On the stability at all times of linearly extrapolated BDF2 timestepping for multiphysics incompressible flow problems

Mine Akbas * Songül Kaya † Leo G. Rebholz ‡

Abstract

We prove long-time stability of linearly extrapolated BDF2 (BDF2LE) timestepping methods, together with finite element spatial discretizations, for incompressible Navier-Stokes equations (NSE) and related multiphysics problems. For the NSE, Boussinesq, and MHD schemes, we prove unconditional long time L^2 stability, provided external forces (and sources) are uniformly bounded in time. We also provide numerical experiments to compare stability of BDF2LE to linearly extrapolated Crank-Nicolson scheme for NSE, and find that BDF2LE has better stability properties, particularly for smaller viscosity values.

1 Introduction

We consider long-time stability of algorithms for multiphysics incompressible flow problems that use second-order, linearly extrapolated BDF2 timestepping together with finite element spatial discretizations. Long time stability is essential for accuracy of longer time simulations, and thus it is important to know whether a discretization possesses such a property, and if there is a timestep restriction for it to hold.

Recent works of F. Tone, X. Wang, S. Gottlieb, D. Wirosoetisno and coworkers have made great strides in understanding long-time stability results for several commonly used discretizations of incompressible flow problems. In the past five years, they have proven long-time stability for several schemes for the incompressible Navier-Stokes equations (NSE), including [1, 6, 13, 15, 16], and also for backward Euler schemes for magnetohydrodynamics (MHD) [14] and the Boussinesq equations [5]. For BDF2 type schemes, unconditional long-time stability has been proven for a Stokes-Darcy system in [3] and by Heister et al. for a particular 2D velocity-vorticity method in [8]. The purpose of this paper is to extend the works above to study the long-time stability of second order, linearized, BDF2 timestepping schemes, together with a finite element spatial discretizations, for 3D incompressible Navier-Stokes equations, Boussinesq equations, and MHD in both primitive and Elsässer variable formulations. To our knowledge, no long time stability results have been proven for the common linearly extrapolated BDF2 scheme for the 3D NSE in velocity-pressure formulation, nor for second order timestepping scheme for multiphysics incompressible flow problems such as magnetohydrodynamics (MHD) and the Boussinesq system. For each of these systems/schemes, we prove unconditional long-time L^2 stability of velocity / temperature /

*Department of Mathematics, Middle East Technical University, Ankara, 06531; miakbasb@metu.edu.tr
†Department of Mathematics, Middle East Technical University, Ankara, 06531; smerdan@metu.edu.tr
‡Department of Mathematical Sciences, Clemson University, Clemson, SC, 29634; rebholz@clemson.edu, partially supported by NSF Grant DMS 1522191.

magnetic field. In addition to our analytical stability results, we also provide numerical tests that compare long-time stability of the linearly extrapolated BDF2 scheme for the NSE to the linearly extrapolated Crank-Nicolson scheme, and find that the BDF2LE scheme is more stable for smaller values of viscosity.

This paper is arranged as follows. Section 2 gives notation and mathematical preliminaries to make for a smoother analysis. Section 3 studies a second order, linearized BDF2 in time, finite element in space, scheme for the primitive variable NSE. Here, we take advantage of the G-norm theory from [7] in order to prove unconditional long time stability of the scheme, and also give numerical experiments that compare stability of this scheme compared to an analogous Crank-Nicolson scheme. In section 4, we prove unconditional long-time stability of a linearized, decoupled, BDF2 scheme for the incompressible Boussinesq system. In section 5, we extend the results to a second order linearized BDF2 scheme for MHD, and finally in section 6, we extend the results to a *decoupled*, linearized BDF2 scheme for MHD in Elsasser variables.

2 Mathematical Preliminaries

Let Ω be an open, connected bounded Lipschitz domain in $\mathbb{R}^d$, $d = 2, 3$. We denote the usual L^2 inner product and its induced norm by $(\cdot,\cdot)$ and $\|\cdot\|$, respectively. We use the following function spaces:

$$\begin{aligned} X &:= H_0^1(\Omega)^d = \{v \in (L^p(\Omega))^d : \nabla v \in L^2(\Omega)^{d\times d}, v = 0 \ on\ \partial\Omega\},\\ Q &:= L_0^2(\Omega) = \{q \in L^2(\Omega) : \int_\Omega q\,dx = 0\},\\ W &:= H_0^1(\Omega). \end{aligned}$$

In the long time stability analysis with respect to the L^2-norm, we frequently use the Poincaré inequality: there exists a constant C_P, depending only the size of the domain, such that

$$\|\psi\| \leq C_P\|\nabla\psi\|, \quad \forall\psi \in X \ \text{ and } \ \psi \in W.$$

Let π_h be a regular, conforming, triangulation of the domain, and $X_h \subset X$, $Q_h \subset Q$, $W_h \subset W$ be conforming finite element spaces defined on π_h. We assume that velocity-pressure finite element spaces (X_h, Q_h) are inf-sup stable, i.e., there exists a constant β, independent of the mesh size h, such that

$$\inf_{q_h\in Q_h} \sup_{v_h\in X_h} \frac{(q_h, \nabla\cdot v_h)}{||\nabla v_h||\,||\,q_h\,||} \geq \beta > 0.$$

Another frequently used inequality in the long time stability analysis is inverse inequality: there exists a constant C_I, dependent on the minimum angle of the mesh but independent of h, such that

$$\|\nabla\phi_h\| \leq C_I h^{-1}\|\phi_h\|, \quad \forall\phi_h \in X_h.$$

The discretely divergence free subspace of X_h is defined as

$$V_h = \{v_h \in X_h : (q_h, \nabla\cdot v_h) = 0 \ \ \forall q_h \in Q_h\}.$$

We denote V_h^* as the dual space of V_h and its norm by $\|\cdot\|_{V_h^*}$:

$$\|\Phi\|_{V_h^*} := \sup_{v_h \in V_h^*} \frac{(\Phi, v_h)}{\|\nabla v_h\|}.$$

We also define the space

$$L^\infty(\mathbb{R}_+, V_h^*) := \{f : \Omega^d \times \mathbb{R}_+ \to \mathbb{R}^d,\ \exists C < \infty \text{ with } \|f(t)\|_{V_h^*} < C \text{ for almost every } t > 0\}.$$

In the discretization of non-linear terms for the proposed flow problems, we use the following skew-symmetric trilinear forms:

$$b^*(u, v, w) := \frac{1}{2}\left((u \cdot \nabla v, w) - (u \cdot \nabla w, v)\right) \quad \forall u, v, w \in X,$$
$$c^*(u, \theta, \psi) := \frac{1}{2}\left((u \cdot \nabla \theta, \psi) - (u \cdot \nabla \psi, \theta)\right) \quad \forall u \in X \text{ and } \theta, \psi \in W.$$

The schemes we consider are of BDF2 type, and the analysis is greatly simplified if we use the G-stability framework, as in [7]. Hence, we define here the G-matrix

$$G := \begin{pmatrix} 1/2 & -1 \\ -1 & 5/2 \end{pmatrix},$$

and its associated G-norm by

$$\|\chi\|_G^2 = (\chi, G\chi), \quad \chi \in \mathbb{R}^{2n}.$$

It is well known from [7] that the L^2-norm and the G-norm are equivalent in the following sense: $\exists\, C_u, C_l > 0$ such that

$$C_l \|\chi\|_G \le \|\chi\| \le C_u \|\chi\|_G. \tag{2.1}$$

Set $\chi_v^n := [v^{n-1}, v^n]^T$ and $\chi_v^{n+1} := [v^n, v^{n+1}]^T$. Then if each $v^i \in L^2(\Omega)$ the following relation holds (see, e.g. [7]):

$$\left(\frac{3}{2}v^{n+1} - 2v^n + \frac{1}{2}v^{n-1}, v^{n+1}\right) = \frac{1}{2}\left(\|\chi_v^{n+1}\|_G^2 - \|\chi_v^n\|_G^2\right) + \frac{\|v^{n+1} - 2v^n + v^{n-1}\|^2}{4}. \tag{2.2}$$

3 Long Time Stability of the BDF2LE for the NSE

In this section, we study a numerical algorithm for the incompressible, viscous NSE, which are given by

$$\begin{aligned} u_t + u \cdot \nabla u + \nabla p - \nu \Delta u &= f, \\ \nabla \cdot u &= 0, \\ u(0, x) &= u_0(x), \\ u|_{\partial\Omega} &= 0, \end{aligned}$$

where u_0 is the initial velocity, f is given forcing, ν is the kinematic viscosity, and u and p are velocity and pressure unknowns.

We first study the long time stability of the BDF2LE finite element for the NSE. This is a widely-used algorithm, and is given as follows:

Algorithm 3.1 *Let the forcing $f \in L^\infty(\mathbb{R}_+; V_h^*)$ and an initial condition $u_0 \in L^2(\Omega)^d$ be given. Define $u_h^{-1} = u_h^0$ to be interpolant of u_0. Choose a time step Δt. For all $n = 0, 1, 2, ..,$ find (u_h^{n+1}, p_h^{n+1}) satisfying*

$$\frac{1}{2\Delta t}(3u_h^{n+1} - 4u_h^n + u_h^{n-1}, v_h) + b^*(2u_h^n - u_h^{n-1}, u_h^{n+1}, v_h) + \nu(\nabla u_h^{n+1}, \nabla v_h) - (p_h^{n+1}, \nabla \cdot v_h) = (f^{n+1}, v_h) \quad (3.1)$$

$$(\nabla \cdot u_h^{n+1}, q_h) = 0, \quad (3.2)$$

for all $(v_h, q_h) \in (X_h, Q_h)$.

We now show the BDF2LE finite element scheme is unconditionally long time stable, i.e., stable independent of end time and the time step Δt.

Theorem 3.2 *Let (u_h^{n+1}, p_h^{n+1}) be the solutions to Algorithm 3.1 for all $n = 0, 1, 2, \ldots$. Then for any $\Delta t > 0$, we have*

$$\|\chi_u^{n+1}\|_G^2 + \frac{\nu\Delta t}{4}\|\nabla u_h^{n+1}\|^2 \le (1+\alpha)^{-(n+1)}\left(\|\chi_u^0\|_G^2 + \frac{\nu\Delta t}{4}\|\nabla u_h^0\|^2\right) + \nu^{-1}\alpha^*\|f\|^2_{L^\infty(\mathbb{R}_+; V_h^*)}, \quad (3.3)$$

where $\alpha^* = \max\{\frac{1}{2}\Delta t, \frac{4C_P^2}{\nu C_l^2}\} > 0$, $\alpha = \min\{2, \frac{\nu \Delta t C_l^2}{4C_P^2}\} > 0$.

Proof: Take as test functions $v_h = 2\Delta t u_h^{n+1}$ in (3.1), $q_h = p_h^{n+1}$ in (3.2). Using relation (2.2), and applying the Cauchy-Schwarz inequality and the Young's inequality yields

$$\left(\|\chi_u^{n+1}\|_G^2 - \|\chi_u^n\|_G^2\right) + \frac{\|u_h^{n+1} - 2u_h^n + u_h^{n-1}\|^2}{2} + \nu\Delta t\|\nabla u_h^{n+1}\|^2 \le \nu^{-1}\Delta t\|f^{n+1}\|^2_{V_h^*}.$$

Drop the non-negative term $\frac{\|u_h^{n+1} - 2u_h^n + u_h^{n-1}\|^2}{2}$ and add $\frac{\nu\Delta t}{4}\|\nabla u_h^n\|^2$ to both sides to produce

$$\left(\|\chi_u^{n+1}\|_G^2 + \frac{\nu\Delta t}{4}\|\nabla u_h^{n+1}\|^2\right) + \frac{\nu\Delta t}{4}\left(\|\nabla u_h^{n+1}\|^2 + \|\nabla u_h^n\|^2\right) + \frac{\nu\Delta t}{2}\|\nabla u_h^{n+1}\|^2 \le \left(\|\chi_u^n\|_G^2 + \frac{\nu\Delta t}{4}\|\nabla u_h^n\|^2\right) + \nu^{-1}\Delta t\|f^{n+1}\|^2_{V_h^*}. \quad (3.4)$$

By using the Poincaré inequality and relation (2.1), the last two terms on the left hand side can be written as

$$\begin{aligned}\frac{\nu\Delta t}{4}\left(\|\nabla u_h^{n+1}\|^2 + \|\nabla u_h^n\|^2\right) + \frac{\nu\Delta t}{2}\|\nabla u_h^{n+1}\|^2 &\ge \frac{\nu\Delta t}{4C_P^2}\|\chi_u^{n+1}\|^2 + \frac{\nu\Delta t}{2}\|\nabla u_h^{n+1}\|^2\\ &\ge \frac{\nu\Delta t C_l^2}{4C_P^2}\|\chi_u^{n+1}\|_G^2 + \frac{\nu\Delta t}{2}\|\nabla u_h^{n+1}\|^2\\ &\ge \alpha\left(\|\chi_u^{n+1}\|_G^2 + \frac{\nu\Delta t}{4}\|\nabla u_h^{n+1}\|^2\right),\end{aligned}$$

where $\alpha := \min\{2, \frac{\nu\Delta t C_l^2}{4C_P^2}\}$. Inserting the last estimate in (3.4) and using induction results gives

$$\|\chi_u^{n+1}\|_G^2 + \frac{\nu\Delta t}{4}\|\nabla u_h^{n+1}\|^2 \le (1+\alpha)^{-(n+1)}\left(\|\chi_u^0\|_G^2 + \frac{\nu\Delta t}{4}\|\nabla u_h^0\|^2\right) + \frac{\nu^{-1}\Delta t}{\alpha}\|f\|^2_{L^\infty(\mathbb{R}_+; V_h^*)}.$$

Setting $\frac{\Delta t}{\alpha} = \max\{\frac{1}{2}\Delta t, \frac{4C_P^2}{\nu C_l^2}\} =: \alpha^*$ and using the equivalence of the L^2 and the G-norms (2.1) completes the proof. $\square$

3.1 Numerical Experiment

In this numerical experiment, we compare the stability of the BDF2LE and the CNLE for the NSE using (P_2, P_1^{disc}) Scott-Vogelius finite elements. The CNLE scheme for the NSE is given below by Algorithm 3.3. This scheme has been studied recently in [1, 9, 11]. In [1], it is shown that the discrete solution of Algorithm 3.3 is conditionally long time stable, which is stated in Lemma 3.4:

Algorithm 3.3 *Let $f \in L^\infty(\mathbb{R}_+; V_h^*)$ and the initial velocity $u_0 \in L^2(\Omega)$ be given. Define $u_h^{-1} = u_h^0$ to be the nodal interpolant of u_0 and choose a time step $\Delta t > 0$. For $n = 0, 1, 2, 3, ...,$ find $(u_h^{n+1}, p_h^{n+1}) \in (X_h, Q_h)$ satisfying for every $(v_h, q_h) \in (X_h, Q_h)$,*

$$\left(\frac{u_h^{n+1} - u_h^n}{\Delta t}, v_h\right) + b^*(u_h^*, u_h^{n+1/2}, v_h) + \nu(\nabla u_h^{n+1/2}, \nabla v_h) - (p_h^{n+1}, \nabla \cdot v_h) = (f(t^{n+1/2}), v_h), \quad (3.5)$$

$$(\nabla \cdot u_h^{n+1}, q_h) = 0, \quad (3.6)$$

where $u_h^* = \frac{3}{2}u_h^n - \frac{1}{2}u_h^{n-1}$, $u_h^{n+1/2} = \frac{1}{2}(u_h^{n+1} + u_h^n)$ and p_h^{n+1} is understood to be an approximation of $p_h^{n+1/2}$.

Lemma 3.4 *Let (X_h, Q_h) be an inf-sup stable, conforming velocity-pressure finite element pair. Set $K := \left(\|u_0\|^2 + \frac{c_P^2}{\nu^2}\|f\|^2_{L^\infty(\mathbb{R}_+;V_h^*)}\right)^{1/2}$. If $\Delta t \le \min\left\{\frac{h^2}{4\nu C_I^2}, \frac{\nu h^d}{20 c_0^2 K^2 C_I^d}\right\}$, then*

$$\|u_h^{n+1}\|^2 \le \left(1 + \frac{\nu}{2c_P^2}\Delta t\right)^{-(n+1)} \|u_0\|^2 + \frac{c_P^2}{\nu^2}\|f\|^2_{L^\infty(0,\infty;V_h^*)} \left[1 - \left(1 + \frac{\nu}{2c_P^2}\right)^{-(n+1)}\right] \le K^2. \quad (3.7)$$

where c_0 is a constant independent of the mesh size h and time step Δt.

We now compare L^2-stability of the BDF2LE and the CNLE. We choose an initial velocity and a body force as follows:

$$u_0 = \begin{pmatrix} \sin(\pi x)\sin(\pi y) \\ \cos(\pi x)\cos(\pi y) \end{pmatrix}, \quad f = \begin{pmatrix} y^2\cos(xy^2) + \sin(x)\sin(y) \\ 2xy\cos(xy^2) + \cos(x)\cos(y) \end{pmatrix},$$

and use the same 16×16 barycenter refined uniform mesh of $\Omega := (0,1)^2$. We calculate approximate velocity solutions using Algorithm 3.1 and Algorithm 3.3, using time step $\Delta t = 0.25$ with end time $T = 500$, and varying ν. The results are shown in Figure 1 and Figure 2 as plots of $\|u_h^n\|$ versus time. We observe that for larger ν ($\nu = 1$ and $\nu = 0.004$), the schemes produce very similar stability properties. For $\nu = 0.002$, the schemes are similar until around $T = 70$, when the CNLE deviates: the L^2-norm of the solution grows for the CNLE but remains constant for the BDF2LE. For $\nu = 0.001$, we observe similar results as for $\nu = 0.002$, but the deviation happens sooner.

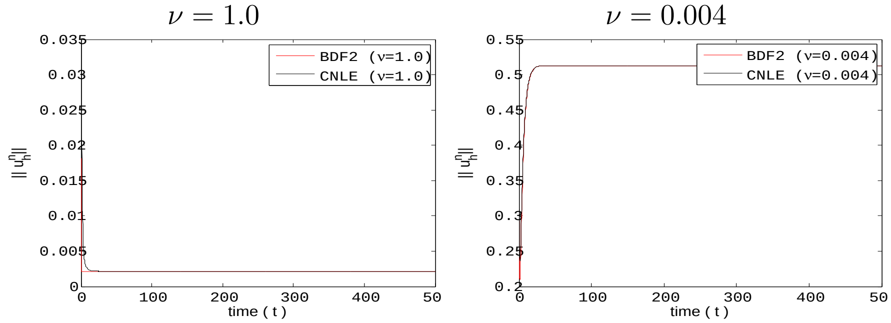

Figure 1: Discrete velocity solutions of the BDF2LE and the CNLE schemes for the NSE with $\Delta t = 0.25$ varying ν.

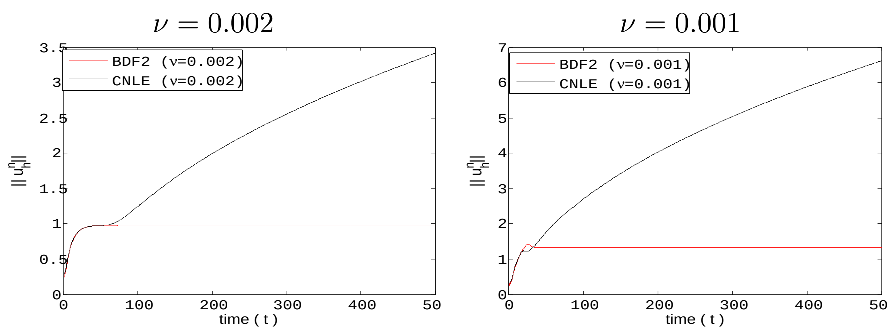

Figure 2: Discrete velocity solutions of the BDF2LE and the CNLE schemes for the NSE with $\Delta t = 0.25$ varying ν.

4 Boussinesq Flow with BDF2LE

In this section, we focus on the Boussinesq system which describes incompressible non-isothermal fluid flow. Its governing equations are given by the incompressible NSE together with the heat transport equation:

$$u_t + (u \cdot \nabla)u - \nu\Delta u + \nabla p = Ri\langle 0, \theta\rangle + f, \tag{4.1}$$

$$\nabla \cdot u = 0, \tag{4.2}$$

$$\theta_t + (u \cdot \nabla)\theta - \kappa\Delta\theta = \gamma, \tag{4.3}$$

$$u(0, x) = u_0 \text{ and } \theta(0, x) = \theta_0, \tag{4.4}$$

$$u|_{\partial\Omega} = 0 \text{ and } \theta|_{\partial\Omega} = 0, \tag{4.5}$$

where $Ri\langle 0, \theta\rangle$ stands for the vector $\langle 0, Ri\theta\rangle^T$, $d = 2, 3$, and u represents the fluid velocity, p the pressure, θ the temperature, f the prescribed forcing, γ the heat source, ν the kinematic viscosity, which is inversely proportional to the Reynolds number $Re = \mathcal{O}(\nu^{-1})$, Ri the Richardson number accounting for the gravitational force, and $\kappa := Re^{-1}Pr^{-1}$, where Pr is the Prandtl number.

The linearly extrapolated BDF2 approximation of the system (4.1)-(4.5) reads as follows:

Algorithm 4.1 *Let the forcing f, the heat source γ and initial conditions u_0, θ_0 be given. Define $u_h^{-1} = u_h^0$ and $\theta_h^{-1} = \theta_h^0$ to be the nodal interpolants of u_0 and θ_0, respectively. Choose a time step $\Delta t > 0$, and for all $n = 0, 1, 2, 3, ...,$ find $(u_h^{n+1}, p_h^{n+1}, \theta_h^{n+1})$ satisfying*

$$\frac{1}{2\Delta t}(3u_h^{n+1} - 4u_h^n + u_h^{n-1}, v_h) + b^*(2u_h^n - u_h^{n-1}, u_h^{n+1}, v_h) + \nu(\nabla u_h^{n+1}, \nabla v_h) - (p_h^{n+1}, \nabla \cdot v_h)$$
$$= (Ri\langle 0, 2\theta_h^n - \theta_h^{n-1}\rangle, v_h) + (f^{n+1}, v_h), \quad (4.6)$$
$$(\nabla \cdot u_h^{n+1}, q_h) = 0, \quad (4.7)$$
$$\frac{1}{2\Delta t}(3\theta_h^{n+1} - 4\theta_h^n + \theta_h^{n-1}, w_h) + c^*(u_h^n, \theta_h^{n+1}, w_h) + \kappa(\nabla\theta_h^{n+1}, \nabla w_h) = (\gamma^{n+1}, w_h). \quad (4.8)$$

for all $(v_h, q_h, w_h) \in (X_h, Q_h, W_h)$.

We now present the long time stability of Algorithm 4.1.

Theorem 4.2 *Let $f \in L^\infty(\mathbb{R}_+; V_h^*)$ and $\gamma \in L^\infty(\mathbb{R}_+; W_h^*)$. Then for any $\Delta t > 0$ and for n sufficiently large, the velocity and temperature solutions of Algorithm 4.1 satisfy :*

$$\|\chi_u^{n+1}\|_G^2 + \frac{\nu\Delta t}{4}\|\nabla u_h^{n+1}\|^2 \le \left(\frac{1}{1+\beta}\right)^{n+1}\left(\|\chi_u^0\|_G^2 + \frac{\nu\Delta t}{4}\|\nabla u_h^0\|^2\right) + \nu^{-1}\beta^* M^*, \quad (4.9)$$
$$\|\chi_\theta^{n+1}\|_G^2 + \frac{\kappa\Delta t}{4}\|\nabla\theta_h^{n+1}\|^2 \le \left(\frac{1}{1+\eta}\right)^{n+1}\left(\|\chi_\theta^0\|_G^2 + \frac{\kappa\Delta t}{4}\|\nabla\theta_h^0\|^2\right) + \kappa^{-1}\eta^*\|\gamma\|^2_{L^\infty(\mathbb{R}_+;W_h^*)}, \quad (4.10)$$

where $\beta^* = \max\{\frac{1}{2}\Delta t, \frac{4C_P^2}{\nu C_l^2}\}$, $\eta^* = \max\{\frac{1}{2}\Delta t, \frac{4C_P^2}{\kappa C_l^2}\}$, $\eta := \min\{2, \frac{\kappa\Delta t C_l^2}{4C_P^2}\}$, $\beta := \min\{2, \frac{\nu\Delta t C_l^2}{4C_P^2}\}$, and

$$M^* = 2\left(10Ri^2C_P^2\max\{\|\theta_h^0\|^2, M^2\} + \|f\|^2_{L^\infty(\mathbb{R}_+;V_h^*)}\right).$$

If

$$n\Delta t > \frac{2}{\kappa\eta_{**}}\ln\left(\frac{\|\chi_\theta^0\|_G^2 + \frac{\kappa\Delta t}{4}\|\nabla\theta_h^0\|^2}{\kappa^{-2}\eta_{**}^{-1}\|\gamma\|^2_{L^\infty(\mathbb{R}_+;W_h^*)}}\right) \quad \text{and} \quad \kappa\Delta t < 1, \quad (4.11)$$

then for all n sufficiently large

$$\|\theta_h^n\| \le \sqrt{2\eta_{**}^{-1}}C_u\kappa^{-1}\|\gamma\|_{L^\infty(\mathbb{R}_+;W_h^*)} =: M, \quad (4.12)$$

where $\eta_{**} = \min\{2, \frac{C_l^2}{4C_P^2}\}$.

Proof: The proof of Theorem 4.2 consists of three steps. In Step 1, we first show the long time stability of the temperature and we obtain the bound on the temperature in Step 2. In Step 3, to obtain the long time stability of the velocity we use the results of Step 1 and Step 2.

Step 1: *(Long-time L^2-stability of the temperature)*

We choose $w_h = 2\Delta t \theta_h^{n+1}$ in (4.8) and use relation (2.2). Application of the Cauchy-Schwarz and the Young's inequalities gives

$$(\|\chi_\theta^{n+1}\|_G^2 - \|\chi_\theta^n\|_G^2) + \frac{\|\theta_h^{n+1} - 2\theta_h^n + \theta_h^{n-1}\|^2}{2} + \kappa\Delta t\|\nabla\theta_h^{n+1}\|^2 \le \kappa^{-1}\Delta t\|\gamma^{n+1}\|_{W_h^*}^2. \tag{4.13}$$

Dropping the non-negative term $\frac{\|\theta_h^{n+1}-2\theta_h^n+\theta_h^{n-1}\|^2}{2}$ and adding the term $\frac{\kappa\Delta t}{4}\|\nabla\theta_h^{n+1}\|^2$ to the both sides of (4.13) leads to

$$\begin{aligned}\left(\|\chi_\theta^{n+1}\|_G^2 + \frac{\kappa\Delta t}{4}\|\nabla\theta_h^{n+1}\|^2\right) + \frac{\kappa\Delta t}{4}\left(\|\nabla\theta_h^{n+1}\|^2 + \|\nabla\theta_h^n\|^2\right) + \frac{\kappa\Delta t}{2}\|\nabla\theta_h^{n+1}\|^2 \\ \le \left(\|\chi_\theta^n\|_G^2 + \frac{\kappa\Delta t}{4}\|\nabla\theta_h^n\|^2\right) + \kappa^{-1}\Delta t\|\gamma^{n+1}\|_{W_h^*}^2.\end{aligned} \tag{4.14}$$

The last two terms on the left hand side of (4.14) are bounded by the Poincaré inequality and relation (2.1):

$$\begin{aligned}\frac{\kappa\Delta t}{4}\left(\|\nabla\theta_h^{n+1}\|^2 + \|\nabla\theta_h^n\|^2\right) + \frac{\kappa\Delta t}{2}\|\nabla\theta_h^{n+1}\|^2 &\ge \frac{\kappa\Delta t}{4C_P^2}(\|\theta_h^{n+1}\|^2 + \|\theta_h^n\|^2) + \frac{\kappa\Delta t}{2}\|\nabla\theta_h^{n+1}\|^2 \\ &= \frac{\kappa\Delta t}{4C_P^2}\|\chi_\theta^{n+1}\|^2 + \frac{\kappa\Delta t}{2}\|\nabla\theta_h^{n+1}\|^2 \\ &\ge \frac{\kappa\Delta t C_l^2}{4C_P^2}\|\chi_\theta^{n+1}\|_G^2 + \frac{\kappa\Delta t}{2}\|\nabla\theta_h^{n+1}\|^2 \\ &\ge \eta\left(\|\chi_\theta^{n+1}\|_G^2 + \frac{\kappa\Delta t}{4}\|\nabla\theta_h^{n+1}\|^2\right),\end{aligned} \tag{4.15}$$

where $\eta := \min\{2, \frac{\kappa\Delta t C_l^2}{4C_P^2}\}$. We then insert (4.15) into (4.14) and use induction to produce

$$\begin{aligned}\|\chi_\theta^{n+1}\|_G^2 + \frac{\kappa\Delta t}{4}\|\nabla\theta_h^{n+1}\|^2 &\le \left(\frac{1}{1+\eta}\right)^{n+1}\left(\|\chi_\theta^0\|_G^2 + \frac{\kappa\Delta t}{4}\|\nabla\theta_h^0\|^2\right) \\ &\quad + \kappa^{-1}\Delta t\|\gamma\|_{L^\infty(\mathbb{R}_+;W_h^*)}^2\left[\frac{1}{1+\eta} + \left(\frac{1}{1+\eta}\right)^2 + \ldots + \left(\frac{1}{1+\eta}\right)^{n+1}\right] \\ &\le \left(\frac{1}{1+\eta}\right)^{n+1}\left(\|\chi_\theta^0\|_G^2 + \frac{\kappa\Delta t}{4}\|\nabla\theta_h^0\|^2\right) + \frac{\kappa^{-1}\Delta t}{\eta}\|\gamma\|_{L^\infty(\mathbb{R}_+;W_h^*)}^2.\end{aligned}$$

Noting that $\frac{\Delta t}{\eta} = \max\{\frac{1}{2}\Delta t, \frac{4C_P^2}{\kappa C_l^2}\} =: \eta^*$, applying the equivalence of the L^2 norm and the G-norm, and dropping the non-negative term $\frac{\kappa\Delta t}{4}\|\nabla\theta_h^n\|^2$ gives the long time stability of temperature (4.10).

Step 2: *(Bound for the temperature)*
Using the Poincaré inequality and relation (2.1) along with the assumption $\kappa\Delta t \le 1$, the last two

terms on the left hand side of (4.14) can be written as:

$$\frac{\kappa\Delta t}{4}\left(\|\nabla\theta_h^{n+1}\|^2+\|\nabla\theta_h^n\|^2\right)+\frac{\kappa\Delta t}{2}\|\nabla\theta_h^{n+1}\|^2 \geq \frac{\kappa\Delta t C_l^2}{4C_P^2}\|\chi_\theta^{n+1}\|_G^2+\frac{(\kappa\Delta t)^2}{2}\|\nabla\theta_h^{n+1}\|^2$$
$$= \kappa\Delta t\left(\frac{C_l^2}{4C_P^2}\|\chi_\theta^{n+1}\|_G^2+\frac{\kappa\Delta t}{2}\|\nabla\theta_h^{n+1}\|^2\right)$$
$$\geq \kappa\Delta t\eta_{**}\left(\|\chi_\theta^{n+1}\|_G^2+\frac{\kappa\Delta t}{4}\|\nabla\theta_h^{n+1}\|^2\right) \quad (4.16)$$

where $\eta_{**}:=\min\{2,\frac{C_l^2}{4C_P^2}\}$. Inserting this into (4.14) and using induction gives

$$\|\chi_\theta^{n+1}\|_G^2+\frac{\kappa\Delta t}{4}\|\nabla\theta_h^{n+1}\|^2$$
$$\leq (1+\kappa\Delta t\eta_{**})^{-(n+1)}\left(\|\chi_\theta^0\|_G^2+\frac{\kappa\Delta t}{4}\|\nabla\theta_h^0\|^2\right)+\kappa^{-2}\eta_{**}^{-1}\|\gamma\|^2_{L^\infty(\mathbb{R}_+;W_h^*)}. \quad (4.17)$$

By changing the index from $(n+1)$ to (n) in (4.17) and using the inequality

$$(1+x)\geq \exp(x/2), \quad x\in(0,1)$$

along with the assumption (4.11), one obtains

$$LHS \leq \exp\left(-\frac{\kappa n\Delta t\eta_{**}}{2}\right)\left(\|\chi_\theta^0\|_G^2+\frac{\kappa\Delta t}{4}\|\nabla\theta_h^0\|^2\right)+\kappa^{-2}\eta_{**}^{-1}\|\gamma\|^2_{L^\infty(\mathbb{R}_+;W_h^*)}$$
$$\leq \exp\left(-ln\left(\frac{\|\chi_\theta^0\|_G^2+\frac{\kappa\Delta t}{4}\|\nabla\theta_h^0\|^2}{\kappa^{-2}\eta_{**}^{-1}\|\gamma\|^2_{L^\infty(\mathbb{R}_+;W_h^*)}}\right)\right)\left(\|\chi_\theta^0\|_G^2+\frac{\kappa\Delta t}{4}\|\nabla\theta_h^0\|^2\right)+\kappa^{-2}\eta_{**}^{-1}\|\gamma\|^2_{L^\infty(\mathbb{R}_+;W_h^*)}$$
$$= 2\kappa^{-2}\eta_{**}^{-1}\|\gamma\|^2_{L^\infty(\mathbb{R}_+;W_h^*)}. \quad (4.18)$$

Using the equivalence of the norms (2.1) gives the required bound on the temperature.

Step-3: *(Long-time L^2-stability of the velocity)*
To obtain the long time stability of velocity, we take $v_h=u_h^{n+1}$ in (4.6), $q_h=p_h^{n+1}$ in (4.7). By using relation (2.2), the standard inequalities, dropping the nonlinear term and multiplying the resulting inequality with $2\Delta t$ gives

$$\|\chi_u^{n+1}\|_G^2+\nu\Delta t\|\nabla u_h^{n+1}\|^2 \leq \|\chi_u^n\|_G^2+2\nu^{-1}\Delta t\left[2C_P^2Ri^2\left(4\|\theta_h^n\|^2+\|\theta_h^{n-1}\|^2\right)+\|f^{n+1}\|^2_{V_h^*}\right]. \quad (4.19)$$

We first add $\frac{\nu\Delta t}{4}\|\nabla u_h^n\|^2$ to both sides of (4.19). Then rearranging terms results in

$$\|\chi_u^{n+1}\|_G^2+\frac{\nu\Delta t}{4}\|\nabla u_h^{n+1}\|^2+\frac{\nu\Delta t}{4}\left(\|\nabla u_h^{n+1}\|^2+\|\nabla u_h^n\|^2\right)+\frac{\nu\Delta t}{2}\|\nabla u_h^{n+1}\|^2$$
$$\leq \|\chi_u^n\|_G^2+\frac{\nu\Delta t}{4}\|\nabla u_h^n\|^2+2\nu^{-1}\Delta t\left[2C_P^2Ri^2\left(4\|\theta_h^n\|^2+\|\theta_h^{n-1}\|^2\right)+\|f^{n+1}\|^2_{V_h^*}\right]. \quad (4.20)$$

Using the same idea used in Step 1, the last two terms on the left hand side can be rewritten as follows:

$$\frac{\nu\Delta t}{4}\Big(\|\nabla u_h^{n+1}\|^2+\|\nabla u_h^n\|^2\Big)+\frac{\nu\Delta t}{2}\|\nabla u_h^{n+1}\|^2\geq\beta\Big(\|\chi_u^{n+1}\|_G^2+\frac{\nu\Delta t}{4}\|\nabla u_h^{n+1}\|^2\Big),$$

where $\beta:=\min\{2,\frac{\nu\Delta tC_l^2}{4C_P^2}\}$. Inserting the last estimate in (4.20), using induction and (4.12) yields

$$\|\chi_u^{n+1}\|_G^2+\frac{\nu\Delta t}{4}\|\nabla u_h^{n+1}\|^2\leq\left(\frac{1}{1+\beta}\right)^{n+1}\Big(\|\chi_u^0\|_G^2+\frac{\nu\Delta t}{4}\|\nabla u_h^0\|^2\Big)+\frac{\nu^{-1}\Delta t}{\beta}M^* \tag{4.21}$$

where $M^*=2\left(10Ri^2C_P^2\max\{\|\theta_h^0\|^2,M^2\}+\|f\|^2_{L^\infty(\mathbb{R}_+;V_h^*)}\right)$. Noting that $\frac{\Delta t}{\beta}=\max\{\frac{1}{2}\Delta t,\frac{4C_P^2}{\nu C_l^2}\}=:\beta^*$ completes the proof. □

5 Long time stability of MHD in primitive variables with BDF2LE

The magnetohydrodynamics equations predict the dynamics of electrically conducting fluids and their interaction. Examples of such fluids include plasmas, liquid metals, salt water, and electrolytes. The governing equations for such flows are a combination of Navier-Stokes equations of fluid dynamics and Maxwell equations of electromagnetism [2, 10]:

$$u_t+(u\cdot\nabla)u-s(B\cdot\nabla)B-\nu\Delta u+\nabla P = f, \tag{5.1}$$
$$\nabla\cdot u = 0, \tag{5.2}$$
$$B_t+(u\cdot\nabla)B-(B\cdot\nabla)u-\nu_m\Delta B = \nabla\times g, \tag{5.3}$$
$$\nabla\cdot B = 0, \tag{5.4}$$

where u is the velocity of the fluid, B is the magnetic field, P is a modified pressure, ν is the kinematic viscosity, ν_m is the magnetic resistivity, s is the coupling number, f is the body force, and $\nabla\times g$ is the forcing on the magnetic field.

The linearized BDF2 finite element of the system (5.1)-(5.4) is as follows:

Algorithm 5.1 *Let the body forces f,g and the initial conditions u_0,B_0 be given. Define $u_h^{-1}=u_h^0$ and $B_h^{-1}=B_h^0$ to be the nodal interpolants of u_0 and B_0, respectively. Set a time step $\Delta t>0$. For all $n=0,1,2,3,...,$ calculate $(u_h^{n+1},P_h^{n+1},B_h^{n+1},\lambda_h^{n+1})\in(X_h,Q_h,X_h,Q_h)$ satisfying for every $(v_h,q_h,w_h,\lambda_h)\in(X_h,Q_h,X_h,Q_h)$,*

$$\frac{1}{2\Delta t}\Big(3u_h^{n+1}-4u_h^n+u_h^{n-1},v_h\Big)+b^*(2u_h^n-u_h^{n-1},u_h^{n+1},v_h)-sb^*(2B_h^n-B_h^{n-1},B_h^{n+1},v_h)$$
$$+\nu(\nabla u_h^{n+1},\nabla v_h)-(P_h^{n+1},\nabla\cdot v_h)=(f^{n+1},v_h) \tag{5.5}$$
$$(\nabla\cdot u_h^{n+1},q_h)=0, \tag{5.6}$$
$$\frac{1}{2\Delta t}\Big(3B_h^{n+1}-4B_h^n+B_h^{n-1},w_h\Big)+b^*(2u_h^n-u_h^{n-1},B_h^{n+1},w_h)-b^*(2B_h^n-B_h^{n-1},u_h^{n+1},w_h)$$
$$+\nu_m(\nabla B_h^{n+1},\nabla w_h)=(\nabla\times g^{n+1},w_h), \tag{5.7}$$
$$(\nabla\cdot B_h^{n+1},l_h)=0. \tag{5.8}$$

Theorem 5.2 *Let $f, \nabla\times g \in L^\infty(\mathbb{R}_+; V_h^*)$, $u_0, B_0 \in L^2(\Omega)^d$. Then the solution of Algorithm 5.1 satisfies*

$$
\begin{aligned}
&\|\chi_u^{n+1}\|_G^2 + s\|\chi_B^{n+1}\|_G^2 + \frac{\Delta t}{4}\Big(\nu\|\nabla u_h^{n+1}\|^2 + s\nu_m\|\nabla B_h^{n+1}\|^2\Big)\\
&\qquad \le \left(\frac{1}{1+\mu}\right)^{n+1}\left[\|\chi_u^0\|_G^2 + s\|\chi_B^0\|_G^2 + \frac{\Delta t}{4}\Big(\nu\|\nabla u_h^0\|^2 + s\nu_m\|\nabla B_h^0\|^2\Big)\right]\\
&\qquad\qquad + \mu^*\Big(\nu^{-1}\|f\|^2_{L^\infty(\mathbb{R}_+;V_h^*)} + s\nu_m^{-1}\|\nabla\times g\|^2_{L^\infty(\mathbb{R}_+;V_h^*)}\Big)
\end{aligned}
$$

where $\mu^* = \max\{\frac{1}{2}\Delta t, \frac{4C_P^2}{\nu C_l^2}, \frac{4C_P^2}{\nu_m C_l^2}\}$, $\mu = \min\{2, \frac{\nu C_l^2\Delta t}{4C_P^2}, \frac{\nu_m C_l^2 \Delta t}{4C_P^2}\}$.

Proof: The proof starts by choosing $v_h = 2\Delta t u_h^{n+1}$ in (5.5), $q_h = P_h^{n+1}$ in (5.6), $w_h = 2\Delta t B_h^{n+1}$ in (5.7) and $l_h = \lambda_h^{n+1}$ in (5.8). A straight forward calculation with (2.2) shows that

$$
\begin{aligned}
\|\chi_u^{n+1}\|_G^2 - \|\chi_u^n\|_G^2 + \frac{\|u_h^{n+1} - 2u_h^n + u_h^{n-1}\|^2}{2} - 2s\Delta t b^*(2B_h^n - B_h^{n-1}, B_h^{n+1}, u_h^{n+1})&\\
+2\nu\Delta t\|\nabla u_h^{n+1}\|^2 = 2\Delta t(f^{n+1}, u_h^{n+1}),& \qquad (5.9)\\
\|\chi_B^{n+1}\|_G^2 - \|\chi_B^n\|_G^2 + \frac{\|B_h^{n+1} - 2B_h^n + B_h^{n-1}\|^2}{2} - 2\Delta t b^*(2B_h^n - B_h^{n-1}, u_h^{n+1}, B_h^{n+1})&\\
+2\nu_m\Delta t\|\nabla B_h^{n+1}\|^2 = 2\Delta t(\nabla\times g^{n+1}, B_h^{n+1}).& \qquad (5.10)
\end{aligned}
$$

Multiplying equation (5.10) by s and adding it to (5.9), taking into account the skew-symmetry of nonlinear term yields

$$
\begin{aligned}
&\Big(\|\chi_u^{n+1}\|_G^2 + s\|\chi_B^{n+1}\|_G^2\Big) + 2\Delta t\Big(\nu\|\nabla u_h^{n+1}\|^2 + s\nu_m\|\nabla B_h^{n+1}\|^2\Big)\\
&\qquad \le \Big(\|\chi_u^n\|_G^2 + s\|\chi_B^n\|_G^2\Big) + 2\Delta t\Big((f^{n+1}, u_h^{n+1}) + s(\nabla\times g^{n+1}, B_h^{n+1})\Big). \qquad (5.11)
\end{aligned}
$$

Applying now the Cauchy-Schwarz and Young's inequalities on the forcing terms gives

$$
\begin{aligned}
&2\Delta t\Big((f^{n+1}, u_h^{n+1}) + s(\nabla\times g^{n+1}, B_h^{n+1})\Big)\\
&\qquad \le \Delta t\Big(\nu\|\nabla u_h^{n+1}\|^2 + s\nu_m\|\nabla B_h^{n+1}\|^2\Big) + \Delta t\Big(\nu^{-1}\|f^{n+1}\|^2_{V_h^*} + s\nu_m^{-1}\|\nabla\times g^{n+1}\|^2_{V_h^*}\Big).
\end{aligned}
$$

Using this estimate in (5.11), and adding $\frac{\Delta t}{4}\Big(\nu\|\nabla u_h^n\|^2 + s\nu_m\|\nabla B_h^n\|^2\Big)$ to the both sides of the inequality produces

$$\|\chi_u^{n+1}\|_G^2 + s\|\chi_B^{n+1}\|_G^2 + \frac{\Delta t}{4}\Big(\nu\|\nabla u_h^{n+1}\|^2 + s\nu_m\|\nabla B_h^{n+1}\|^2\Big)$$
$$+ \frac{\Delta t}{4}\Big(\nu\Big(\|\nabla u_h^{n+1}\|^2 + \|\nabla u_h^n\|^2\Big) + s\nu_m\Big(\|\nabla B_h^{n+1}\|^2 + \|\nabla B_h^n\|^2\Big)\Big)$$
$$+ \frac{\Delta t}{2}\Big(\nu\|\nabla u_h^{n+1}\|^2 + s\nu_m\|\nabla B_h^{n+1}\|^2\Big)$$
$$\leq \Big(\|\chi_u^n\|_G^2 + s\|\chi_B^n\|_G^2\Big) + \frac{\Delta t}{4}\Big(\nu\|\nabla u_h^n\|^2 + s\nu_m\|\nabla B_h^n\|^2\Big)$$
$$+ \Delta t\Big(\nu^{-1}\|f^{n+1}\|_{V_h^*}^2 + s\nu_m^{-1}\|\nabla \times g^{n+1}\|_{V_h^*}^2\Big). \quad (5.12)$$

Next, we rewrite the last two terms on the left hand side of (5.12) by using the Poincaré inequality and relation (2.1):

$$\frac{\Delta t}{4}\Big(\nu\Big(\|\nabla u_h^{n+1}\|^2 + \|\nabla u_h^n\|^2\Big) + s\nu_m\Big(\|\nabla B_h^{n+1}\|^2 + \|\nabla B_h^n\|^2\Big)\Big)$$
$$+ \frac{\Delta t}{2}\Big(\nu\|\nabla u_h^{n+1}\|^2 + s\nu_m\|\nabla B_h^{n+1}\|^2\Big)$$
$$\geq \frac{\nu C_l^2\Delta t}{4C_P^2}\|\chi_u^{n+1}\|_G^2 + \frac{\nu\Delta t}{2}\|\nabla u_h^{n+1}\|^2 + \frac{s\nu_m C_l^2\Delta t}{4C_P^2}\|\chi_B^{n+1}\|_G^2 + \frac{s\nu_m\Delta t}{2}\|\nabla B_h^{n+1}\|^2$$
$$\geq \mu\Big(\|\chi_u^{n+1}\|_G^2 + s\|\chi_B^{n+1}\|_G^2 + \frac{\Delta t}{4}\Big(\nu\|\nabla u_h^{n+1}\|^2 + s\nu_m\|\nabla B_h^{n+1}\|^2\Big)\Big),$$

where $\mu := \min\{2, \frac{\nu C_l^2\Delta t}{4C_P^2}, \frac{\nu_m C_l^2\Delta t}{4C_P^2}\}$. The last estimate produces

$$(1+\mu)\Big(\Big(\|\chi_u^{n+1}\|_G^2 + \frac{\nu\Delta t}{4}\|\nabla u_h^{n+1}\|^2\Big) + s\Big(\|\chi_B^{n+1}\|_G^2 + \frac{\nu_m\Delta t}{4}\|\nabla B_h^{n+1}\|^2\Big)\Big)$$
$$\leq \|\chi_u^n\|_G^2 + \frac{\nu\Delta t}{4}\|\nabla u_h^n\|^2 + s\Big(\|\chi_B^n\|_G^2 + \frac{\nu_m\Delta t}{4}\|\nabla B_h^n\|^2\Big)$$
$$+ \Delta t\Big(\nu^{-1}\|f^{n+1}\|_{V_h^*}^2 + s\nu_m^{-1}\|\nabla \times g^{n+1}\|_{V_h^*}^2\Big), \quad (5.13)$$

and then we use induction to get

$$LHS \leq \Big(\frac{1}{1+\mu}\Big)^{n+1}\Big(\|\chi_u^0\|_G^2 + s\|\chi_B^0\|_G^2 + \frac{\Delta t}{4}\Big(\nu\|\nabla u_h^0\|^2 + s\nu_m\|\nabla B_h^0\|^2\Big)\Big)$$
$$+ \frac{\Delta t}{\mu}\Big(\nu^{-1}\|f\|_{L^\infty(\mathbb{R}_+;V_h^*)}^2 + s\nu_m^{-1}\|\nabla \times g\|_{L^\infty(\mathbb{R}_+;V_h^*)}^2\Big). \quad (5.14)$$

The final step of the proof consists of considering $\mu^* := \frac{\Delta t}{\mu} = \max\{\frac{1}{2}\Delta t, \frac{4C_P^2}{\nu C_l^2}, \frac{4C_P^2}{\nu_m C_l^2}\}$ and using the equivalence of the G-norm and L^2-norm. □

6 The long time stability of MHD in Elsässer variables with BDF2LE

In this section, we study the long time behavior of MHD in Elsässer variables, which was first proposed by W. Elsässer in 1950 [4]:

$$v_t + w\cdot\nabla v - (\tilde{B}_0\cdot\nabla)v - \frac{\nu+\nu_m}{2}\Delta v - \frac{\nu-\nu_m}{2}\Delta w + \nabla q = f_1, \quad (6.1)$$

$$\nabla\cdot v = 0, \quad (6.2)$$

$$w_t + v\cdot\nabla w + (\tilde{B}_0\cdot\nabla)w - \frac{\nu+\nu_m}{2}\Delta w - \frac{\nu-\nu_m}{2}\Delta v + \nabla r = f_2, \quad (6.3)$$

$$\nabla\cdot w = 0. \quad (6.4)$$

The linearized BDF2 algorithm for the system (6.1)-(6.4) is given as below. It is a very interesting algorithm since it decouples the equations, which seemingly cannot be done in an unconditionally stable way for primitive variable MHD.

Algorithm 6.1 *Let f_1, f_2 and the initial conditions v_0, w_0 be given. Define $v_h^{-1} = v_h^0$, $w_h^{-1} = w_h^0$ to be nodal interpolants of v_0, w_0 and choose a time step $\Delta t > 0$. For $n = 0, 1, 2, 3, \ldots$ find $(v_h^{n+1}, q_h^{n+1}, w_h^{n+1}, r_h^{n+1}) \in (X_h, Q_h, X_h, Q_h)$ such that*

$$\frac{1}{2\Delta t}\Big(3v_h^{n+1} - 4v_h^n + v_h^{n-1}, \chi_h\Big) + b^*(w_h^n, v_h^{n+1}, \chi_h) + b^*(\tilde{B}_0, v_h^{n+1}, \chi_h)$$
$$+\frac{\nu+\nu_m}{2}(\nabla v_h^{n+1}, \nabla\chi_h) + \frac{\nu-\nu_m}{2}(\nabla(2w_h^n - w_h^{n-1}), \nabla\chi_h) - (q_h^{n+1}, \chi_h) = (f_1^{n+1}, \chi_h), \quad (6.5)$$

$$(\nabla\cdot v_h^{n+1}, p_h) = 0, \quad (6.6)$$

$$\frac{1}{2\Delta t}\Big(3w_h^{n+1} - 4w_h^n + w_h^{n-1}, \phi_h\Big) + b^*(v_h^n, w_h^{n+1}, \phi_h) + b^*(\tilde{B}_0, w_h^{n+1}, \phi_h)$$
$$+\frac{\nu+\nu_m}{2}(\nabla w_h^{n+1}, \nabla\phi_h) + \frac{\nu-\nu_m}{2}(\nabla(2v_h^n - v_h^{n-1}), \nabla\phi_h) - (r_h^{n+1}, \phi_h) = (f_2^{n+1}, \phi_h), \quad (6.7)$$

$$(\nabla\cdot w_h^{n+1}, l_h) = 0. \quad (6.8)$$

for all $(\chi_h, p_h, \phi_h, l_h) \in (X_h, Q_h, X_h, Q_h)$.

Theorem 6.2 *Let $f_1, f_2 \in L^\infty(\mathbb{R}_+; V_h^*)$, $u_0, B_0 \in L^2(\Omega)^d$ and $(v_h^{n+1}, q_h^{n+1}, w_h^{n+1}, r_h^{n+1})$ be the solution to the Algorithm 6.1. If*

$$\Delta t < \frac{2h^2\nu\nu_m}{C_I{}^2(\nu-\nu_m)^2(\nu+\nu_m)}, \quad (6.9)$$

then we have

$$\Big(\|\chi_v^{n+1}\|_G^2 + \|\chi_w^{n+1}\|_G^2\Big) + \frac{\nu\nu_m}{8(\nu+\nu_m)}\Delta t\Big(\|\nabla v_h^{n+1}\|^2 + \|\nabla w_h^{n+1}\|^2\Big)$$
$$\le \left(\frac{1}{1+\gamma}\right)^{n+1}\Big(\|\chi_v^0\|_G^2 + \|\chi_w^0\|_G^2 + \frac{\nu\nu_m}{8(\nu+\nu_m)}\Delta t(\|\nabla v_h^0\|^2 + \|\nabla w_h^0\|^2)\Big)$$
$$+ \frac{2\gamma^*(\nu+\nu_m)}{\nu\nu_m}\Big(\|f_1\|_{L^\infty(\mathbb{R}_+;V_h^*)} + \|f_2\|_{L^\infty(\mathbb{R}_+;V_h^*)}\Big),$$

where $\gamma^* = \max\{\frac{1}{2}\Delta t, \frac{8C_P^2(\nu+\nu_m)}{C_I^2\nu\nu_m}\}$, $\gamma = \min\{2, \frac{C_I^2\nu\nu_m\Delta t}{8C_P^2(\nu+\nu_m)}\}$.

Proof: First take $\chi_h = v_h^{n+1}$ in (6.5), $p_h = q_h^{n+1}$ in (6.6), $\phi_h = w_h^{n+1}$ in (6.7), $l_h = r_h^{n+1}$ in (6.8) and use relation (2.2) to produce

$$\frac{1}{2\Delta t}\left(\|\chi_v^{n+1}\|_G^2 - \|\chi_v^n\|_G^2\right) + \frac{1}{4\Delta t}\|v_h^{n+1} - 2v_h^n + v_h^{n-1}\|^2 + \frac{\nu+\nu_m}{2}\|\nabla v_h^{n+1}\|^2$$
$$= -\frac{\nu-\nu_m}{2}(\nabla w_h^{n+1}, \nabla v_h^{n+1}) + \frac{\nu-\nu_m}{2}(\nabla(w_h^{n+1} - 2w_h^n + w_h^{n-1}), \nabla v_h^{n+1}) + (f_1^{n+1}, v_h^{n+1}) \quad (6.10)$$

and

$$\frac{1}{2\Delta t}\left(\|\chi_w^{n+1}\|_G^2 - \|\chi_w^n\|_G^2\right) + \frac{1}{4\Delta t}\|w_h^{n+1} - 2w_h^n + w_h^{n-1}\|^2 + \frac{\nu+\nu_m}{2}\|\nabla w_h^{n+1}\|^2$$
$$= -\frac{\nu-\nu_m}{2}(\nabla v_h^{n+1}, \nabla w_h^{n+1}) + \frac{\nu-\nu_m}{2}(\nabla(v_h^{n+1} - 2v_h^n + v_h^{n-1}), \nabla w_h^{n+1}) + (f_2^{n+1}, w_h^{n+1}). \quad (6.11)$$

To bound the left hand side of (6.10), we apply Cauchy-Schwarz and Young's inequalities with $\varepsilon = \frac{\nu+\nu_m}{2}$ on the first right hand side term which gives:

$$\frac{\nu-\nu_m}{2}(\nabla w_h^{n+1}, \nabla v_h^{n+1}) \le \frac{|\nu-\nu_m|}{2}\|\nabla w_h^{n+1}\|\|\nabla v_h^{n+1}\|$$
$$\le \frac{\nu+\nu_m}{4}\|\nabla v_h^{n+1}\|^2 + \frac{(\nu-\nu_m)^2}{4(\nu+\nu_m)}\|\nabla w_h^{n+1}\|^2,$$

Similarly, the second term on the right hand side of (6.10) can be bounded with $\varepsilon = \frac{\nu\nu_m}{\nu+\nu_m}$ and inverse inequality:

$$\frac{\nu-\nu_m}{2}\left(\nabla(w_h^{n+1} - 2w_h^n + w_h^{n-1}), \nabla v_h^{n+1}\right)$$
$$\le C_I h^{-1}\frac{|\nu-\nu_m|}{2}\|w_h^{n+1} - 2w_h^n + w_h^{n-1}\|\|\nabla v_h^{n+1}\|$$
$$\le \frac{\nu\nu_m}{2(\nu+\nu_m)}\|\nabla v_h^{n+1}\|^2 + \frac{{C_I}^2 h^{-2}(\nu-\nu_m)^2(\nu+\nu_m)}{8\nu\nu_m}\|w_h^{n+1} - 2w_h^n + w_h^{n-1}\|^2.$$

With the choice of $\varepsilon = \frac{\nu\nu_m}{2(\nu+\nu_m)}$, the forcing term is estimated with

$$(f_1^{n+1}, v_h^{n+1}) \le \|f_1^{n+1}\|_{V_h^*}\|\nabla v_h^{n+1}\|$$
$$\le \frac{\nu\nu_m}{4(\nu+\nu_m)}\|\nabla v_h^{n+1}\|^2 + \frac{\nu+\nu_m}{\nu\nu_m}\|f_1^{n+1}\|_{V_h^*}^2. \quad (6.12)$$

Plug these estimates into (6.10) to produce

$$\frac{1}{2\Delta t}\left(\|\chi_v^{n+1}\|_G^2 - \|\chi_v^n\|_G^2\right) + \frac{1}{4\Delta t}\|v_h^{n+1} - 2v_h^n + v_h^{n-1}\|^2 + \frac{\nu+\nu_m}{4}\|\nabla v_h^{n+1}\|^2$$
$$\le \frac{3\nu\nu_m}{4(\nu+\nu_m)}\|\nabla v_h^{n+1}\|^2 + \frac{(\nu-\nu_m)^2}{4(\nu+\nu_m)}\|\nabla w_h^{n+1}\|^2$$
$$+\frac{{C_I}^2 h^{-2}(\nu-\nu_m)^2(\nu+\nu_m)}{8\nu\nu_m}\|w_h^{n+1} - 2w_h^n + w_h^{n-1}\|^2 + \frac{\nu+\nu_m}{\nu\nu_m}\|f_1^{n+1}\|_{V_h^*}^2. \quad (6.13)$$

Using the same technique, the left hand side of (6.11) can be bounded as follows:

$$\frac{1}{2\Delta t}\Big(\|\chi_w^{n+1}\|_G^2 - \|\chi_w^n\|_G^2\Big) + \frac{1}{4\Delta t}\|w_h^{n+1} - 2w_h^n + w_h^{n-1}\|^2 + \frac{\nu+\nu_m}{4}\|\nabla w_h^{n+1}\|^2$$
$$\le \frac{3\nu\nu_m}{4(\nu+\nu_m)}\|\nabla w_h^{n+1}\|^2 + \frac{(\nu-\nu_m)^2}{4(\nu+\nu_m)}\|\nabla v_h^{n+1}\|^2$$
$$+\frac{{C_I}^2h^{-2}(\nu-\nu_m)^2(\nu+\nu_m)}{8\nu\nu_m}\|v_h^{n+1} - 2v_h^n + v_h^{n-1}\|^2 + \frac{\nu+\nu_m}{\nu\nu_m}\|f_2^{n+1}\|_{V_h^*}^2. \tag{6.14}$$

Now adding these two equations results

$$\frac{1}{2\Delta t}\Big(\|\chi_v^{n+1}\|_G^2 + \|\chi_w^{n+1}\|_G^2\Big) + \frac{\nu\nu_m}{4(\nu+\nu_m)}\Big(\|\nabla v_h^{n+1}\|^2 + \|\nabla w_h^{n+1}\|^2\Big)$$
$$+\frac{1}{2\Delta t}\left(\frac{1}{2} - \frac{{C_I}^2h^{-2}(\nu-\nu_m)^2(\nu+\nu_m)}{4\nu\nu_m}\Delta t\right)\Big(\|v_h^{n+1} - 2v_h^n + v_h^{n-1}\|^2 + \|w_h^{n+1} - 2w_h^n + w_h^{n-1}\|^2\Big)$$
$$\le \frac{1}{2\Delta t}\Big(\|\chi_v^n\|_G^2 + \|\chi_w^n\|_G^2\Big) + \frac{\nu+\nu_m}{\nu\nu_m}\Big(\|f_1^{n+1}\|_{V_h^*}^2 + \|f_2^{n+1}\|_{V_h^*}^2\Big), \tag{6.15}$$

and multiplying by $2\Delta t$ and dropping the non-negative left hand side second term by using Δt restriction (6.9) gives

$$\Big(\|\chi_v^{n+1}\|_G^2 + \|\chi_w^{n+1}\|_G^2\Big) + \frac{\nu\nu_m\Delta t}{2(\nu+\nu_m)}\Big(\|\nabla v_h^{n+1}\|^2 + \|\nabla w_h^{n+1}\|^2\Big)$$
$$\le \Big(\|\chi_v^n\|_G^2 + \|\chi_w^n\|_G^2\Big) + \frac{2(\nu+\nu_m)\Delta t}{\nu\nu_m}\Big(\|f_1^{n+1}\|_{V_h^*}^2 + \|f_2^{n+1}\|_{V_h^*}^2\Big). \tag{6.16}$$

The last step follows from the application of the same technique as in Theorem 3.2. □

Remark 6.3 *It is proven in [12] that if $1/2 < \nu/\nu_m < 2$, then there is no timestep restriction for stability.*

6.1 Numerical Experiment

We now present numerical experiments for the linearized BDF2 scheme for MHD in Elsasser variables, given by Algorithm 6.1. The goal of the numerical experiment is to test the stability time step restriction of Theorem 6.2, and the result from [12] which states the method is unconditionally stable when $1/2 < \nu/\nu_m < 2$. We test values of ν/ν_m inside and outside of this range, and find that this restriction is sharp. As a test problem, we select initial conditions and forcing terms as follows

$$v_0 = (\cos(y), \sin(x))^T, \quad w_0 = (\sin(y), cos(x))^T$$
$$f_1 = (\sin(x+y), \cos(x-y))^T, \quad f_2 = (\cos(x-y), \sin(x+y))^T$$

We then calculate the approximate solutions of Algorithm 6.1 on 16×16 barycenter uniform mesh of the domain $\Omega = (0,1)^2$ by taking (P_2, P_1^{disc}) Scott-Vogelius finite elements. Computations are run to $T = 500$ for varying Δt and ν, ν_m. The results are seen below as plots of $(\|v_h^{n+1}\|^2 + \|w_h^{n+1}\|^2)$.

In Figure 3 and Figure 4, we observe that the solutions are stable in the case $1/2 < \nu/\nu_m < 2$. However, the linearized BDF2 discrete solutions fail to be stable outside of this interval as in Figure 5 and Figure 6. The solutions remain stable in the limit case $\nu/\nu_m = 2$ as in Figure 7 but become unstable for $\nu/\nu_m = 2.1$ (Figure 8).

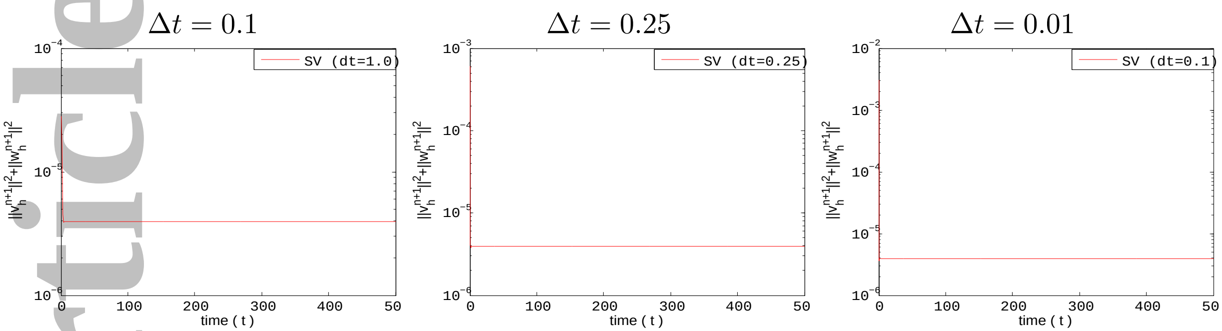

Figure 3: Energy vs. time for discrete velocity solutions of MHD in Elsässer variables with $\nu = 1.0, \nu_m = 1.0$ $(1/2 < \nu/\nu_m < 2)$.

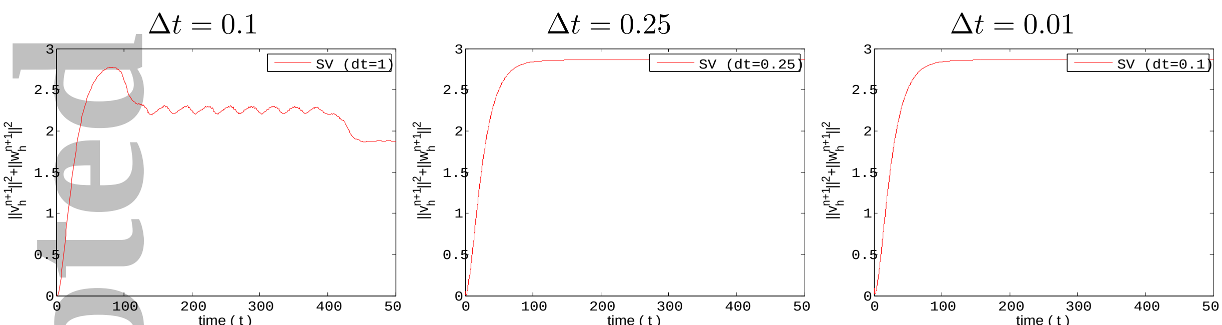

Figure 4: Energy vs. time for discrete velocity solutions of MHD in Elsässer variables with $\nu = 0.0125, \nu_m = 0.01$ $(1/2 < \nu/\nu_m < 2)$.

7 Conclusions

We have studied the long time behavior of the NSE and related multiphysics problems using the BDF2LE timestepping scheme together with finite element method. We prove that the approximate solutions of the proposed algorithm for the NSE, Boussinesq and MHD in primitive variables are uniformly bounded at all time without time step restriction. For MHD in Elsässer variables we obtain conditionally long time stability. We also provided numerical tests that showed the stability properties of the BDF2LE scheme for the NSE has better stability properties than CNLE for smaller viscosities, and for MHD we numerically showed that the restriction of the magnetic Prandtl number to $1/2 < \nu/\nu_m < 2$ is (at least) very close to being sharp.

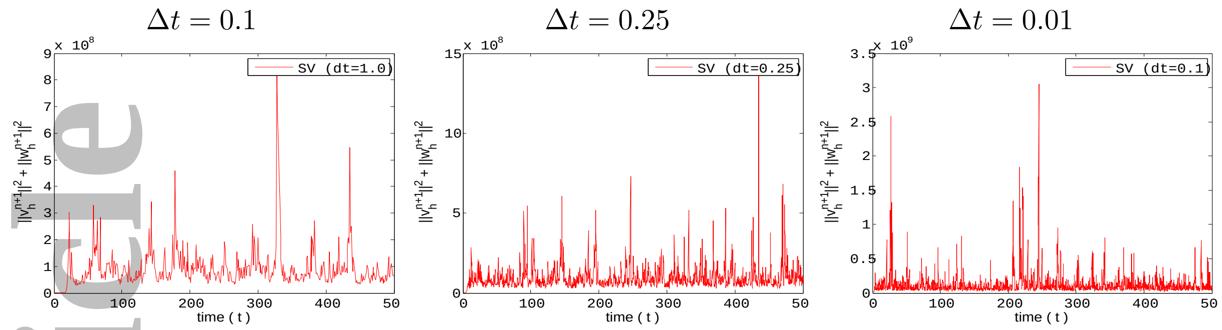

Figure 5: Energy vs. time for discrete velocity solutions of MHD in Elsässer variables with $\nu = 0.01$ and $\nu_m = 1$ ($\nu/\nu_m < 1/2$).

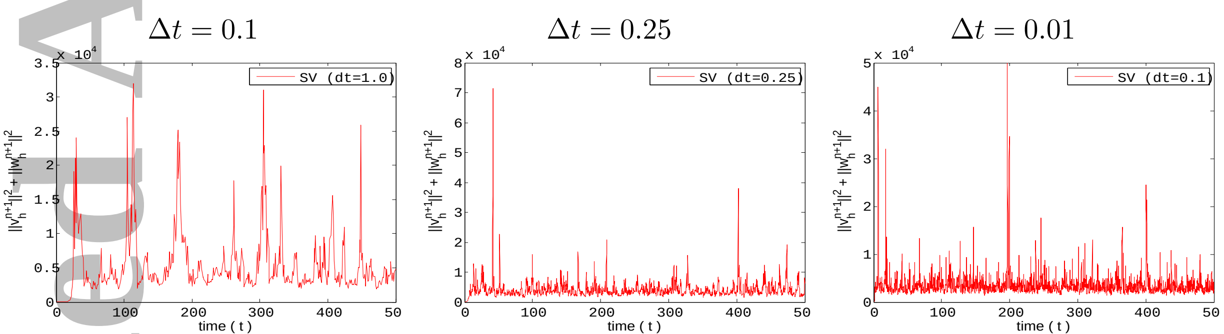

Figure 6: Energy vs. time for discrete velocity solutions of MHD in Elsässer variables with $\nu = 0.01$ and $\nu_m = 0.001$ ($\nu/\nu_m > 2$).

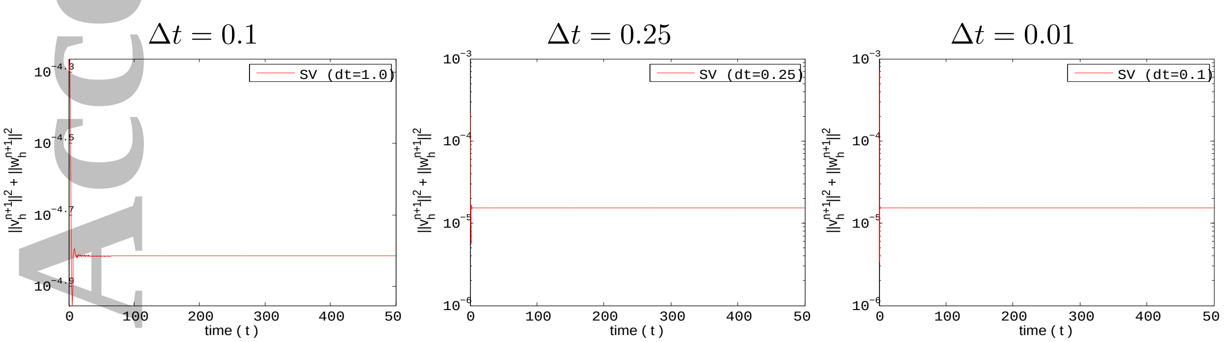

Figure 7: Energy vs. time for discrete velocity solutions of MHD in Elsässer variables with $\nu = 1$ and $\nu_m = 0.5$ ($\nu/\nu_m = 2$).

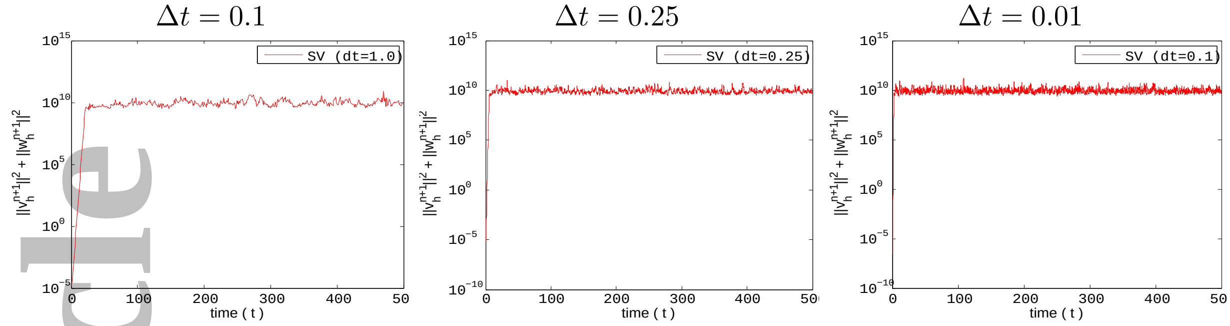

Figure 8: Energy vs. time for discrete velocity solutions of MHD in Elsässer variables with $\nu = 0.2$ and $\nu_m = .095$ $(\nu/\nu_m = 2.105)$.

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