Conditions for synchronizability in arrays of coupled linear systems

S. Emre Tuna
tuna@eee.metu.edu.tr

November 4, 2018

Abstract

Synchronization control in arrays of identical output-coupled continuous-time linear systems is studied. Sufficiency of new conditions for the existence of a synchronizing feedback law are analyzed. It is shown that for neutrally stable systems that are detectable form their outputs, a linear feedback law exists under which any number of coupled systems synchronize provided that the (directed, weighted) graph describing the interconnection is fixed and connected. An algorithm generating one such feedback law is presented. It is also shown that for critically unstable systems detectability is not sufficient, whereas full-state coupling is, for the existence of a linear feedback law that is synchronizing for all connected coupling configurations.

1 Introduction

Synchronization is, at one hand, a desired behaviour in many dynamical systems related to numerous technological applications [10, 11, 9]; and, at the other, a frequently-encountered phenomenon in biology [23, 28, 1]. Aside from that, it is an important system theoretical topic on its own right. For instance, there is nothing keeping us from seeing a simple Luenberger observer [15] with decaying error dynamics as a two-agent system where the agents globally synchronize [12, Ch. 5]. Due to abundance of motivation to investigate synchronization, a wealth of literature has been formed, mostly in recent times [7, Sec. 5], [24, 29].

An essential problem in control theory is to find general conditions that imply synchronization of a number of coupled individual systems. This problem, which is usually studied under the name *synchronization stability*, has been attacked by many and from various angles. Two cases, which partly overlap, are of particular interest: (i) where the dynamics of individual systems are relatively primitive (such as that of an integrator) yet the coupling between them satisfies but minimum regularity stipulations (e.g. time-varying, delayed, directed, need not be connected at all times); and (ii) where the individual systems are let be more complex, but the structure of their interaction resides in a more stringent set (e.g. fixed, balanced, symmetric, with coupling strength greater than some threshold.) The studies concentrated on the first case have resulted in the emergence of the area now known as consensus in multi-agent systems [17, 16, 8, 13, 20, 2, 6, 25] where fairly weak conditions on the interconnection have been established under which the states of individual systems converge to a common point fixed in space. The second case has also accommodated important theoretical developments especially by the use of tools from algebraic graph theory [30]. Using Lyapunov functions, it has been shown that spectrum of the coupling matrix plays a crucial role in determining the stability of synchronization [33, 18] notwithstanding it need not be explicitly known [5]. It has also been shown that passivity theory can be useful in studying stability provided that the coupling is symmetric [4, 19, 22].

As alluded to earlier, Luenberger observer makes a particular example for synchronization. The well-known dynamics read

$$\begin{aligned} \dot{x} &= Ax \\ \dot{\hat{x}} &= A\hat{x} + L(y - C\hat{x}) \end{aligned}$$

where $y = Cx$ is the output of the observed system $\dot{x} = Ax$ and $\hat{x}$ is the estimate of the actual state x. Taking observer gain as a design parameter, detectability of the pair (C, A) is necessary and sufficient for the existence of a linear feedback law L that ensures $|x(t) - \hat{x}(t)| \to 0$ as $t \to \infty$. Once the above dynamics are rewritten as

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \left(I \otimes A + \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \otimes LC \right) \begin{bmatrix} x \\ \hat{x} \end{bmatrix}$$

a natural generalization of the (observer design) problem becomes apparent. Namely, for the following system

$$\begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_p \end{bmatrix} = (I \otimes A + \Gamma \otimes LC) \begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix}$$

investigate conditions on (coupling) matrix Γ and pair (C, A) under which one can find an L (synchronizing feedback law) yielding $|x_i(t) - x_j(t)| \to 0$ for all i, j (synchronization.) Most applications related to this question render the assumption that coupling matrix is known too restrictive. Even the size of Γ (hence the number of coupled systems) may sometimes be unknown. Therefore when designing a synchronizing feedback law, one is forced to not count on the exact knowledge of coupling matrix (interconnection.) However, something has to be assumed and how loose that assumption can get is yet a partly open problem. To be precise, given a set of possible interconnections, where each element of the set determines not only the coupling between the systems but also the size of the array, a fundamental question is the following. What condition on pair (C, A) guarantees the existence of an L under which the systems synchronize for all coupling configurations that belong to the given set? Clearly, the choice of the set shapes the possible answers to this question. When the set is the collection of all interconnections with coupling strength greater than some (*a priori* known) positive threshold, the answer (which can be extracted, for instance, from [32]) is as simple as one can expect: that (C, A) is detectable, just as is the case with Luenberger observer. Unlike Luenberger observer though, mere that $A - LC$ is Hurwitz is not sufficient for synchronization. In attempt to make the general picture closer to complete (see Theorem 1), in this paper we focus on the *set of all interconnections with connected graphs* (no assumption on the strength of coupling nor on the symmetry or balancedness of the graph) and study new conditions on (C, A) which do (not) guarantee the existence of a synchronizing feedback law with respect to that set. The main contribution of this paper is in establishing the following results:

(b) If (C, A) is detectable with A neutrally stable then there exists a synchronizing linear feedback law. (We also provide an algorithm to explicitly compute one such feedback law.)

(c) If A is critically unstable (e.g. double integrator) and C is full column rank then there exists a synchronizing linear feedback law.

(f) If (C, A) is detectable with A critically unstable then, in general, there does not exist a synchronizing linear feedback law.

For the discrete-time version of (b) see [26].

In the remainder of the paper we first provide basic notation and definitions. In Section 3 we define *synchronizability with respect to a set of interconnections* and formalize the problem through that definition. Following the problem statement we present the main theorem as a 8-item list of sufficient and nonsufficient conditions for synchronizability. After stating the main theorem (Theorem 1) we set out to prove what it claims. The sufficiency statements of Theorem 1 are demonstrated in Section 4 where we also show that coupled harmonic oscillators synchronize if the coupling is via a connected graph (Corollary 1.) To the best of our knowledge, establishing synchronization of harmonic oscillators without any symmetry, balancedness, or strong coupling assumption on the interconnection is new. Section 5 includes the proofs of nonsufficiency statements of Theorem 1, which are given in order to give a measure on how tight (close to necessary) the sufficient conditions are. We spend a few words on a dual problem in Section 6 after which we conclude.

2 Notation and definitions

Let $\mathbb{N}$ denote the set of nonnegative integers and $\mathbb{R}_{\geq 0}$ set of nonnegative real numbers. Let $|\cdot|$ denote 2-norm. For $\lambda \in \mathbb{C}$ let $\mathrm{Re}(\lambda)$ denote the real part of λ. Identity matrix in $\mathbb{R}^{n\times n}$ is denoted by I_n and zero matrix in $\mathbb{R}^{m\times n}$ by $0_{m\times n}$. Conjugate transpose of a matrix A is denoted by A^H. A matrix $A \in \mathbb{C}^{n\times n}$ is *Hurwitz* if all of its eigenvalues have strictly negative real parts. A matrix $S \in \mathbb{R}^{n\times n}$ is *skew-symmetric* if $S + S^T = 0$. For $A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{m\times n}$, pair

(C, A) is *detectable* (in the continuous-time sense) if that $Ce^{At}x = 0$ for some $x \in \mathbb{R}^n$ and for all $t \geq 0$ implies $\lim_{t\to\infty} e^{At}x = 0$. Matrix $A \in \mathbb{R}^{n\times n}$ is *neutrally stable* (in the continuous-time sense) if it has no eigenvalue with positive real part and the Jordan block corresponding to any eigenvalue on the imaginary axis is of size one. Let $\mathbf{1} \in \mathbb{R}^p$ denote the vector with all entries equal to one.

Kronecker product of $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$ is

$$A \otimes B := \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}$$

Kronecker product comes with the properties $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$ (provided that products AC and BD are allowed) $A \otimes B + A \otimes C = A \otimes (B + C)$ (for B and C that are of same size) and $(A \otimes B)^T = A^T \otimes B^T$.

A (*directed*) *graph* is a pair $(\mathcal{N}, \mathcal{E})$ where $\mathcal{N}$ is a nonempty finite set (of *nodes*) and $\mathcal{E}$ is a finite collection of ordered pairs (*edges*) (n_i, n_j) with $n_i, n_j \in \mathcal{E}$. A *path* from n_1 to n_ℓ is a sequence of nodes $(n_1, n_2, \ldots, n_\ell)$ such that (n_i, n_{i+1}) is an edge for $i \in \{1, 2, \ldots, \ell - 1\}$. A graph is *connected* if it has a node to which there exists a path from every other node.[1] Fig. 1 illustrates two graphs, where one is connected and the other is not.

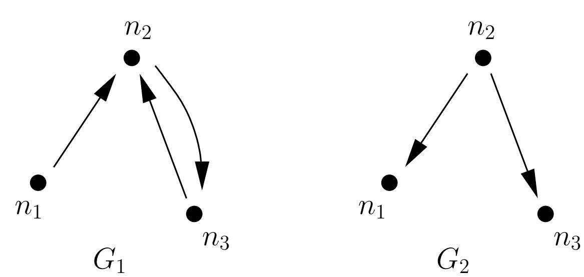

Figure 1: Schematic representations of two graphs $G_1 = (\mathcal{N}, \mathcal{E}_1)$ and $G_2 = (\mathcal{N}, \mathcal{E}_2)$ where $\mathcal{N} = \{n_1, n_2, n_3\}$, $\mathcal{E}_1 = \{(n_1, n_2), (n_2, n_3), (n_3, n_2)\}$ and $\mathcal{E}_2 = \{(n_2, n_1), (n_2, n_3)\}$. G_1 is connected, G_2 is not.

A matrix $\Gamma := [\gamma_{ij}] \in \mathbb{R}^{p\times p}$ describes (is) an *interconnection* if $\gamma_{ij} \geq 0$ for $i \neq j$ and $\gamma_{ii} = -\sum_{j\neq i} \gamma_{ij}$. It immediately follows that $\lambda = 0$ is an eigenvalue with eigenvector $\mathbf{1}$ (i.e. $\Gamma\mathbf{1} = 0$.) The graph of Γ is the pair $(\mathcal{N}, \mathcal{E})$ where $\mathcal{N} = \{n_1, n_2, \ldots, n_p\}$ and $(n_i, n_j) \in \mathcal{E}$ iff $\gamma_{ij} > 0$. Interconnection Γ is said to be *connected* if its graph is connected.

For connected Γ, eigenvalue $\lambda = 0$ is distinct and all the other eigenvalues have real parts strictly negative. When we write $\mathrm{Re}(\lambda_2(\Gamma))$ we mean the real part of a nonzero eigenvalue of Γ closest to the imaginary axis. Let r^T be the

[1] Note that this definition of connectedness for directed graphs is weaker than strong connectivity and stronger than weak connectivity.

left eigenvector of eigenvalue $\lambda = 0$ (i.e. $r^T\Gamma = 0$) with $r^T\mathbf{1} = 1$. Then r^T is unique and satisfies $\lim_{t\to\infty} e^{\Gamma t} = \mathbf{1}r^T$.

Given maps $\xi_i : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ for $i = 1, 2, \ldots, p$ and a map $\bar{\xi} : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$, the elements of the set $\{\xi_i(\cdot) : i = 1, 2, \ldots, p\}$ are said to *synchronize to* $\bar{\xi}(\cdot)$ if $|\xi_i(t) - \bar{\xi}(t)| \to 0$ as $t \to \infty$ for all i. The elements of the set $\{\xi_i(\cdot) : i = 1, 2, \ldots, p\}$ are said to synchronize if they synchronize to some $\bar{\xi}(\cdot)$.

Let $\mathcal{P}$ denote the set of all pairs (C, A) where matrix C and square matrix A, both real, have the same number of columns. We define the following subsets of $\mathcal{P}$.

- $\mathcal{A}_{\rm H}$: set of all pairs (C, A) with A Hurwitz.
- $\mathcal{A}_{\rm N}$: set of all pairs (C, A) with A neutrally stable.
- $\mathcal{A}_{\rm J}$: set of all pairs (C, A) with A having no eigenvalue with positive real part.
- $\mathcal{O}_{\rm F}$: set of all pairs (C, A) with C full column rank.
- $\mathcal{O}_{\rm P}$: set of all detectable pairs.

Few remarks are in order regarding the above definitions. Note that $\mathcal{A}_{\rm H} \subset \mathcal{A}_{\rm N} \subset \mathcal{A}_{\rm J}$ and $\mathcal{O}_{\rm F} \subset \mathcal{O}_{\rm P}$. Set $\mathcal{A}_{\rm J}$ allows A matrices with *Jordan* blocks of arbitrary size with eigenvalues on the imaginary axis. (Hence the subscript J.) Therefore there are pairs (C, A) in $\mathcal{A}_{\rm J}$ with critically unstable A matrices. For a pair $(C, A) \in \mathcal{O}_{\rm F}$ if C is an output matrix of some system of order n then there exists matrix L such that $LC = I_n$, i.e. *full* state information is instantly available at the output. (Hence the subscript F.) For an arbitrary $(C, A) \in \mathcal{O}_{\rm P}$, however, the state information is only *partially* available at the output. (Hence the subscript P.) We also need the following definitions.

- $\mathcal{G}_{\geq\delta}$: set of all connected interconnections with $|{\rm Re}(\lambda_2(\Gamma))| \geq \delta > 0$.
- $\mathcal{G}_{>0}$: set of all connected interconnections.
- $\mathcal{G}_{\geq 0}$: set of all interconnections.

Note that $\mathcal{G}_{\geq\delta} \subset \mathcal{G}_{>0} \subset \mathcal{G}_{\geq 0}$.

3 Problem statement and main theorem

For a given interconnection $\Gamma \in \mathbb{R}^{p\times p}$, let an array of p linear systems be

$$\dot{x}_i = Ax_i + u_i \quad (1a)$$
$$y_i = Cx_i \quad (1b)$$
$$z_i = \sum_{j\neq i} \gamma_{ij}(y_j - y_i) \quad (1c)$$

where $x_i \in \mathbb{R}^n$ is the *state*, $u_i \in \mathbb{R}^n$ is the *input*, $y_i \in \mathbb{R}^m$ is the *output*, and $z_i \in \mathbb{R}^m$ is the *coupling* of the ith system for $i = 1, 2, \ldots, p$. Matrices A and C are of proper dimensions. The solution of ith system at time $t \geq 0$ is denoted by $x_i(t)$.

Definition 1 *Given $A \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{m\times n}$, and set of interconnections $\mathcal{S}$; pair (C, A) is said to be* synchronizable with respect to $\mathcal{S}$ *if there exists a linear feedback law $L \in \mathbb{R}^{n\times m}$ such that for each $\Gamma \in \mathcal{S}$ solutions $x_i(\cdot)$ of array (1) with $u_i = Lz_i$ synchronize for all initial conditions.*

When is pair (C, A) synchronizable with respect to interconnection set $\mathcal{G}_{>0}$?

The above question is what we are mainly concerned with in this paper. Our objective is to find sufficient conditions on (C, A) for synchronizability with respect to set of all connected interconnections and examine their slackness. To this end, we adopt a constructive approach, i.e. we compute a synchronizing feedback law L whenever possible. We also provide the explicit trajectory that the solutions synchronize to (if they do synchronize.) By adding our findings to already known results on synchronization of coupled linear systems we aim to reach a clearer picture of the problem with fewer missing pieces, which is recapitulated in the below theorem.

Theorem 1 *Let us be given $\delta > 0$. We have the following.*

(a) All $(C, A) \in \mathcal{A}_{\rm H}$ are synchronizable with respect to $\mathcal{G}_{\geq 0}$.

(b) All $(C, A) \in \mathcal{A}_{\rm N} \cap \mathcal{O}_{\rm P}$ are synchronizable with respect to $\mathcal{G}_{>0}$.

(c) All $(C, A) \in \mathcal{A}_{\rm J} \cap \mathcal{O}_{\rm F}$ are synchronizable with respect to $\mathcal{G}_{>0}$.

(d) All $(C, A) \in \mathcal{O}_{\rm P}$ are synchronizable with respect to $\mathcal{G}_{\geq\delta}$.

(e) Not all $(C, A) \in \mathcal{A}_{\rm N} \cap \mathcal{O}_{\rm F}$ are synchronizable with respect to $\mathcal{G}_{\geq 0}$.

(f) Not all $(C, A) \in \mathcal{A}_{\rm J} \cap \mathcal{O}_{\rm P}$ are synchronizable with respect to $\mathcal{G}_{>0}$.

(g) Not all $(C, A) \in \mathcal{O}_{\rm F}$ are synchronizable with respect to $\mathcal{G}_{>0}$.

(h) Not all $(C, A) \in \mathcal{P}$ are synchronizable with respect to $\mathcal{G}_{\geq\delta}$.

As mentioned earlier, statement (d) has been known for a while. General results pertaining to that case go as far back as [33]. Among the remaining statements, (a), (e), (g), and (h) are straightforward if not obvious. Still, for the sake of completeness, we establish each of the eight statements of Theorem 1 by a series of lemmas.

4 Cases when synchronizing feedback exists

This section is dedicated to prove the existence statements of Theorem 1, which are analysis results. However, what is in here is more than mere showing the existence of some synchronizing feedback law L under different assumptions. For each case we actually propose/compute an L. We also explicitly provide the trajectory that systems will synchronize to under the proposed feedback law. Therefore what will have been solved at the end is a synthesis problem.

4.1 System matrix Hurwitz

Let us be given some interconnection Γ. Consider (1) under the feedback connection $u_i = Lz_i$ for some L. When system matrices A are Hurwitz, the trivial choice $L = 0$ decouples the systems and $x_i(t) \to 0$ as $t \to \infty$ for all initial conditions. Therefore synchronization accrues. We formalize this trivial observation to the following result and, consequently, establish Theorem 1(a).

Lemma 1 *Given $(C,\, A) \in \mathcal{A}_{\rm H}$, let $L := 0$. Then for all $\Gamma \in \mathcal{G}_{\geq 0}$ solutions $x_i(\cdot)$ of systems* (1) *with $u_i = Lz_i$ synchronize to $\bar{x}(t) \equiv 0$.*

4.2 System matrix neutrally stable

The second statement of Theorem 1 is concerned with neutrally stable system matrices. Here we choose to first study a subcase (Proposition 1) where the system matrices are skew-symmetric and then generalize what we obtain. For some interconnection $\Gamma = [\gamma_{ij}]$ consider the following coupled systems

$$\dot{\xi}_i = S\xi_i + H^T H \sum_{j=1}^{p} \gamma_{ij}(\xi_j - \xi_i)\,, \quad i = 1,\, 2,\, \ldots,\, p \qquad (2)$$

where $\xi_i \in \mathbb{R}^n$ is the state of the ith system, $S \in \mathbb{R}^{n\times n}$, and $H \in \mathbb{R}^{m\times n}$. We make the following assumptions on systems (2) which will henceforth hold.
(A1) S is skew-symmetric.
(A2) $(H,\, S)$ is observable.
(A3) Γ is connected.

Proposition 1 *Consider systems* (2)*. Solutions $\xi_i(\cdot)$ synchronize to*

$$\bar{\xi}(t) := (r^T \otimes e^{St}) \begin{bmatrix} \xi_1(0) \\ \vdots \\ \xi_p(0) \end{bmatrix}$$

where $r \in \mathbb{R}^p$ is such that $r^T\Gamma = 0$ and $r^T\mathbf{1} = 1$.

Proof. Consider matrix $\Gamma - \mathbf{1}r^T$. Observe that $(\Gamma - \mathbf{1}r^T)^k = \Gamma^k + (-1)^k\mathbf{1}r^T$ for $k \in \mathbb{N}$. For $t \in \mathbb{R}$ therefore we can write

$$\begin{aligned} e^{(\Gamma-1r^T)t} &= I_p + t(\Gamma - 1r^T) + \frac{t^2}{2}(\Gamma - 1r^T)^2 + \ldots \\ &= \left(I_p + t\Gamma + \frac{t^2}{2}\Gamma^2 + \ldots\right) - \left(t\mathbf{1}r^T - \frac{t^2}{2}\mathbf{1}r^T + \ldots\right) \\ &= e^{\Gamma t} - (1 - e^{-t})\mathbf{1}r^T \,. \end{aligned}$$

Consequently $\lim_{t\to\infty} e^{(\Gamma-\mathbf{1}r^T)t} = 0$. We deduce therefore that $\Gamma - \mathbf{1}r^T$ is Hurwitz. Since $\Gamma - \mathbf{1}r^T$ is Hurwitz, there exist symmetric positive definite matrices P, $Q \in \mathbb{R}^{p\times p}$ such that

$$-Q = (\Gamma - \mathbf{1}r^T)^T P + P(\Gamma - \mathbf{1}r^T)\,. \tag{3}$$

Define positive semidefinite matrices $\widehat{P} := (I_p - \mathbf{1}r^T)^T P(I_p - \mathbf{1}r^T)$ and $\widehat{Q} := (I_p-\mathbf{1}r^T)^T Q(I_p-\mathbf{1}r^T)$. Now pre- and post-multiply equation (3) by $(I_p-\mathbf{1}r^T)^T$ and $(I_p - \mathbf{1}r^T)$, respectively. We obtain

$$\begin{aligned} -\widehat{Q} &= (I_p - \mathbf{1}r^T)^T(\Gamma - \mathbf{1}r^T)^T P(I_p - \mathbf{1}r^T) \\ &\quad +(I_p - \mathbf{1}r^T)^T P(\Gamma - \mathbf{1}r^T)(I_p - \mathbf{1}r^T) \\ &= \Gamma^T P(I_p - \mathbf{1}r^T) + (I_p - \mathbf{1}r^T)^T P\Gamma \\ &= \Gamma^T(I_p - \mathbf{1}r^T)^T P(I_p - \mathbf{1}r^T) + (I_p - \mathbf{1}r^T)^T P(I_p - \mathbf{1}r^T)\Gamma \\ &= \Gamma^T\widehat{P} + \widehat{P}\Gamma\,. \end{aligned}$$

We now stack the individual system states to obtain $\mathbf{x} := [\xi_1^T\ \xi_2^T\ \cdots\ \xi_p^T]^T$. We can then cast (2) into

$$\dot{\mathbf{x}} = (I_p \otimes S + \Gamma \otimes H^T H)\mathbf{x}\,. \tag{4}$$

Define $V : \mathbb{R}^{pn} \to \mathbb{R}_{\geq 0}$ as $V(\mathbf{x}) := \mathbf{x}^T(\widehat{P} \otimes I_n)\mathbf{x}$. Differentiating $V(\mathbf{x}(t))$ with respect to time we obtain

$$\begin{aligned} \dot{V}(\mathbf{x}) &= \mathbf{x}^T(I_p \otimes S^T + \Gamma^T \otimes H^T H)(\widehat{P} \otimes I_n)\mathbf{x} \\ &\quad +\mathbf{x}^T(\widehat{P} \otimes I_n)(I_p \otimes S + \Gamma \otimes H^T H)\mathbf{x} \\ &= \mathbf{x}^T(\widehat{P} \otimes (S^T + S) + (\Gamma^T\widehat{P} + \widehat{P}\Gamma) \otimes H^T H)\mathbf{x} \\ &= -\mathbf{x}^T(\widehat{Q} \otimes H^T H)\mathbf{x}\,. \end{aligned} \tag{5}$$

Thence $\dot{V}(\mathbf{x}) \leq 0$ for both $\widehat{Q}$ and $H^T H$ (and consequently their Kronecker product) are positive semidefinite.

Given some $\zeta \in \mathbb{R}^{pn}$, let $\mathcal{X} \subset \mathbb{R}^{pn}$ be the closure of the set of all points η such that $\eta = (\mathbf{1}r^T \otimes e^{St})\zeta$ for some $t \geq 0$. Set $\mathcal{X}$ is compact for it is closed by definition and bounded due to that ζ is fixed and S is a neutrally-stable matrix. Having defined $\mathcal{X}$, we now define

$$\Omega := \{\eta \in \mathbb{R}^{pn} : (\mathbf{1}r^T \otimes I_n)\eta \in \mathcal{X}\,, V(\eta) \leq V(\zeta)\}\,.$$

Let us show that Ω is forward invariant. Observe that

$$\begin{aligned}\frac{d}{dt}\big((\mathbf{1}r^T \otimes I_n)\mathbf{x}(t)\big) &= (\mathbf{1}r^T \otimes I_n)(I_p \otimes S + \Gamma \otimes H^T H)\mathbf{x}(t)\\ &= (\mathbf{1}r^T \otimes S + \mathbf{1}r^T\Gamma \otimes H^T H)\mathbf{x}(t)\\ &= (\mathbf{1}r^T \otimes S)\mathbf{x}(t)\\ &= (I_p \otimes S)(\mathbf{1}r^T \otimes I_n)\mathbf{x}(t)\,.\end{aligned}$$

We therefore have

$$(\mathbf{1}r^T \otimes I_n)\mathbf{x}(t) = (\mathbf{1}r^T \otimes e^{St})\mathbf{x}(0) \tag{6}$$

which in turn implies that if $(\mathbf{1}r^T \otimes I_n)\mathbf{x}(0) \in \mathcal{X}$ then $(\mathbf{1}r^T \otimes I_n)\mathbf{x}(t) \in \mathcal{X}$ for all $t \geq 0$. Likewise, if $V(\mathbf{x}(0)) \leq V(\zeta)$ then $V(\mathbf{x}(t)) \leq V(\zeta)$ for all $t \geq 0$ thanks to (5). As a result, if $\mathbf{x}(0) \in \Omega$ then $\mathbf{x}(t) \in \Omega$ for all $t \geq 0$, that is, Ω is forward invariant with respect to (4).

Set Ω is closed by construction. To show that it is compact therefore all we need to do is to establish its boundedness. Let

$$a := \sup_{V(\eta)\leq V(\zeta)} |\eta - (\mathbf{1}r^T \otimes I_n)\eta|\,.$$

If we go back to the definition of V we immediately see that $a < \infty$. Now let

$$b := \sup_{\omega\in\mathcal{X}} |\omega|\,.$$

Since $\mathcal{X}$ is bounded, $b < \infty$ as well. Now, given any $\eta \in \Omega$ we have $|\eta - (\mathbf{1}r^T \otimes I_n)\eta| \leq a$. Hence we can write

$$\begin{aligned}|\eta| &\leq a + |(\mathbf{1}r^T \otimes I_n)\eta|\\ &\leq a + \sup_{\omega\in\mathcal{X}} |\omega|\\ &= a + b\,.\end{aligned}$$

Therefore Ω is bounded. Having shown that Ω is forward invariant and compact, we can now invoke LaSalle's invariance principle [14, Thm. 3.4] and claim that any solution starting in Ω approaches to the largest invariant set $\mathcal{W} \subset \{\eta \in \Omega : \dot{V}(\eta) = 0\}$.

Let now $\eta(\cdot)$ be a solution of (4) such that $\eta(t) \in \mathcal{W}$ for all $t \geq 0$. Given some $\tau \geq 0$, since $\dot{V}(\eta(\tau)) = 0$, we can write

$$\begin{aligned}0 &= \eta(\tau)^T(\widehat{Q} \otimes H^T H)\eta(\tau)\\ &= \eta(\tau)^T((I_p - \mathbf{1}r^T)^T Q(I_p - \mathbf{1}r^T) \otimes H^T H)\eta(\tau)\end{aligned}$$

which implies, since Q is positive definite, that either $((I_p - \mathbf{1}r^T) \otimes I_n)\eta(\tau) = 0$ or $(I_p \otimes H)\eta(\tau) = 0$. Suppose now that

$$((I_p - \mathbf{1}r^T) \otimes I_n)\eta(\tau) \neq 0\,. \tag{7}$$

Continuity of $\eta(\cdot)$ implies that there exists $\delta > 0$ such that $((I_p - \mathbf{1}r^T)\otimes I_n)\eta(t) \neq 0$ for $t \in [\tau,\, \tau+\delta]$. Therefore we must have $(I_p \otimes H)\eta(t) = 0$ for $t \in [\tau,\, \tau+\delta]$. However, observability of pair $(H,\, S)$ stipulates that $\eta(t) = 0$ for $t \in [\tau,\, \tau+\delta]$ which contradicts (7). We then deduce $((I_p - \mathbf{1}r^T)\otimes I_n)\eta(t) = 0$ for all $t \geq 0$. Therefore $\mathcal{W} \subset \{\omega \in \Omega : \omega = (\mathbf{1}r^T \otimes I_n)\omega\} = \mathcal{X}$.

Let us now be given any solution $\mathbf{x}(\cdot)$ of (4). Since ζ that we used to construct Ω was arbitrary, without loss of generality, we can take $\mathbf{x}(0) = \zeta$. That $\mathbf{x}(0) \in \Omega$ implies that $\mathbf{x}(t)$ approaches $\mathcal{X}$ as $t \to \infty$. Therefore we are allowed to write

$$\begin{aligned} 0 &= \lim_{t\to\infty} \big(\mathbf{x}(t) - (\mathbf{1}r^T \otimes I_n)\mathbf{x}(t)\big) \\ &= \lim_{t\to\infty} \big(\mathbf{x}(t) - (\mathbf{1}r^T \otimes e^{St})\mathbf{x}(0)\big) \end{aligned}$$

where we used (6). ■

The following result, which establishes synchronization of coupled harmonic oscillators, comes as a byproduct of Proposition 1.

Corollary 1 *Consider p coupled harmonic oscillators (in $\mathbb{R}^2$) described by*

$$\begin{aligned} \dot{x}_i &= y_i \\ \dot{y}_i &= -x_i + \sum_{j=1}^{p} \gamma_{ij}(y_j - y_i)\,. \end{aligned}$$

Oscillators synchronize for all connected interconnections $[\gamma_{ij}]$.

We will use the next fact in Algorithm 1 to construct a feedback law for synchronization.

Fact 1 *Let $F \in \mathbb{R}^{n\times n}$ be a neutrally-stable matrix with all its eigenvalues residing on the imaginary axis. Then*

$$P := \lim_{t\to\infty}\; t^{-1}\int_0^t e^{F^T\tau}e^{F\tau}d\tau \qquad (8)$$

is well-defined and symmetric positive definite. It also satisfies $PF + F^TP = 0$.

Proof. Matrix F is similar to a skew-symmetric matrix. Therefore e^{Ft} is (almost) periodic [27]. Periodicity directly yields that limit in (8) exists, that is, P is well-defined. Similarity to a skew-symmetric matrix also brings that $\inf_{t\in\mathbb{R}} |e^{Ft}| > 0$ and $\sup_{t\in\mathbb{R}} |e^{Ft}| < \infty$. Same goes for F^T. Therefore there exist scalars $a,\, b > 0$ such that $aI_n \leq e^{F^Tt}e^{Ft} \leq bI_n$ for all $t \in \mathbb{R}$. We can then write

$$aI_n \leq t^{-1}\int_0^t e^{F^T\tau}e^{F\tau}d\tau \leq bI_n$$

for all $t \geq 0$. Therefore P is positive definite. Symmetricity of P comes by construction. Finally, observe that

$$\begin{aligned}
|PF + F^T P| &= \lim_{t\to\infty} t^{-1} \left| \int_0^t \left(e^{F^T\tau} e^{F\tau} F + F^T e^{F^T\tau} e^{F\tau} \right) d\tau \right| \\
&= \lim_{t\to\infty} t^{-1} \left| \int_0^t d\left(e^{F^T\tau} e^{F\tau} \right) \right| \\
&\leq \lim_{t\to\infty} t^{-1} \left(\left| e^{F^T t} e^{Ft} \right| + \left| e^{F^T 0} e^{F0} \right| \right) \\
&\leq \lim_{t\to\infty} t^{-1}(b+1) \\
&= 0
\end{aligned}$$

whence the result follows. ■

Algorithm 1 *Given $A \in \mathbb{R}^{n\times n}$ that is neutrally stable and $C \in \mathbb{R}^{m\times n}$, we obtain $L \in \mathbb{R}^{n\times m}$ as follows. Let $n_1 \leq n$ be the number of eigenvalues of A that reside on the imaginary axis. Let $n_2 := n - n_1$. If $n_1 = 0$, then let $L := 0$; else construct L according to the following steps.*
Step 1: *Choose $U \in \mathbb{R}^{n\times n_1}$ and $W \in \mathbb{R}^{n\times n_2}$ satisfying*

$$[U\ W]^{-1} A [U\ W] = \begin{bmatrix} F & 0 \\ 0 & G \end{bmatrix}$$

where all the eigenvalues of $F \in \mathbb{R}^{n_1\times n_1}$ have zero real parts.
Step 2: *Obtain $P \in \mathbb{R}^{n_1\times n_1}$ from F by* (8).
Step 3: *Finally let $L := UP^{-1}(CU)^T$.*

Below result establishes Theorem 1(b).

Lemma 2 *Given $(C, A) \in \mathcal{A}_{\rm N} \cap \mathcal{O}_{\rm P}$, let L be constructed according to Algorithm 1. Then for all $\Gamma \in \mathcal{G}_{>0}$ solutions $x_i(\cdot)$ of systems* (1) *with $u_i = Lz_i$ synchronize to*

$$\bar{x}(t) := (r^T \otimes e^{At}) \begin{bmatrix} x_1(0) \\ \vdots \\ x_p(0) \end{bmatrix}$$

where $r \in \mathbb{R}^p$ is such that $r^T \Gamma = 0$ and $r^T \mathbf{1} = 1$.

Proof. Let the variables that are not introduced here be defined as in Algorithm 1. Let $H := CUP^{-1/2}$ and $S := P^{1/2} F P^{-1/2}$. Then (H, S) is observable for (C, A) is detectable. Also, note that S is skew-symmetric due to $PF + F^T P = 0$.

We let $U^\dagger \in \mathbb{R}^{n_1\times n}$ and $W^\dagger \in \mathbb{R}^{n_2\times n}$ be such that

$$\begin{bmatrix} U^\dagger \\ W^\dagger \end{bmatrix} = [U\ W]^{-1}.$$

Note then that $U^\dagger U = I_{n_1}$, $W^\dagger W = I_{n_2}$, $U^\dagger W = 0$, and $W^\dagger U = 0$. Since $u_i = Lz_i$, from (1) we obtain

$$\dot{x}_i = Ax_i + LC\sum_{j=1}^{p}\gamma_{ij}(x_j - x_i) \tag{9}$$

Let now $\xi_i \in \mathbb{R}^{n_1}$ and $\eta_i \in \mathbb{R}^{n_2}$ be

$$\begin{bmatrix}\xi_i\\ \eta_i\end{bmatrix} := \begin{bmatrix}P^{1/2} & 0\\ 0 & I_{n_2}\end{bmatrix}\begin{bmatrix}U^\dagger\\ W^\dagger\end{bmatrix} x_i \tag{10}$$

Combining (9) and (10) we can write

$$\dot{\xi}_i = S\xi_i + H^T H\sum_{j=1}^{p}\gamma_{ij}(\xi_j - \xi_i) + H^T CW\sum_{j=1}^{p}\gamma_{ij}(\eta_j - \eta_i) \tag{11}$$

$$\dot{\eta}_i = G\eta_i\,. \tag{12}$$

Let $\Gamma \in \mathbb{R}^{p\times p}$ be connected and $r \in \mathbb{R}^p$ be such that $r^T\Gamma = 0$. Then define $\omega_i : \mathbb{R}_{\geq 0} \to \mathbb{R}^{n_1}$ as $\omega_i(t) := e^{-St}\xi_i(t)$ for $i = 1, 2, \ldots, p$. Let $\mathbf{w} := [\omega_1^T\ \omega_2^T\ \ldots\ \omega_p^T]^T$ and $\mathbf{v} := [\eta_1^T\ \eta_2^T\ \ldots\ \eta_p^T]^T$. Starting from (11) and (12) we can write

$$\dot{\mathbf{w}}(t) = (\Gamma \otimes e^{-St}H^T He^{St})\mathbf{w}(t) + (\Gamma \otimes e^{-St}H^T CWe^{Gt})\mathbf{v}(0)\,.$$

Thence

$$\mathbf{w}(t) = \Phi(t,\,0)\mathbf{w}(0) + \left[\int_0^t \Phi(t,\,\tau)(\Gamma \otimes e^{-S\tau}H^T CWe^{G\tau})d\tau\right]\mathbf{v}(0) \tag{13}$$

where

$$\Phi(t,\,\tau) := \exp\left(\int_\tau^t \left(\Gamma \otimes e^{-S\alpha}H^T He^{S\alpha}\right) d\alpha\right)$$

is the state transition matrix [3]. From Proposition 1 we can deduce that $\Phi(t,\,\tau)$ is uniformly bounded for all t and τ. Also, for any fixed τ we have $\lim_{t\to\infty}\Phi(t,\,\tau) = \mathbf{1}r^T \otimes I_{n_1}$. Moreover, e^{St} is uniformly bounded for all t, and e^{Gt} decays exponentially as $t \to \infty$ for G is Hurwitz. Therefore we can write

$$\begin{aligned}\lim_{t\to\infty}\int_0^t &\Phi(t,\,\tau)(\Gamma \otimes e^{-S\tau}H^T CWe^{G\tau})d\tau\\ &= \int_0^\infty \left(\lim_{t\to\infty}\Phi(t,\,\tau)\right)(\Gamma \otimes e^{-S\tau}H^T CWe^{G\tau})d\tau\\ &= \int_0^\infty (\mathbf{1}r^T \otimes I_{n_1})(\Gamma \otimes e^{-S\tau}H^T CWe^{G\tau})d\tau\\ &= 0\,.\end{aligned}$$

Then, by (13), we can write

$$\lim_{t\to\infty}\mathbf{w}(t) = (\mathbf{1}r^T \otimes I_{n_1})\mathbf{w}(0)\,.$$

Therefore solutions $\xi_i(\cdot)$ synchronize to $(r^T \otimes e^{St})\mathbf{w}(0)$. Moreover, $\lim_{t\to\infty} \mathbf{v}(t) = 0$ for G is Hurwitz. Hence we can say that solutions $\eta_i(\cdot)$ synchronize to $(r^T \otimes e^{Gt})\mathbf{v}(0)$. As a result, solutions $x_i(\cdot)$ synchronize to

$$\left(r^T \otimes \begin{bmatrix} UP^{-1/2} & W \end{bmatrix} \begin{bmatrix} e^{St} & 0 \\ 0 & e^{Gt} \end{bmatrix} \begin{bmatrix} P^{1/2}U^\dagger \\ W^\dagger \end{bmatrix}\right) \begin{bmatrix} x_1(0) \\ \vdots \\ x_p(0) \end{bmatrix}$$
$$= (r^T \otimes e^{At}) \begin{bmatrix} x_1(0) \\ \vdots \\ x_p(0) \end{bmatrix}$$

Hence the result. ■

4.3 System matrix critically unstable

The third statement of Theorem 1 is not surprising. The naive reasoning, which we will later have no difficulty in formalizing, is that when the output matrix C is full column rank, there exists an L such that LC equals the identity matrix. Then, see (9), there will be two *forces* driving the array. One is due to system matrices A, the other due to coupling between the systems. When A matrices have no eigenvalues with positive real part, even if the systems tend to move away from each other, that tendency will not be of exponential nature. However, if Γ is connected, that apart-drifting behaviour will be dominated by the exponential attraction introduced by the coupling between the systems. The result will be synchronization of the solutions. Having said that, now we can state our main reason for including this existence statement in Theorem 1: to emphasize the sixth statement of Theorem 1, which is a nonexistence result. When availability of full-state information is replaced with detectability, which in all the other cases we study causes no problem, synchronizability is lost for critically unstable systems. That is, Theorem 1(f) holds *despite* Theorem 1(c), which we find counterintuitive. Below we establish Theorem 1(c).

Lemma 3 *Given* $(C, A) \in \mathcal{A}_{\rm J} \cap \mathcal{O}_{\rm F}$*, let* $L := (C^TC)^{-1}C^T$*. Then for all* $\Gamma \in \mathcal{G}_{>0}$ *solutions* $x_i(\cdot)$ *of systems* (1) *with* $u_i = Lz_i$ *synchronize to*

$$\bar{x}(t) := (r^T \otimes e^{At}) \begin{bmatrix} x_1(0) \\ \vdots \\ x_p(0) \end{bmatrix}$$

where $r \in \mathbb{R}^p$ *is such that* $r^T\Gamma = 0$ *and* $r^T\mathbf{1} = 1$*.*

Proof. For $u_i = Lz_i$ and some connected $\Gamma \in \mathbb{R}^{p\times p}$, with $r \in \mathbb{R}^p$ such that $r^T\Gamma = 0$ and $r^T\mathbf{1} = 1$, from (1) one can write

$$\begin{aligned}\dot{x}_i &= Ax_i + LC\sum_{j=1}^{p}\gamma_{ij}(x_j - x_i)\\ &= Ax_i + \sum_{j=1}^{p}\gamma_{ij}(x_j - x_i)\end{aligned}$$

since $LC = I_n$. Letting $\mathbf{x} := [x_1^T\ x_2^T\ \ldots\ x_p^T]^T$ we obtain

$$\dot{\mathbf{x}} = (I_p \otimes A + \Gamma \otimes I_n)\mathbf{x}$$

which implies

$$\mathbf{x}(t) = (e^{\Gamma t} \otimes e^{At})\mathbf{x}(0)\,.$$

Let us define

$$\begin{aligned}\tilde{\mathbf{x}}(t) &:= \mathbf{x}(t) - (\mathbf{1}r^T \otimes e^{At})\mathbf{x}(0)\\ &= (e^{\Gamma t} \otimes e^{At})\mathbf{x}(0) - (\mathbf{1}r^T \otimes e^{At})\mathbf{x}(0)\\ &= ((e^{\Gamma t} - \mathbf{1}r^T) \otimes e^{At})\mathbf{x}(0)\,. \qquad (14)\end{aligned}$$

From the proof of Theorem 2 we know that $e^{\Gamma t} - \mathbf{1}r^T = e^{(\Gamma - \mathbf{1}r^T)t} + e^{-t}\mathbf{1}r^T$ and that $\Gamma - \mathbf{1}r^T$ is Hurwitz. Also recall that A does not have any eigenvalues with positive real part. Therefore there exist M_g, M_a, $\lambda > 0$, and integer $k \leq n-1$ such that $|e^{(\Gamma-\mathbf{1}r^T)t}| \leq M_g e^{-\lambda t}$ and $|e^{At}| \leq M_a t^k$. Then we can proceed from (14) as

$$\begin{aligned}\tilde{\mathbf{x}}(t) &= ((e^{\Gamma t} - \mathbf{1}r^T) \otimes e^{At})\mathbf{x}(0)\\ &= (e^{(\Gamma-\mathbf{1}r^T)t} \otimes e^{At})\mathbf{x}(0) + e^{-t}(\mathbf{1}r^T \otimes e^{At})\mathbf{x}(0)\end{aligned}$$

whence

$$\begin{aligned}|\tilde{\mathbf{x}}(t)| &\leq |e^{(\Gamma-\mathbf{1}r^T)t}| \cdot |e^{At}| \cdot |\mathbf{x}(0)| + e^{-t}|\mathbf{1}r^T| \cdot |e^{At}| \cdot |\mathbf{x}(0)|\\ &\leq M_g e^{-\lambda t} M_a t^k |\mathbf{x}(0)| + e^{-t} M_a t^k |\mathbf{1}r^T| \cdot |\mathbf{x}(0)|\end{aligned}$$

whence $\lim_{t\to\infty} |\tilde{\mathbf{x}}(t)| = 0$. Hence the result. ∎

4.4 System matrix arbitrary

The fourth statement of Theorem 1 is a special case of a well-studied problem [33, 18, 5, 32, 31] which is based on the idea that *identical systems synchronize under strong enough coupling.* For coherence we regenerate a proof here. We first borrow a well-known result from optimal control theory [21]: Given a detectable pair (C, A), where $C \in \mathbb{R}^{m\times n}$ and $A \in \mathbb{R}^{n\times n}$, the following algebraic Riccati equation

$$AP + PA^T + I_n - PC^TCP = 0 \qquad (15)$$

has a (unique) solution $P = P^T > 0$. One can rewrite (15) as

$$(A - PC^TC)P + P(A - PC^TC)^T + (I_n + PC^TCP) = 0$$

whence we infer that $A - PC^TC$ is Hurwitz.

Claim 1 *Let $C \in \mathbb{R}^{m\times n}$ and $A \in \mathbb{R}^{n\times n}$ satisfy (15) for some symmetric positive definite P. Then for all $\sigma \geq 1$ and $\omega \in \mathbb{R}$ matrix $A-(\sigma+j\omega)PC^TC$ is Hurwitz.*

Proof. Let $\varepsilon := \sigma - 1 \geq 0$. Write

$$\begin{aligned}
&(A - (\sigma + j\omega)PC^TC)P + P(A - (\sigma + j\omega)PC^TC)^H \\
&\quad = (A - (\sigma + j\omega)PC^TC)P + P(A - (\sigma - j\omega)PC^TC)^T \\
&\quad = (A - (1 + \varepsilon)PC^TC)P + P(A - (1 + \varepsilon)PC^TC)^T \\
&\quad = (A - PC^TC)P + P(A - PC^TC)^T - 2\varepsilon PC^TCP \\
&\quad = -I_n - (1 + 2\varepsilon)PC^TCP\,. \qquad (16)
\end{aligned}$$

Observe that (16) is nothing but (complex) Lyapunov equation. ■

Below result establishes Theorem 1(d).

Lemma 4 *Given $\delta > 0$ and $(C, A) \in \mathcal{O}_{\mathrm{P}}$, let $L := \max\{1, \delta^{-1}\}PC^T$ where P is the solution to (15). Then for all $\Gamma \in \mathcal{G}_{\geq\delta}$ solutions $x_i(\cdot)$ of systems (1) with $u_i = Lz_i$ synchronize to*

$$\bar{x}(t) := (r^T \otimes e^{At}) \begin{bmatrix} x_1(0) \\ \vdots \\ x_p(0) \end{bmatrix}$$

where $r \in \mathbb{R}^p$ is such that $r^T\Gamma = 0$ and $r^T\mathbf{1} = 1$.

Proof. Given $\Gamma \in \mathcal{G}_{\geq\delta}$, from (1) we can write

$$\dot{x}_i = Ax_i + LC\sum_{j=1}^{p} \gamma_{ij}(x_j - x_i)\,. \qquad (17)$$

Stack individual system states as $\mathbf{x} := [x_1^T\ x_2^T\ \ldots\ x_p^T]^T$. Then from (17) we can write

$$\dot{\mathbf{x}} = (I_p \otimes A + \Gamma \otimes LC)\mathbf{x}\,. \qquad (18)$$

Now let $Y \in \mathbb{C}^{p\times(p-1)}$, $W \in \mathbb{C}^{(p-1)\times p}$, $V \in \mathbb{C}^{p\times p}$, and upper triangular $\Delta \in \mathbb{C}^{(p-1)\times(p-1)}$ be such that

$$V = [\mathbf{1}\ Y]\ , \quad V^{-1} = \begin{bmatrix} r^T \\ W \end{bmatrix}$$

and

$$V^{-1}\Gamma V = \begin{bmatrix} 0 & 0 \cdots 0 \\ 0 & \\ \vdots & \Delta \\ 0 & \end{bmatrix}$$

Note that the diagonal entries of Δ are nothing but the nonzero eigenvalues of Γ which we know have real parts no greater than $-\delta$. Engage the change of variables $\mathbf{v} := (V^{-1} \otimes I_n)\mathbf{x}$ and modify (18) first into

$$\dot{\mathbf{v}} = (I_p \otimes A + V^{-1}\Gamma V \otimes LC)\mathbf{v}$$

and then into

$$\dot{\mathbf{v}} = \begin{bmatrix} A & 0_{n\times(p-1)n} \\ 0_{(p-1)n\times n} & I_{p-1} \otimes A + \Delta \otimes LC \end{bmatrix} \mathbf{v} \tag{19}$$

Observe that $I_{p-1} \otimes A + \Delta \otimes LC$ is upper block triangular with (block) diagonal entries of the form $A + \lambda_i \max\{1,\, \delta^{-1}\} PC^T C$ for $i = 2,\, 3,\, \ldots,\, p$ with $\mathrm{Re}(\lambda_i) \leq -\delta$. Claim 1 implies therefore that $I_{p-1} \otimes A + \Delta \otimes LC$ is Hurwitz. Thus (19) implies

$$\lim_{t\to\infty} \left| \mathbf{v}(t) - \begin{bmatrix} e^{At} & 0_{n\times(p-1)n} \\ 0_{(p-1)n\times n} & 0_{(p-1)n\times(p-1)n} \end{bmatrix} \mathbf{v}(0) \right| = 0$$

which yields

$$\lim_{t\to\infty} \left| \mathbf{x}(t) - (\mathbf{1}r^T \otimes e^{At})\mathbf{x}(0) \right| = 0\,.$$

Hence the result. ■

5 Cases when no synchronizing feedback exists

In this section the nonsufficiency statements of the main theorem are demonstrated. Below we establish Theorem 1(e).

Lemma 5 *There exists $(C,\, A) \in \mathcal{A}_{\mathrm{N}} \cap \mathcal{O}_{\mathrm{F}}$ that is not synchronizable with respect to $\mathcal{G}_{\geq 0}$.*

Proof. Suppose not. Then for $C = 1$ and $A = 0$ (note that $(C,\, A) \in \mathcal{A}_{\mathrm{N}} \cap \mathcal{O}_{\mathrm{F}}$) there exists $L \in \mathbb{R}$ such that for all $\Gamma \in \mathcal{G}_{\geq 0}$ solutions of systems (1) with $u_i = Lz_i$ synchronize for all initial conditions. However for $\Gamma := 0_{2\times 2} \in \mathcal{G}_{\geq 0}$ we have $\dot{x}_i = 0$, for $i = 1,\, 2$, regardless of L. Therefore solutions $x_i(\cdot)$ do not synchronize for $x_1(0) \neq x_2(0)$. The result follows by contradiction. ■

Next result establishes Theorem 1(f). It emphasizes the nonsufficiency of partial-state coupling for synchronizability of critically unstable systems with respect to set all connected interconnections. As shown by Lemma 3 this lost synchronizability can be restored under full-state coupling.

Lemma 6 *There exists* $(C,\, A) \in \mathcal{A}_{\rm J} \cap \mathcal{O}_{\rm P}$ *that is not synchronizable with respect to* $\mathcal{G}_{>0}$*.*

Proof. Suppose not. Let

$$C := [1 \;\; 0]\,, \qquad A := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Note that $(C,\, A) \in \mathcal{A}_{\rm J} \cap \mathcal{O}_{\rm P}$. Then there exists $L \in \mathbb{R}^{2\times 1}$

$$L := \begin{bmatrix} \ell \\ \rho \end{bmatrix}$$

such that for all $\Gamma \in \mathcal{G}_{>0}$ solutions of systems (1) with $u_i = Lz_i$ synchronize for all initial conditions. That is equivalent, by [18], to that $A + \lambda LC$ is Hurwitz for all $\lambda \in \{\gamma \in \mathbb{C} : 0 \neq \gamma \text{ is an eigenvalue of some } \Gamma \in \mathcal{G}_{>0}\} =: \mathcal{L}$. Note that

$$\Gamma \in \mathcal{G}_{>0} \implies r\Gamma \in \mathcal{G}_{>0} \quad \forall r > 0$$

whence

$$\lambda \in \mathcal{L} \implies r\lambda \in \mathcal{L} \quad \forall r > 0\,. \tag{20}$$

Observe that any $\Gamma \in \mathcal{G}_{>0} \cap \mathbb{R}^{2\times 2}$ has a negative (real) eigenvalue. By (20) therefore $\mathcal{L}$ includes negative real line. Then a simple calculation shows that both ℓ and ρ need be strictly positive which lets us, thanks to (20), take $\ell = 2$ without loss of generality. Thence $A + \lambda LC$ reduces to the following matrix

$$\begin{bmatrix} 2\lambda & 1 \\ \lambda\rho & 0 \end{bmatrix} \tag{21}$$

which must be Hurwitz for all $\lambda \in \mathcal{L}$. The roots of the characteristic equation of the matrix in (21) is

$$s_{1,2} = \lambda \pm \sqrt{\lambda^2 + \lambda\rho}\,.$$

Then for $\theta \in (\pi/2,\, 3\pi/2)$ we can write

$$\begin{aligned} \mathrm{Re}(s_{1,2})\big|_{\lambda = e^{j\theta}} &= \cos\theta \pm \frac{1}{\sqrt{2}}\sqrt{\cos 2\theta + \rho\cos\theta + \sqrt{(\cos 2\theta + \rho\cos\theta)^2 + (\sin 2\theta + \rho\sin\theta)^2}} \\ &= \cos\theta \pm \frac{1}{\sqrt{2}}\sqrt{\cos 2\theta + \rho\cos\theta + \sqrt{1 + \rho^2 + 2\rho\cos\theta}}\,. \end{aligned}$$

Observe that we can choose $\bar\theta \in (\pi/2,\, \pi]$ and $m > 0$ such that the following inequalities are satisfied

$$\begin{aligned} 1 + \frac{9\rho^2}{m^2} &\leq \sqrt{1 + \rho^2 + 2\rho\cos\theta} \\ -\rho\cos\theta &\leq \frac{\rho^2}{m^2} \\ -\cos\theta &\leq \frac{\rho}{m} \end{aligned}$$

for all $\theta \in (\pi/2, \bar{\theta}]$. Then for all $\theta \in (\pi/2, \bar{\theta}]$ one of the roots, say s_1, satisfies

$$\begin{aligned}
\mathrm{Re}(s_1) &= \cos\theta + \frac{1}{\sqrt{2}}\sqrt{\cos 2\theta + \rho\cos\theta + \sqrt{1+\rho^2+2\rho\cos\theta}} \\
&\geq \cos\theta + \frac{1}{\sqrt{2}}\sqrt{\cos 2\theta - \frac{\rho^2}{m^2} + 1 + \frac{9\rho^2}{m^2}} \\
&= \cos\theta + \frac{1}{\sqrt{2}}\sqrt{2\cos^2\theta + \frac{8\rho^2}{m^2}} \\
&\geq \cos\theta + \frac{2\rho}{m} \\
&\geq \frac{\rho}{m} \\
&> 0
\end{aligned}$$

which implies, since no eigenvalue of matrix (21) can have positive real part, $e^{j\theta} \notin \mathcal{L}$ for $\theta \in (\pi/2, \bar{\theta}]$. Due to (20) therefore the wedge $\mathcal{W} := \{re^{j\theta} : \theta \in (\pi/2, \bar{\theta}],\, r > 0\}$ and $\mathcal{L}$ are disjoint, i.e.

$$\mathcal{W} \cap \mathcal{L} = \emptyset\,. \tag{22}$$

For $p \in \mathbb{N}_{\geq 1}$ let $\Gamma_p \in \mathbb{R}^{p\times p}$ be

$$\Gamma_p := \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & -1 \end{bmatrix}$$

Observe for all p that we have $\Gamma_p \in \mathcal{G}_{>0}$ and that $\lambda_p := e^{j\frac{2\pi}{p}} - 1$ is an eigenvalue of Γ_p with corresponding eigenvector

$$v_p = \begin{bmatrix} (\lambda_p + 1)^1 \\ (\lambda_p + 1)^2 \\ \vdots \\ (\lambda_p + 1)^p \end{bmatrix}$$

Note that for p large enough $\lambda_p \in \mathcal{W}$ which contradicts (22) for $\Gamma_p \in \mathcal{G}_{>0}$. Hence the result. ■

Below we establish Theorem 1(g).

Lemma 7 *There exists $(C,\, A) \in \mathcal{O}_{\mathrm{F}}$ that is not synchronizable with respect to $\mathcal{G}_{>0}$.*

Proof. Suppose not. Then for $C = 1$ and $A = 1$ (note that $(C,\, A) \in \mathcal{O}_{\mathrm{F}}$) there exists $L \in \mathbb{R}$ such that for all $\Gamma \in \mathcal{G}_{>0}$ solutions of systems (1) with $u_i = Lz_i$

synchronize for all initial conditions. Choose $\varepsilon > 0$ such that $1 - \varepsilon L > 0$. Then let

$$\Gamma := \begin{bmatrix} -\varepsilon & \varepsilon \\ 0 & 0 \end{bmatrix}$$

Observe that $\Gamma \in \mathcal{G}_{>0}$. Consider systems (1) under interconnection Γ with $u_i = Lz_i$, for $i = 1, 2$. We can write $\dot{x}_1 - \dot{x}_2 = (1 - \varepsilon L)(x_1 - x_2)$. Therefore solutions $x_i(\cdot)$ do not synchronize unless $x_1(0) = x_2(0)$. Hence the result by contradiction. ■

Below lemma yields Theorem 1(h).

Lemma 8 *For no $\delta > 0$ pair $(0, 0)$ is synchronizable with respect to $\mathcal{G}_{\geq\delta}$.*

6 Dual problem

For a given interconnection $\Gamma \in \mathbb{R}^{p\times p}$, consider the following array of p identical linear systems

$$\dot{x}_i = Ax_i + Bu_i \tag{23a}$$

$$z_i = \sum_{j\neq i} \gamma_{ij}(x_j - x_i) \tag{23b}$$

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$, and $z_i \in \mathbb{R}^n$ for $i = 1, 2, \ldots, p$. Matrices A and B are of proper dimensions. The duality between observability and controllability for linear systems readily yields the following theorem by which the results in this paper can be extended for the synchronization of the arrays of coupled linear systems depicted by (23).

Theorem 2 *Let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and set of interconnections $\mathcal{S}$ be such that (B^T, A^T) is synchronizable with respect to $\mathcal{S}$. Then and only then there exists a linear feedback law $K \in \mathbb{R}^{m\times n}$ such that for each $\Gamma \in \mathcal{S}$ solutions $x_i(\cdot)$ of array (23) with $u_i = Kz_i$ synchronize for all initial conditions.*

7 Conclusion

For arrays of identical output-coupled linear systems we have investigated the sufficiency of certain conditions on system matrix A and output matrix C for existence of a feedback law under which the systems synchronize for all coupling configurations with connected graphs. Namely, we have filled in the previously missing pieces (the boxes indicated with a question mark) of the chart given in Fig. 2. In addition, for each case corresponding to a box with "✓" we have designed a synchronizing feedback law.

Γ \ $(C,\,A)$	$\mathcal{A}_{\mathrm{H}}$	$\mathcal{A}_{\mathrm{N}} \cap \mathcal{O}_{\mathrm{P}}$	$\mathcal{A}_{\mathrm{J}} \cap \mathcal{O}_{\mathrm{F}}$	$\mathcal{A}_{\mathrm{J}} \cap \mathcal{O}_{\mathrm{P}}$	$\mathcal{O}_{\mathrm{P}}$	$\mathcal{P}$
$\mathcal{G}_{\geq 0}$	✓	✗	✗	✗	✗	✗
$\mathcal{G}_{>0}$	✓	✓ (?)	✓ (?)	✗ (?)	✗	✗
$\mathcal{G}_{\geq \delta}$	✓	✓	✓	✓	✓	✗

Figure 2: Sufficiency of certain conditions on $(C,\,A)$ for synchronizability with respect to different sets of interconnections Γ.

References

[1] J. Aldridge and E.K. Pye. Cell density dependence of oscillatory metabolism. *Nature*, 259:670–671, 1976.

[2] D. Angeli and P.-A. Bliman. Stability of leaderless discrete-time multi-agent systems. *Mathematics of Control, Signals & Systems*, 18:293–322, 2006.

[3] P.J. Antsaklis and A.N. Michel. *Linear Systems*. McGraw-Hill, 1997.

[4] M. Arcak. Passivity as a design tool for group coordination. *IEEE Transactions on Automatic Control*, 52:1380–1390, 2007.

[5] I. Belykh, V. Belykh, and M. Hasler. Generalized connection graph method for synchronization in asymmetrical networks. *Physica D*, 224:42–51, 2006.

[6] V.D. Blondel, J.M. Hendrickx, A. Olshevsky, and J.N. Tsitsiklis. Convergence in multiagent coordination, consensus, and flocking. In *Proc. of the 44th IEEE Conference on Decision and Control*, pages 2996–3000, 2005.

[7] S. Boccaletti, V. Latora, Y. Moreno, M. Chavez, and D.-U. Hwang. Complex networks: structure and dynamics. *Physics Reports-Review Section of Physics Letters*, 424:175–308, 2006.

[8] J. Cortes. Distributed algorithms for reaching consensus on general functions. *Automatica*, 44:726–737, 2007.

[9] J. Cortes, S. Martinez, T. Karatas, and F. Bullo. Coverage control for mobile sensing networks. *IEEE Transactions on Robotics and Automation*, 20:243–255, 2004.

[10] L. Fabiny, P. Colet, R. Roy, and D. Lenstra. Coherence and phase dynamics of spatially coupled solid-state lasers. *Physical Review A*, 47:4287–4296, 1993.

[11] J.A. Fax and R.M. Murray. Information flow and cooperative control of vehicle formations. *IEEE Transactions on Automatic Control*, 49:1465–1476, 2004.

[12] A.L. Fradkov. *Cybernetical Physics: From Control of Chaos to Quantum Control (Understanding Complex Systems)*. Springer, 2007.

[13] A. Jadbabaie, J. Lin, and A.S. Morse. Coordination of groups of mobile autonomous agents using nearest neighbor rules. *IEEE Transactions on Automatic Control*, 48:988–1001, 2003.

[14] H.K. Khalil. *Nonlinear Systems*. Prentice Hall, 1996.

[15] D.G. Luenberger. Observing the state of a linear system. *IEEE Transactions on Military Electronics*, pages 74–80, April 1964.

[16] L. Moreau. Stability of multi-agent systems with time-dependent communication links. *IEEE Transactions on Automatic Control*, 50:169–182, 2005.

[17] R. Olfati-Saber, J.A. Fax, and R.M. Murray. Consensus and cooperation in networked multi-agent systems. *Proceedings of the IEEE*, 95:215–233, 2007.

[18] L.M. Pecora and T.L. Carroll. Master stability functions for synchronized coupled systems. *Physical Review Letters*, 80:2109–2112, 1998.

[19] A. Pogromsky and H. Nijmeijer. Cooperative oscillatory behavior of mutually coupled dynamical systems. *IEEE Transactions on Circuits and Systems-I*, 48:152–162, 2001.

[20] W. Ren and R.W. Beard. Consensus seeking in multiagent systems under dynamically changing interaction topologies. *IEEE Transactions on Automatic Control*, 50:655–661, 2005.

[21] E. Sontag. *Mathematical Control Theory: Deterministic Finite Dimensional Systems*. Springer, 1998.

[22] G.-B. Stan and R. Sepulchre. Analysis of interconnected oscillators by dissipativity theory. *IEEE Transactions on Automatic Control*, 52:256–270, 2007.

[23] R. Stoop, K. Schindler, and L.A. Bunimovich. Neocortical networks of pyramidal neurons: from local locking and chaos to macroscopic chaos and synchronization. *Nonlinearity*, 13:1515–1529, 2000.

[24] S.H. Strogatz. Exploring complex networks. *Nature*, 410:268–276, 2001.

[25] J.N. Tsitsiklis, D.P. Bertsekas, and M. Athans. Distributed asynchronous deterministic and stochastic gradient optimization algorithms. *IEEE Transactions on Automatic Control*, 31:803–812, 1986.

[26] S.E. Tuna. Synchronizing linear systems via partial-state coupling. *Automatica*, 2008.

[27] F.S. Van Vleck. A note on the relation between periodic and orthogonal fundamental solutions of linear systems. *The American Mathematical Monthly*, 71:406–408, 1964.

[28] T.J. Walker. Acoustic synchrony: two mechanisms in the snowy tree cricket. *Science*, 166:891–894, 1969.

[29] X.F. Wang. Complex networks: topology, dynamics and synchronization. *International Journal of Bifurcation and Chaos*, 12:885–916, 2002.

[30] C.W. Wu. Algebraic connectivity of directed graphs. *Linear and Multilinear Algebra*, 53:203–223, 2005.

[31] C.W. Wu. Synchronization in arrays of coupled nonlinear systems with delay and nonreciprocal time-varying coupling. *IEEE Transactions on Circuits and Systems-II*, 52:282–286, 2005.

[32] C.W. Wu. Synchronization in networks of nonlinear dynamical systems coupled via a directed graph. *Nonlinearity*, 18:1057–1064, 2005.

[33] C.W. Wu and L.O. Chua. Synchronization in an array of linearly coupled dynamical systems. *IEEE Transactions on Circuits and Systems-I*, 42:430–447, 1995.