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FINITE TYPE POINTS ON SUBSETS OF $\mathbb{C}^n$

OZCAN YAZICI

Abstract. In [4], D'Angelo introduced the notion of points of finite type for a real hypersurface $M \subset \mathbb{C}^n$ and showed that the set of points of finite type in M is open. Later, Lamel-Mir [8] considered a natural extension of D'Angelo's definition for an arbitrary set $M \subset \mathbb{C}^n$. Building on D'Angelo's work, we prove the openness of the set of points of finite type for any subset $M \subset \mathbb{C}^n$.

1. Introduction

Let M be a smooth real hypersurface of $\mathbb{C}^n$, $p \in M$ and r be a defining function for M such that $dr(p)$ does not vanish. In his fundamental work [4], D'Angelo introduced the notion of type of M at p with respect to (possibly singular) holomorphic curves. More precisely, the type of M at p is defined by

$$\Delta(M,p) = \sup_{\gamma \in \mathcal{C}} \frac{\nu(r \circ \gamma)}{\nu(\gamma)}$$

where $\mathcal{C}$ is the set of non-constant holomorphic germs γ at $0 \in \mathbb{C}$ so that $\gamma(0) = p$ and $\nu(r \circ \gamma)$ denotes the order of vanishing of the function $r \circ \gamma$ at 0. A point p is called finite type if $\Delta(M,p) < \infty$. In [4], D'Angelo proved the crucial property that the set of points of finite type forms an open subset of the hypersurface M. This condition of finite type has been central in Catlin's work [3] on subelliptic estimates for the $\bar{\partial}$-Neumann problem. See [6] for a more recent discussion of the relationship between finite type and subellipticity.

More recently, in their study of the C^∞ regularity problem for CR maps between smooth CR manifolds, Lamel and Mir [8] considered finite type points for arbitrary sets M in $\mathbb{C}^n$. Building on D'Angelo's notion, they defined the type of an arbitrary set $M \subset \mathbb{C}^n$ as follows:

$$\Delta(M,p) = \sup_{\gamma \in \mathcal{C}} \inf_{r \in I_M(p)} \frac{\nu(r \circ \gamma)}{\nu(\gamma)}$$

where $I_M(p)$ is the set of germs at p of real valued C^∞ functions defined in a neighborhood of p and vanishing on M near p. This definition of type coincides with the D'Angelo's definition for hypersurfaces, since the ideal $I_M(p)$ is generated by the defining function of M when M is a hypersurface.

As before, if $p \in M$, we say that p is a point of finite type if $\Delta(M,p) < \infty$. The goal of this note is to show that D'Angelo's arguments can be used to establish the openness of the set of points of finite type for any set $M \subset \mathbb{C}^n$.

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Let $\mathcal{O}_p$ denote the ring of holomorphic germs at p in $\mathbb{C}^n$. Following [4], for a proper ideal $I \subset \mathcal{O}_p$, we define

$$\tau(I) = \sup_{\gamma\in\mathcal{C}} \inf_{\varphi\in I} \frac{\nu(\varphi\circ\gamma)}{\nu(\gamma)} \text{ and } D(I) = \dim_{\mathbb{C}}(\mathcal{O}_p/I)$$

If I contains q independent linear functions, then it follows from Theorem 2.7 in [4] that

$$\tau(I) \leq D(I) \leq (\tau(I))^{n-q}.$$

For any $r \in I_M(p)$ and $k \in \mathbb{Z}^+$, we denote by r_k the Taylor polynomial of r of order k at p. We define

$$\Delta(M_k, p) = \sup_{\gamma\in\mathcal{C}} \inf_{r\in I_M(p)} \frac{\nu(r_k\circ\gamma)}{\nu(\gamma)}.$$

Here $\nu(r_k \circ \gamma)$ denotes the multiplicity of $r_k \circ \gamma(t) - r_k(p)$ at 0. By Proposition 3.1 in [4], we can decompose the polynomial r_k as $r_k = \mathsf{Re}\,(h^{r,k}) + ||f^{r,k}||^2 - ||g^{r,k}||^2$ where $h^{r,k}$ is a holomorphic polynomial and $f^{r,k} = (f_1^{r,k}, \dots, f_N^{r,k})$ and $g^{r,k} = (g_1^{r,k}, \dots, g_N^{r,k})$ are holomorphic polynomial mappings. Here $N = N_k$ is independent of the polynomial and only depends on the degree k and n. Let $\mathcal{U}(N_k)$ denote the group of unitary matrices on $\mathbb{C}^{N_k}$. For any $U \in \mathcal{U}(N_k)$, we denote by $I(U, k, p)$, the ideal generated by the set

$$\{h^{r,k}, \text{ components of } f^{r,k} - Ug^{r,k} : r \in I_M(p)\}.$$

We should note that the decomposition of r_k is not unique and $I(U, k, p)$ depends on the choice of decomposition. Here, we use the one in the proof of Proposition 3.1 in [4].

Lemma 1.1. *Let $M \subset \mathbb{C}^n$ be a subset containing p. If $\Delta(M_k, p) < k$ for some positive integer k, then $\Delta(M, p) = \Delta(M_k, p)$.*

Proof. For any curve $\gamma \in \mathcal{C}$ and $r \in I_M(p)$ we write

$$r\circ\gamma = r_k\circ\gamma + e_k\circ\gamma \tag{1.1}$$

$$\nu(e_k(\gamma)) \geq (k+1)\nu(\gamma) \tag{1.2}$$

where $e_k = r - r_k$. Since $\Delta(M_k, p) < k$, for all $\gamma \in \mathcal{C}$ there exists $r^0 \in I_M(p)$ such that $\frac{\nu(r_k^0\circ\gamma)}{\nu(\gamma)} < k$. Then (1.1) and (1.2) imply that $\frac{\nu(r^0\circ\gamma)}{\nu(\gamma)} < k$. By taking infimum over r and supremum over $\gamma \in \mathcal{C}$ we obtain that $\Delta(M, p) \leq k$. For all $\gamma \in \mathcal{C}$ and $\epsilon > 0$ small enough, we can find $r^1 \in I_M(p)$ such that

$$\frac{\nu(r^1\circ\gamma)}{\nu(\gamma)} < \inf_{r\in I_M(p)} \frac{\nu(r\circ\gamma)}{\nu(\gamma)} + \epsilon < k+1.$$

It follows from (1.1) and (1.2) that

$$\frac{\nu(r_k^1\circ\gamma)}{\nu(\gamma)} = \frac{\nu(r^1\circ\gamma)}{\nu(\gamma)} < \inf_{r\in I_M(p)} \frac{\nu(r\circ\gamma)}{\nu(\gamma)} + \epsilon.$$

As $\epsilon > 0$ is arbitrary, $\inf_{r\in I_M(p)} \frac{\nu(r_k\circ\gamma)}{\nu(\gamma)} \leq \inf_{r\in I_M(p)} \frac{\nu(r\circ\gamma)}{\nu(\gamma)}$. By taking supremum over $\gamma \in \mathcal{C}$, we get that $\Delta(M_k, p) \leq \Delta(M, p)$.

We will show the other side of inequality in a similar way. Since we assume that $\Delta(M_k, p) < k$, for all $\gamma \in \mathcal{C}$ and $\epsilon > 0$ there exists a $r^2 \in I_M(p)$ such that

$$\frac{\nu(r_k^2 \circ \gamma)}{\nu(\gamma)} < \inf_{r \in I_M(p)} \frac{\nu(r_k \circ \gamma)}{\nu(\gamma)} + \epsilon < k.$$

It follows from (1.1) and (1.2) that

$$\frac{\nu(r^2 \circ \gamma)}{\nu(\gamma)} = \frac{\nu(r_k^2 \circ \gamma)}{\nu(\gamma)} < \inf_{r \in I_M(p)} \frac{\nu(r_k \circ \gamma)}{\nu(\gamma)} + \epsilon.$$

As $\epsilon > 0$ is arbitrary, $\inf_{r \in I_M(p)} \frac{\nu(r \circ \gamma)}{\nu(\gamma)} \leq \inf_{r \in I_M(p)} \frac{\nu(r_k \circ \gamma)}{\nu(\gamma)}$. By taking supremum over $\gamma \in \mathcal{C}$, we get that $\Delta(M, p) \leq \Delta(M_k, p)$. □

Lemma 1.2. *Let $M \subset \mathbb{C}^n$ be a subset containing p. Then*

$$\sup_{U \in \mathcal{U}(N_k)} \tau(I(U, k, p)) \leq \Delta(M_k, p) \leq 2 \sup_{U \in \mathcal{U}(N_k)} \tau(I(U, k, p)).$$

Proof. For any $r \in I_M(p)$, we can write $r_k = \mathsf{Re}\,(h^{r,k}) + ||f^{r,k}||^2 - ||g^{r,k}||^2$ where $h^{r,k}$ is a holomorphic polynomial function and $f^{r,k}$ $g^{r,k}$ are holomorphic polynomial mappings. We denote by $I(r, U, k, p)$ the ideal generated by $h^{r,k}$ and the components of $f^{r,k} - Ug^{r,k}$ where $U \in \mathcal{U}(N_k)$. Since the supremum over $\gamma \in \mathcal{C}$ in the definition of $\tau(I(r, U, k, p))$ is attained for some $\gamma \in \mathcal{C}$ such that $h^{r,k} \circ \gamma = 0$, it is enough to work with such curves.(See the proof of Theorem 1 on page 127 in [5]). As in the proof of Theorem 3.4 in [6],

$$\nu((f^{r,k} - Ug^{r,k}) \circ \gamma) \leq \nu(r_k \circ \gamma)$$

for any $\gamma \in \mathcal{C}$ such that $h^{r,k} \circ \gamma = 0$. This implies that,

$$\inf_{\varphi \in I(r,U,k,p)} \nu(\varphi \circ \gamma) \leq \nu(r_k \circ \gamma).$$

Dividing by $\nu(\gamma)$ and taking infimum over $r \in I_M(p)$, supremum over $\gamma \in \mathcal{C}$ and over $U \in \mathcal{U}(N_k)$, we obtain that

$$\sup_{U \in \mathcal{U}(N_k)} \tau(I(U, k, p)) \leq \Delta(M_k, p).$$

Let $r \in I_M(p)$, $\gamma \in \mathcal{C}$ and $2l + 1 \leq \nu(r_k \circ \gamma) \leq 2l + 2$ for some $l \in \mathbb{Z}^+$. By Theorem 3.5 [4], there exists $U \in \mathcal{U}(N_k)$ such that $(h^{r,k} \circ \gamma)_{2l} = ((f^{r,k} - Ug^{r,k}) \circ \gamma)_l = 0$ where $(h^{r,k} \circ \gamma)_{2l}$ and $((f^{r,k} - Ug^{r,k}) \circ \gamma)_l$ are the Taylor polynomials of $h^{r,k} \circ \gamma$ and the components of $(f^{r,k} - Ug^{r,k}) \circ \gamma$ of degree $2l$ and l, respectively. This implies that

$$\frac{\nu(r_k \circ \gamma)}{\nu(\gamma)} \leq 2 \inf_{\varphi \in I(r,U,k,p)} \frac{\nu(\varphi \circ \gamma)}{\nu(\gamma)}. \tag{1.3}$$

When $\nu(r_k \circ \gamma) = \infty$, by Theorem 3.5 in [4], there exists a $U \in \mathcal{U}(N_k)$ such that $\nu(\varphi \circ \gamma) = \infty$ for all $\varphi \in I(r, U, k, p)$ and the inequality (1.3) still holds. By taking infimum over $r \in I_M(p)$ and supremum over $\gamma \in \mathcal{C}$ and over $U \in \mathcal{U}(N_k)$, we obtain that $\Delta(M_k, p) \leq 2 \sup_{U \in \mathcal{U}(N_k)} \tau(I(U, k, p))$. □

Theorem 1.3. *Let M be subset of $\mathbb{C}^n$ and p_0 be a point of finite type. Then there exists a neighborhood V_0 of p_0 such that*

$$\Delta(M,p) \leq 2(\Delta(M,p_0))^n, \tag{1.4}$$

for all $p \in V_0$. In particular, the set of points of finite type is an open subset of M.

Proof. We note that the coefficients of the generators of $I(U,k,p)$ depend smoothly on p. Then by Proposition 2.15 in [4], $D(I(U,k,p))$ is an upper-semicontinuous function of p. Since $\mathcal{U}(N_k)$ is compact, upper-semicontinuity of D implies that there exists a neighborhood V_0 of p_0 such that

$$\sup_{U\in\mathcal{U}(N_k)} D(I(U,k,p)) \leq \sup_{U\in\mathcal{U}(N_k)} D(I(U,k,p_0))$$

for all $p \in V_0$. By Theorem 2.7 in [4],

$$\tau(I(U,k,p_0)) \leq D(I(U,k,p_0)) \leq (\tau(I(U,k,p_0)))^n. \tag{1.5}$$

By Lemma 1.2 we have $\Delta(M_k,p) \leq 2\sup_{U\in\mathcal{U}(N_k)} \tau(I(U,k,p)$. Now we have the following chain of inequalities. For all $p \in V_0$,

$$\begin{aligned}
\Delta(M_k,p) &\leq 2\sup_{U\in\mathcal{U}(N_k)} \tau(I(U,k,p)) \leq 2\sup_{U\in\mathcal{U}(N_k)} D(I(U,k,p) \\
&\leq 2\sup_{U\in\mathcal{U}(N_k)} D(I(U,k,p_0) \leq 2\left(\sup_{U\in\mathcal{U}(N_k)} \tau(I(U,k,p_0))\right)^n \\
&\leq 2(\Delta(M_k,p_0))^n = 2(\Delta(M,p_0))^n.
\end{aligned}$$

The fifth inequality above follows from Lemma 1.2 and the last equality follows from Lemma 1.1 for k large enough. For any $k > 2(\Delta(M,p_0))^n$, by Lemma 1.1,

$$\Delta(M,p) = \Delta(M_k,p) \leq 2(\Delta(M,p_0))^n,$$

for all $p \in V_0$. □

Remark 1.4. If M is contained in a generic submanifold of real codimension d, then we have a better estimate than (1.4). In that case, there are d local defining functions which vanish on M near p_0 and their complex differentials are linearly independent in a neighborhood V_1 of p_0. After a change of coordinates on V_1, we may assume that $I(U,k,p_0)$ contains d independent linear functions. Thus by Theorem 2.7 in [4],

$$D(I(U,k,p_0)) \leq (\tau(I(U,k,p_0)))^{n-d}. \tag{1.6}$$

We should note that after local biholomorphic change of coordinates, although the ideal $I(U,k,p_0)$ changes, the inequality (1.6) still holds. Indeed, Corollary 3 on page 65 in [5] implies that $D(I)$ is invariant under a local biholomorphic change of coordinates. Also, $\tau(I)$ is invariant under a local biholomorphic change of coordinates, (see the remarks after Definition 8 on page 72 in [5]).Then it follows from a similar chain of inequalities as above that $\Delta(M,p) \leq 2(\Delta(M,p_0))^{n-d}$ for all $p \in V := V_0 \cap V_1$.

Remark 1.5. If a type property fails for hypersurfaces, it also fails in higher codimension. For example, in [5], D'Angelo gave an example of the hypersurface

$$M = \{z \in \mathbb{C}^3 : 2\mathsf{Re}\,(z_3) + |z_1^2 - z_2 z_3|^2 + |z_2|^4 = 0\}$$

to show that the type $\Delta(M,p)$ is not an upper semicontinuous function of p. Here $\Delta(M,0) = 4$ and $\Delta(M,p) = 8$ where $p = (0,0,ia)$. Following this example, we consider the set

$$M' = \{z \in \mathbb{C}^4 : \mathsf{Re}\,(z_4) = 2\mathsf{Re}\,(z_3) + |z_1^2 - z_2 z_3|^2 + |z_2|^4 = 0\}$$

of real codimension 2 in $\mathbb{C}^4$ which has the same types as M. Hence, upper semicontinuity of type fails in higher codimension as well.

2. Points of finite q-type

In [4], D'Angelo defined the q-type of a hypersurface $M \subset \mathbb{C}^n$, which possibly contains $q-1$ dimensional complex analytic varieties. In a natural way, we can define q-type of an arbitrary subset $M \subset \mathbb{C}^n$. The q-type of M at $p \in M$ is defined by

$$\Delta^q(M,p) = \inf\{\Delta(M \cap P, p) : P \text{ is any } n-q+1 \text{ dimensional complex affine subspace of } \mathbb{C}^n\}.$$

More precisely,

$$\Delta^q(M,p) = \inf\{\Delta(\varphi^{-1}(M), \varphi^{-1}(p)) : \varphi : \mathbb{C}^{n-q+1} \to \mathbb{C}^n \text{ is any linear imbedding}\}.$$

In [3], Catlin defined q-type, $D_q(M,p)$, of a hypersurface M at p by considering generic $(n-q+1)$-dimensional complex affine subspaces of $\mathbb{C}^n$ through p. When $q = 1$, $D_1(M,p) = \Delta^1(M,p) = \Delta(M,p)$. For a long time, Catlin's q-type, $D_q(M,p)$, and D'Angelo's q-type, $\Delta^q(M,p)$, were believed to be equal. In [7], Fassina gave examples of ideals and hypersurfaces to show that these two types are different when $q \geq 2$. This result also points out some mistakes in [1], where the authors claimed that infimum in the definition of $\Delta^q(M,p)$ is achieved by the generic value with respect to choices of $(n-q+1)$-dimensional complex affine subspaces of $\mathbb{C}^n$. Later, in [2], the authors corrected their results in [1], by replacing $\Delta^q(M,p)$ with another invariant β_q which is defined in terms of the generic value.

With the D'Angelo's definition of q-type, openness of the set of points of finite q-type easily follows from Theorem 1.3.

Corollary 2.1. *Let M be a subset of $\mathbb{C}^n$. The set of points on M for which $\Delta^q(M,p) < \infty$ is an open subset of M.*

Proof. It follows from the definition of q-type that

$$\{p \in M : \Delta^q(M,p) < \infty\} = \cup_\varphi A_\varphi$$

where the union is taken over any imbedding $\varphi : \mathbb{C}^{n-q+1} \to \mathbb{C}^n$ and

$$A_\varphi = \{z \in M : \Delta(\varphi^{-1}(M), \varphi^{-1}(p)) < \infty\}.$$

Let $p_0 \in A_\varphi$. By Theorem 1.3, there exists an open subset $U \subset \mathbb{C}^{n-q+1}$ around $\varphi^{-1}(p_0)$ such that, for all $w \in U \cap \varphi^{-1}(M)$, $\Delta(\varphi^{-1}(M), w) < \infty$. Since φ is an imbedding, $\varphi(U) = V \cap \varphi(\mathbb{C}^{n-q+1})$ for some open subset $V \subset \mathbb{C}^n$ containing p_0. This implies that $M \cap V \subset A_\varphi$ and A_φ is open in M. Thus $\{p \in M : \Delta^q(M,p) < \infty\}$ is an open set in M. $\square$

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Department of Mathematics, Middle East Technical University, 06800 Ankara, Turkey
E-mail address: oyazici@metu.edu.tr