

ON ABELIAN GROUP ACTIONS WITH TNI-CENTRALIZERS

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ABSTRACT. A subgroup H of a group G is said to be a TNI-subgroup if $N_G(H) \cap H^g = 1$ for any $g \in G \setminus N_G(H)$. Let A be an abelian group acting coprimely on the finite group G by automorphisms in such a way that $C_G(A) = \{g \in G : g^a = g \text{ for all } a \in A\}$ is a solvable TNI-subgroup of G . We prove that G is a solvable group with Fitting length $h(G)$ is at most $h(C_G(A)) + \ell(A)$. In particular $h(G) \leq \ell(A) + 3$ whenever $C_G(A)$ is nonnormal. Here, $h(G)$ is the Fitting length of G and $\ell(A)$ is the number of primes dividing A counted with multiplicities.

1. INTRODUCTION

Throughout the paper all groups are finite and $h(G)$ denotes the Fitting length of the group G . A subgroup H of a group G is said to be a TNI-subgroup if $N_G(H) \cap H^g = 1$ for any $g \in G \setminus N_G(H)$. In particular, every normal subgroup is a TNI-subgroup. In [1], we studied the consequences of the action of a group A on the group G in case where $C_G(A)$ is a TNI-subgroup, and obtained the following two results:

Theorem A. *Let A be a group that acts coprimely on the group G by automorphisms. If $C_G(A)$ is a solvable TNI-subgroup of G , then G is solvable.*

Theorem B. *Let A be a coprime automorphism of prime order of a finite solvable group G such that $C_G(A)$ is a TNI-subgroup. Then $h(G) \leq h(C_G(A)) + 1$. In particular, $h(G) \leq 4$ when $C_G(A)$ is nonnormal.*

In the present paper we extend Theorem B to the case where A is abelian, namely we prove

Theorem. *Let A be an abelian group acting coprimely on the finite group G in such a way that $C_G(A)$ is a solvable TNI-subgroup of G . Then G is a solvable group with $h(G) \leq h(C_G(A)) + \ell(A)$ where $\ell(A)$ is the number of primes dividing A counted with multiplicities. In particular $h(G) \leq \ell(A) + 3$ whenever $C_G(A)$ is nonnormal.*

This is achieved by applying the same techniques used in [2] in order to prove that $h(G) \leq \ell(A)$ if A acts coprimely and fixed point freely on the group G and for every proper subgroup D and every D -invariant section S of G such that D

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acts irreducibly on S , there is $v \in S$ with $C_D(S) = C_D(v)$, that is, A acts with regular orbit on G . In the light of this result we asked whether the conclusion of Theorem B is true if A is not necessarily abelian, but acts with regular orbits on G . The main difficulty forcing us to study under the assumption that A is abelian arises from the fact that a homomorphic image of a TNI -subgroup need not be a TNI -subgroup.

2. PROOF OF THE THEOREM

The group G is solvable by Theorem A in [1]. It remains to show that $h(G) \leq h(C_G(A)) + \ell(A)$. Suppose false and let G, A be a counterexample with $|G|$ minimum.

Suppose that $C_G(A)$ is normal in G . Then the fixed point free action of A on the group $G/C_G(A)$ yields that $h(G/C_G(A)) \leq \ell(A)$ by the main theorem of [2]. So $h(G) \leq h(C_G(A)) + \ell(A)$, a contradiction. Therefore by Theorem 2.2 in [1] we may assume that

(1) $C_G(A)$ is a nonnormal subgroup of G acting Frobeniusly on a section M/N of G .

We also have

(2) There is an A -tower $\hat{P}_i, i = 1, \dots, t$ where $t = h(G)$ satisfying the following conditions (see [3]):

- (a) \hat{P}_i is an A -invariant p_i -subgroup, p_i is a prime, $p_i \neq p_{i+1}$, for $i = 1, \dots, t-1$;
- (b) $\hat{P}_i \leq N_G(\hat{P}_j)$ whenever $i \leq j$;
- (c) $P_t = \hat{P}_t$ and $P_i = \hat{P}_i/C_{\hat{P}_i}(P_{i+1})$ for $i = 1, \dots, t-1$ and $P_i \neq 1$ for $i = 1, \dots, t$;
- (d) $\Phi(\Phi(P_i)) = 1$, $\Phi(P_i) \leq Z(P_i)$, and $\exp(P_i) = p_i$ when p_i is odd for $i = 1, \dots, t$;
- (e) $[\Phi(P_i), \hat{P}_{i-1}] = 1$ and $[P_i, P_{i-1}] = P_i$ for $i = 1, \dots, t$;
- (f) If $S \leq \hat{P}_i$ for some i , S is normalized by $\hat{P}_{i-1} \dots \hat{P}_1 A$ and its image in P_i is not contained in $\Phi(P_i)$, then $S = \hat{P}_i$.

By Lemma 2.1 in [1] the group $\prod_{i=1}^t \hat{P}_i$ is of Fitting length t and it satisfies the hypothesis of the theorem. It follows now by induction that

$$(3) \quad G = \prod_{i=1}^t \hat{P}_i.$$

Suppose that $C_{\hat{P}_t}(A) \neq 1$. Then we have $M/N = [M, C_{\hat{P}_t}(A)]N/N$ due to the Frobenius action of $C_{\hat{P}_t}(A)$ on M/N . It follows that $M/N \leq \hat{P}_t N/N \cap M/N = 1$ as $\hat{P}_t \triangleleft G$ and p_t is coprime to $|M/N|$. This contradiction shows that

$$(4) \quad C_{\hat{P}_t}(A) = 1.$$

Set $H = \prod_{i=1}^{t-1} \hat{P}_i$. Pick a nontrivial subgroup C of A . Set $S = [\hat{P}_{t-1}, C]^H$. Clearly $S \leq \hat{P}_{t-1}$ is normalized by $\hat{P}_{t-1} \dots \hat{P}_1 A$. Now the image of S in P_{t-1} is $[P_{t-1}, C]^H$. Suppose that $[P_{t-1}, C]^H$ is contained in $\Phi(P_{t-1})$. It follows that $[P_{t-1}, C] \leq \Phi(P_{t-1})$ and so $[P_{t-1}, C] = 1$ due to coprimeness. By the three subgroup lemma $[P_{t-2}, C, P_{t-1}] = 1$ whence $[P_{t-2}, C] = 1$. Repeating the same argument one gets $[P_i, C] = 1$ for each $i < t$. Now the group $X = \prod_{i=1}^{t-1} C_{\hat{P}_i}(C)$ is of Fitting length $t - 1$ on which A/C acts in such a way that $C_X(A/C)$ is a TNI-subgroup. By induction we get $t - 1 \leq f(C_X(A)) + \ell(A/C)$. It then follows that $t \leq f(C_G(A)) + \ell(A)$. This contradiction shows that $[P_{t-1}, C]^H$ is not contained in $\Phi(P_{t-1})$. By (2) part f we have $S = \hat{P}_{t-1}$. This shows that $[\hat{P}_{t-1}, C]^{\hat{P}_{t-1} \dots \hat{P}_1} = \hat{P}_{t-1}$ for every subgroup C of A with $\ell(C) \geq 1$.

Next let $D \leq A$ with $\ell(D) \geq 2$ and $Y = \prod_{i=1}^{t-2} \hat{P}_i$. Set $T = [\hat{P}_{t-2}, D]^Y$. Clearly T is YA -invariant. If the image $[P_{t-2}, C]^Y$ of T in P_{t-2} is contained in $\Phi(P_{t-2})$, then we can show by an argument similar as in the above paragraph that $[P_i, D] = 1$ for each $i < t - 1$. Then $Z = \prod_{i=1}^{t-2} C_{\hat{P}_i}(D)$ is a group of Fitting length $t - 2$ on which A/D acts in such a way that $C_Z(A/D)$ is a TNI-subgroup. It follows by induction that $t \leq f(C_G(A)) + \ell(A)$, which is not the case. Therefore $T = \hat{P}_{t-2}$ by (2) part (f) . Thus we have

(5) $[\hat{P}_{t-1}, C]^{\hat{P}_{t-1} \dots \hat{P}_1} = \hat{P}_{t-1}$ for every subgroup C of A with $\ell(C) \geq 1$ and $[\hat{P}_{t-2}, D]^{\hat{P}_{t-2} \dots \hat{P}_1} = \hat{P}_{t-2}$ for every subgroup D of A with $\ell(D) \geq 2$.

Let now S be an H -homogeneous component of the irreducible HA -module $V = P_t/\Phi(P_t)$. Notice that \hat{P}_{t-1} acts nontrivially on each H -homogeneous component of V . Set $B = N_A(S)$. Then S is an irreducible HB -module such that $C_S(B) = 0$. By the Fong-Swan theorem

(6) *We may take an irreducible CHB -module M such that $M|_H$ is homogeneous, $\text{Ker}_H(M) = \text{Ker}_H(S)$ and $C_M(B) = 0$. Among all pairs (M_α, C_α) such that $1 \neq C_\alpha \leq B$, M_α is an irreducible HC_α submodule of $M|_{HC_\alpha}$ and $C_{M_\alpha}(C_\alpha) = 0$, choose (M_1, C) with $|C|$ minimum. Then $C_{M_1}(C_0) \neq 0$ for $1 \neq C_0 < C$ and $\text{Ker}_H(M) = \text{Ker}_H(M_1)$.*

Set $\bar{H} = H/\text{Ker}_H(M)$. Suppose that $\bar{U} \neq 1$ is an abelian subgroup of \bar{H} and that $\bar{U} \triangleleft \bar{H}C$. Let U be the preimage in H of \bar{U} . Since $M_1|_H$ is homogeneous, by Glauberman's lemma there is a homogeneous component N_1 of $M_1|_U$ such that $C \leq N_{HC}(N_1)$. Set $H_1 = N_H(N_1)$. Then we have $H_1C = N_{HC}(N_1)$. Now $[U, C] \leq \text{Ker}_H(N_1)$. By Proposition 4.1 in [5], $H = (\bigcap_{x \in HC} H_1^x)C_H(C)$. It follows that $M_1 = \Sigma(N_1)^x$ for $x \in C_H(C)$. Notice that $[U, C] \leq \text{Ker}_H(N_1)$. Thus $[U, C] = [U, C]^x \leq \text{Ker}_H(N_1)^x$ and so $[U, C] \leq \text{Ker}_H(M_1)$. Then we have

(7) *If \bar{U} is an abelian subgroup of \bar{H} such that $\bar{U} \triangleleft \bar{H}C$ where $\bar{H} = H/\text{Ker}_H(M)$ then $[\bar{U}, C] = 1$.*

By (5) we have $[\hat{P}_{t-1}, C]^H C_{\hat{P}_{t-1}}(P_t) = \hat{P}_{t-1}$ and hence $[\hat{P}_{t-1}, C] \not\leq \text{Ker}(M)$. Therefore by (7), $\hat{P}_{t-1} \text{Ker}_H(M)/\text{Ker}_H(M)$ is nonabelian. As P_1 is elementary

abelian we have $t > 2$. Set $\Phi/Ker_H(M) = \Phi(\hat{P}_{t-1}Ker_H(M)/Ker_H(M))$ and let $H_1 = C_{\hat{P}_{t-2}\dots\hat{P}_1}(\bar{\Phi})$. By (7), $[\bar{\Phi}, C] = 1$. Now $H_1C = C_{\hat{P}_{t-2}\dots\hat{P}_1C}(\bar{\Phi}) \triangleleft \hat{P}_{t-2}\dots\hat{P}_1C$. Hence $[\hat{P}_{t-2}\dots\hat{P}_1, C] \leq H_1$. By the coprimeness we have

(8) $t > 2$ and $H_1C_{\hat{P}_{t-2}\dots\hat{P}_1}(C) = \hat{P}_{t-2}\dots\hat{P}_1$ where $\bar{\Phi} = \Phi(\hat{P}_{t-1}Ker(M)/Ker(M))$ and $H_1 = C_{\hat{P}_{t-2}\dots\hat{P}_1}(\bar{\Phi})$.

We have $\hat{P}_{t-2} \leq H_1$ by (2) part (e). Set $Q = \hat{P}_{t-2}$. Let $D \leq C$ be such that $\ell(D) \geq 2$. Then by (8), $[Q, D]^{H_1} = [Q, D]^{\hat{P}_{t-2}\dots\hat{P}_1}$. By (5) we have

(9) If $D \leq C$ with $\ell(D) \geq 2$ then $Q = [Q, D]^{H_1}$ where $Q = \hat{P}_{t-2}$.

Let N be a homogeneous component of $M_1|_S$. Then N is normalized by $\hat{P}_{t-1}H_1C$ since $H_1C = C_{\hat{P}_{t-2}\dots\hat{P}_1C}(\bar{\Phi})$. Set $P_0 = \hat{P}_{t-1}/\hat{P}_{t-1} \cap Ker_H(N)$. Then N is a P_0H_1C -module. Notice that H_1C centralizes $\Phi(P_0)$ and hence $N|_{\Phi(P_0)}$ is homogeneous. Then $\Phi(P_0)$ is cyclic. We also have $\Phi(P_0)$ is elementary abelian by (2) part (d) and hence $|\Phi(P_0)| \leq p$. Recall that $(\hat{P}_{t-1})' \not\leq Ker_H(M_1)$ and hence P_0 is nonabelian. Thus we have $P_0' = \Phi(P_0)$. Note also that $P_0/Z(P_0)$ is elementary abelian. As $P_{t-1}/\Phi(P_{t-1})$ is irreducible $\hat{P}_{t-2}\dots\hat{P}_1A$ -module, it is completely reducible as H_1 -module because H_1 is subnormal in $\hat{P}_{t-2}\dots\hat{P}_1A$. It follows that $P_0/\Phi(P_0)$ is H_1 -completely reducible. It follows by Maschke's theorem that it is H_1C -completely reducible.

Suppose that $1 \neq D \leq C$. Then $[P_0, D] \leq [P_0, D]^{P_0H_1} \triangleleft P_0H_1C_{\hat{P}_{t-2}\dots\hat{P}_1}(C)$. By (8), $P_0H_1C_{\hat{P}_{t-2}\dots\hat{P}_1}(C) = P_0\hat{P}_{t-2}\dots\hat{P}_1$ and hence by (5) we get $P_0 = [P_0, D]^{H_1}$. Now we apply Theorem 1.1 in [4] by letting $G = P_0H_1$, $A = C$, $P = P_0$, $Q = \hat{P}_{t-2}$ and χ as the character afforded by N . This leads to $C_N(C) \neq 0$, which is a contradiction completing the proof.

REFERENCES

- [1] G. Ercan, İ. Ş. Güloğlu, Groups of automorphisms with TNI-centralizers, J.Algebra. 498 (2018), 38–46.
- [2] A. Turull, Fixed point free action with regular orbits, J. Reine Angew. Math. 371 (1986), 67–91.
- [3] A. Turull, Fitting height of groups and of fixed points, J. Algebra 86 (1984) 555–556.
- [4] A.Turull, Groups of automorphisms and centralizers, Math. Proc. Cambridge Philos. Soc. 107 (1990), no. 2, 227–238.
- [5] A.Turull, Supersolvable Automorphism Groups of Solvable Groups. Math. Z. 183 (1983), 47–73.

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