv c \in T$$

Thus for every closed term t and for every $c \in C$

$$\mathfrak{A} \models t \equiv c \; iff \; t \equiv c \in T.$$

It is also easy to show that $\mathfrak{A} \models t_1 \equiv t_2 \; iff \; t_1 \equiv t_2 \in T$ for every pair of closed terms t_1 and t_2. Since,

$$T \vdash \exists x t_1 \equiv x$$

since $\exists x t_1 \equiv x$ is a theorem of quantificational logic and T is maximally consistent. Since T has a set of witnesses C, and by the above remark,

$$T \vdash \exists x(t_1 \equiv x) \rightarrow t_1 \equiv c_{t_1}$$

$$T \vdash t_1 \equiv c_{t_1}$$

Thus we reduce the problem to the $t \equiv c$ case.

For the second case of atomic sentences, we have the following equivalent assertions;

$$\mathfrak{A} \models P(t_1, \ldots, t_n)$$

$$P^{\mathfrak{A}}(\tilde{c}_{t_1}, \ldots, \tilde{c}_{t_n})$$

$$P(c_{t_1}, \ldots, c_{t_n}) \in T$$

For compound sentences the result easily follows;

$$\mathfrak{A} \models \neg\varphi$$

it is not the case that $\mathfrak{A} \models \varphi$

it is not the case that $\varphi \in T$

$$\neg\varphi \in T$$

$$\mathfrak{A} \models (\varphi \wedge \vartheta)$$

$\mathfrak{A} \models \varphi$ and $\mathfrak{A} \models \vartheta$

$\varphi \in T$ and $\vartheta \in T$

$$(\varphi \wedge \vartheta) \in T$$

Let $\mathfrak{A} \models \exists x\varphi$ then there is $\tilde{c} \in A$ such that $\mathfrak{A} \models \varphi[\tilde{c}]$. Then there is $c \in C$ such that $\mathfrak{A} \models \varphi(c)$. Thus, by inductive assumption, $\varphi(c) \in T$. Since $\varphi(c) \rightarrow \exists x\varphi$ is a theorem of quantificational logic, and T is maximally consistent, $\exists x\varphi \in T$.

Let $\exists x\varphi \in T$. Since C is a set of witnesses for T, there is $c \in C$ such that

$$\exists x\varphi \rightarrow \varphi(c) \in T$$

Thus $\varphi(c) \in T$ since T is maximally consistent. Then, by inductive assumption, $\mathfrak{A} \models \varphi(c)$. Thus, $\mathfrak{A} \models \varphi[\tilde{c}]$ and then we have $\mathfrak{A} \models \exists x\varphi$ □

From the previous two results, we may infer the desired result that a consistent set of sentences has a model. Note that, the reduction of the model $\mathfrak{A}$ to $\mathcal{L}$ will give a model of T since T does not include new constants.

CHAPTER 4

CONCLUSION

Model Theory is one of the most rapidly growing areas of mathematics. In recent years, model theorists witnessed a sudden diversification of this field. The importance of model theory is twofold: it enables us to comprehend the nature of mathematical truth and the nature of mathematical structures. Model theory has also proved to be a strong tool with many applications starting from its applications to theoretical computer science.

I believe that, this thesis provides a brief account of basic notions of model theory. First, a general disscussion on formal languages and formal systems was given in order to give the motivation behind the way of construction of specific formal languages and systems which were introduced immediately after this general presentation. The content of the following sections were Chang and Keisler's book 'Model Theory.' In addition to presentation of what is in that book, I gave full proofs of partly or completely unproven results. The formal definitions of free variable, bound variable and substitution of a term for a variable were defined in order to contribute to the understanding of the concept of satisfaction, which is one of the basic notions of model theory.

In the last chapter of the thesis the completeness theorem for quantificational logic was given. In the proof, a model for a consistent theory was constructed. The method of constructing a model in the way described in the proof is called 'model construction by means of constants.' This is by no means the only way of constructing a model. The method of ultraproducts, the method of elementary chains and the method of using Skolem functions are the other methods widely used in model theory. Chang and Keisler's book is a good exposition of these and further classical notions of model theory. I hope that, this work is a contribution to the reading of that book.

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