# A dual pair of optimization-based formulations for estimation and control 

S. Emre Tuna*

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#### Abstract

A finite-horizon optimal estimation problem for discrete-time linear systems is formulated and solved. The formulation is a natural extension of that which yields a deadbeat observer. The resultant observer is the dual of the controller produced by the finite-horizon minimum energy control problem with terminal equality constraint. Nonlinear extensions of this dual pair are also considered and sufficient conditions are provided for stability and convergence.


## 1 Introduction

One of the earliest things that students of control theory are taught is that for linear systems controllability and observability are dual concepts. Very few doubt this because it is in every linear systems textbook. Interestingly, what is usually not in all those books is a clear definition of duality [9. A possibility is that no one wants to confine the notion into the precision required by a definition. Or, perhaps, it is too obvious a thing to define. Either way, people do not seem to need its exact description in order to make use of or enjoy duality; for once a dual pair emerges, the human eye is very quick to recognize it.

An intriguing example of duality is between the problems of linear quadratic regulation (LQR) and linear quadratic estimation (LQE, Kalman-Bucy filter). These celebrated optimization problems, which are very different conceptually and formulation-wise, yield sets of parameters (matrices) that are associated via formal rules that transform one set to another [6] The problems of linear deadbeat control and linear deadbeat estimation make another example of a

[^0]dual pair. Let us recall the former. Consider the below systems, both $n$th order,
\[

$$
\begin{align*}
x_{k+1} & =A x_{k}  \tag{1}\\
\hat{x}_{k+1} & =A \hat{x}_{k}+B u_{k} \tag{2}
\end{align*}
$$
\]

where the system (2) is to track the system (1) by choosing suitable control inputs $u_{0}, u_{1}, \ldots$ (Let us assume for now that the controllability condition is satisfied, input $u$ is scalar, and the full state information $(\hat{x}, x)$ of both systems is available to the controller.) To turn the system (2) into a deadbeat tracker for the system (11), i.e., to achieve $\hat{x}_{k}=x_{k}$ for $k \geq n$, one can follow either of the below methods.
(M1) Apply $u_{k}=K\left(x_{k}-\hat{x}_{k}\right)$ where the row vector $K$ is such that all the eigenvalues of $A-B K$ are at the origin.
(M2) Apply $u_{k}$ from the sequence of inputs $\left(u_{k}, u_{k+1}, \ldots, u_{k+n-1}\right)$ obtained by solving $\hat{x}_{k+n}=A^{n} x_{k}$.

These methods are mathematically equivalent since, in the end, they result in the same thing. However, the latter is superior to the former in the following sense. Firstly, the feedback gain $K$ naturally comes out of the solution of $A^{n} x_{k}=\hat{x}_{k+n}=A^{n} \hat{x}_{k}+A^{n-1} B u_{k}+\ldots+A B u_{k+n-2}+B u_{k+n-1}$. Note that the first method does not give any clues as regards to the computation of $K$. Secondly, and more importantly, the second method is meaningful also for nonlinear systems, which is not the case with the first one.

If we now move to the dual problem, linear deadbeat estimation, the translation of the first statement (M1) is well known. It boils down to something like "Choose an observer gain (say $L$ ) such that all the eigenvalues of the matrix describing the error dynamics (say $A-L C$ ) are at the origin." However, how to translate the more valuable second statement (M2) is not immediately clear. Motivated by the historical pattern that beautiful things tend to come in dual pairs for linear systems, our work here starts with a search for this missing twin of (M2). In more exact terms, guided by linear duality, we look for some sort of a principle that not only leads to linear deadbeat observer but also is useful for nonlinear deadbeat observer design. This search is nothing but a simple linear algebra exercise, but its outcome turns out to have some interesting consequences that go beyond linear and deadbeat. Those consequences are what we report in this paper. In particular, three things are done:

First. In Theorem 2 observer design for linear systems is formulated as a finite-horizon optimization problem. The formulation concerns a movinghorizon type observer (whose order matches that of the system being observed) where at each time an estimate of the system state is generated based solely on the current output (instead of a larger collection of data comprising previous measurements) of the system and the current observer state. Convergence is guaranteed for all horizon lengths no smaller than the order of the system being observed. Interestingly, the formulation presented here turns out to be the dual
of a classic result (Theorem 3) by Kleinman [8], who is acknowledged to be the first to consider moving-horizon feedback [7].

Second. In Theorem 4 a nonlinear generalization of the linear optimal observer construction of Theorem 2 is provided, where convergence is established under certain conditions inspired by those that hold in the linear problem. The resulting nonlinear moving-horizon observer, like its above-mentioned linear version, is driven only by the current output value of the system being observed. This constitutes a conceptual difference between the construction in this paper and the majority of the work on moving horizon estimation [12, 1, 11], the basic philosophy of which is summarized in [3] as: The estimates of the states are obtained by solving a least squares problem, which penalizes the deviation between measurements and predicted outputs of a system. The data considered for the optimization is laying in a window of fixed finite length, which slides forward in time.

Third. For the sake of symmetry we present in Theorem 5 a possible nonlinear extension of Kleinman's optimal controller (Theorem 3). More specifically, a moving-horizon optimal tracking problem is considered, where convergence is established mainly through terminal equality constraint. We note that more general results, i.e., ones that do not require terminal equality constraint or terminal cost, have long existed in the receding horizon control literature [10, 5].

## 2 Notation

$\mathbb{N}$ denotes the set of nonnegative integers and $\mathbb{R}_{\geq 0}$ the set of nonnegative real numbers. For a mapping $f: \mathcal{X} \rightarrow \mathcal{X}$ let $f^{0}(x)=x$ and $f^{k+1}(x)=f\left(f^{k}(x)\right)$. Euclidean norm in $\mathbb{R}^{n}$ is denoted by $\|\cdot\|$. For a symmetric positive definite matrix $Q \in \mathbb{R}^{n \times n}$ the smallest and largest eigenvalues of $Q$ are respectively denoted by $\lambda_{\min }(Q)$ and $\lambda_{\max }(Q)$. Also, $\|x\|_{Q}^{2}=x^{T} Q x$. A function $\alpha: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is said to belong to class- $\mathcal{K}_{\infty}\left(\alpha \in \mathcal{K}_{\infty}\right)$ if it is continuous, zero at zero, strictly increasing, and unbounded.

## 3 An optimal observer

We begin this section by an attempt to obtain the dual of the statement (M2), i.e., some method to construct deadbeat observer, which is meaningful also for nonlinear systems. Consider the discrete-time linear system

$$
\begin{equation*}
x^{+}=A x, \quad y=C x \tag{3}
\end{equation*}
$$

where $x \in \mathbb{R}^{n}$ is the state, $y \in \mathbb{R}^{m}$ is the output, and $x^{+}$is the state at the next time instant. The matrices $A$ and $C$ belong to $\mathbb{R}^{n \times n}$ and $\mathbb{R}^{m \times n}$, respectively. We will denote the solution of the system (3) by $x_{k}$ for $k \in \mathbb{N}$. Driven by the output $y$ of the system (3) suppose that the below system, for $N \geq 1$,

$$
\begin{equation*}
z^{+}=A \eta(z, y) \tag{4}
\end{equation*}
$$

produces at each time $k$ an estimate $z_{k}$ of $x_{k-N+1}$ (the $N-1$ steps earlier value of the current state $x$ ) based on $z_{k-1}$ and $y_{k-1}$. That is, the vector $\eta \in \mathbb{R}^{n}$ is a function of the state $z$ and the output $y$. Note that the system (4) can be used in the following observer

$$
\begin{equation*}
z^{+}=A \eta, \quad \hat{x}=A^{N-1} z \tag{5}
\end{equation*}
$$

where $\hat{x}$ is the estimate of the current state $x$. Assuming for now that the system (3) is observable and its output $y$ is scalar, we now ask the following question. How should $\eta$ be chosen such that the system (5) is a deadbeat observer for the system (3), i.e., $\hat{x}_{k}=x_{k}$ for $k \geq n$ ?

To answer the question we recall the deadbeat tracker, the dual of deadbeat observer. From (M2) it follows that the dynamics of the deadbeat tracker read

$$
\hat{x}^{+}=A \hat{x}+B K(x-\hat{x})
$$

with the feedback gain

$$
K=e_{n}^{T} \mathcal{C}^{-1} A^{n}
$$

where $\mathcal{C}=\left[\begin{array}{llll}B A B \ldots & A^{n-1} B\end{array}\right]$ is the controllability matrix and $e_{n}=\left[\begin{array}{llll}0 & \ldots & 0 & 1\end{array}\right]^{T}$. By duality the dynamics of the deadbeat observer should read

$$
\begin{equation*}
\hat{x}^{+}=A \hat{x}+L(y-C \hat{x}) \tag{6}
\end{equation*}
$$

with the observer gain

$$
\begin{equation*}
L=A^{n} \mathcal{O}^{-1} e_{n} \tag{7}
\end{equation*}
$$

where $\mathcal{O}=\left[\begin{array}{llll}C^{T} & A^{T} C^{T} \ldots & A^{(n-1) T} C^{T}\end{array}\right]^{T}$ is the observability matrix. Now, combining (15), (6), and (7) we can write

$$
\begin{aligned}
A^{N} \eta & =A^{N-1} z^{+} \\
& =\hat{x}^{+} \\
& =A \hat{x}+A^{n} \mathcal{O}^{-1} e_{n}(y-C \hat{x}) \\
& =A^{N} z+A^{n} \mathcal{O}^{-1} e_{n}\left(y-C A^{N-1} z\right)
\end{aligned}
$$

If we let $N=n$ we can write

$$
A^{n} \eta=A^{n}\left(z+\mathcal{O}^{-1} e_{n}\left(y-C A^{n-1} z\right)\right)
$$

which suggests we choose $\eta$ as

$$
\begin{equation*}
\eta=z+\mathcal{O}^{-1} e_{n}\left(y-C A^{n-1} z\right) \tag{8}
\end{equation*}
$$

Equation (8) is not directly generalizable to nonlinear systems so we rewrite it as the following set of equations

$$
\left.\begin{array}{rl}
C \eta & =C z  \tag{9}\\
C A \eta & =C A z \\
& \vdots \\
C A^{n-2} \eta & =C A^{n-2} z \\
C A^{n-1} \eta & =y
\end{array}\right\}
$$

Therefore, to turn the system (5) (for $N=n$ ) into a deadbeat observer for the system (3) one can use the below algorithm.
(M3) Choose $\eta_{k}$ such that the would-be future output values $C A^{i} \eta_{k}$ match the would-be future output values of the current observer state $C A^{i} z_{k}$ for $i=0,1, \ldots, n-2$; and the would-be future output value $C A^{n-1} \eta_{k}$ matches the current measurement $y_{k}$.

The statement (M3) seems to be the dual of (M2). Happily, it serves our purpose in the sense that it allows one to construct nonlinear deadbeat observers. The formal treatment of the case is as follows.

Consider the system

$$
\begin{equation*}
x^{+}=f(x), \quad y=h(x) \tag{10}
\end{equation*}
$$

with $f: \mathcal{X} \rightarrow \mathcal{X}$ and $h: \mathcal{X} \rightarrow \mathcal{Y}$. Now consider the observer system

$$
\begin{equation*}
z^{+}=f(\eta), \quad \hat{x}=f^{N-1}(z) \tag{11}
\end{equation*}
$$

for some integer $N \geq 1$.
Assumption 1 For each $\xi \in \mathcal{Y}^{N}$ the equation

$$
\left[\begin{array}{c}
h(\eta) \\
h(f(\eta)) \\
\vdots \\
h\left(f^{N-1}(\eta)\right)
\end{array}\right]=\xi
$$

has a unique solution $\eta \in \mathcal{X}$.
Note that when the system (10) is linear with scalar output, Assumption 1 (with $N=n$ ) is equivalent to observability. The linear statement (M3) leads to the following result by Glad [4]. For a geometric interpretation see [13].

Theorem 1 Consider the system (10) and the observer (11). Suppose Assumption 1 holds and let $\eta$ be chosen to satisfy

$$
\begin{aligned}
h(\eta) & =h(z) \\
h(f(\eta)) & =h(f(z)) \\
& \vdots \\
h\left(f^{N-2}(\eta)\right) & =h\left(f^{N-2}(z)\right) \\
h\left(f^{N-1}(\eta)\right) & =y
\end{aligned}
$$

Then, for all initial conditions, $\hat{x}_{k}=x_{k}$ for all $k \geq N$.

Proof. The result follows trivially for $N=1$. Suppose now $N \geq 2$ and for some $p \in\{1,2, \ldots, N-1\}$ and some $k \geq 0$ we have

$$
\begin{equation*}
h\left(f^{N-q}\left(\eta_{k}\right)\right)=y_{k-q+1} \quad \forall q \in\{1,2, \ldots, p\} \tag{12}
\end{equation*}
$$

Then we can write

$$
\begin{aligned}
h\left(f^{N-q-1}\left(\eta_{k+1}\right)\right) & =h\left(f^{N-q-1}\left(z_{k+1}\right)\right) \\
& =h\left(f^{N-q-1}\left(f\left(\eta_{k}\right)\right)\right) \\
& =h\left(f^{N-q}\left(\eta_{k}\right)\right) \\
& =y_{k-q+1}
\end{aligned}
$$

Also, $h\left(f^{N-1}\left(\eta_{k+1}\right)\right)=y_{k+1}$ holds by definition. Hence (12) implies

$$
h\left(f^{N-q}\left(\eta_{k+1}\right)\right)=y_{(k+1)-q+1} \quad \forall q \in\{1,2, \ldots, p+1\}
$$

Now, (12) holds at time $k=0$ for $p=1$. By induction therefore we can write

$$
\left[\begin{array}{c}
h\left(\eta_{k}\right) \\
\vdots \\
h\left(f^{N-2}\left(\eta_{k}\right)\right) \\
h\left(f^{N-1}\left(\eta_{k}\right)\right)
\end{array}\right]=\left[\begin{array}{c}
y_{k-N+1} \\
\vdots \\
y_{k-1} \\
y_{k}
\end{array}\right]=\left[\begin{array}{c}
h\left(x_{k-N+1}\right) \\
\vdots \\
h\left(f^{N-2}\left(x_{k-N+1}\right)\right) \\
h\left(f^{N-1}\left(x_{k-N+1}\right)\right)
\end{array}\right]
$$

for all $k \geq N-1$. Then by Assumption 1 we have $\eta_{k}=x_{k-N+1}$ for all $k \geq N-1$. The result follows since $\hat{x}_{k+1}=f^{N}\left(\eta_{k}\right)$.

As Theorem 1 depicted, the rationale behind the set of linear equations (9) allows us to construct a nonlinear deadbeat observer. What else can we get out of (9)? Now we attempt to answer this question.

Let us once again consider the system (3) together with the observer (5) and write the general version of (9)

$$
\left.\begin{array}{rl}
C \eta & =C z  \tag{13}\\
C A \eta & =C A z \\
& \vdots \\
C A^{N-2} \eta & =C A^{N-2} z \\
C A^{N-1} \eta & =y
\end{array}\right\}
$$

where $N$ need not equal the order of the system (3). Suppose now the set of equations (13) is overdetermined and does not admit a solution $\eta$. How to choose $\eta$ then? Any textbook on linear algebra would suggest the least squares approximation, which leads to the following result.

Theorem 2 Consider the system (31). Let $N \geq 1$ be such that the matrix $\left[\begin{array}{llll}C^{T} & A^{T} C^{T} \ldots & A^{(N-1) T} C^{T}\end{array}\right]$ is full row rank. Let $R \in \mathbb{R}^{m \times m}$ be a symmetric
positive definite matrix and consider the observer (5) with $\eta=\arg \min _{\xi} J(\xi, z, y)$ where

$$
\begin{equation*}
J(\xi, z, y):=\left\|C A^{N-1} \xi-y\right\|_{R}^{2}+\sum_{i=0}^{N-2}\left\|C A^{i} \xi-C A^{i} z\right\|_{R}^{2} \tag{14}
\end{equation*}
$$

Then $\left\|\hat{x}_{k}-x_{k}\right\| \rightarrow 0$ as $k \rightarrow \infty$.
Proof. Let us define the symmetric matrices

$$
\begin{aligned}
Q & :=C^{T} R C+A^{T} C^{T} R C A+\ldots+A^{(N-2) T} C^{T} R C A^{N-2} \\
H & :=A^{(N-1) T} C^{T} R C A^{N-1}
\end{aligned}
$$

Note that by rank assumption the matrix $Q+H$ is nonsingular. Solving $\partial J / \partial \xi=$ 0 we obtain

$$
\begin{equation*}
\eta=(Q+H)^{-1} A^{(N-1) T} C^{T} R y+(Q+H)^{-1} Q z \tag{15}
\end{equation*}
$$

Let us define the shorthand notation $\tilde{x}_{k}:=x_{k-N+1}$ for $k \geq N-1$. Then we have $x=A^{N-1} \tilde{x}$ and $y=C A^{N-1} \tilde{x}$. We can now rewrite (15) as

$$
\begin{align*}
\eta & =(Q+H)^{-1} H \tilde{x}+(Q+H)^{-1} Q z \\
& =z+(Q+H)^{-1} H(\tilde{x}-z)  \tag{16}\\
& =\tilde{x}+(Q+H)^{-1} Q(z-\tilde{x})
\end{align*}
$$

Then

$$
\begin{align*}
\eta-\tilde{x} & =(Q+H)^{-1} Q(z-\tilde{x})  \tag{17}\\
\eta-z & =(Q+H)^{-1} H(\tilde{x}-z)
\end{align*}
$$

and we can write

$$
\begin{align*}
J(\eta, z, y)= & (\eta-\tilde{x})^{T} H(\eta-\tilde{x})+(\eta-z)^{T} Q(\eta-z) \\
= & (z-\tilde{x})^{T} Q(Q+H)^{-1} H(Q+H)^{-1} Q(z-\tilde{x}) \\
& +(z-\tilde{x})^{T} H(Q+H)^{-1} Q(Q+H)^{-1} H(z-\tilde{x}) \tag{18}
\end{align*}
$$

Note that

$$
\begin{align*}
Q(Q+H)^{-1} H & =(Q+H-H)(Q+H)^{-1}(Q+H-Q) \\
& =H(Q+H)^{-1} Q \tag{19}
\end{align*}
$$

Combining (18) and (19) we can write

$$
\begin{align*}
J(\eta, z, y)= & (z-\tilde{x})^{T} Q(Q+H)^{-1} H(Q+H)^{-1} Q(z-\tilde{x}) \\
& +(z-\tilde{x})^{T} H(Q+H)^{-1} H(Q+H)^{-1} Q(z-\tilde{x}) \\
= & (z-\tilde{x})^{T}\left(Q(Q+H)^{-1}+H(Q+H)^{-1}\right) H(Q+H)^{-1} Q(z-\tilde{x}) \\
= & (z-\tilde{x})^{T}(Q+H)(Q+H)^{-1} H(Q+H)^{-1} Q(z-\tilde{x}) \\
= & (z-\tilde{x})^{T} H(Q+H)^{-1} Q(z-\tilde{x}) \tag{20}
\end{align*}
$$

Now, by (17) we can write

$$
\begin{equation*}
(\eta-\tilde{x})^{T}(Q+H)(\eta-\tilde{x})=(z-\tilde{x})^{T} Q(Q+H)^{-1} Q(z-\tilde{x}) \tag{21}
\end{equation*}
$$

Then, by (20) and (21) we have

$$
\begin{equation*}
J(\eta, z, y)+(\eta-\tilde{x})^{T}(Q+H)(\eta-\tilde{x})=(z-\tilde{x})^{T} Q(z-\tilde{x}) \tag{22}
\end{equation*}
$$

Since

$$
\begin{aligned}
A^{T} Q A & =Q+H-C^{T} R C \\
& \leq Q+H
\end{aligned}
$$

we can write by (22)

$$
\begin{align*}
\left(z^{+}-\tilde{x}^{+}\right)^{T} Q\left(z^{+}-\tilde{x}^{+}\right) & =(\eta-\tilde{x})^{T} A^{T} Q A(\eta-\tilde{x}) \\
& \leq(\eta-\tilde{x})^{T}(Q+H)(\eta-\tilde{x}) \\
& =(z-\tilde{x})^{T} Q(z-\tilde{x})-J(\eta, z, y) . \tag{23}
\end{align*}
$$

Note that (23) could serve as a Lyapunov inequality if $Q$ were positive definite, which we do not assume. Still, (23) is whence we extract stability. First we need to demonstrate the following.

Claim: For each $\varepsilon \geq 0$ there exists $\delta \geq 0$ such that for all $k_{1} \in \mathbb{N}$

$$
\begin{equation*}
J\left(\eta_{k}, z_{k}, y_{k}\right) \leq \delta \quad \forall k \geq k_{1} \Longrightarrow\left\|\hat{x}_{k}-x_{k}\right\| \leq \varepsilon \quad \forall k \geq k_{1}+N \tag{24}
\end{equation*}
$$

We prove this claim as follows. Let us for some $\delta$ and $k_{1}$ have $J\left(\eta_{k}, z_{k}, y_{k}\right) \leq \delta$ for all $k \geq k_{1}$, which by (14) implies

$$
\max \left\{\left\|C A^{N-1} \eta_{k}-y_{k}\right\|,\left\|C A^{N-2}\left(\eta_{k}-z_{k}\right)\right\|, \ldots,\left\|C\left(\eta_{k}-z_{k}\right)\right\|\right\} \leq \delta_{1}
$$

with $\delta_{1}=\sqrt{\delta / \lambda_{\min }(R)}$. The claim is evident for $N=1$. Consider now $N \geq 2$ and suppose for some $p \in\{1,2, \ldots, N-1\}$ and some $k_{p} \geq k_{1}$ we have

$$
\begin{equation*}
\left\|C A^{N-q} \eta_{k}-y_{k-q+1}\right\| \leq q \delta_{1} \quad \forall q \in\{1,2, \ldots, p\} \quad \forall k \geq k_{p} \tag{25}
\end{equation*}
$$

Then we can write for $q \in\{1,2, \ldots, p\}$ and $k \geq k_{p}$

$$
\begin{aligned}
\left\|C A^{N-(q+1)} \eta_{k+1}-y_{(k+1)-(q+1)+1}\right\| \leq & \left\|C A^{N-q-1}\left(\eta_{k+1}-z_{k+1}\right)\right\| \\
& +\left\|C A^{N-q-1} z_{k+1}-y_{k-q+1}\right\| \\
= & \left\|C A^{N-q-1}\left(\eta_{k+1}-z_{k+1}\right)\right\| \\
& +\left\|C A^{N-q} \eta_{k}-y_{k-q+1}\right\| \\
\leq & \delta_{1}+q \delta_{1} \\
= & (q+1) \delta_{1}
\end{aligned}
$$

which allows us to assert

$$
\left\|C A^{N-q} \eta_{k}-y_{k-q+1}\right\| \leq q \delta_{1} \quad \forall q \in\{1,2, \ldots, p+1\} \quad \forall k \geq k_{p}+1
$$

Since (25) holds with $p=1$, by induction we can write

$$
\begin{equation*}
\left\|C A^{N-q} \eta_{k}-y_{k-q+1}\right\| \leq q \delta_{1} \quad \forall q \in\{1,2, \ldots, N\} \quad \forall k \geq k_{1}+N-1 \tag{26}
\end{equation*}
$$

Define the matrix $W:=\left[\begin{array}{llll}C^{T} & A^{T} C^{T} \ldots A^{(N-1) T} C^{T}\end{array}\right]^{T}$. Now by (26) we can write

$$
\begin{aligned}
\left\|\hat{x}_{k+1}-x_{k+1}\right\|^{2} & =\left\|A^{N}\left(\eta_{k}-\tilde{x}_{k}\right)\right\|^{2} \\
& \leq \lambda_{\max }\left(A^{N T} A^{N}\right)\left\|\left(\eta_{k}-\tilde{x}_{k}\right)\right\|^{2} \\
& \leq \lambda_{\max }\left(A^{N T} A^{N}\right) \lambda_{\min }^{-1}\left(W^{T} W\right)\left\|W\left(\eta_{k}-\tilde{x}_{k}\right)\right\|^{2} \\
& =\lambda_{\max }\left(A^{N T} A^{N}\right) \lambda_{\min }^{-1}\left(W^{T} W\right) \sum_{q=1}^{N}\left\|C A^{N-q} \eta_{k}-y_{k-q+1}\right\|^{2} \\
& \leq \lambda_{\max }\left(A^{N T} A^{N}\right) \lambda_{\min }^{-1}\left(W^{T} W\right) \delta_{1}^{2} \sum_{q=1}^{N} q^{2} \\
& =\frac{\left(2 N^{3}+3 N^{2}+N\right) \lambda_{\max }\left(A^{N T} A^{N}\right) \delta}{6 \lambda_{\min }\left(W^{T} W\right) \lambda_{\min }(R)} .
\end{aligned}
$$

This proves our claim because given any $\varepsilon$, we can choose

$$
\delta \leq \frac{6 \lambda_{\min }\left(W^{T} W\right) \lambda_{\min }(R) \varepsilon^{2}}{\left(2 N^{3}+3 N^{2}+N\right) \lambda_{\max }\left(A^{N T} A^{N}\right)}
$$

to satisfy (24). Now we return to the proof of the theorem. Observe that the inequality (23) implies that the sum $\sum_{k=0}^{\infty} J\left(\eta_{k}, z_{k}, y_{k}\right)$ is bounded. Since the terms being summed are all nonnegative we must have $J\left(\eta_{k}, z_{k}, y_{k}\right) \rightarrow 0$ as $k \rightarrow \infty$, which by (24) yields $\left\|\hat{x}_{k}-x_{k}\right\| \rightarrow 0$ as $k \rightarrow \infty$.

We note that the optimal observer coming out of the formulation depicted in Theorem 2 enjoys the classic linear observer structure $\hat{x}^{+}=A \hat{x}+L(y-C \hat{x})$ with the observer gain

$$
\begin{equation*}
L=A^{N}\left(C^{T} R C+\ldots+A^{(N-1) T} C^{T} R C A^{N-1}\right)^{-1} A^{(N-1) T} C^{T} R \tag{27}
\end{equation*}
$$

following from (5) and (16). Theorem 2 then implies that the eigenvalues of the matrix $A-L C$ (the system matrix of the error dynamics) must all be within the open unit disc.

## 4 An optimal tracker

The previous section started with a search for the principle behind deadbeat observer. Our search was driven by the question what method would lead to the observer gain given in (7), where the gain (7) was obtained by duality from the feedback gain of the deadbeat tracker. In this section we will employ duality once again, this time however in the other direction. In particular, we ask
the following question. What is the dual of the optimal observer described in Theorem 2? Or, more directly, what is the optimization problem that leads to the following feedback gain?

$$
\begin{equation*}
K=R B^{T} A^{(N-1) T}\left(B R B^{T}+\ldots+A^{N-1} B R B^{T} A^{(N-1) T}\right)^{-1} A^{N} \tag{28}
\end{equation*}
$$

which we obtain from (27) by duality. The answer is the below result by Kleinman [8], which is sometimes called the minimum energy control problem.

Theorem 3 Consider the system (11) and the tracker (2) with $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$. Let $N \geq 1$ be such that the matrix $\left[B A B \ldots A^{N-1} B\right]$ is full row rank and let $R \in \mathbb{R}^{m \times m}$ be a symmetric positive definite matrix. Let the control input of the tracker be $u=v_{0}(\hat{x}, x)$ where $v_{0}$ is the first term of the sequence $\left(v_{0}, v_{1}, \ldots, v_{N-1}\right)$ satisfying
$\left(v_{i}\right)_{i=0}^{N-1}=\arg \min _{\left(w_{i}\right)_{i=0}^{N-1}} \sum_{i=0}^{N-1}\left\|w_{i}\right\|_{R^{-1}}^{2} \quad$ subject to $\quad\left\{\begin{aligned} z_{0} & =\hat{x} \\ z_{i+1} & =A z_{i}+B w_{i} \\ z_{N} & =A^{N} x\end{aligned}\right.$
Then $\left\|\hat{x}_{k}-x_{k}\right\| \rightarrow 0$ as $k \rightarrow \infty$.
Proof. Given $\hat{x}$ and $x$, one can show that $v_{0}=K(x-\hat{x})$ with $K$ given in (28). In the light of duality convergence then follows from Theorem 2 and (27).

## 5 Nonlinear formulations

In this section we present possible nonlinear extensions of the linear formulations described earlier in the paper. First, for the observer design problem we will propose an optimization-based formulation that leads to desired observer behavior under certain sufficient conditions. Then we will repeat the procedure for the tracker design problem. Throughout this section the pairs $\left(\mathcal{X}, \rho_{\mathrm{x}}\right)$ and $\left(\mathcal{Y}, \rho_{\mathrm{y}}\right)$ will denote finite-dimensional complete metric spaces [2].

Caveat. Henceforth we will avoid the standard use of parentheses when the risk of confusion is negligible. For instance, $h(f(x))$ will be replaced by $h f x$.

### 5.1 Observer design

Consider the system (10) and the observer (11). We let $f$ and $h$ be uniformly continuous functions. Let $\ell: \mathcal{Y} \times \mathcal{Y} \rightarrow \mathbb{R}_{>0}$ and $\alpha_{1}, \alpha_{2} \in \mathcal{K}_{\infty}$ satisfy $\alpha_{1} \rho_{\mathrm{y}}(v, w) \leq \ell(v, w) \leq \alpha_{2} \rho_{\mathrm{y}}(v, w)$ for every $v, w \in \mathcal{Y}$. There is no harm in assuming the symmetry $\ell(v, w)=\ell(w, v)$. Now we define the cost function $J: \mathcal{X} \times \mathcal{X} \times \mathcal{Y} \rightarrow \mathbb{R}_{\geq 0}$ as

$$
\begin{equation*}
J(\xi, z, y):=\ell\left(h f^{N-1} \xi, y\right)+\sum_{i=0}^{N-2} \ell\left(h f^{i} \xi, h f^{i} z\right) \tag{29}
\end{equation*}
$$

## Assumption 2 The following hold.

1. There exists $\alpha_{3} \in \mathcal{K}_{\infty}$ such that for all $z, \tilde{x} \in \mathcal{X}$ we have

$$
\begin{equation*}
\sum_{i=0}^{N-1} \ell\left(h f^{i} z, h f^{i} \tilde{x}\right) \geq \alpha_{3} \rho_{\mathrm{x}}(z, \tilde{x}) \tag{30}
\end{equation*}
$$

2. There exists $\eta: \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{X}$ such that $J(\eta(z, y), z, y)<J(\xi, z, y)$ for all $\xi \neq \eta(z, y)$. Moreover, there exists $\alpha_{4} \in \mathcal{K}_{\infty}$ such that for all $z, \tilde{x} \in \mathcal{X}$ we have

$$
\begin{equation*}
\alpha_{4} J\left(\eta, z, h f^{N-1} \tilde{x}\right)+\sum_{i=0}^{N-1} \ell\left(h f^{i} \eta, h f^{i} \tilde{x}\right) \leq \sum_{i=0}^{N-2} \ell\left(h f^{i} z, h f^{i} \tilde{x}\right) \tag{31}
\end{equation*}
$$

where $\eta=\eta\left(z, h f^{N-1} \tilde{x}\right)$.
Remark 1 The linear case studied in Theorem inspires the conditions listed in Assumption 2. In particular, the first condition is a characterization of (uniform) observability and the second condition attempts to translate (22) to the nonlinear setting.

Theorem 4 Consider the system (10). Let $N \geq 1$ and Assumption 2 hold. Consider the observer (11) with $\eta=\arg \min _{\xi} J(\xi, z, y)$. Then for each $\varepsilon>0$ there exists $\delta>0$ such that $\rho_{\mathrm{x}}\left(\hat{x}_{0}, x_{0}\right) \leq \delta$ implies $\rho_{\mathrm{x}}\left(\hat{x}_{k}, x_{k}\right) \leq \varepsilon$ for all $k \in \mathbb{N}$. Moreover, for all initial conditions, $\rho_{\mathrm{x}}\left(\hat{x}_{k}, x_{k}\right) \rightarrow 0$ as $k \rightarrow \infty$.

Proof. Note that if $N=1$ then $\eta=\arg \min _{\xi} \ell(h \xi, h x)$ and by uniqueness assumption we must have $\eta=x$. Therefore $\hat{x}_{k}=x_{k}$ for all $k \geq 1$ and the result follows trivially. In the sequel we suppose $N \geq 2$.

We begin by stability. Since $f$ and $h$ are uniformly continuous there exist $\alpha_{5}, \alpha_{6} \in \mathcal{K}_{\infty}$ such that $\rho_{\mathrm{x}}(f \xi, f \zeta) \leq \alpha_{5} \rho_{\mathrm{x}}(\xi, \zeta)$ and $\rho_{\mathrm{y}}(h \xi, h \zeta) \leq \alpha_{6} \rho_{\mathrm{x}}(\xi, \zeta)$ for all $\xi, \zeta \in \mathcal{X}$. Then we have

$$
\begin{align*}
\rho_{\mathrm{x}}\left(\hat{x}^{+}, x^{+}\right) & =\rho_{\mathrm{x}}\left(f^{N} \eta, f x\right) \\
& \leq \alpha_{5} \rho_{\mathrm{x}}\left(f^{N-1} \eta, x\right) \\
& \leq \alpha_{5}\left(\rho_{\mathrm{x}}\left(f^{N-1} \eta, f^{N-1} z\right)+\rho_{\mathrm{x}}\left(f^{N-1} z, x\right)\right) \\
& =\alpha_{5}\left(\rho_{\mathrm{x}}\left(f^{N-1} \eta, f^{N-1} z\right)+\rho_{\mathrm{x}}(\hat{x}, x)\right) \\
& =\alpha_{5}\left(\alpha_{5}^{N-1} \rho_{\mathrm{x}}(\eta, z)+\rho_{\mathrm{x}}(\hat{x}, x)\right) \tag{32}
\end{align*}
$$

By (30) and the fact that $J(\eta, z, y) \leq J(z, z, y)=\ell(h \hat{x}, h x)$ we can proceed as

$$
\begin{align*}
\rho_{\mathrm{x}}(\eta, z) & \leq \alpha_{3}^{-1} \sum_{i=0}^{N-1} \ell\left(h f^{i} \eta, h f^{i} z\right) \\
& =\alpha_{3}^{-1}\left(J(\eta, z, y)+\ell\left(h f^{N-1} \eta, h f^{N-1} z\right)-\ell\left(h f^{N-1} \eta, h x\right)\right) \\
& \leq \alpha_{3}^{-1}\left(\ell(h \hat{x}, h x)+\ell\left(h f^{N-1} \eta, h f^{N-1} z\right)\right) \\
& \leq \alpha_{3}^{-1}\left(\alpha_{2} \alpha_{6} \rho_{\mathrm{x}}(\hat{x}, x)+\ell\left(h f^{N-1} \eta, h f^{N-1} z\right)\right) \tag{33}
\end{align*}
$$

Moreover,

$$
\begin{align*}
\ell\left(h f^{N-1} \eta, h f^{N-1} z\right) & \leq \alpha_{2} \rho_{\mathrm{y}}\left(h f^{N-1} \eta, h f^{N-1} z\right) \\
& \leq \alpha_{2}\left(\rho_{\mathrm{y}}\left(h f^{N-1} \eta, h x\right)+\rho_{\mathrm{y}}\left(h x, h f^{N-1} z\right)\right) \\
& \leq \alpha_{2}\left(\alpha_{1}^{-1} \ell\left(h f^{N-1} \eta, h x\right)+\rho_{\mathrm{y}}(h x, h \hat{x})\right) \\
& \leq \alpha_{2}\left(\alpha_{1}^{-1} J(\eta, z, y)+\alpha_{6} \rho_{\mathrm{x}}(\hat{x}, x)\right) \\
& \leq \alpha_{2}\left(\alpha_{1}^{-1} \ell(h \hat{x}, h x)+\alpha_{6} \rho_{\mathrm{x}}(\hat{x}, x)\right) \\
& \leq \alpha_{2}\left(\alpha_{1}^{-1} \alpha_{2} \alpha_{6} \rho_{\mathrm{x}}(\hat{x}, x)+\alpha_{6} \rho_{\mathrm{x}}(\hat{x}, x)\right) \tag{34}
\end{align*}
$$

Now let us define $\alpha_{7} \in \mathcal{K}_{\infty}$ as

$$
\alpha_{7} s:=\alpha_{5}\left(\alpha_{5}^{N-1} \alpha_{3}^{-1}\left(\alpha_{2} \alpha_{6} s+\alpha_{2}\left(\alpha_{1}^{-1} \alpha_{2} \alpha_{6} s+\alpha_{6} s\right)\right)+s\right)
$$

Then by (32), (33), and (34) we can write

$$
\begin{equation*}
\rho_{\mathrm{x}}\left(\hat{x}^{+}, x^{+}\right) \leq \alpha_{7} \rho_{\mathrm{x}}(\hat{x}, x) \tag{35}
\end{equation*}
$$

which tells us that if $x$ and its estimate $\hat{x}$ are close to each other at some instant then they will stay close at the next instant. Now we direct our attention to (31). Before however let us let $\tilde{x}$ indicate the $N-1$ time steps earlier value of the state $x$, i.e., $\tilde{x}_{k}:=x_{k-N+1}$ for $k \geq N-1$. Then we can write

$$
\begin{aligned}
\alpha_{4} J\left(\eta, z, h f^{N-1} \tilde{x}\right) & \leq \sum_{i=0}^{N-2} \ell\left(h f^{i} z, h f^{i} \tilde{x}\right)-\sum_{i=0}^{N-1} \ell\left(h f^{i} \eta, h f^{i} \tilde{x}\right) \\
& \leq \sum_{i=0}^{N-2} \ell\left(h f^{i} z, h f^{i} \tilde{x}\right)-\sum_{i=1}^{N-1} \ell\left(h f^{i} \eta, h f^{i} \tilde{x}\right) \\
& =\sum_{i=0}^{N-2} \ell\left(h f^{i} z, h f^{i} \tilde{x}\right)-\sum_{i=0}^{N-2} \ell\left(h f^{i} z^{+}, h f^{i} \tilde{x}^{+}\right)
\end{aligned}
$$

which implies

$$
\begin{equation*}
\sum_{k=k_{0}}^{\infty} \alpha_{4} J\left(\eta_{k}, z_{k}, y_{k}\right) \leq \sum_{i=0}^{N-2} \ell\left(h f^{i} z_{k_{0}}, h f^{i} \tilde{x}_{k_{0}}\right) \tag{36}
\end{equation*}
$$

for all $k_{0} \geq N-1$. We now demonstrate the following.
Claim: There exists $\alpha_{8} \in \mathcal{K}_{\infty}$ such that for all $k_{1} \in \mathbb{N}$

$$
\begin{equation*}
J\left(\eta_{k}, z_{k}, y_{k}\right) \leq \delta \quad \forall k \geq k_{1} \Longrightarrow \rho_{\mathrm{x}}\left(\hat{x}_{k}, x_{k}\right) \leq \alpha_{8} \delta \quad \forall k \geq k_{1}+N \tag{37}
\end{equation*}
$$

We prove this claim as follows. Let us for some $\delta$ and $k_{1}$ have $J\left(\eta_{k}, z_{k}, y_{k}\right) \leq \delta$ for all $k \geq k_{1}$, which by (29) implies

$$
\max \left\{\rho_{\mathrm{y}}\left(h f^{N-1} \eta_{k}, y_{k}\right), \rho_{\mathrm{y}}\left(h f^{N-2} \eta_{k}, h f^{N-2} z_{k}\right), \ldots, \rho_{\mathrm{y}}\left(h \eta_{k}, h z_{k}\right)\right\} \leq \delta_{1}
$$

with $\delta_{1}=\alpha_{1}^{-1} \delta$. Suppose for some $p \in\{1,2, \ldots, N-1\}$ and some $k_{p} \geq k_{1}$ we have

$$
\begin{equation*}
\rho_{\mathrm{y}}\left(h f^{N-q} \eta_{k}, y_{k-q+1}\right) \leq q \delta_{1} \quad \forall q \in\{1,2, \ldots, p\} \quad \forall k \geq k_{p} \tag{38}
\end{equation*}
$$

Then we can write for all $q \in\{1,2, \ldots, p\}$ and $k \geq k_{p}$

$$
\begin{aligned}
\rho_{\mathrm{y}}\left(h f^{N-(q+1)} \eta_{k+1}, y_{(k+1)-(q+1)+1}\right) \leq & \rho_{\mathrm{y}}\left(h f^{N-q-1} \eta_{k+1}, h f^{N-q-1} z_{k+1}\right) \\
& +\rho_{\mathrm{y}}\left(h f^{N-q-1} z_{k+1}, y_{k-q+1}\right) \\
= & \rho_{\mathrm{y}}\left(h f^{N-q-1} \eta_{k+1}, h f^{N-q-1} z_{k+1}\right) \\
& +\rho_{\mathrm{y}}\left(h f^{N-q} \eta_{k}, y_{k-q+1}\right) \\
\leq & \delta_{1}+q \delta_{1} \\
= & (q+1) \delta_{1}
\end{aligned}
$$

which allows us to assert

$$
\rho_{\mathrm{y}}\left(h f^{N-q} \eta_{k}, y_{k-q+1}\right) \leq q \delta_{1} \quad \forall q \in\{1,2, \ldots, p+1\} \quad \forall k \geq k_{p}+1
$$

Since (38) holds with $p=1$, by induction we can write

$$
\begin{equation*}
\rho_{\mathrm{y}}\left(h f^{N-q} \eta_{k}, y_{k-q+1}\right) \leq q \delta_{1} \quad \forall q \in\{1,2, \ldots, N\} \quad \forall k \geq k_{1}+N-1 \tag{39}
\end{equation*}
$$

Now by (30) and (39) we can write for $k \geq k_{1}+N-1$

$$
\begin{aligned}
\rho_{\mathrm{x}}\left(\hat{x}_{k+1}, x_{k+1}\right) & =\rho_{\mathrm{x}}\left(f^{N} \eta_{k}, f^{N} \tilde{x}_{k}\right) \\
& \leq \alpha_{5}^{N} \rho_{\mathrm{x}}\left(\eta_{k}, \tilde{x}_{k}\right) \\
& \leq \alpha_{5}^{N} \alpha_{3}^{-1} \sum_{i=0}^{N-1} \ell\left(h f^{i} \eta_{k}, h f^{i} \tilde{x}_{k}\right) \\
& =\alpha_{5}^{N} \alpha_{3}^{-1} \sum_{q=1}^{N} \ell\left(h f^{N-q} \eta_{k}, y_{k-q+1}\right) \\
& \leq \alpha_{5}^{N} \alpha_{3}^{-1} \sum_{q=1}^{N} \alpha_{2} \rho_{\mathrm{y}}\left(h f^{N-q} \eta_{k}, y_{k-q+1}\right) \\
& \leq \alpha_{5}^{N} \alpha_{3}^{-1} \sum_{q=1}^{N} \alpha_{2} q \delta_{1} \\
& =\alpha_{5}^{N} \alpha_{3}^{-1} \sum_{q=1}^{N} \alpha_{2} q \alpha_{1}^{-1} \delta .
\end{aligned}
$$

This proves our claim since we can define $\alpha_{8} \in \mathcal{K}_{\infty}$ as

$$
\alpha_{8} s:=\alpha_{5}^{N} \alpha_{3}^{-1} \sum_{q=1}^{N} \alpha_{2} q \alpha_{1}^{-1} s .
$$

Now we return to the proof of the theorem. By (36) we can write for all $k \geq N-1$

$$
\begin{align*}
\alpha_{4} J\left(\eta_{k}, z_{k}, y_{k}\right) & \leq \sum_{i=0}^{N-2} \ell\left(h f^{i} z_{N-1}, h f^{i} \tilde{x}_{N-1}\right) \\
& \leq \sum_{i=0}^{N-2} \alpha_{2} \alpha_{6} \rho_{\mathrm{x}}\left(f^{i} z_{N-1}, f^{i} \tilde{x}_{N-1}\right) \\
& \leq \sum_{i=0}^{N-2} \alpha_{2} \alpha_{6} \alpha_{5}^{i} \rho_{\mathrm{x}}\left(z_{N-1}, x_{0}\right) \\
& \leq \sum_{i=0}^{N-2} \alpha_{2} \alpha_{6} \alpha_{5}^{i}\left(\rho_{\mathrm{x}}\left(z_{N-1}, \hat{x}_{0}\right)+\rho_{\mathrm{x}}\left(\hat{x}_{0}, x_{0}\right)\right) \\
& =\sum_{i=0}^{N-2} \alpha_{2} \alpha_{6} \alpha_{5}^{i}\left(\rho_{\mathrm{x}}\left(z_{N-1}, f^{N-1} z_{0}\right)+\rho_{\mathrm{x}}\left(\hat{x}_{0}, x_{0}\right)\right) \tag{40}
\end{align*}
$$

Observe that for all $k \geq 1$ we can write

$$
\begin{align*}
\rho_{\mathrm{x}}\left(z_{k}, f^{k} z_{0}\right) & =\rho_{\mathrm{x}}\left(f \eta_{k-1}, f^{k} z_{0}\right) \\
& \leq \alpha_{5} \rho_{\mathrm{x}}\left(\eta_{k-1}, f^{k-1} z_{0}\right) \\
& \leq \alpha_{5}\left(\rho_{\mathrm{x}}\left(\eta_{k-1}, z_{k-1}\right)+\rho_{\mathrm{x}}\left(z_{k-1}, f^{k-1} z_{0}\right)\right) \tag{41}
\end{align*}
$$

By (33) and (34) we can write

$$
\rho_{\mathrm{x}}(\eta, z) \leq \alpha_{9} \rho_{\mathrm{x}}(\hat{x}, x)
$$

where we define $\alpha_{9} \in \mathcal{K}_{\infty}$ as

$$
\alpha_{9} s:=\alpha_{3}^{-1}\left(\alpha_{2} \alpha_{6} s+\alpha_{2}\left(\alpha_{1}^{-1} \alpha_{2} \alpha_{6} s+\alpha_{6} s\right)\right)
$$

Hence we can proceed from (41) as

$$
\begin{align*}
\rho_{\mathrm{x}}\left(z_{k}, f^{k} z_{0}\right) & \leq \alpha_{5}\left(\alpha_{9} \rho_{\mathrm{x}}\left(\hat{x}_{k-1}, x_{k-1}\right)+\rho_{\mathrm{x}}\left(z_{k-1}, f^{k-1} z_{0}\right)\right) \\
& \leq \alpha_{5}\left(\alpha_{9} \alpha_{7}^{k-1} \rho_{\mathrm{x}}\left(\hat{x}_{0}, x_{0}\right)+\rho_{\mathrm{x}}\left(z_{k-1}, f^{k-1} z_{0}\right)\right) \tag{42}
\end{align*}
$$

where we used (35). Then (42) implies

$$
\begin{equation*}
\rho_{\mathrm{x}}\left(z_{N-1}, f^{N-1} z_{0}\right) \leq \gamma_{N-1} \rho_{\mathrm{x}}\left(\hat{x}_{0}, x_{0}\right) \tag{43}
\end{equation*}
$$

where we define $\gamma_{j} \in \mathcal{K}_{\infty}$ recursively through

$$
\gamma_{j+1} s:=\alpha_{5}\left(\alpha_{9} \alpha_{7}^{j} s+\gamma_{j} s\right)
$$

for $j \in\{1,2, \ldots\}$ with $\gamma_{1} s:=\alpha_{5} \alpha_{9} s$. Now, by (40) and (43) we can write for all $k \geq N-1$

$$
\begin{equation*}
J\left(\eta_{k}, z_{k}, y_{k}\right) \leq \alpha_{10} \rho_{\mathrm{x}}\left(\hat{x}_{0}, x_{0}\right) \tag{44}
\end{equation*}
$$

once we define $\alpha_{10} \in \mathcal{K}_{\infty}$ as

$$
\alpha_{10} s:=\alpha_{4}^{-1} \sum_{i=0}^{N-2} \alpha_{2} \alpha_{6} \alpha_{5}^{i}\left(\gamma_{N-1} s+s\right)
$$

Note that (37) and (44) allow us to write

$$
\rho_{\mathrm{x}}\left(\hat{x}_{k}, x_{k}\right) \leq \alpha_{8} \alpha_{10} \rho_{\mathrm{x}}\left(\hat{x}_{0}, x_{0}\right) \quad \forall k \geq 2 N-1
$$

Moreover, by (35) we have

$$
\rho_{\mathrm{x}}\left(\hat{x}_{k}, x_{k}\right) \leq \alpha_{7}^{k} \rho_{\mathrm{x}}\left(\hat{x}_{0}, x_{0}\right) \quad \forall k \in\{0,1, \ldots, 2 N-2\} .
$$

Hence by defining $\alpha_{11} \in \mathcal{K}_{\infty}$ as

$$
\alpha_{11} s:=\max \left\{\alpha_{8} \alpha_{10} s, \max _{k \in\{0, \ldots, 2 N-2\}} \alpha_{7}^{k} s\right\}
$$

we can write

$$
\rho_{\mathrm{x}}\left(\hat{x}_{k}, x_{k}\right) \leq \alpha_{11} \rho_{\mathrm{x}}\left(\hat{x}_{0}, x_{0}\right) \quad \forall k \in \mathbb{N}
$$

which establishes the stability.
Now we prove convergence. From (36) we deduce that $J\left(\eta_{k}, z_{k}, y_{k}\right) \rightarrow 0$ as $k \rightarrow \infty$. Then (37) implies $\rho_{\mathrm{x}}\left(\hat{x}_{k}, x_{k}\right) \rightarrow 0$ as $k \rightarrow \infty$.

### 5.2 Tracker design

Here we attempt to generalize the linear result of Theorem 3, where an optimal tracker was constructed through solving the following problem
$V(\hat{x}, x):=\min _{\left(w_{i}\right)_{i=0}^{N-1}} \sum_{i=0}^{N-1} w_{i}^{T} R^{-1} w_{i} \quad$ subject to $\quad\left\{\begin{aligned} z_{0} & =\hat{x} \\ z_{i+1} & =A z_{i}+B w_{i} \\ z_{N} & =A^{N} x\end{aligned}\right.$
where the control inputs $w_{0}, w_{1}, \ldots, w_{N-1}$ are penalized via the quadratic stage $\operatorname{cost} w \mapsto w^{T} R^{-1} w$. Note that in a general nonlinear setting, imposing a direct penalty on the control input may not be meaningful. For instance, $w$ may just be an index that belongs to a finite set. For this reason we will now try to express $V(\hat{x}, x)$ in a different way that is more welcoming to generalization. If we assume that $B$ is full column rank and let $Q:=B\left(B^{T} B\right)^{-1} R^{-1}\left(B^{T} B\right)^{-1} B^{T}$, we can write

$$
\begin{aligned}
w_{i}^{T} R^{-1} w_{i} & =w_{i}^{T} B^{T} Q B w_{i} \\
& =\left(z_{i+1}-A z_{i}\right)^{T} Q\left(z_{i+1}-A z_{i}\right)
\end{aligned}
$$

Then we have

$$
\begin{aligned}
V(\hat{x}, x)=\min _{\left(z_{i}\right)_{i=0}^{N}} \sum_{i=0}^{N-1}\left(z_{i+1}-A z_{i}\right)^{T} Q\left(z_{i+1}-A z_{i}\right) \\
\text { subject to }\left\{\begin{array}{rll}
z_{0} & =\hat{x} \\
z_{i+1} & \in & A z_{i}+\operatorname{range}(B) \\
z_{N} & = & A^{N} x
\end{array}\right.
\end{aligned}
$$

which is the form we adopt for generalization. We remind the reader that the optimal cost (45) enjoys the following analytical expression

$$
V(\hat{x}, x)=(\hat{x}-x)^{T} A^{N T}\left(\sum_{i=0}^{N-1} A^{i} B R B^{T} A^{i T}\right)^{-1} A^{N}(\hat{x}-x)
$$

which is positive definite (for nonsingular $A$ ) with respect to the error $e=\hat{x}-x$.
Consider now the system

$$
\begin{equation*}
x^{+}=f x \tag{46}
\end{equation*}
$$

with $f: \mathcal{X} \rightarrow \mathcal{X}$ continuous and the tracker

$$
\begin{equation*}
\hat{x}^{+}=F(\hat{x}, u) \tag{47}
\end{equation*}
$$

with $F: \mathcal{X} \times \mathcal{U} \rightarrow \mathcal{X}$. We assume $f \hat{x} \in F(\hat{x}, \mathcal{U})$ for all $\hat{x} \in \mathcal{X}$. Let $\ell:$ $\mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}_{\geq 0}$ and $\alpha_{1}, \alpha_{2} \in \mathcal{K}_{\infty}$ satisfy $\alpha_{1} \rho_{\mathrm{x}}(\xi, \zeta) \leq \ell(\xi, \zeta) \leq \alpha_{2} \rho_{\mathrm{x}}(\xi, \zeta)$ for every $\xi, \zeta \in \mathcal{X}$. Since there is no significant reason against it, we assume the symmetry $\ell(\xi, \zeta)=\ell(\zeta, \xi)$. Let $N \geq 1$ and $\left(\phi_{i}\right)_{i=0}^{N}$ denote a sequence of functions $\phi_{i}: \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{X}$ satisfying

$$
\begin{align*}
\left(\phi_{i}(\hat{x}, x)\right)_{i=0}^{N}=\arg \min _{\left(z_{i}\right)_{i=0}^{N}} \sum_{i=0}^{N-1} \ell\left(z_{i+1}, f z_{i}\right) \\
\text { subject to }\left\{\begin{array}{rll}
z_{0} & =\hat{x} \\
z_{i+1} & \in & F\left(z_{i}, \mathcal{U}\right) \\
z_{N} & = & f^{N} x
\end{array}\right. \tag{48}
\end{align*}
$$

Note that $\phi_{0}(\hat{x}, x)=\hat{x}$ and $\phi_{N}(\hat{x}, x)=f^{N} x$. Assuming $\left(\phi_{i}\right)_{i=0}^{N}$ exists we can run the tracker (47) so as to satisfy $\hat{x}^{+}=\phi_{1}(\hat{x}, x)$ since by (48) we have $\phi_{1}(\hat{x}, x) \in F(\hat{x}, \mathcal{U})$, i.e., for each pair $(\hat{x}, x)$ we can find $u \in \mathcal{U}$ such that $\phi_{1}(\hat{x}, x)=F(\hat{x}, u)$. Determining whether this construction will actually work or not requires some analysis. As usual in moving horizon feedback systems [7] the optimal cost $V: \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}_{\geq 0}$ defined below will be of key importance in the analysis.

$$
V(\hat{x}, x):=\min _{\left(z_{i}\right)_{i=0}^{N}} \sum_{i=0}^{N-1} \ell\left(z_{i+1}, f z_{i}\right) \quad \text { subject to } \quad\left\{\begin{align*}
z_{0} & =\hat{x}  \tag{49}\\
z_{i+1} & \in F\left(z_{i}, \mathcal{U}\right) \\
z_{N} & =f^{N} x
\end{align*}\right.
$$

Assumption 3 The following hold.

1. The optimal cost (49) is continuous and there exist $\alpha_{3}, \alpha_{4} \in \mathcal{K}_{\infty}$ such that

$$
\begin{equation*}
\alpha_{3} \rho_{\mathrm{x}}(\hat{x}, x) \leq V(\hat{x}, x) \leq \alpha_{4} \rho_{\mathrm{x}}(\hat{x}, x) \tag{50}
\end{equation*}
$$

for all $\hat{x}, x \in \mathcal{X}$.
2. The sequence of functions (48) is unique and its second element $\phi_{1}$ is continuous.

Remark 2 Note that Assumption 3 comes for free for linear systems under the conditions of Theorem 3, provided that the system matrix $A$ is nonsingular.

Theorem 5 Consider the system (46) and the tracker (47). Let $N \geq 1$ and the control input $u$ of the tracker satisfy $F(\hat{x}, u)=\phi_{1}(\hat{x}, x)$ where $\phi_{1}$ is the second term of the sequence of functions (48). If Assumption 3 holds, then for each $\varepsilon>0$ there exists $\delta>0$ such that $\rho_{\mathrm{x}}\left(\hat{x}_{0}, x_{0}\right) \leq \delta$ implies $\rho_{\mathrm{x}}\left(\hat{x}_{k}, x_{k}\right) \leq \varepsilon$ for all initial conditions and $k \in \mathbb{N}$. Moreover, if the solutions $\hat{x}_{k}$ and $x_{k}$ belong to a bounded region $\mathcal{D} \subset \mathcal{X}$ for all $k \in \mathbb{N}$, then $\rho_{\mathrm{x}}\left(\hat{x}_{k}, x_{k}\right) \rightarrow 0$ as $k \rightarrow \infty$.

Proof. The tracker dynamics being $\hat{x}^{+}=\phi_{1}(\hat{x}, x)$, we can write by optimality

$$
\begin{equation*}
V\left(\hat{x}^{+}, x^{+}\right)-V(\hat{x}, x) \leq-\ell\left(\phi_{1}(\hat{x}, x), f \hat{x}\right) \tag{51}
\end{equation*}
$$

whence it follows that $V\left(\hat{x}_{k}, x_{k}\right) \leq V\left(\hat{x}_{0}, x_{0}\right)$ for all $k$. Hence the stability is established by the assumed positive definiteness (50) of the optimal cost $V$.

Now we show convergence under the assumption that both $\hat{x}_{k}$ and $x_{k}$ belong to a bounded region $\mathcal{D}$ for all $k$. By (51) we can write $0 \leq V\left(\hat{x}_{k+1}, x_{k+1}\right) \leq$ $V\left(\hat{x}_{k}, x_{k}\right)$ for all $k$. Therefore there exists $\bar{V} \geq 0$ such that $V\left(\hat{x}_{k}, x_{k}\right) \rightarrow \bar{V}$ as $k \rightarrow \infty$. Note that establishing the convergence $\rho_{\mathrm{x}}\left(\hat{x}_{k}, x_{k}\right) \rightarrow 0$ is equivalent to showing that $\bar{V}=0$ thanks to (50).

By $\mathbf{x} \in \mathcal{X}^{2}$ let us denote the aggregate state $(\hat{x}, x)$. Then we can write $\mathbf{x}^{+}=\left(\hat{x}^{+}, x^{+}\right)=\left(\phi_{1}(\hat{x}, x), f x\right)=: \mathbf{f x}$. Since both $f$ and $\phi_{1}$ are continuous, so is $\mathbf{f}: \mathcal{X}^{2} \rightarrow \mathcal{X}^{2}$. Also, the solution $\mathbf{x}_{k}$ belongs to the bounded region $\mathcal{D}^{2}$ for all $k$. Since the sequence $\left(\mathbf{x}_{k}\right)_{k=0}^{\infty}$ is bounded it must have an accumulation point $\mathbf{x}^{*}=\left(\hat{x}^{*}, x^{*}\right)$. That is, $\left(\mathbf{x}_{k}\right)_{k=0}^{\infty}$ must have a convergent subsequence $\left(\mathbf{x}_{k_{j}}\right)_{j=0}^{\infty}$ satisfying $\mathbf{x}_{k_{j}} \rightarrow \mathbf{x}^{*}$ as $j \rightarrow \infty$ [2]. Note that $V \mathbf{x}^{*}=V\left(\hat{x}^{*}, x^{*}\right)=\bar{V}$ because $V$ is continuous. Since $\mathbf{x}^{+}=\mathbf{f x}$ the sequence $\left(\mathbf{f} \mathbf{x}_{k_{j}}\right)_{j=0}^{\infty}$ must also be a subsequence of $\left(\mathbf{x}_{k}\right)_{k=0}^{\infty}$. Moreover, $\mathbf{f x}^{*}$ has to be an accumulation point because $\mathbf{f}$ is continuous. By induction $\mathbf{f}^{q} \mathbf{x}^{*}$ is an accumulation point of $\left(\mathbf{x}_{k}\right)_{k=0}^{\infty}$ for all $q \in \mathbb{N}$. Consequently

$$
\begin{equation*}
V \mathbf{f}^{q} \mathbf{x}^{*}=\bar{V} \tag{52}
\end{equation*}
$$

for all $q \in \mathbb{N}$. By (51) and (52) we can write

$$
\begin{aligned}
\alpha_{1} \rho_{\mathrm{x}}\left(\phi_{1}\left(\hat{x}^{*}, x^{*}\right), f \hat{x}^{*}\right) & \leq \ell\left(\phi_{1}\left(\hat{x}^{*}, x^{*}\right), f \hat{x}^{*}\right) \\
& \leq V \mathbf{x}^{*}-V \mathbf{f \mathbf { x } ^ { * }} \\
& =\bar{V}-\bar{V} \\
& =0
\end{aligned}
$$

Therefore $\phi_{1}\left(\hat{x}^{*}, x^{*}\right)=f \hat{x}^{*}$, which means $\mathbf{f x}{ }^{*}=\left(f \hat{x}^{*}, f x^{*}\right)$. Employing induction we can thus write

$$
\begin{equation*}
\mathbf{f}^{q} \mathbf{x}^{*}=\left(f^{q} \hat{x}^{*}, f^{q} x^{*}\right) \tag{53}
\end{equation*}
$$

i.e., $\left(f^{q} \hat{x}^{*}, f^{q} x^{*}\right)$ is an accumulation point. Hence

$$
\begin{equation*}
\phi_{1}\left(f^{q} \hat{x}^{*}, f^{q} x^{*}\right)=f^{q+1} \hat{x}^{*} \tag{54}
\end{equation*}
$$

for all $q \in \mathbb{N}$.
Now, given a pair $(\hat{x}, x)$ let a sequence $\left(z_{i}\right)_{i=0}^{N}=\left(\hat{x}, z_{1}, \ldots, z_{N-1}, f^{N} x\right)$ be said to be feasible with respect to $(\hat{x}, x)$ if it respects the constraints in (48). Also, we define

$$
J\left(z_{i}\right)_{i=0}^{N}:=\sum_{i=0}^{N-1} \ell\left(z_{i+1}, f z_{i}\right)
$$

Then we can write by (54)

$$
\begin{align*}
& V\left(f^{q} \hat{x}^{*}, f^{q} x^{*}\right) \\
& \quad=J\left(\phi_{0}\left(f^{q} \hat{x}^{*}, f^{q} x^{*}\right), \ldots, \phi_{N}\left(f^{q} \hat{x}^{*}, f^{q} x^{*}\right)\right) \\
& \quad=J\left(f^{q} \hat{x}^{*}, f^{q+1} \hat{x}^{*}, \phi_{2}\left(f^{q} \hat{x}^{*}, f^{q} x^{*}\right), \ldots, \phi_{N-1}\left(f^{q} \hat{x}^{*}, f^{q} x^{*}\right), f^{N+q} x^{*}\right) \\
& \quad=J\left(f^{q+1} \hat{x}^{*}, \phi_{2}\left(f^{q} \hat{x}^{*}, f^{q} x^{*}\right), \ldots, \phi_{N-1}\left(f^{q} \hat{x}^{*}, f^{q} x^{*}\right), f^{N+q} x^{*}, f^{N+q+1} x^{*}\right) \\
& \quad=J\left(\phi_{1}\left(f^{q} \hat{x}^{*}, f^{q} x^{*}\right), \ldots, \phi_{N}\left(f^{q} \hat{x}^{*}, f^{q} x^{*}\right), f^{N+q+1} x^{*}\right) . \tag{55}
\end{align*}
$$

By (52) and (53) we have $V\left(f^{q+1} \hat{x}^{*}, f^{q+1} x^{*}\right)=V\left(f^{q} \hat{x}^{*}, f^{q} x^{*}\right)$, which allows us by (55) to write

$$
\begin{aligned}
V\left(f^{q+1} \hat{x}^{*}, f^{q+1} x^{*}\right) & =J\left(\phi_{0}\left(f^{q+1} \hat{x}^{*}, f^{q+1} x^{*}\right), \ldots, \phi_{N}\left(f^{q+1} \hat{x}^{*}, f^{q+1} x^{*}\right)\right) \\
& =J\left(\phi_{1}\left(f^{q} \hat{x}^{*}, f^{q} x^{*}\right), \ldots, \phi_{N}\left(f^{q} \hat{x}^{*}, f^{q} x^{*}\right), f^{N+q+1} x^{*}\right)
\end{aligned}
$$

By (54) the sequence $\left(\phi_{1}\left(f^{q} \hat{x}^{*}, f^{q} x^{*}\right), \ldots, \phi_{N}\left(f^{q} \hat{x}^{*}, f^{q} x^{*}\right), f^{N+q+1} x^{*}\right)$ is feasible with respect to $\left(f^{q+1} \hat{x}^{*}, f^{q+1} x^{*}\right)$. Therefore the uniqueness condition stated in Assumption 3 implies the following equality of sequences for all $q \in \mathbb{N}$

$$
\begin{aligned}
& \left(\phi_{1}\left(f^{q} \hat{x}^{*}, f^{q} x^{*}\right), \ldots, \phi_{N}\left(f^{q} \hat{x}^{*}, f^{q} x^{*}\right), f^{N+q+1} x^{*}\right) \\
& \quad=\left(\phi_{0}\left(f^{q+1} \hat{x}^{*}, f^{q+1} x^{*}\right), \ldots, \phi_{N}\left(f^{q+1} \hat{x}^{*}, f^{q+1} x^{*}\right)\right)
\end{aligned}
$$

whence we can write $\phi_{i}\left(f^{q} \hat{x}^{*}, f^{q} x^{*}\right)=f^{q+i} \hat{x}^{*}$ for $i=1, \ldots, N$. That means we have $\phi_{i+1}\left(f^{q} \hat{x}^{*}, f^{q} x^{*}\right)=f \phi_{i}\left(f^{q} \hat{x}^{*}, f^{q} x^{*}\right)$ for $i=0, \ldots, N-1$. Finally, by (52) and (53)

$$
\begin{aligned}
\bar{V} & =V\left(f^{q} \hat{x}^{*}, f^{q} x^{*}\right) \\
& =\sum_{i=0}^{N-1} \ell\left(\phi_{i+1}\left(f^{q} \hat{x}^{*}, f^{q} x^{*}\right), f \phi_{i}\left(f^{q} \hat{x}^{*}, f^{q} x^{*}\right)\right) \\
& =\sum_{i=0}^{N-1} \ell\left(f \phi_{i}\left(f^{q} \hat{x}^{*}, f^{q} x^{*}\right), f \phi_{i}\left(f^{q} \hat{x}^{*}, f^{q} x^{*}\right)\right) \\
& \leq \sum_{i=0}^{N-1} \alpha_{2} \rho_{\mathrm{x}}\left(f \phi_{i}\left(f^{q} \hat{x}^{*}, f^{q} x^{*}\right), f \phi_{i}\left(f^{q} \hat{x}^{*}, f^{q} x^{*}\right)\right) \\
& =0
\end{aligned}
$$

which was to be shown.

Remark 3 For the special (yet important) case where the trajectory of the system (46) to be tracked is constant, i.e., $x_{k}=x_{\mathrm{eq}}$ for all $k$, the boundedness condition required in Theorem 5 to establish convergence $\rho_{\mathrm{x}}\left(\hat{x}_{k}, x_{k}\right) \rightarrow 0$ need not be explicitly assumed for it is implied by (50) and (51). In other words, to establish the regulation of an equilibrium point $x_{\mathrm{eq}}=f x_{\mathrm{eq}}$ Assumption 3 is sufficient.

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[^0]:    *The author is with Department of Electrical and Electronics Engineering, Middle East Technical University, 06800 Ankara, Turkey. Email: tuna@eee.metu.edu.tr
    ${ }^{\dagger}$ Though LQR and LQE are acknowledged as a dual pair, nowhere (to the best of our knowledge) it is mentioned whether duality played much (if any) role in their discoveries. In other words, there seems to be no evidence to suggest that the birth of LQE was a consequence of the pressing fact that LQR must have a twin.

