KOMLÓS PROPERTIES IN BANACH LATTICES

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ABSTRACT. Several Komlós like properties in Banach lattices are investigated. We prove that C(K) fails the *oo*-pre-Komlós property, assuming that the compact Hausdorff space K has a nonempty separable open subset U without isolated points such that every $u \in U$ has countable neighborhood base. We prove also that for any infinite dimensional Banach lattice E there is an unbounded convex *uo*-pre-Komlós set $C \subseteq E_+$ which is not *uo*-Komlós.

1. INTRODUCTION

In recent paper [5], the unbounded order convergence in Banach lattices was deeply studied. Among other things, this development has lead to study of generalizations of the Komlós celebrated theorem [7] to the Banach lattice setting. The authors of [5] did their generalization through ALrepresentations of a Banach lattice with a strictly positive order continuous functional, replacing almost everywhere convergence by unbounded order convergence. Beside such a natural extension, many questions on generalized Komlós properties are still requiring an investigation.

In the present paper, we study Komlós properties in more breadth settings, than for the *uo*-convergence. In Section 2, we define and investigate several Komlós properties for different modes of boundedness and convergence in a Banach lattice. Section 3 is completely devoted to Komlós properties in Banach lattices of continuous functions. Section 4 is dealing with so-called Komlós sets.

As the nature of the Komlós theorem is sequential, we restrict ourselves to sequential convergences in Banach lattices. For unexplained terminology and notations we refer the reader to [1, 2, 3, 4, 5, 6]. In the present paper, E stands for a real Banach lattice.

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2. Komlós like properties in Banach lattices

Let x_n be a sequence in E and $x \in E$. Recall that:

- (1) x_n converges in order to x (we write $x_n \xrightarrow{o} x$), if there is a sequence y_n decreasing to 0 (we write $y_n \downarrow 0$) with $|x_n x| \le y_n$ for all n;
- (2) if x_n converges in norm to x, we write $x_n \xrightarrow{n} x$;
- (3) x_n is unbounded order convergent to x (we write $x_n \xrightarrow{u_0} x$) if $|x_n x| \wedge u \xrightarrow{o} 0$ for every $u \in E_+$;
- (4) if $|x_n x| \wedge u \xrightarrow{n} 0$ for every $u \in E_+$, we write $x_n \xrightarrow{un} x$ and say that x_n is unbounded norm convergent to x.

The main motivation for the present paper is the following classical result [7].

Theorem 1 (Komlós). Let $E = L_1(\mu)$, where μ is a probability measure. Then, for every norm bounded sequence x_n , there is a subsequence x_{n_k} such that the Cesáro means $\frac{1}{m} \sum_{j=1}^m x_{n_{k_j}}$ of any further subsequence $x_{n_{k_j}}$ converge almost everywhere to some $x \in X$.

The Komlós theorem has initiated many investigations and was extended recently for Banach lattices [5] by replacing a.e.-convergence with *uo*-convergence. This development has inspired the following definition.

Definition 1. A Banach lattices E is said to have ab-Komlós property (respectively, ab-pre-Komlós property) if, for every a-bounded sequence x_n in E, there exist a subsequence x_{n_k} and an element $x \in E$ such that

$$x = b - \lim_{m \to \infty} \frac{1}{m} \sum_{j=1}^{m} x_{n_{k_j}}$$

for any subsequence $x_{n_{k_j}}$ of x_{n_k} (respectively, the sequence $\frac{1}{m} \sum_{j=1}^m x_{n_{k_j}}$ is b-Cauchy for any subsequence $x_{n_{k_i}}$ of x_{n_k}).

Here, a-boundedness stands for o- or n-boundedness; and b-convergence stands for o-, uo-, n-, or un-convergence.

Clearly, it suffices to check *ab*-Komlós and *ab*-pre-Komlós properties only for sequences in E_+ . Furthermore, *no*-Komlós implies *oo*-Komlós and, if E is σ -Dedekind complete, they coincide with *no*-pre-Komlós and *oo*-pre-Komlós properties respectively.

nuo-Komlós and *nuo*-pre-Komlós properties were introduced in [5, Def.5.1] under the names of Komlós and pre-Komlós properties respectively. The Komlós Theorem 1 has been extended further in [5, Prop.5.13] as follows. **Proposition 1** (Gao–Troitsky–Xanthos). Let E be a regular sublattice of an order continuous Banach lattice F. Then E has the nuo-pre-Komlós property. Moreover, E has the nuo-Komlós property iff it is sequentially boundedly uo-complete.

Let us mention also the next corollary (see [5, Cor.5.14]) of [5, Prop.5.13].

Corollary 1 (Gao–Troitsky–Xanthos). Every order continuous Banach lattice E has the nuo-pre-Komlós property. Moreover, E has the nuo-Komlós property iff it is a KB-space.

Proposition 2. Every o-continuous Banach lattice E has the nun-pre-Komlós property. Moreover, if E is a KB-space then E has the nun-Komlós property.

Proof. It follows from Corollary 1 since *uo*-convergence implies *un*-convergence (see [4, Prop.2.5]). \Box

Example 1. The Banach lattice c of real convergent sequences fails the oo-Komlós (and hence no-Komlós) property. To see this, take the sequence $x_n = \sum_{j=1}^n e_{2j}$ in $[0, 1] \subset c$. Clearly, for any subsequence $x_{n_{k_j}}$, the sequence $\frac{1}{m} \sum_{j=1}^m x_{n_{k_j}}$ is uo-divergent, and also o-divergent, since it is order bounded. It shows also that c fails the nuo-Komlós property. Since the sequence $\frac{1}{m} \sum_{j=1}^m x_{n_{k_j}}$ is not n-Cauchy, c fails the nn-pre-Komlós property as well.

It can be easily seen that c has oo-, ouo-, and nuo-pre-Komlós property (cf. also [5, Cor.5.10]).

Proposition 3. Any o-continuous Banach lattice E has oo- and on-Komlós property.

Proof. Let $x_n \in [-u, u]$ for all n. By [5, Cor.5.14], E has nuo-pre-Komlós property, and hence ouo-pre-Komlós property. So, there exists a subsequence x_{n_k} such that any sequence $\frac{1}{m} \sum_{j=1}^m x_{n_{k_j}} \subset [-u, u]$ is uo-Cauchy, and hence o-Cauchy. By o-continuity of the norm, E is Dedekind complete. It follows that there exist $y \in E$ with $y = o - \lim_{m \to \infty} \frac{1}{m} \sum_{j=1}^m x_{n_{k_j}}$ for any subsequence $x_{n_{k_j}}$ of x_{n_k} . Using o-continuity once more, we get $y = n - \lim_{m \to \infty} \frac{1}{m} \sum_{j=1}^m x_{n_{k_j}}$.

3. Komlós properties in Banach lattices of continuous functions

Notice that, in E = C(K): *oo*-Komlós, *no*-Komlós, *ouo*-Komlós, and *nuo*-Komlós properties coincide. The same is true for *oo*-, *no*-, *ouo*-, and *nuo*-pre-Komlós properties. Furthermore, *nn*-Komlós property coincides with both *nn*-pre-Komlós and *on*-pre-Komlós properties.

In view of Example 1, the Banach lattice $c \cong C(\mathbb{N} \cup \{\infty\})$ fails the *oo*-Komlós property, but still has the *oo*-pre-Komlós property. We point out that the one-point compactification $\mathbb{N} \cup \{\infty\}$ of \mathbb{N} is a separable compact metric space in which all points except ∞ are isolated.

It was mentioned in [5, Ex.5.3] that it was still unknown whether or not C[0, 1] has the *nuo*-pre-Komlós property. Here, we clarify the situation with Banach lattices C(K) for a large class of compact Hausdorff spaces.

Theorem 2. Let K be a compact Hausdorff space without isolated points in which there exist two distinct sequences t_n and t'_n such that $cl\{t_n\}_{n=1}^{\infty} = cl\{t'_n\}_{n=1}^{\infty} = K$. Then C(K) fails the oo-pre-Komlós property.

Proof. Define $f_k(t)$ on $\{t_n\}_{n=1}^k \cup \{t'_n\}_{n=1}^k$ to be equal to 1 if $t = t_1, ..., t_k$ and $f_k(t) = 0$ if $t = t'_1, ..., t'_k$. Then extend each $f_k(t)$ continuously to whole K so that $f_k(K) \subseteq [0, 1]$. It is easy to see that, for any subsequence f_{k_j} the sequence

$$g_m = \frac{1}{m} \sum_{j=1}^m f_{k_j} \quad (m \in \mathbb{N})$$

of Cesáro means is not *oo*-Cauchy.

Theorem 3. Let K be a compact Hausdorff space with a nonempty separable open subset $U \subset K$ without isolated points such that every $u \in U$ has countable neighborhood base. Then C(K) fails the oo-pre-Komlós property.

Proof. Let $D = \{d_k\}_{k=1}^{\infty}$ be a countable dense subset of a nonempty open subset $U \subset K$ without isolated points. Without lost of generality, we may choose countable neighborhood bases $B_d = \{U_d^n\}_{n=1}^{\infty}$ of elements $d \in D$ such that

$$U_d^{n+1} \subseteq U_d^n \quad (\forall d \in D, n \in \mathbb{N})$$

and $d_m \notin U_{d_k}^k$ for m < k.

We choose a sequence $\{d_{n_k}\}_{k=1}^{\infty}$ of distinct elements of D as follows

$$d_{n_1} \in U_{d_1}^1, d_{n_2} \in U_{d_1}^2, d_{n_3} \in U_{d_2}^2, d_{n_4} \in U_{d_1}^3, d_{n_5} \in U_{d_2}^3, d_{n_6} \in U_{d_3}^3, \dots$$

$$\square$$

It is an easy exercise to show that $\operatorname{cl}\{d_{n_{2k-1}}\}_{k=1}^{\infty} = \operatorname{cl}\{d_{n_{2k}}\}_{k=1}^{\infty} = \operatorname{cl} U$.

By Theorem 2, there is a sequence $f_k \in \operatorname{cl} U$ such that $f_k(\operatorname{cl} U) \subseteq [0,1]$ for all k with the property that for any subsequence f_{k_i} the sequence

$$g_m = \frac{1}{m} \sum_{j=1}^m f_{k_j} \quad (m \in \mathbb{N})$$

of Cesáro means is not *oo*-Cauchy in $C(\operatorname{cl} U)$. Now, extend f_k to $\overline{f}_k \in C(K)$ so that $\overline{f}_k(K) \subseteq [0, 1]$ for all k. Clearly, for any subsequence \overline{f}_{k_i} the sequence

$$y_m = \frac{1}{m} \sum_{j=1}^m \bar{f}_{k_j} \quad (m \in \mathbb{N})$$

is not *oo*-Cauchy in C(K).

Corollary 2. For a compact metric space K possessing a nonempty separable open subset without isolated points, the Banach lattice C(K) fails the oo-pre-Komlós property.

Note that $\ell_{\infty} \cong C(\beta\mathbb{N})$ has *oo*-Komlós property (cf. e.g., [5, Ex.5.11]) From the other hand side, [5, Ex.5.2] implies that $\ell_{\infty}(\Gamma)$ fails the *oo*-pre-Komlós property, whenever $Card(\Gamma) \geq c$.

4. Komlós sets

The converse of the Komlós Theorem 1 has been proved in [8, Thm.2.1], namely Lennard has proved that: for a probability measure μ , every convex $C \subset L_1(\mu)$ must be norm bounded provided that C satisfies the property: for every sequence x_n in C there exist a subsequence x_{n_k} and an $x \in E$, with

$$x = uo - \lim_{m \to \infty} \frac{1}{m} \sum_{j=1}^{m} x_{n_{k_j}}$$

for any subsequence $x_{n_{k_j}}$ of x_{n_k} . Subsets of any Banach lattice E satisfying above property are called *Komlós sets* in [5, Def.5.22]. This motivates the following definition.

Definition 2. We say that $C \subset E$ is an o-, uo-, n, or un-Komlós set (respectively, o-, uo-, n-, or un-pre-Komlós set) if, for every sequence x_n in C, there exist a subsequence x_{n_k} and $x \in E$ such that, for any further subsequence $x_{n_{k_j}}$, the sequence $g_m = \frac{1}{m} \sum_{j=1}^m x_{n_{k_j}}$ is o-, uo-, n-, or un-convergent to x (respectively, g_m is o-, uo-, n-, or un-Cauchy).

The main result of paper [5] concerning Komlós sets [5, Thm.5.23] can be read as follows.

Theorem 4 (Gao–Troitsky–Xanthos). Let E be a Banach lattice with the projection property. If E_{oc} is a norming subspace of E^* , then any convex uo-Komlós set C in E is norm bounded.

Below, in Proposition 4 we show that in arbitrary Banach lattice E every convex *uo*-Komlós set $C \subseteq E_+$ is norm bounded.

The following result shows that in [8, Thm.2.1] and in [5, Thm.5.23] *uo*-Komlós sets can not be replaced by *uo*-pre-Komlós set.

Theorem 5. In any infinite dimensional Banach lattice E, there is an unbounded convex uo-pre-Komlós set $C \subseteq E_+$.

Proof. By [5, Cor.3.6, Cor.3.13], any disjoint sequence is a *uo*-Komlós set. Take a disjoint sequence d_n in E_+ such that $||d_n|| = n$ for all n. Let $x_i = \sum_{n=1}^{\infty} \alpha_n^i d_n$ be a sequence in the convex hull $C = \operatorname{co}\{d_n\}_{n=1}^{\infty}$. By diagonal argument, it is easy to find a subsequence x_{i_i} satisfying

$$\lim_{n \to \infty} \alpha_n^{i_j} = \beta_n \quad (\forall n). \tag{1}$$

By choosing further subsequence, if necessary, we may suppose that

$$\alpha_n^{i_j} - \beta_n | \le 1 \quad (\forall j \ge n).$$
⁽²⁾

For $u = o - \sum_{n=1}^{\infty} nd_n$, $y = o - \sum_{n=1}^{\infty} \beta_n d_n$ in the universal completion E^u (cf. [1, Def.7.20]) of E, by using (1) and (2) we get that

$$|x_{i_j} - y| \le \frac{1}{n}u\tag{3}$$

for big enough j. In view of (3), the sequence x_{i_j} o-converges to $y = o - \sum_{n=1}^{\infty} \beta_n d_n$ in E^u , and hence is *uo*-Cauchy in E by [5, 3.12]. It follows that every further subsequence $x_{i_{j_l}}$ is *uo*-Cauchy, and hence, by [5, 3.13], the sequence $g_m = \frac{1}{m} \sum_{l=1}^m x_{i_{j_l}}$ of its Cesáro means is *uo*-Cauchy as well.

Thus, C is a norm unbounded convex uo-pre-Komlós set. \Box

Proposition 4. In any Banach lattice E every convex uo-Komlós set $C \subseteq E_+$ is norm bounded.

Proof. Let $C \subseteq E_+$ be a norm unbounded convex set. We are going to show that C is not *uo*-Komlós.

Choose a sequence $x_n \in C$ so that $||x_1|| \ge 4$ and

$$\frac{1}{2^n} \|x_n\| \ge 2 \sum_{k=1}^{n-1} \frac{1}{2^k} \|x_k\| \quad (\forall n > 1).$$

Define an increasing sequence $z_n \in E_+$ as follows:

$$z_n = \sum_{k=1}^n \frac{1}{2^k} x_k.$$

Clearly, every subsequence z_{n_k} of z_n is not *uo*-convergent, since otherwise $z_{n_k} \xrightarrow{\text{uo}} z \in E_+$ and $z - z_{n_k} = |z - z_{n_k}| \wedge z \xrightarrow{\text{o}} 0$, which means that $z_{n_k} \uparrow z$, and hence $||z_{n_k}|| \leq ||z|| < \infty$, violating

$$||z_n|| \ge \frac{1}{2^n} ||x_n|| \ge 2\sum_{k=1}^{n-1} \frac{1}{2^k} ||x_k|| \ge 2||z_{n-1}|| \ge 2^n \quad (\forall n \ge 1).$$

The sequence $w_m = \frac{1}{m} \sum_{j=1}^m z_{n_k}$ is also increasing and $||w_m|| \to \infty$. The similar argument shows that w_m is not *uo*-convergent. Let

$$y_n = \frac{1}{2^n} x_1 + z_n \in C \quad (\forall n \ge 1).$$

Since $\frac{1}{m} \sum_{k=1}^{m} \frac{1}{2^{n_k}} x_1 \leq \frac{1}{m} x_1 \xrightarrow{\mathbf{o}} 0$, the sequence of the Cesáro means

$$\frac{1}{m}\sum_{k=1}^{m} y_{n_k} = \frac{1}{m}\sum_{k=1}^{m} \frac{1}{2^{n_k}}x_1 + \frac{1}{m}\sum_{k=1}^{m} z_{n_k} = w_m + \frac{1}{m}\sum_{k=1}^{m} \frac{1}{2^{n_k}}x_1 \quad (m \in \mathbb{N})$$

is not uo-convergent. It shows that C is not uo-Komlós.

Note that the similar argument, as in the proof of Proposition 4, shows that in any Banach lattice E every convex un-Komlós set $C \subseteq E_+$ is norm bounded.

The next result follows directly from Theorem 5 and Proposition 4.

Theorem 6. Let E be an infinite dimensional Banach lattice. Then there exists a norm unbounded convex uo-pre-Komlós set $C \in E_+$ which is not uo-Komlós.

Corollary 3. Let E be an Banach lattice. The following conditions are equivalent:

- (1) $\dim(E) < \infty;$
- (2) E is uo-complete;
- (3) E is sequentially uo-complete;
- (4) every uo-pre-Komlós set is uo-Komlós;

(5) every convex uo-pre-Komlós set $C \in E_+$ is norm bounded.

Proof. Implications $(1) \Rightarrow (2) \Rightarrow (3)$ are trivial, and $(3) \Rightarrow (4)$ easily follows from Definition 2.

 $(4) \Rightarrow (5)$: Let $C \in E_+$ be a convex *uo*-pre-Komlós set. Then C is *uo*-Komlós by the assumption. Proposition 4 ensures that C is norm bounded. $(5) \Rightarrow (1)$: It is exactly Theorem 5.

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