# Synchronization of linear systems via relative actuation

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**Abstract**

Synchronization in networks of discrete-time linear time-invariant systems is considered under relative actuation. Neither input nor output matrices are assumed to be commensurable. A distributed algorithm that ensures synchronization via dynamic relative output feedback is presented.

## 1 Introduction

Theory on synchronization in networks of linear systems with general dynamics has reached a certain maturity over the last decade; see, for instance, [7, 9, 6, 13, 1]. A significant part of this theory is founded on the following setup. The nominal individual agent dynamics reads $\dot{x}_i = Ax_i + Bu_i$ with $y_i = Cx_i$. And, the signals available for decision are of the form $z_i = \sum_j a_{ij}(y_i - y_j)$, where the nonnegative scalars $a_{ij}$ describe the so-called communication topology. Within the boundary of this setup different approaches have yielded various interesting solutions to the synchronization problem, where the universal goal is to drive the agents' states $x_i(t)$ to a common trajectory. E.g., communication delays are considered in [15], $\mathcal{L}_2$-gain output-feedback is employed in [2], distributed containment problem is studied in [4]. Among many other works contributing to our wealth of knowledge are [5] on adaptive protocols, [8] on switching topologies, and [14] on optimal state feedback and observer design.

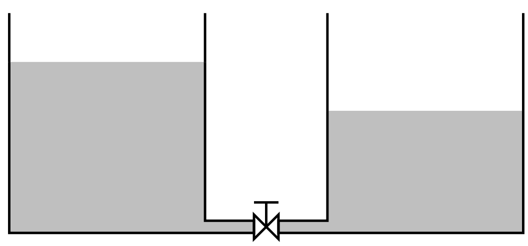


Figure 1: Water tanks with a shared actuator (pump).

Despite their differences the above-mentioned works allow each agent to have its own independent input $u_i$. In this paper we shed this independence. Instead of each agent having its own input we look at the case where each input ($u_{ij}$) is shared by a pair of agents ($i$th and $j$th systems) in the sense displayed in Fig. 1. In particular, we consider the agent dynamics $x_i^+ = Ax_i + \sum_j B_{ij}u_{ij}$ with $y_{ij} = C_{ij}(x_i - x_j)$, where (i) the actuation is *relative* (i.e., $B_{ij}u_{ij} + B_{ji}u_{ji} = 0$) and (ii) the signals available for decision read $z_i = \sum_j C_{ij}^T y_{ij}$. In our setup the input matrices $B_{ij}$ are allowed to be *incommensurable* in the sense that there need not exist a common $B$ satisfying $B_{ij} = a_{ij}B$ with scalar $a_{ij}$. In fact, two input matrices do not even have to be of the same size. The same goes for the output matrices $C_{ij}$. The problem we study here is that of decentralized stabilization (of the synchronization subspace) by choosing appropriate inputs $u_{ij}$ based on the relative measurements $y_{ij}$. As a solution to this problem we construct a distributed algorithm that achieves synchronization via dynamic relative output feedback.

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Let us now illustrate our setup on an example network. Consider the array of identical electrical oscillators shown in Fig. 2, where each oscillator (of order $2p$) has $p$ nodes (excepting the ground node) and the $k$th node of the $i$th oscillator is denoted by $n_k^{(i)}$. The actuation is achieved through current sources while the measurements are collected through voltmeters. Each current source/voltmeter connects a pair of nodes (belonging to two separate oscillators) with the same index number, say $n_k^{(i)}$ and $n_k^{(j)}$. It is not difficult to see that this architecture enjoys the form $\dot{x}_i = Ax_i + \sum_j B_{ij}u_{ij}$ with $y_{ij} = C_{ij}(x_i - x_j)$ and, since each current source connects two nodes with the same index number, the actuation throughout the network is relative. Furthermore, the input matrices $B_{ij}$ are incommensurable. For instance, while the current source $u_{32}$ connects $n_1^{(3)}$ and $n_1^{(2)}$, the current source $u_{12}$ connects $n_p^{(1)}$ and $n_p^{(2)}$. Since for these two current sources the indices (1 and $p$) of the nodes they are associated to are different, we cannot find a scalar $a$ that satisfies $B_{12} = aB_{32}$. Likewise, the output matrices $C_{ij}$ too are incommensurable.

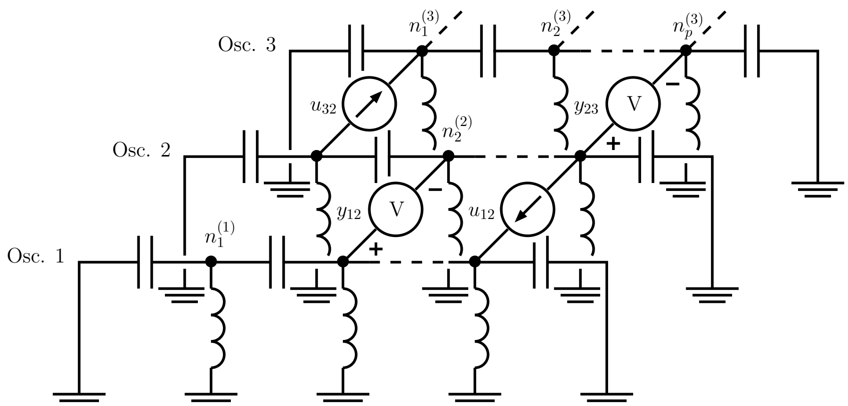


Figure 2: Network of electrical oscillators.

We begin the remainder of the paper by providing the formal description of the *array* we study. After that we present a distributed *algorithm* that generates control inputs through dynamic output feedback, followed by our main (and only) theorem, which states that this algorithm with suitable parameter choice achieves synchronization. To prove the theorem we first obtain the explicit expression of the *closed-loop* system the array becomes under the algorithm. Once the righthand side of the closed loop is computed we proceed to establish *stability* and thus complete the proof of the main result.

## 2 Array

Consider an *array* of $q$ discrete-time linear time-invariant systems

$$x_i^+ = Ax_i + \sum_{j=1}^{q} B_{ij}u_{ij} \tag{1a}$$

$$y_{ij} = C_{ij}(x_i - x_j)\,; \qquad i,\, j = 1,\, 2,\, \ldots,\, q \tag{1b}$$

where $x_i \in \mathbb{R}^n$ is the state of the $i$th system with $A \in \mathbb{R}^{n\times n}$, $x_i^+$ denotes the state at the next time instant, $u_{ij} = u_{ji} \in \mathbb{R}^{p_{ij}}$ is the $ij$th input with $B_{ij} = -B_{ji} \in \mathbb{R}^{n\times p_{ij}}$, and $y_{ij} = y_{ji} \in \mathbb{R}^{m_{ij}}$ is the $ij$th (relative) output with $C_{ij} = -C_{ji} \in \mathbb{R}^{m_{ij}\times n}$. We interpret the equality $u_{ij} = u_{ji}$ as that $u_{ij}$ and $u_{ji}$ are different notations for the same single variable. Same goes for the oneness of $y_{ij}$ and $y_{ji}$. Note that we have to have $B_{ii} = 0$ and $C_{ii} = 0$. Note also that the actuation is relative because $B_{ij}u_{ij} + B_{ji}u_{ji} = 0$. Hence the average of the states $x_{\rm av} = q^{-1}\sum x_i$ evolves independently of the inputs driving the array, i.e., we have $x_{\rm av}^+ = Ax_{\rm av}$. The ordered collections $(B_{ij})_{i,j=1}^q$ and $(C_{ij})_{i,j=1}^q$ are denoted by $(B_{::})$ and $(C_{::})$, respectively. The value of the solution of the $i$th system at the $k$th time instant ($k = 0,\, 1,\, \ldots$) is denoted by $x_i[k]$. The meanings of $u_{ij}[k]$ and $y_{ij}[k]$ should be clear.

The array (1) gives rise to the following single big system

$$\mathbf{x}^+ = \mathbf{Ax} + \mathbf{Bu} \tag{2a}$$

$$\mathbf{y} = \mathbf{Cx} \tag{2b}$$

where $\mathbf{x} = [x_1^T \ x_2^T \ \cdots \ x_q^T]^T$ is the state, $\mathbf{u} = [u_{12}^T \ u_{13}^T \ \cdots \ u_{1q}^T \,|\, u_{23}^T \ u_{24}^T \ \cdots \ u_{2q}^T \,|\, \cdots \,|\, u_{(q-1)q}^T]^T$ is the input, and $\mathbf{y} = [y_{12}^T \ y_{13}^T \ \cdots \ y_{1q}^T \,|\, y_{23}^T \ y_{24}^T \ \cdots \ y_{2q}^T \,|\, \cdots \,|\, y_{(q-1)q}^T]^T$ is the output. Clearly, we have

$$\mathbf{A} = [I_q \otimes A]$$

where $I_q \in \mathbb{R}^{q\times q}$ is the identity matrix (which we may also denote by $I$ when its dimension is either clear from the context or immaterial),

$$\mathbf{B} = \mathrm{inc}\,(B_{::}) := \left[\begin{array}{cccc|cccc|c|c} B_{12} & B_{13} & \cdots & B_{1q} & 0 & 0 & \cdots & 0 & \cdots & 0 \\ -B_{12} & 0 & \cdots & 0 & B_{23} & B_{24} & \cdots & B_{2q} & \cdots & 0 \\ 0 & -B_{13} & \cdots & 0 & -B_{23} & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & -B_{24} & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & B_{(q-1)q} \\ 0 & 0 & \cdots & -B_{1q} & 0 & 0 & \cdots & -B_{2q} & \cdots & -B_{(q-1)q} \end{array}\right]$$

and

$$\mathbf{C} = [\mathrm{inc}\,(C_{::}^T)]^T\,.$$

The notational choice "inc" has to do with that the structure of $\mathbf{B}$ resembles that of the *incidence matrix* of a graph. Let $\mathcal{S}_n = \mathrm{range}\,[\mathbf{1}_q \otimes I_n]$, where $\mathbf{1}_q \in \mathbb{R}^q$ is the vector of all ones. The set $\mathcal{S}_n \subset (\mathbb{R}^n)^q$ is called the *synchronization subspace*, whose orthogonal complement, the *disagreement subspace*, is denoted by $\mathcal{S}_n^\perp$. Let us construct the matrices one is all too familiar with

$$\mathbf{W}_{\mathrm{c}} = [\mathbf{B} \ \mathbf{AB} \ \cdots \ \mathbf{A}^{n-1}\mathbf{B}]\,, \quad \mathbf{W}_{\mathrm{o}} = \left[\begin{array}{c} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-1} \end{array}\right].$$

We have $\mathrm{range}\,\mathbf{W}_{\mathrm{c}} \subset \mathcal{S}_n^\perp$ and $\mathrm{null}\,\mathbf{W}_{\mathrm{c}} \supset \mathcal{S}_n$ by construction. Now we define (relative) controllability and (relative) observability concerning the array (1).

**Definition 1** *The array* (1) *(or the pair* $[A,\, (B_{::})]$*) is said to be* controllable *if* $\mathrm{range}\,\mathbf{W}_{\mathrm{c}} \supset \mathcal{S}_n^\perp$.

**Definition 2** *The array* (1) *(or the pair* $[(C_{::}),\, A]$*) is said to be* observable *if* $\mathrm{null}\,\mathbf{W}_{\mathrm{o}} \subset \mathcal{S}_n$.

Note that $[A,\, (B_{::})]$ is controllable if and only if $[(B_{::}^T),\, A^T]$ is observable. Necessary and sufficient conditions for controllability and observability (in the above sense) are reported in [12] and [11], respectively. Henceforth we assume:

The array (1) is both controllable and observable.

In the next section we present a distributed synchronization algorithm that generates input signals $u_{ij}$ for the array (1) based on the measurements $y_{ij}$. This algorithm is meant to achieve convergence $\|x_i[k] - x_j[k]\| \to 0$ for all pairs $(i,\, j)$ and all initial conditions.

# 3 Algorithm

There are four design parameters to be chosen for the algorithm: the integers $N_{\mathrm{c}},\, N_{\mathrm{o}} \geq n$ and the real numbers $\tau_{\mathrm{c}},\, \tau_{\mathrm{o}} > 0$. The variables employed are denoted by $\lambda_i,\, \hat{x}_i,\, \xi_i \in \mathbb{R}^n$, and $w_{ij} \in (\mathbb{R}^{p_{ij}})^{N_{\mathrm{c}}}$ for $i,\, j = 1,\, 2,\, \ldots,\, q$. The variables $\hat{x}_i$ are purely discrete and their values at the $k$th discrete time instant is denoted by $\hat{x}_i[k]$. The remaining variables, at each $k$, solve certain differential equations, the solutions of which are denoted by $\lambda_i[k,\, t)$, $\xi_i[k,\, t)$, and $w_{ij}[k,\, t)$ with $t \in \mathbb{R}$ being the continuous time variable. Now, the algorithm generating the control inputs $u_{ij}[k]$ for the array (1) is as follows.

$$\begin{aligned}
\dot{w}_{ij}[k,\,t) &= -w_{ij}[k,\,t) - [B_{ij}\ AB_{ij}\ \cdots\ A^{N_{\rm c}-1}B_{ij}]^T(\lambda_i[k,\,t) - \lambda_j[k,\,t)) && \text{(3a)}\\
\dot{\lambda}_i[k,\,t) &= A^{N_{\rm c}}\hat{x}_i[k] + \sum_{j=1}^{q}[B_{ij}\ AB_{ij}\ \cdots\ A^{N_{\rm c}-1}B_{ij}]w_{ij}[k,\,t) && \text{(3b)}\\
\hat{x}_i[k+1] &= A\hat{x}_i[k] + A^{N_{\rm o}}\xi_i[k,\,\tau_{\rm o}) + \sum_{j=1}^{q}B_{ij}u_{ij}[k] && \text{(3c)}\\
\dot{\xi}_i[k,\,t) &= -\sum_{\ell=0}^{N_{\rm o}-1}\sum_{j=1}^{q}A^{\ell T}C_{ij}^TC_{ij}A^{\ell}(\xi_i[k,\,t) - \xi_j[k,\,t)) \\
&\quad + \sum_{j=1}^{q}A^{(N_{\rm o}-1)T}C_{ij}^T(y_{ij}[k] - C_{ij}(\hat{x}_i[k] - \hat{x}_j[k])) && \text{(3d)}\\
u_{ij}[k] &= [\,\underbrace{[0\ \cdots\ 0\ 1]}_{N_{\rm c}\text{ terms}}\otimes I_{p_{ij}}]w_{ij}[k,\,\tau_{\rm c}) && \text{(3e)}
\end{aligned}$$

where, for all $k$, the initial conditions for integrations are set as

$$\lambda_i[k,\,0) = 0\,,\quad \xi_i[k,\,0) = 0\,,\quad w_{ij}[k,\,0) = 0\,.$$

As for $\hat{x}_i$, the initial conditions $\hat{x}_i[0]$ can be chosen arbitrarily. Having described our algorithm, we can now state what it does. Below is our main result.

**Theorem 1** *Consider the array* (1) *under the control inputs* (3e)*. There exist real numbers $\bar{\tau}_{\rm c}$ and $\bar{\tau}_{\rm o}$ such that if $\tau_{\rm c} > \bar{\tau}_{\rm c}$ and $\tau_{\rm o} > \bar{\tau}_{\rm o}$, then the systems synchronize, i.e., $\|x_i[k] - x_j[k]\| \to 0$ as $k \to \infty$ for all pairs $(i,\,j)$ and all initial conditions $x_1[0],\,x_2[0],\,\ldots,\,x_q[0]$.*

In the remainder of the paper we construct the proof of Theorem 1. To this end, we first obtain the discrete-time closed-loop dynamics explicitly. Then we study its stability.

# 4 Closed loop

In this section we compute the closed-loop dynamics governing the system (2) under the algorithm (3). Namely, we obtain explicit expressions for the matrices $\mathbf{K}$ and $\mathbf{L}$ which should appear as

$$\begin{aligned}
\mathbf{x}^+ &= \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{K}\hat{\mathbf{x}} && \text{(4a)}\\
\hat{\mathbf{x}}^+ &= \mathbf{A}\hat{\mathbf{x}} - \mathbf{B}\mathbf{K}\hat{\mathbf{x}} + \mathbf{L}(\mathbf{y} - \mathbf{C}\hat{\mathbf{x}}) && \text{(4b)}
\end{aligned}$$

where $\hat{\mathbf{x}} = [\hat{x}_1^T\ \hat{x}_2^T\ \cdots\ \hat{x}_q^T]^T$ and $\hat{x}_i$ are updated via (3c). We begin with $\mathbf{K}$.

## 4.1 Gain K

We denote by $e_{N_{\rm c}} \in \mathbb{R}^{N_{\rm c}}$ the unit vector whose last entry is one, i.e., $e_{N_{\rm c}} = [0\ \cdots\ 0\ 1]^T$. Recall that the vector $\mathbf{1}_q$ spans the synchronization subspace $\mathcal{S}_1$. Let $S$ denote its normalization, i.e., $S = \mathbf{1}_q/\sqrt{q}$ and hence $S^TS = 1$. Also, let $D \in \mathbb{R}^{q\times(q-1)}$ be some matrix whose columns make an orthonormal basis for $\mathcal{S}_1^{\perp}$. Note that $D^TD = I_{q-1}$ and the columns of the matrix $[D\ S]$ make an orthonormal basis for $\mathbb{R}^n$. We let $\mathbf{D} = [D \otimes I_n]$ and $\mathbf{S} = [S \otimes I_n]$. The following identities are easy to show and find use in the sequel.

(i) $\mathbf{D}\mathbf{D}^T + \mathbf{S}\mathbf{S}^T = I_{qn}$.

(ii) $\text{range}\,[D \otimes I_n] = \mathcal{S}_n^{\perp}$.

(iii) $[S^T \otimes I_n]\mathbf{B} = 0$.

(iv) $\mathbf{C}[S \otimes I_n] = 0$.

Recall that $w_{ij} \in (\mathbb{R}^{p_{ij}})^{N_c}$ are the variables in (3a). Note that $B_{ij} = -B_{ji}$ yields $\dot{w}_{ij}+w_{ij} = \dot{w}_{ji}+w_{ji}$. Then $w_{ij}[k, 0) = w_{ji}[k, 0)$ implies $w_{ij}[k, t) \equiv w_{ji}[k, t)$. This allows us to consider in our analysis only $w_{ij}$ with $i < j$. Now, let us partition $w_{ij}$ as $w_{ij} = [w_{ij}^{[N_c-1]T}\ w_{ij}^{[N_c-2]T}\ \cdots\ w_{ij}^{[0]T}]^T$ with $w_{ij}^{[\ell]} \in \mathbb{R}^{p_{ij}}$. Then gather $w_{ij}^{[\ell]}$ as $\mathbf{w}^{[\ell]} = [w_{12}^{[\ell]T}\ w_{13}^{[\ell]T}\ \cdots\ w_{1q}^{[\ell]T}\,|\,w_{23}^{[\ell]T}\ w_{24}^{[\ell]T}\ \cdots\ w_{2q}^{[\ell]T}\,|\cdots|\,w_{(q-1)q}^{[\ell]T}]^T$ to construct $\mathbf{w} = [\mathbf{w}^{[N_c-1]T}\ \mathbf{w}^{[N_c-2]T}\ \cdots\ \mathbf{w}^{[0]T}]^T$. Also, let $\lambda = [\lambda_1^T\ \lambda_2^T\ \cdots\ \lambda_q^T]^T$ where $\lambda_i \in \mathbb{R}^n$ are the variables in (3b). Finally define

$$\mathbf{R} = [\mathbf{B}\ \mathbf{AB}\ \cdots\ \mathbf{A}^{N_c-1}\mathbf{B}]\,.$$

This new set of notation allows us to put the dynamics (3a)-(3b) into the following compact form

$$\begin{bmatrix} \dot{\mathbf{w}}[k, t) \\ \dot{\lambda}[k, t) \end{bmatrix} = \begin{bmatrix} -I & -\mathbf{R}^T \\ \mathbf{R} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{w}[k, t) \\ \lambda[k, t) \end{bmatrix} + \begin{bmatrix} 0 \\ \mathbf{A}^{N_c}\hat{\mathbf{x}}[k] \end{bmatrix}, \quad \mathbf{w}[k, 0) = 0\,, \quad \lambda[k, 0) = 0\,. \tag{5}$$

Solving (5) allows us to obtain the inputs generated by the algorithm (3) because $\mathbf{u}[k] = \mathbf{w}^{[0]}[k, \tau_c)$. Since we are not interested in the solution $\lambda[k, t)$ let us consider another differential equation, which in certain ways is more convenient:

$$\begin{bmatrix} \dot{\mathbf{v}}[k, t) \\ \dot{\mu}[k, t) \end{bmatrix} = \underbrace{\begin{bmatrix} -I & -\mathbf{R}^T\mathbf{D} \\ \mathbf{D}^T\mathbf{R} & 0 \end{bmatrix}}_{\Lambda} \begin{bmatrix} \mathbf{v}[k, t) \\ \mu[k, t) \end{bmatrix} + \begin{bmatrix} 0 \\ \mathbf{D}^T\mathbf{A}^{N_c}\hat{\mathbf{x}}[k] \end{bmatrix}, \quad \mathbf{v}[k, 0) = 0\,, \quad \mu[k, 0) = 0 \tag{6}$$

where the size of the vector $\mathbf{v}$ is same as that of $\mathbf{w}$ and the vector $\mu$ is of appropriate size. We now make a succession of simple observations that eventually lead us to an explicit expression for the gain $\mathbf{K}$.

**Lemma 1** *We have* $\mathbf{D}\mathbf{D}^T\mathbf{A}^\ell\mathbf{B} = \mathbf{A}^\ell\mathbf{B}$ *for any integer* $\ell \geq 0$*.*

**Proof.** Observe that

$$\begin{aligned} \mathbf{S}^T\mathbf{A}^\ell\mathbf{B} &= [S^T \otimes I_n][I_q \otimes A]^\ell\mathbf{B} \\ &= [S^T \otimes I_n][I_q \otimes A^\ell]\mathbf{B} \\ &= A^\ell \underbrace{[S^T \otimes I_n]\mathbf{B}}_{0} \\ &= 0\,. \end{aligned}$$

Therefore we can write

$$\begin{aligned} \mathbf{D}\mathbf{D}^T\mathbf{A}^\ell\mathbf{B} &= \mathbf{D}\mathbf{D}^T\mathbf{A}^\ell\mathbf{B} + \mathbf{S}(\mathbf{S}^T\mathbf{A}^\ell\mathbf{B}) \\ &= (\underbrace{\mathbf{D}\mathbf{D}^T + \mathbf{S}\mathbf{S}^T}_{I})\mathbf{A}^\ell\mathbf{B} \\ &= \mathbf{A}^\ell\mathbf{B}\,. \end{aligned}$$

Hence the result. ■

**Lemma 2** *Consider the differential equations* (5) *and* (6)*. We have* $\mathbf{w}[k, t) = \mathbf{v}[k, t)$*.*

**Proof.** Consider (5). We can write

$$\begin{aligned} \ddot{\mathbf{w}} &= -\dot{\mathbf{w}} - \mathbf{R}^T\dot{\lambda} \\ &= -\dot{\mathbf{w}} - \mathbf{R}^T(\mathbf{R}\mathbf{w} + \mathbf{A}^{N_c}\hat{\mathbf{x}})\,. \end{aligned}$$

Also, $\dot{\mathbf{w}}[k, 0) = -\mathbf{w}[k, 0) - \mathbf{R}^T\lambda[k, 0) = 0$. Hence the solution $t \mapsto \mathbf{w}[k, t)$ should satisfy

$$\ddot{\mathbf{w}}[k, t) + \dot{\mathbf{w}}[k, t) + \mathbf{R}^T\mathbf{R}\mathbf{w}[k, t) + \mathbf{R}^T\mathbf{A}^{N_c}\hat{\mathbf{x}}[k] = 0\,, \quad \mathbf{w}[k, 0) = 0\,, \quad \dot{\mathbf{w}}[k, 0) = 0\,. \tag{7}$$

Similarly, (6) implies

$$\ddot{\mathbf{v}}[k, t) + \dot{\mathbf{v}}[k, t) + \mathbf{R}^T\mathbf{D}\mathbf{D}^T\mathbf{R}\mathbf{v}[k, t) + \mathbf{R}^T\mathbf{D}\mathbf{D}^T\mathbf{A}^{N_c}\hat{\mathbf{x}}[k] = 0\,, \quad \mathbf{v}[k, 0) = 0\,, \quad \dot{\mathbf{v}}[k, 0) = 0\,. \tag{8}$$

Lemma 1 allows us to write

$$\begin{aligned}
\mathbf{R}^T\mathbf{D}\mathbf{D}^T &= \left(\mathbf{D}\mathbf{D}^T[\mathbf{B}\ \mathbf{A}\mathbf{B}\ \cdots\ \mathbf{A}^{N_\mathrm{c}-1}\mathbf{B}]\right)^T \\
&= [\mathbf{B}\ \mathbf{A}\mathbf{B}\ \cdots\ \mathbf{A}^{N_\mathrm{c}-1}\mathbf{B}]^T \\
&= \mathbf{R}^T\,. \qquad (9)
\end{aligned}$$

Combining (7), (8), and (9) yields the result. ■

**Lemma 3** *The matrix* $\mathbf{D}^T\mathbf{R}$ *is full row rank.*

**Proof.** Suppose not. Then we can find a nonzero vector $\eta \in (\mathbb{R}^n)^{q-1}$ satisfying $\eta^T\mathbf{D}^T\mathbf{R} = 0$. Let $\zeta = \mathbf{D}\eta$, which belongs to $\mathcal{S}_n^\perp$ due to range $[D \otimes I_n] = \mathcal{S}_n^\perp$. Also, $\zeta \neq 0$ because $\eta \neq 0$ and $\mathbf{D}$ is full column rank. Thence $\zeta \notin \mathcal{S}_n$. This implies null $\mathbf{R}^T \not\subset \mathcal{S}_n$ due to $\mathbf{R}^T\zeta = 0$. Hence range $\mathbf{R} \not\supset \mathcal{S}_n^\perp$. Consequently range $\mathbf{W}_\mathrm{c} \not\supset \mathcal{S}_n^\perp$ because range $\mathbf{R} \supset$ range $\mathbf{W}_\mathrm{c}$ thanks to $N_\mathrm{c} \geq n$. But range $\mathbf{W}_\mathrm{c} \not\supset \mathcal{S}_n^\perp$ contradicts that the array (1) is controllable. ■

**Lemma 4** *The matrix* $\Lambda$ *defined in* (6) *is Hurwitz, i.e, all its eigenvalues are on the open left half-plane.*

**Proof.** It is easy to see that $\Lambda^T + \Lambda \leq 0$. Therefore $\Lambda$ is at least neutrally stable. In particular, it cannot have any eigenvalues with positive real part. To show that it can neither have any eigenvalues on the imaginary axis let us suppose the contrary. That is, assume $j\omega$ with $\omega \in \mathbb{R}$ is an eigenvalue of $\Lambda$. Then we could find two vectors $v_1$, $v_2$, at least one of them nonzero, satisfying

$$\begin{bmatrix} -I & -\mathbf{R}^T\mathbf{D} \\ \mathbf{D}^T\mathbf{R} & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = j\omega\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

which yields $v_1 = -(1+j\omega)^{-1}\mathbf{R}^T\mathbf{D}v_2$ and $\mathbf{D}^T\mathbf{R}v_1 = j\omega v_2$. Note that $v_2$ cannot be zero, for otherwise $v_1$ would also have to be zero and by assumption it cannot be that both are zero. Hence we combine the two equations and write

$$[\mathbf{D}^T\mathbf{R}\mathbf{R}^T\mathbf{D}]v_2 = -\frac{j\omega}{1+j\omega}\,v_2\,.$$

That is, $v_2$ is an eigenvector of $\mathbf{D}^T\mathbf{R}\mathbf{R}^T\mathbf{D}$. Since $\mathbf{D}^T\mathbf{R}\mathbf{R}^T\mathbf{D}$ is a real symmetric matrix, its eigenvalues are real. Therefore we have to have $\omega = 0$. Thence $[\mathbf{D}^T\mathbf{R}\mathbf{R}^T\mathbf{D}]v_2 = 0$, i.e., $\mathbf{D}^T\mathbf{R}\mathbf{R}^T\mathbf{D}$ is singular, which however cannot be true because $\mathbf{D}^T\mathbf{R}$ is full row rank by Lemma 3. Hence $\Lambda$ has no eigenvalue on the imaginary axis, which completes the proof. ■

**Lemma 5** *The matrix* $\mathbf{K}$ *in the closed-loop system* (4) *reads*

$$\begin{aligned}
\mathbf{K} &= \Big[\mathbf{B}^T\mathbf{A}^{(N_\mathrm{c}-1)T}\mathbf{D}[\mathbf{D}^T\mathbf{R}\mathbf{R}^T\mathbf{D}]^{-1}\mathbf{D}^T\mathbf{R} - [e_{N_\mathrm{c}}^T \otimes I] \quad \mathbf{B}^T\mathbf{A}^{(N_\mathrm{c}-1)T}\mathbf{D}[\mathbf{D}^T\mathbf{R}\mathbf{R}^T\mathbf{D}]^{-1}\Big] \\
&\qquad \times [I - e^{\Lambda\tau_\mathrm{c}}] \times \begin{bmatrix} 0 \\ \mathbf{D}^T\mathbf{A}^{N_\mathrm{c}} \end{bmatrix}\,.
\end{aligned}$$

**Proof.** Consider (6). It can be verified by direct substitution that the solution reads

$$\begin{aligned}
\begin{bmatrix} \mathbf{v}[k,\,t) \\ \mu[k,\,t) \end{bmatrix} &= \Lambda^{-1}[e^{\Lambda t} - I]\begin{bmatrix} 0 \\ \mathbf{D}^T\mathbf{A}^{N_\mathrm{c}}\hat{\mathbf{x}}[k] \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{R}^T\mathbf{D}[\mathbf{D}^T\mathbf{R}\mathbf{R}^T\mathbf{D}]^{-1}\mathbf{D}^T\mathbf{R} - I & \mathbf{R}^T\mathbf{D}[\mathbf{D}^T\mathbf{R}\mathbf{R}^T\mathbf{D}]^{-1} \\ -[\mathbf{D}^T\mathbf{R}\mathbf{R}^T\mathbf{D}]^{-1}\mathbf{D}^T\mathbf{R} & -[\mathbf{D}^T\mathbf{R}\mathbf{R}^T\mathbf{D}]^{-1} \end{bmatrix}[e^{\Lambda t} - I]\begin{bmatrix} 0 \\ \mathbf{D}^T\mathbf{A}^{N_\mathrm{c}} \end{bmatrix}\hat{\mathbf{x}}[k]
\end{aligned}$$

where $\Lambda^{-1}$ exists because $\Lambda$ is Hurwitz by Lemma 4 and $[\mathbf{D}^T\mathbf{R}\mathbf{R}^T\mathbf{D}]^{-1}$ exists because $\mathbf{D}^T\mathbf{R}$ is full row rank by Lemma 3. Using $\mathbf{u}[k] = \mathbf{w}^{[0]}[k,\,\tau_\mathrm{c})$ and Lemma 2 we can now write

$$\begin{aligned}
\mathbf{K}\hat{\mathbf{x}}[k] &= -\mathbf{u}[k] \\
&= -\mathbf{w}^{[0]}[k,\,\tau_\mathrm{c}) \\
&= -[e_{N_\mathrm{c}}^T \otimes I]\mathbf{w}[k,\,\tau_\mathrm{c}) \\
&= -[e_{N_\mathrm{c}}^T \otimes I]\mathbf{v}[k,\,\tau_\mathrm{c}) \\
&= [e_{N_\mathrm{c}}^T \otimes I]\left[\mathbf{R}^T\mathbf{D}[\mathbf{D}^T\mathbf{R}\mathbf{R}^T\mathbf{D}]^{-1}\mathbf{D}^T\mathbf{R} - I \quad \mathbf{R}^T\mathbf{D}[\mathbf{D}^T\mathbf{R}\mathbf{R}^T\mathbf{D}]^{-1}\right][I - e^{\Lambda\tau_\mathrm{c}}]\begin{bmatrix} 0 \\ \mathbf{D}^T\mathbf{A}^{N_\mathrm{c}} \end{bmatrix}\hat{\mathbf{x}}[k]\,.
\end{aligned}$$

The result then follows since $\mathbf{R}[e_{N_\mathrm{c}} \otimes I] = \mathbf{A}^{N_\mathrm{c}-1}\mathbf{B}$. ■

## 4.2 Gain L

Having computed $\mathbf{K}$ of (4), we now focus on $\mathbf{L}$. Let $\xi = [\xi_1^T \; \xi_2^T \; \cdots \; \xi_q^T]^T$, where $\xi_i \in \mathbb{R}^n$ are the variables in (3d), and define

$$\mathbf{Q} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{N_o - 1} \end{bmatrix} .$$

This allows the dynamics (3d) to be compactly expressed as

$$\dot{\xi}[k,\, t) = -\mathbf{Q}^T\mathbf{Q}\xi[k,\, t) + \mathbf{A}^{(N_o-1)T}\mathbf{C}^T(\mathbf{y}[k] - \mathbf{C}\hat{\mathbf{x}}[k]) , \quad \xi[k,\, 0) = 0 . \tag{10}$$

Finally, we define $\Gamma = \mathbf{D}^T\mathbf{Q}^T\mathbf{QD}$. Let us make a few observations before we attempt to solve (10).

**Lemma 6** *The matrix* $\mathbf{D}^T\mathbf{Q}^T$ *is full row rank.*

**Proof.** This is a consequence of the observability of the array (1). See the dual result Lemma 3. ■

**Lemma 7** *The matrix* $-\Gamma$ *is Hurwitz.*

**Proof.** The matrix $\Gamma$ is symmetric positive semidefinite because we can write $\Gamma = (\mathbf{QD})^T\mathbf{QD}$. Also, it is nonsingular because $(\mathbf{QD})^T$ is full row rank by Lemma 6. Hence $\Gamma$ is symmetric positive definite. Then $-\Gamma$ is symmetric negative definite and consequently all its eigenvalues are real and strictly negative. Hence the result. ■

**Lemma 8** *We have* $\mathbf{DD}^T\mathbf{A}^{\ell T}\mathbf{C}^T = \mathbf{A}^{\ell T}\mathbf{C}^T$ *for any integer* $\ell \geq 0$.

**Proof.** Like Lemma 1. ■

**Lemma 9** *The solution to* (10) *reads* $\xi[k,\, t) = \mathbf{D}\Gamma^{-1}[I - e^{-\Gamma t}]\mathbf{D}^T\mathbf{A}^{(N_o-1)T}\mathbf{C}^T(\mathbf{y}[k] - \mathbf{C}\hat{\mathbf{x}}[k])$.

**Proof.** First note that the initial condition constraint $\xi[k,\, 0) = 0$ is satisfied. Now we show that this $\xi[k,\, t)$ also satisfies the differential equation. For compactness let $\beta = \mathbf{A}^{(N_o-1)T}\mathbf{C}^T(\mathbf{y}[k] - \mathbf{C}\hat{\mathbf{x}}[k])$. Note that $\mathbf{DD}^T\beta = \beta$ and $\mathbf{DD}^T\mathbf{Q}^T = \mathbf{Q}^T$ by Lemma 8. Hence putting our candidate solution into the righthand side of (10) yields

$$\begin{aligned} -\mathbf{Q}^T\mathbf{Q}\xi[k,\, t) + \beta &= -\mathbf{Q}^T\mathbf{QD}\Gamma^{-1}[I - e^{-\Gamma t}]\mathbf{D}^T\beta + \beta \\ &= -\mathbf{D}\underbrace{\mathbf{D}^T\mathbf{Q}^T\mathbf{QD}}_{\Gamma}\Gamma^{-1}[I - e^{-\Gamma t}]\mathbf{D}^T\beta + \beta \\ &= -\mathbf{D}[I - e^{-\Gamma t}]\mathbf{D}^T\beta + \beta \\ &= \mathbf{D}e^{-\Gamma t}\mathbf{D}^T\beta - \mathbf{DD}^T\beta + \beta \\ &= \mathbf{D}e^{-\Gamma t}\mathbf{D}^T\beta . \end{aligned}$$

Now, by differentiating the candidate solution we obtain

$$\begin{aligned} \dot{\xi}[k,\, t) &= \frac{d}{dt}\left\{\mathbf{D}\Gamma^{-1}[I - e^{-\Gamma t}]\mathbf{D}^T\beta\right\} \\ &= \mathbf{D}\Gamma^{-1}[\Gamma e^{-\Gamma t}]\mathbf{D}^T\beta \\ &= \mathbf{D}e^{-\Gamma t}\mathbf{D}^T\beta \\ &= -\mathbf{Q}^T\mathbf{Q}\xi[k,\, t) + \beta \end{aligned}$$

which was to be shown. ■

**Lemma 10** *The matrix* $\mathbf{L}$ *in the closed-loop system* (4) *reads*

$$\mathbf{L} = \mathbf{A}^{N_o}\mathbf{D}\Gamma^{-1}[I - e^{-\Gamma\tau_o}]\mathbf{D}^T\mathbf{A}^{(N_o-1)T}\mathbf{C}^T .$$

**Proof.** Using (3c) and Lemma 9 we can write

$$\begin{aligned} \hat{\mathbf{x}}[k+1] &= \mathbf{A}\hat{\mathbf{x}}[k] - \mathbf{BK}\hat{\mathbf{x}}[k] + \mathbf{A}^{N_o}\xi[k,\, \tau_o) \\ &= \mathbf{A}\hat{\mathbf{x}}[k] - \mathbf{BK}\hat{\mathbf{x}}[k] + \mathbf{A}^{N_o}\mathbf{D}\Gamma^{-1}[I - e^{-\Gamma\tau_o}]\mathbf{D}^T\mathbf{A}^{(N_o-1)T}\mathbf{C}^T(\mathbf{y}[k] - \mathbf{C}\hat{\mathbf{x}}[k]) . \end{aligned}$$

Comparing this to (4b) yields the result. ■

## 5 Stability

In the previous section we obtained the explicit expression for the righthand side of (4) by computing the gains $\mathbf{K}$ and $\mathbf{L}$. Now we study the behavior of the closed-loop system. Our goal is to show that (under appropriate choices of the parameters $\tau_{\rm c}, \tau_{\rm o}$) for all initial conditions $\mathbf{x}[0]$ and $\hat{\mathbf{x}}[0]$ the solution $\mathbf{x}[k]$ enjoys the convergence $\|\mathbf{x}[k]\|_{\mathcal{S}_n} \to 0$, where $\|\cdot\|_{\mathcal{S}_n}$ denotes the Euclidean distance to the subspace $\mathcal{S}_n$. By proving this convergence we will have established Theorem 1. For our analysis we borrow the following result due to Kleinman [3].[1]

**Proposition 1** *Let $N \geq 1$ be an integer, $F \in \mathbb{R}^{n\times n}$, $G \in \mathbb{R}^{n\times p}$, and $\operatorname{rank}[G\ FG\ \cdots\ F^{N-1}G] = n$. Then the matrix*

$$H = F - GG^T F^{(N-1)T}\left(\sum_{\ell=0}^{N-1} F^\ell GG^T F^{\ell T}\right)^{-1} F^N$$

*is Schur, i.e., all the eigenvalues of $H$ are on the open unit disc.*

Define the reduced state $\mathbf{x}_{\rm r} = \mathbf{D}^T\mathbf{x}$ and the error $\mathbf{e}_{\rm r} = \mathbf{D}^T(\hat{\mathbf{x}} - \mathbf{x})$. Note that $\|\mathbf{x}_{\rm r}\| = \|\mathbf{x}\|_{\mathcal{S}_n}$. Also, define the following reduced parameters

$$\mathbf{A}_{\rm r} = \mathbf{D}^T\mathbf{A}\mathbf{D}\,,\quad \mathbf{B}_{\rm r} = \mathbf{D}^T\mathbf{B}\,,\quad \mathbf{C}_{\rm r} = \mathbf{C}\mathbf{D}\,,\quad \mathbf{K}_{\rm r} = \mathbf{K}\mathbf{D}\,,\quad \mathbf{L}_{\rm r} = \mathbf{D}^T\mathbf{L}\,.$$

**Lemma 11** *We have $\mathbf{D}^T\mathbf{A}^\ell = \mathbf{D}^T\mathbf{A}^\ell\mathbf{D}\mathbf{D}^T$ for any integer $\ell \geq 0$.*

**Proof.** Like Lemma 1. ■

Note that Lemma 5 and Lemma 11 imply $\mathbf{K} = \mathbf{K}\mathbf{D}\mathbf{D}^T$. Using the structural properties of our matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ emphasized in Lemmas 1, 8, and 11, we now proceed to obtain the dynamics for $\mathbf{x}_{\rm r}$ and $\mathbf{e}_{\rm r}$. Consider (4a). We can write

$$\begin{aligned}
\mathbf{x}_{\rm r}^+ &= \mathbf{D}^T\mathbf{x}^+ \\
&= \mathbf{D}^T\mathbf{A}\mathbf{x} - \mathbf{D}^T\mathbf{B}\mathbf{K}\hat{\mathbf{x}} \\
&= \mathbf{D}^T\mathbf{A}\mathbf{D}\mathbf{D}^T\mathbf{x} - \mathbf{D}^T\mathbf{B}\mathbf{K}\mathbf{D}\mathbf{D}^T\hat{\mathbf{x}} \\
&= \mathbf{A}_{\rm r}\mathbf{x}_{\rm r} - \mathbf{B}_{\rm r}\mathbf{K}_{\rm r}(\mathbf{x}_{\rm r} + \mathbf{e}_{\rm r}) \\
&= [\mathbf{A}_{\rm r} - \mathbf{B}_{\rm r}\mathbf{K}_{\rm r}]\mathbf{x}_{\rm r} - \mathbf{B}_{\rm r}\mathbf{K}_{\rm r}\mathbf{e}_{\rm r}\,.
\end{aligned}$$

As for $\mathbf{e}_{\rm r}$, the dynamics (4) yields

$$\begin{aligned}
\mathbf{e}_{\rm r}^+ &= \mathbf{D}^T(\hat{\mathbf{x}}^+ - \mathbf{x}^+) \\
&= \mathbf{D}^T(\mathbf{A}\hat{\mathbf{x}} + \mathbf{L}(\mathbf{C}\mathbf{x} - \mathbf{C}\hat{\mathbf{x}}) - \mathbf{A}\mathbf{x}) \\
&= [\mathbf{D}^T\mathbf{A} - \mathbf{D}^T\mathbf{L}\mathbf{C}](\hat{\mathbf{x}} - \mathbf{x}) \\
&= [\mathbf{D}^T\mathbf{A}\mathbf{D}\mathbf{D}^T - \mathbf{D}^T\mathbf{L}\mathbf{C}\mathbf{D}\mathbf{D}^T](\hat{\mathbf{x}} - \mathbf{x}) \\
&= [\mathbf{D}^T\mathbf{A}\mathbf{D} - \mathbf{D}^T\mathbf{L}\mathbf{C}\mathbf{D}]\mathbf{D}^T(\hat{\mathbf{x}} - \mathbf{x}) \\
&= [\mathbf{A}_{\rm r} - \mathbf{L}_{\rm r}\mathbf{C}_{\rm r}]\mathbf{e}_{\rm r}\,.
\end{aligned}$$

Hence the overall dynamics for the pair $(\mathbf{x}_{\rm r}, \mathbf{e}_{\rm r})$ reads

$$\begin{bmatrix}\mathbf{x}_{\rm r}\\ \mathbf{e}_{\rm r}\end{bmatrix}^+ = \underbrace{\begin{bmatrix}\mathbf{A}_{\rm r} - \mathbf{B}_{\rm r}\mathbf{K}_{\rm r} & -\mathbf{B}_{\rm r}\mathbf{K}_{\rm r}\\ 0 & \mathbf{A}_{\rm r} - \mathbf{L}_{\rm r}\mathbf{C}_{\rm r}\end{bmatrix}}_{\mathbf{\Phi}_{\rm r}}\begin{bmatrix}\mathbf{x}_{\rm r}\\ \mathbf{e}_{\rm r}\end{bmatrix}\,. \tag{11}$$

Next, we show that the block diagonal entries in (11) can be made Schur by choosing $\tau_{\rm c}$ and $\tau_{\rm o}$ large enough. To this end, let us define the following $(q-1)n \times (q-1)n$ matrices.

$$\begin{aligned}
\theta_{\rm c}(\tau) &= \mathbf{D}^T\mathbf{B}\left[\mathbf{B}^T\mathbf{A}^{(N_{\rm c}-1)T}\mathbf{D}[\mathbf{D}^T\mathbf{R}\mathbf{R}^T\mathbf{D}]^{-1}\mathbf{D}^T\mathbf{R} - [e_{N_{\rm c}}^T \otimes I]\quad \mathbf{B}^T\mathbf{A}^{(N_{\rm c}-1)T}\mathbf{D}[\mathbf{D}^T\mathbf{R}\mathbf{R}^T\mathbf{D}]^{-1}\right] \\
&\quad \times [e^{\Lambda\tau}] \times \begin{bmatrix} 0 \\ \mathbf{D}^T\mathbf{A}^{N_{\rm c}}\end{bmatrix}\mathbf{D} \\
\theta_{\rm o}(\tau) &= \mathbf{D}^T\mathbf{A}^{N_{\rm o}}\mathbf{D}\Gamma^{-1}[e^{-\Gamma\tau}]\mathbf{D}^T\mathbf{A}^{(N_{\rm o}-1)T}\mathbf{C}^T\mathbf{C}\mathbf{D}\,.
\end{aligned}$$

[1] Kleinman assumes that the matrix $F$ is invertible. This assumption however is superfluous; see [10].

Now we can write by Lemmas 1, 5, and 11

$$\begin{aligned}\mathbf{A}_{\rm r}-\mathbf{B}_{\rm r}\mathbf{K}_{\rm r} &= \mathbf{A}_{\rm r}-\mathbf{D}^T\mathbf{B}\mathbf{K}\mathbf{D}\\ &= \mathbf{A}_{\rm r}-\mathbf{D}^T\mathbf{B}\mathbf{B}^T\mathbf{A}^{(N_{\rm c}-1)T}\mathbf{D}[\mathbf{D}^T\mathbf{R}\mathbf{R}^T\mathbf{D}]^{-1}\mathbf{D}^T\mathbf{A}^{N_{\rm c}}\mathbf{D}+\theta_{\rm c}(\tau_{\rm c})\\ &= \mathbf{A}_{\rm r}-\mathbf{B}_{\rm r}\mathbf{B}_{\rm r}^T\mathbf{A}_{\rm r}^{(N_{\rm c}-1)T}[\mathbf{D}^T\mathbf{R}\mathbf{R}^T\mathbf{D}]^{-1}\mathbf{A}_{\rm r}^{N_{\rm c}}+\theta_{\rm c}(\tau_{\rm c})\\ &= \underbrace{\mathbf{A}_{\rm r}-\mathbf{B}_{\rm r}\mathbf{B}_{\rm r}^T\mathbf{A}_{\rm r}^{(N_{\rm c}-1)T}\left(\sum_{\ell=0}^{N_{\rm c}-1}\mathbf{A}_{\rm r}^{\ell}\mathbf{B}_{\rm r}\mathbf{B}_{\rm r}^T\mathbf{A}_{\rm r}^{\ell T}\right)^{-1}\mathbf{A}_{\rm r}^{N_{\rm c}}}_{\mathbf{H}_{\rm c}}+\theta_{\rm c}(\tau_{\rm c}) \qquad (12)\end{aligned}$$

where we used $\mathbf{D}^T\mathbf{R}=[\mathbf{B}_{\rm r}\ \mathbf{A}_{\rm r}\mathbf{B}_{\rm r}\ \cdots\ \mathbf{A}_{\rm r}^{N_{\rm c}-1}\mathbf{B}_{\rm r}]$ and $\mathbf{A}_{\rm r}^{\ell}=\mathbf{D}^T\mathbf{A}^{\ell}\mathbf{D}$. Similarly, by Lemmas 8, 10, and 11 we obtain

$$\begin{aligned}\mathbf{A}_{\rm r}-\mathbf{L}_{\rm r}\mathbf{C}_{\rm r} &= \mathbf{A}_{\rm r}-\mathbf{D}^T\mathbf{L}\mathbf{C}\mathbf{D}\\ &= \mathbf{A}_{\rm r}-\mathbf{D}^T\mathbf{A}^{N_{\rm o}}\mathbf{D}\Gamma^{-1}\mathbf{D}^T\mathbf{A}^{(N_{\rm o}-1)T}\mathbf{C}^T\mathbf{C}\mathbf{D}+\theta_{\rm o}(\tau_{\rm o})\\ &= \mathbf{A}_{\rm r}-\mathbf{A}_{\rm r}^{N_{\rm o}}[\mathbf{D}^T\mathbf{Q}^T\mathbf{Q}\mathbf{D}]^{-1}\mathbf{A}_{\rm r}^{(N_{\rm o}-1)T}\mathbf{C}_{\rm r}^T\mathbf{C}_{\rm r}+\theta_{\rm o}(\tau_{\rm o})\\ &= \underbrace{\mathbf{A}_{\rm r}-\mathbf{A}_{\rm r}^{N_{\rm o}}\left(\sum_{\ell=0}^{N_{\rm o}-1}\mathbf{A}_{\rm r}^{\ell T}\mathbf{C}_{\rm r}^T\mathbf{C}_{\rm r}\mathbf{A}_{\rm r}^{\ell}\right)^{-1}\mathbf{A}_{\rm r}^{(N_{\rm o}-1)T}\mathbf{C}_{\rm r}^T\mathbf{C}_{\rm r}}_{\mathbf{H}_{\rm o}}+\theta_{\rm o}(\tau_{\rm o}) \qquad (13)\end{aligned}$$

where we used $\mathbf{D}^T\mathbf{Q}^T=[\mathbf{C}_{\rm r}^T\ \mathbf{A}_{\rm r}^T\mathbf{C}_{\rm r}^T\ \cdots\ \mathbf{A}_{\rm r}^{(N_{\rm o}-1)T}\mathbf{C}_{\rm r}^T]$.

**Lemma 12** *There exists $\bar{\tau}>0$ such that the matrix $[\mathbf{H}_{\rm c}+\theta_{\rm c}(\tau)]$ is Schur for all $\tau>\bar{\tau}$.*

**Proof.** By Lemma 3 the matrix $[\mathbf{B}_{\rm r}\ \mathbf{A}_{\rm r}\mathbf{B}_{\rm r}\ \cdots\ \mathbf{A}_{\rm r}^{N_{\rm c}-1}\mathbf{B}_{\rm r}]=\mathbf{D}^T\mathbf{R}$ is full row rank. Then $\mathbf{H}_{\rm c}$ is Schur by Proposition 1. Recall that $\Lambda$ is Hurwitz by Lemma 4. Therefore $\theta_{\rm c}(\tau)\to 0$ because $e^{\Lambda\tau}\to 0$ as $\tau\to\infty$. Now, since small perturbations on a square matrix mean small changes on its eigenvalues, we can find $\varepsilon>0$ such that $[\mathbf{H}_{\rm c}+\theta]$ is Schur for all $\|\theta\|<\varepsilon$. Then, thanks to $\theta_{\rm c}(\tau)\to 0$, we can choose some $\bar{\tau}>0$ such that $\|\theta_{\rm c}(\tau)\|<\varepsilon$ for all $\tau>\bar{\tau}$. Hence the result. ■

**Lemma 13** *There exists $\bar{\tau}>0$ such that the matrix $[\mathbf{H}_{\rm o}+\theta_{\rm o}(\tau)]$ is Schur for all $\tau>\bar{\tau}$.*

**Proof.** By Lemma 6 the matrix $[\mathbf{C}_{\rm r}^T\ \mathbf{A}_{\rm r}\mathbf{C}_{\rm r}^T\ \cdots\ \mathbf{A}_{\rm r}^{(N_{\rm o}-1)T}\mathbf{C}_{\rm r}^T]=\mathbf{D}^T\mathbf{Q}^T$ is full row rank. Then $\mathbf{H}_{\rm o}^T$ is Schur by Proposition 1, meaning $\mathbf{H}_{\rm o}$ is also Schur. Lemma 7 tells us that $-\Gamma$ is Hurwitz. Hence $e^{-\Gamma\tau}\to 0$ and, consequently, $\theta_{\rm o}(\tau)\to 0$ as $\tau\to\infty$. The rest is like the proof of Lemma 12. ■

Our preparations for the proof of Theorem 1 are now complete:

**Proof of Theorem 1.** Consider the array (1) under the control inputs (3e) and arbitrary initial conditions $x_1[0]$, $x_2[0]$, …, $x_q[0]$. Let $\bar{\tau}_{\rm c}>0$ be such that $[\mathbf{H}_{\rm c}+\theta_{\rm c}(\tau)]$ is Schur for all $\tau>\bar{\tau}_{\rm c}$. Likewise, let $\bar{\tau}_{\rm o}>0$ be such that $[\mathbf{H}_{\rm o}+\theta_{\rm o}(\tau)]$ is Schur for all $\tau>\bar{\tau}_{\rm o}$. Such $\bar{\tau}_{\rm c}$ and $\bar{\tau}_{\rm o}$ exist thanks, respectively, to Lemma 12 and Lemma 13. Suppose now the parameters $\tau_{\rm c}$ and $\tau_{\rm o}$ in the algorithm (3) satisfy $\tau_{\rm c}>\bar{\tau}_{\rm c}$ and $\tau_{\rm o}>\bar{\tau}_{\rm o}$. Then, by (12) the matrix $[\mathbf{A}_{\rm r}-\mathbf{B}_{\rm r}\mathbf{K}_{\rm r}]$ is Schur. Likewise, $[\mathbf{A}_{\rm r}-\mathbf{L}_{\rm r}\mathbf{C}_{\rm r}]$ is Schur by (13). Therefore the system matrix $\mathbf{\Phi}_{\rm r}$ in (11) is Schur because it is upper block triangular with Schur block diagonal entries. This implies that the solution of the system (11) must converge to the origin regardless of the initial conditions. In particular, we have $\mathbf{x}_{\rm r}[k]\to 0$ as $k\to\infty$. Then the solution of the closed-loop system (4) must satisfy $\|\mathbf{x}[k]\|_{\mathcal{S}_n}\to 0$ because $\|\mathbf{x}\|_{\mathcal{S}_n}=\|\mathbf{D}^T\mathbf{x}\|=\|\mathbf{x}_{\rm r}\|$. Clearly, $\|\mathbf{x}[k]\|_{\mathcal{S}_n}\to 0$ means $\|x_i[k]-x_j[k]\|\to 0$ as $k\to\infty$ for all pairs $(i,\,j)$. Hence the result. ■

# 6 Conclusion

In this paper we studied relatively actuated arrays of discrete-time linear time-invariant systems with incommensurable coupling parameters. For this general class of arrays we presented a distributed algorithm that achieved synchronization through dynamic output feedback. In the case we studied, even though the array evolved in discrete time, part of the algorithm required integration in continuous time so that the overall process remained decentralized. At this point it is not clear how to construct a purely discrete-time algorithm for a discrete-time array. As for continuous-time arrays with incommensurable input/output matrices, the problem of distributed synchronization under relative actuation seems still to be open.

# References


[1] Y. Cao, W. Yu, W. Ren, and G. Chen. An overview of recent progress in the study of distributed multi-agent coordination. *IEEE Transactions on Industrial Informatics*, 9:427–438, 2013.

[2] Q. Jiao, H. Modares, F.L. Lewis, S. Xu, and L. Xie. Distributed $\mathrm{Ł}_2$-gain output-feedback control of homogeneous and heterogeneous systems. *Automatica*, 71:361–368, 2016.

[3] D.L. Kleinman. Stabilizing a discrete constant linear system with application to iterative methods for solving the Riccati equation. *IEEE Transactions on Automatic Control*, 19:252–254, 1974.

[4] Z. Li, W. Ren, X. Liu, and M. Fu. Distributed containment control of multi-agent systems with general linear dynamics in the presence of multiple leaders. *International Journal of Robust and Nonlinear Control*, 23:534–547, 2013.

[5] Z. Li, W. Ren, X. Liu, and L. Hie. Distributed consensus of linear multi-agent systems with adaptive dynamic protocols. *Automatica*, 49:1986–1995, 2013.

[6] Z.K. Li, Z.S. Duan, G.R. Chen, and L. Huang. Consensus of multiagent systems and synchronization of complex networks: A unified viewpoint. *IEEE Transactions on Circuits and Systems I: Regular Papers*, 57:213–224, 2010.

[7] L. Scardovi and R. Sepulchre. Synchronization in networks of identical linear systems. *Automatica*, 45:2557–2562, 2009.

[8] J.H. Seo, J. Back, H. Kim, and H. Shim. Output feedback consensus for high-order linear systems having uniform ranks under switching topology. *IET Control Theory and Applications*, 6:1118–1124, 2012.

[9] J.H. Seo, H. Shim, and J. Back. Consensus of high-order linear systems using dynamic output feedback compensator: Low gain approach. *Automatica*, 45:2659–2664, 2009.

[10] S.E. Tuna. A dual pair of optimization-based formulations for estimation and control. *Automatica*, 51:18–26, 2015.

[11] S.E. Tuna. Observability through matrix-weighted graph, 2016. arXiv:1603.07637 [math.DS].

[12] S.E. Tuna. Positive controllability of networks under relative actuation, 2017. arXiv:1707.05502 [math.DS].

[13] T. Yang, S. Roy, and A. Saberi. Constructing consensus controllers for networks with identical general linear agents. *International Journal of Robust and Nonlinear Control*, 21:1237–1256, 2011.

[14] H. Zhang, F.L. Lewis, and A. Das. Optimal design for synchronization of cooperative systems: State feedback, observer and output feedback. *IEEE Transactions on Automatic Control*, 56:1948–1952, 2011.

[15] B. Zhou and Z. Lin. Consensus of high-order multi-agent systems with large input and communication delays. *Automatica*, 50:452–464, 2014.